Stake-governed tug-of-war
and the biased infinity Laplacian

Alan Hammond and Gábor Pete

June 15, 2022

Abstract

We introduce a two-person zero-sum game that we call stake-governed tug-of-war. The game is a development of the classic tug-of-war random-turn game that [PSSW09] analysed. In tug-of-war, two players compete by iteratively moving a counter along adjacent edges of a graph, each winning the right to move at a given turn according to the outcome of the flip of a fair coin; a payment is made from one player to the other when the counter reaches a boundary set on whose vertices the terminal payment value is specified. The player Mina who makes the payment seeks to minimize its mean; her opponent Maxine seeks to maximize it. The game’s value is the infinity harmonic extension of the payment boundary data. In the stake-governed version, Mina and Maxine each receive a limited budget at the outset. At the start of any given turn, each stakes an amount drawn from her present budget, and the right to move at the turn is won randomly by a player with probability equal to the ratio of her stake and the combined stake just offered. For certain graphs, we present the solution of a leisurely version of the stake-governed game, in which, after stakes are offered at a given turn, the upcoming move is cancelled, independently of other randomness, with probability $1 - \epsilon \in (0, 1)$. With the parameter $\epsilon$ small enough, and for finite trees whose leaves are the boundary set and whose payment function is the indicator on a given leaf, we determine the value of the game and the set of Nash equilibria. When the ratio of the initial fortunes of Maxine and Mina is $\lambda \in (0, \infty)$, Maxine wins each turn with a constant probability $\lambda / (\lambda + 1)$ under optimal play, and game value is a biased version $h(\lambda, v)$ of the infinity harmonic function; each player stakes a shared and non-random proportion of her present fortune, for which proportion we derive a formula in terms of the spatial gradient and $\lambda$-derivative of $h(\lambda, v)$. We also indicate examples in which the solution takes a different form when the parameter $\epsilon$ equals one.

Contents

1 Introduction .......................................................... 2
  1.1 The classical tug-of-war game .................................. 4
  1.2 The stake-governed game ...................................... 5
  1.3 The rules of the game ........................................... 6
  1.4 Principal definitions ........................................... 8
  1.5 Main results ...................................................... 10

2 The big picture: the global saddle hope and the Poisson game 13
  2.1 Strategy mimicry: proving Proposition 2.1 .................................. 14
  2.2 The stake formula argued via perturbation ..................... 16
1 Introduction

We introduce and analyse a class of two-player zero-sum games with several turns of random outcome in which each player spends a precious resource in order to improve her chances of winning the right to move. Each may be tempted to spend big at the next turn, in an effort to secure a gain in position; but each must also restrain this urge because enough funds must be held over that later turns may plausibly be won. Our game is a direct descendent of the random turn games of Peres, Schramm, Sheffield and Wilson [PSSW07, PSSW09]; nevertheless, let us begin with a brief but wider overview of some similar ideas found in the game theory literature.

Differential games. Rufus Isaacs [Isa65] introduced a class of two-person games in continuous time that he called differential games. In an archetypal example, an evader $E$ and a pursuer $P$ occupy distinct point positions in the plane at an initial time. It is the pursuer’s aim to capture the evader, in the shortest time possible; it is the evader’s aim to ensure that capture, if it must occur at all,
takes place after the greatest possible delay. The pursuer is faster but more ponderous than the evader: $P$’s constant speed is higher than $E$’s, but $P$ may alter his direction only gradually. In such differential games, state variables describe the present state of play: the locations of $P$ and $E$, in this example. The respective players may instantaneously adjust the values of certain control variables, such as their directions of movement; $P$ must respect constraints, say on the derivative of directional angle. The evolution of the state variables is given by kinematic differential equations specified in terms of the choices of the controls. Isaacs proved in many examples that optimal strategies are solutions of a minimax problem that may be expressed in terms of a PDE that now bears his name.

**Stochastic games.** Lloyd Shapley [Sha53] studied two-person zero-sum games in which play proceeds step-by-step from position to position according to transition probabilities controlled jointly by the players. At each of finitely many positions, each player may select from a finite collection of options; the position then changes according to probabilities dictated by the present position and the two selected options; among these positions is a terminal state, which is selected with positive probability, with the game ending when this state is reached. At each turn, a payment is made from the second player to the first, also as a function of present position and the pair of selected options. Shapley used the Banach fixed point theorem to exhibit the existence of the game’s value and he proved the existence of optimal strategies. Both differential and stochastic games are the object of much recent study: see several of the chapters in a recent handbook [Han20] treating dynamic game theory, and a collection [Sto91] of contributions concerning stochastic games.

**Richman games.** In many recreational or combinatorial games, two players make alternating moves. In the late 1980s, David Richman proposed a variant in which players bid for the right to make the next move. The players each have some money at the start of the game. At each move, they bid some part of their reserves; the player who bids more wins the right to move, but pays this higher bid amount to the opposing player. (Should the bids be equal, the outcome may be decided by a fair coin flip.) In the late 1990s, optimal bid amounts and conditions for victory were identified by [LLPU94, LLP+99] in terms of the Richman cost function, a min/max average related to the upcoming \[.\]

**Random-turn selection games.** Hex is an alternating move two-player game that was invented by Piet Hein in 1942 and rediscovered by John Nash in 1948. A finite domain in the hexagonal lattice is delimited by four boundary segments, consecutively labelled red–blue–red–blue. Red and blue players alternately place like-coloured hexagons on as-yet-unplayed faces in the domain, in an effort to secure a crossing in the given colour between the opposing boundary segments of that colour. It is a classical fact that, in a suitably symmetric domain, the player who moves first has a winning strategy, but this strategy is unknown except on the smallest of gameboards; indeed, the game is played on boards of given height and width such as eleven in international competitions. In 2007, Peres, Schramm, Sheffield and Wilson [PSSW07] introduced a variant of Hex, in which the right to move at any given turn is awarded to the red or blue player according to the flip of a fair coin. A simple and striking analysis reveals the existence and explicit form of the optimal strategies for this random turn variant of Hex. Namely, the victorious player at a given turn should choose the hexagon that is most likely to be pivotal for forging the desired path when the unplayed gameboard is completed by an independent critical percolation (where each unplayed face is coloured red or blue according to a fair coin flip). Thus, in jointly optimal play, the players always choose the same hexagon, coloured according to the winner of the coin flip, and hence the gameplay can be
considered as a carefully optimized random order in which an unknown fair random colouring of the board is revealed. This form of the explicit strategy holds true for any random turn selection game (i.e., where the winner is given by a Boolean function of the two-colourings of a base set), and therefore, jointly optimal play is an adaptive algorithm (also called a randomized decision tree) to determine the output of a Boolean function on independent fair random input. These random-turn-game algorithms often appear to have interesting properties [PSSW07], such as low revealment, which is important, for instance, in proving sharp thresholds and noise sensitivity for Boolean functions (see [OSSS05, DCRT19, SS10, GS14]). However, these algorithmic connections are far from being well-understood in general.

Random tug-of-war. In [PSSW09], Peres, Schramm, Sheffield and Wilson introduced a further two-person random-turn game, which they called tug-of-war. On a finite or infinite graph, a counter is randomly moved at each turn to a neighbour of the current location, and a payment is made from one player to the other when the counter reaches a fixed subset of the vertices, called the boundary, according to a boundary condition that is imposed there. A particularly interesting case is when the vertices are the points in a bounded domain in Euclidean space, and the adjacency relation is given by being closer than a given small distance \( \delta > 0 \). The existence of the game-theoretic value of tug-of-war on finite graphs, and in the Euclidean setting, was proved in [PSSW09]. Moreover, in the Euclidean setting, in the \( \delta \to 0 \) limit, this value, viewed as a function of the initial counter location, was shown to be an infinity harmonic function. Namely, this value \( h \) solves the equation \( \Delta_\infty h = 0 \) on the domain, where the operator \( \Delta_\infty \) is the infinity Laplacian. This is a degenerate second-order elliptic operator; formally at least, a solution \( h \) of the stated equation has vanishing second derivative in the direction of its gradient. The infinity Laplacian is a subtle and beautiful object, whose study was given impetus by this game theoretic perspective. There is a notion of viscosity solution for \( \Delta_\infty h = 0 \) that is a form of weak solution governed by the maximum principle and that is characterized as the absolutely Lipschitz minimizing function that interpolates given boundary data. Solutions enjoy uniqueness [Jen93] but limited regularity [Sav03, ES08]. The game theoretic point-of-view that [PSSW09] offers led to simpler and intuitive proofs of uniqueness [AS12]; this point-of-view may be expressed by a dynamic programming principle [MPR12]. The abundant connections between tug-of-war and PDE are reviewed in the book [BR19].

Stake-governed tug-of-war. Here we introduce a development of the classical tug-of-war game, which may be interpreted as incorporating the bidding mechanism of Richman games in a probabilistic guise. Two players compete iteratively for the right to move the counter by spending limited resources allocated to them at the outset. At each turn, they will offer stakes, drawn from their reserves: these choices are entirely a matter for the respective players, and play the role of control variables in the context of differential games; while the counter’s location and the players’ relative fortune describe the state of play and may be taken to be the state variables in Isaacs’ framework. This agency granted to the players is akin to that seen in Shapley’s stochastic games. By analysing the new game, we will explore how the geometry of a spatial game influences players’ decisions and the strength of their positions. Such themes have been explored for iterative network-bargaining problems in [KT08, ABC+09, KBB+11].

1.1 The classical tug-of-war game

Tug-of-war may be played on domains in Euclidean space, or on graphs. The discrete context is our object of attention in this article and we start by reviewing the simple definition of tug-of-war for
finite graphs.

**Definition 1.1.** Let a finite graph \( G = (V, E) \) be given, with the vertex set \( V \) written as the disjoint union of two non-empty sets: the field of *open* play, \( V_O \), and the set of *boundary* vertices, \( V_B \). The payoff function is a given map \( p : V_B \to [0, \infty) \). We will refer to the triple \( (V, E, p) \) as a *boundary-payment graph.*

We specify tug-of-war with a biased coin, and its counter evolution \( X : [0, F] \to V \). (We denote by \( [i, j] \) the integer interval \( \{k \in \mathbb{Z} : i \leq k \leq j\} \).)

**Definition 1.2.** The game \( \text{TugOfWar}(q, v) \) has parameters \( q \in [0, 1] \) and \( v \in V \). Set \( X_0 = v \), so that the counter starts at \( v \). Let \( i \in \mathbb{N}_+ \) (where \( \mathbb{N}_+ \) denotes the natural numbers excluding zero). At the start of the \( i \)th turn, the counter is at \( X_{i-1} \). A coin is flipped that lands head with probability \( q \).

If it lands heads, Maxine wins the right to move the counter to a vertex in \( V \) adjacent to its present location \( X_{i-1} \); if not, Mina wins the same right. The resulting location is \( X_i \). The finish time \( F \) is the minimum value of \( i \in \mathbb{N}_+ \) such that \( X_i \) lies in the boundary \( V_B \). The game ends when turn \( F \) does. A final payment of \( p(X_F) \) units is made by Mina to Maxine. (Our currency is the ‘unit’.)

The notion of game value will be recalled in Definition 1.4; in essence, it is the mean final payment when the game is correctly played. It is quite intuitive that the value \( V(q, v) \) of \( \text{TugOfWar}(q, v) \) exists and solves the system of equations

\[
V(q, v) = (1 - q) \min_{w \sim v} V(q, w) + q \max_{w \sim v} V(q, w),
\]

subject to \( V(q, w) = p(w) \) for \( w \in V_B \). Tug-of-war was considered in the symmetric case where \( q = 1/2 \) by [PSSW09]. The biased game, with \( q \neq 1/2 \), was treated by [PPS10, PS19]; we will later review the fast algorithm given in [PS19] for computing biased infinity harmonic functions (which solve the displayed equations).

### 1.2 The stake-governed game

Maxine wins each turn with probability \( q \in (0, 1) \) by fiat in \( \text{TugOfWar}(q, v) \). In many applications, decision makers face choices of how to spend a precious and limited resource in order to gain strategic advantage in an evolving random situation. Our development of tug-of-war offers a model for such choices. Maxine and Mina will each be given a certain amount of money at the start of *stake-governed tug-of-war.* Our convention will be that Maxine holds \( \lambda \) units and Mina one unit at this time. A counter is placed on the game-board \( (V, E) \) at \( v \in V \), as it was in classic tug-of-war. Each turn opens with a request that the two players offer stakes. These are amounts that each player selects, drawn from her remaining reserves. The amounts are withdrawn from the respective reserves and are not returned. Each player will win the right to move the counter to an adjacent location of her choosing with a probability that equals the ratio of the amount that she has just staked and the total amount just staked by the two players. The game ends as it does in tug-of-war, with the arrival of the counter in \( V_B \) (at \( w \), say). The remaining reserves of the players are swept from them and, as in tug-of-war, Mina pays Maxine \( p(w) \) units. Thus the initial funds allocated to the players are a non-renewable resource that each player must spend during the lifetime of the game in an effort to gain control on the resting place of the counter. The funds dictate the players’ capacity to control play and have no other utility, because they cannot be saved beyond the end of the game.
How should stake-governed tug-of-war be played? What is the value of this game? What is the relation of the new game to classic tug-of-war on a finite graph? These are the principal questions that we seek to address. Players may stake random amounts, but an inkling of a solution is offered by restricting to deterministic stakes. If Maxine always stakes the $\lambda$-multiple of the non-random stake offered by Mina, then the win probability at every turn is $\frac{1}{1+\lambda}$. The relative fortune of the players holds steady at $\lambda$; the game reduces to a copy of TugOfWar($q,v$) with $q = \frac{\lambda}{1+\lambda}$; and the game’s value is the last displayed $V(q,v)$ (which corresponds to $h(\lambda,v)$ in the notation we will adopt, as set out in Definition 1.6). So the answers to the questions seem simple, in the sense that the new game appears to project onto the old one. But the picture is more complicated. If Mina stakes randomly, then Maxine cannot reliably maintain the fortune ratio by the strategy of stake-proportion mimicry. And, if stakes are in fact non-random under optimal play, so that the projection to tug-of-war occurs, there is a further natural question. The players are staking a shared and non-random proportion of their present reserves at any given turn. We call this proportion the stake function. What is it?

We will see that the suggested, simple-minded, projection to classic tug-of-war is false for several simple graphs. Our principal results show nonetheless that the picture sketched above is correct for a certain class of graphs and payment functions, and we identify the stake function and explain how to play the game optimally in these cases (by characterizing the Nash equilibria). And in fact these results will be shown to hold not for stake-governed tug-of-war but for a leisurely variant thereof, in which each move in the game is cancelled by an independent event of probability $1 - \epsilon \in [0,1)$ whose occurrence is revealed to the players after stakes for the turn in question have been submitted. Our formula (5) for the stake function is a ratio of a spatial derivative of $h(\lambda,v)$ and a multiple of a $\lambda$-derivative. This form reflects the competition between the desire to gain territorial advantage, which pushes for a high stake value via the numerator, and the need to keep money, which places downward pressure on the stake via the denominator. It is equally a competition between the short and the long term: spend now and push the counter as you wish; but pay for this tomorrow, with a depleted budget for later gameplay. An alternative stake formula (6) will be proved in which the latter interpretation is manifest. A battle between space and money is waged in this two-person zero-sum game, which is fought over territory and mediated by the initial provision of limited finances. This interpretation perhaps offers a signpost to broad and attractive applications of stake-governed random-turn games.

We turn next to specifying the stake-governed tug-of-war more carefully. This will permit us to state our main conclusions later in the introduction.

1.3 The rules of the game

A parameter $\epsilon \in (0,1]$ called the move probability is given. The starting fortune $\lambda$ is a parameter taking a value in $[0,\infty)$.

Maxine and Mina prepare for the start of play. Maxine receives $\lambda$ units and Mina, one unit. A counter is placed at a given vertex $v \in V_O$ in the field of open play. In a sequence of turns, the counter will move along adjacent vertices in $G$ until it reaches a boundary vertex, in $V_B$. If $X_i$ denotes the counter location at the end of the $i^{th}$ turn, then the process $X : [0, F] \rightarrow V, X_0 = v$, encodes the counter locations throughout gameplay. Here, $F$, the finish time, is $\min \{ i \in \mathbb{N}_+ : X_i \in V_B \}$.  

---

1Roughly comparable forces compete in the War of Attrition and Attack, a differential game in [Isa65, Section 5.4].
Maxine’s fortune at the start of the $i^{th}$ turn is recorded as $\lambda_{i-1}$ units, where $\lambda_{i-1} \in [0, \infty)$. Mina’s fortune at the start of this, and every, turn equals one unit. As we describe gameplay, note that the randomness constituting any tossed coin is independent of any other randomness.

Let $i \in \mathbb{N}_+$. The $i^{th}$ turn is now about to begin. The data $\text{StateOfPlay} = (\lambda_{i-1}, X_{i-1}) \in (0, \infty) \times V_B$ encodes the present state of play of the game. The upcoming turn takes place in four steps.

First step: bidding. Maxine stakes a value $a_{i-1}$ that lies in $[0, \lambda_{i-1}]$. Mina stakes a value $b_{i-1} \in [0, 1]$. These amounts are deducted from the fortunes of the two players, who now hold the remaining amounts $\lambda_{i-1} - a_{i-1}$ and $1 - b_{i-1}$ in reserve. Maxine’s relative fortune is now $\frac{\lambda_{i-1} - a_{i-1}}{1 - b_{i-1}}$, and we denote this by $\lambda_i$. The value of $\text{StateOfPlay}$ is updated to be $(\lambda_i, X_{i-1})$.

Second step: determining if a move will take place. The croupier tosses a coin that lands heads with probability $\epsilon$. If it lands heads, Maxine is the turn victor; otherwise, Mina is. Thus each player wins with probability proportional to her stake at the present turn.

Third step: selecting who wins the turn. A coin is tossed that lands heads with probability $\frac{a_{i-1}}{a_{i-1} + b_{i-1}}$. If it lands heads, Maxine is the turn victor; otherwise, Mina is. Thus each player wins with probability proportional to her stake at the present turn.

Fourth step: the counter moves. The victorious player selects an element $X_i$ in $V$ that is a neighbour of $X_{i-1}$—so that $(X_{i-1}, X_i)$ is an element of the edge-set $E$ (a notion we record with the notation $X_{i-1} \sim X_i$).

If $X_i \in V_O$, the game is declared over at the end of the $i^{th}$ turn. In this case, we set $F = i$. Any remaining holdings of the two players are swept from them. Mina makes a payment of $\text{Pay}$ units to Maxine. The value of $\text{Pay}$ is the evaluation $p(X_F)$ of the payoff function at the terminal vertex.

Before the next move. If $X_i \in V_O$, play continues to the $(i + 1)^{st}$ turn. A simple coordinate change—a currency revaluation—is now made. The change does not affect later gameplay but it makes for simpler notation. Recall that Maxine has $\lambda_{i-1} - a_{i-1}$ units in reserve, and Mina has $1 - b_{i-1}$. We revalue so that Mina has one unit. Under this accounting device, the already specified $\lambda_i = \frac{\lambda_{i-1} - a_{i-1}}{1 - b_{i-1}}$ is equal to Maxine’s fortune as the $(i + 1)^{st}$ turn arrives. In short, $\text{StateOfPlay}$ is now set equal to $(\lambda_i, X_i)$.

The zero-bid and reset rules. The attentive reader may have noticed two problems in the specification of gameplay. First, if the players stake $a_{i-1} = b_{i-1} = 0$, who wins the right to make any move that takes place at the $i^{th}$ turn? The rule presented in the third step above involves a badly specified 0/0 probability. Second, if the players both go for broke, so that $a_{i-1} = \lambda_{i-1}$ and $b_{i-1} = 1$, and the game continues via $X_i \in V_O$, how can play continue given two bankrupt players? Some arbitrary rules are needed to permit play to continue in the two scenarios: a zero-bid rule to cope with the first problem, and a reset rule to address a pair of players with zero combined fortune. In the status quo reset rule applied as the $i^{th}$ turn begins, we set $\lambda_{i-1} = \lambda_{i-2}$ if the amounts held in reserve, $\lambda_{i-2} - a_{i-2}$ and $1 - b_{i-2}$, were both zero at the $(i - 1)^{st}$ turn. (The rule makes sense and resolves a problem only when $i \geq 2$.) In the status quo zero-bid rule, we take the win probability $\frac{a_{i-1}}{a_{i-1} + b_{i-1}}$ (in the third step above) at the $i^{th}$ turn (for $i \geq 1$) equal to $\frac{\lambda_{i-1}}{1 + \lambda_{i-1}}$ in the case that $a_{i-1} = b_{i-1} = 0$. Note that it may be that both rules are invoked at the $i^{th}$ turn, if $i$ is at least two: two bankrupt players will have their fortunes restored, but they may then both stake nothing. We will work with
the two status quo rules. Doing so slightly simplifies some proofs, but our results may be proved under a broad range of zero-bid and reset rules. We will briefly return to this matter in some closing comments, in Section S1.

The payment rule for unfinished games. A remaining ambiguity is that gameplay may never finish due to the counter failing to reach $V_B$. We stipulate that, in this event, the value of $\text{Pay}$ equals 1. This choice is more prescriptive than the mere need to render play well-defined dictates: see Section S2.

If the move probability $\epsilon$ equals one, then a move takes place at every turn; we may simply omit the second step to obtain a description of this form of the game, which will call the regular game and denote by $\text{Game}(1, \lambda, v)$. The first argument is the value of $\epsilon$; the latter two give the initial value of $\text{StateOfPlay}$. When $\epsilon \in (0,1)$, we call the game the leisurely game (with move probability $\epsilon$, of course), and denote it by $\text{Game}(\epsilon, \lambda, v)$. We may thus refer to all the games by the condition that $\epsilon \in (0,1]$. We occasionally refer to $\text{Game}(\epsilon)$ in verbal summary or when the value of $\text{StateOfPlay}$ is variable.

1.4 Principal definitions

We specify basic notions concerning strategy, game value, biased infinity harmonic functions, Nash equilibria and graphs.

1.4.1 Strategy

Before any given turn (in either form of the game), Maxine and Mina glance at the balance sheet and at the board. They see Maxine’s fortune and the counter location encoded in the vector $\text{StateOfPlay} \in (0,\infty) \times V_O$ and (let us say) they know how many turns have passed. How should they play?

Definition 1.3. Let $\mathcal{P}[0,\infty)$ denote the space of probability measures on $[0,\infty)$. We view this set as a metric space given by total variation distance; and as a measurable space by endowing it with the Borel $\sigma$-algebra that arises from the system of metric open sets. A mixed strategy for Maxine is a pair of measurable maps, the stake $s_+: [0,\infty) \times V_O \times N_+ \to \mathcal{P}(0,\infty)$ and the move $m_+: [0,\infty) \times V_O \times N_+ \to V$, that satisfy that the supremum of the support of $s_+(x,v,i)$ is at most $x$, and $m_+(x,v,i) \sim v$, for $x \geq 0$ and $v \in V_O$ and $i \in N_+$. Suppose that Maxine adheres to the strategy $(s_+,m_+)$. To play at the $i$th turn, she randomly selects her stake of $a_{i-1}$ units by setting $a_{i-1}$ equal to an independent sample of the law $s_+(\lambda_{i-1},X_{i-1},i)$. She nominates the move $m_+(\lambda_i,X_{i-1},i)$.

A mixed strategy for Mina is a pair $s_-: [0,\infty) \times V_O \times N_+ \to \mathcal{P}(0,\infty)$ and $m_-: [0,\infty) \times V_O \times N_+ \to V$, with the supremum of the support of $s_-(x,v,i)$ being at most one, and with $m_-(x,v,i) \sim v$, for $x \geq 0$ and $v \in V_O$. In following this strategy, Mina stakes $b_{i-1}$, an independent sample of the law $s_-(\lambda_{i-1},X_{i-1},i)$, and nominates the move $m_-(\lambda_i,X_{i-1},i)$, at the $i$th turn.

Note that, by use of mixed strategies, the players are permitted to randomize their stakes at any given turn. The definition does not permit random choices of move nomination, however.

If every image probability measure in a strategy is a Dirac delta, so that the concerned player never places a randomly chosen stake, then the strategy is called pure.
Let \( \mathcal{P}_+ \) and \( \mathcal{P}_- \) denote Maxine’s and Mina’s spaces of pure strategies, and let \( S_+ \) and \( S_- \) denote the counterpart mixed strategy spaces. Thus, \( \mathcal{P}_+ \subseteq S_+ \) and \( \mathcal{P}_- \subseteq S_- \). Each element \((S_-, S_+) \in S_- \times S_+\) specifies the evolution of gameplay begun from any given state of play \((\lambda, v) \in [0, \infty) \times V_O\), where of course the two players follow their selected strategies. Let \( \epsilon \in (0, 1) \). The mean payoff \( M(\epsilon, S_-, S_+) \) of the mixed strategy pair \((S_-, S_+)\) in Game(\( \epsilon \)) equals the mean value of the payoff Pay made at the end of the game dictated by \((S_-, S_+)\).

The basic perspective of Definition 1.3 is that players’ decisions may be made in light only of the present state of play of the game and the turn index. Note how the stake component at the \( i \)-th turn depends on the input \((\lambda_{i-1}, X_{i-1}, i)\), while move nomination depends on \((\lambda_i, X_{i-1}, i)\). The rules of the game stipulate that stakes are made at the start of the turn, with StateOfPlay equal to \((\lambda_{i-1}, X_{i-1})\). The players hand over their stakes and the turn victor is decided, with StateOfPlay updated to \((\lambda_i, X_{i-1})\), before the move victor makes her move; thus she does so in view of the relative fortune \( \lambda_i \) at this time.

### 1.4.2 Game value and Nash equilibria

**Definition 1.4.** Consider Game(\( \epsilon \)) for \( \epsilon \in (0, 1) \). The value \( \text{Val}(\epsilon, \lambda, v) \in [0, \infty) \) of the game begun with StateOfPlay = \((\lambda, v)\) satisfies the conditions that

\[
\sup_{S_+ \in S_+} \inf_{S_- \in S_-} M(\epsilon, S_-, S_+) \geq \text{Val}(\epsilon, \lambda, v),
\]

and

\[
\inf_{S_- \in S_-} \sup_{S_+ \in S_+} M(\epsilon, S_-, S_+) \leq \text{Val}(\epsilon, \lambda, v).
\]

When the value exists, the displayed bounds in the concerned case are equalities, because the left-hand side of (2) is at most that of (3).

**Definition 1.5.** A pair \((S_-, S_+) \in S_- \times S_+\) is a Nash equilibrium if \( M(\epsilon, S'_-, S_+) \geq M(\epsilon, S_-, S_+) \) and \( M(\epsilon, S_-, S'_+) \leq M(\epsilon, S_-, S_+) \) whenever \( S'_- \in S_- \) and \( S'_+ \in S_+ \). A Nash equilibrium is called pure if it lies in \( \mathcal{P}_- \times \mathcal{P}_+ \) and mixed if it does not.

It is easy to see that, at every Nash equilibrium \((S_-, S_+)\), the payoff \( M(\epsilon, S_-, S_+) \) is the same, and it is the value of the game.

### 1.4.3 Biased infinity harmonic functions

**Definition 1.6.** For \( \lambda \in [0, \infty) \), let \( h(\lambda, \cdot) : V \rightarrow [0, 1] \) denote the \( \lambda \)-biased infinity harmonic function on \( V \) with boundary data \( p : V_B \rightarrow [0, \infty) \). This function is the solution to the system of equations

\[
h(\lambda, v) = \begin{cases} 
p(v) \text{ for } v \in V_B, \\
\frac{\lambda}{\lambda + 1} \max_{u \sim v} h(\lambda, u) + \frac{1}{\lambda + 1} \min_{u \sim v} h(\lambda, u) \text{ for } v \in V_O,
\end{cases}
\]

subject to \( h(v) = p(v) \) for \( v \in V_B \). Recall that, by \( u \sim v \), we mean that \( u \in V \) with \( (u, v) \in E \).

\(^2\)Peres–Šunić would prefer to call this function \( \lambda^{-1} \)-biased. We adopt the name \( \lambda \)-biased because it is not only simpler but also consistent with an albeit-perhaps-unfair tendency to adopt the point of view of the maximizing player, who may interpret a high probability of winning any given turn as arising from a highly positive bias.
The solution exists and is unique: see [PS19, Section 1.5]. We will recall the effective algorithm of Peres and Šunić [PS19] to compute this value in Section 3. The structure of this solution is particularly simple for root-reward trees, which we now define.

1.4.4 Root-reward trees

The focus will lie on a class of boundary-payment graphs. These are trees with a unique leaf supporting the payment function.

**Definition 1.7.** Recall that a boundary-payment graph comprises a triple \((V, E, p)\) where the finite graph \((V, E)\) has vertex set \(V\) written as a disjoint union \(V_O \cup V_B\) and where the payment function \(p\) maps \(V_B\) to \([0, \infty)\). Such a triple is called a root-reward tree when these conditions are met:

- the graph \((V, E)\) is a rooted tree whose root \(r\) is a leaf in the tree;
- the boundary set \(V_B\) is equal to the set of leaves of the tree; and
- the payment function equals \(1\) at \(r\).

In this way, the game ends with a payout, of one unit, only if the counter reaches the set of leaves at location \(r\)—the root, or, if a name that reflects Maxine’s wishes is preferred, the reward vertex. We call \(V_B \setminus \{r\}\) the non-reward boundary: no payment is made if the game ends with the arrival of the counter in this set.

Let \((V, E, p), p = 1_r\) be a root-reward tree. Each vertex \(v \in V\) except the root has a unique neighbour that lies closer to \(r\). This is the parent vertex of \(v\), to be denoted by \(v_+\). The remaining neighbours of \(v\) lie further from \(r\) than does \(v\). These are the children of \(v\). The set of children of \(v\) that attain the minimal distance among these children to the non-reward boundary is denoted by \(V_-(v)\). Formally, \(V_-(v) = \{u \in V : u \sim v, u \neq v_+, d(u, V_B \setminus \{r\}) = d - 1\}\), where \(d = d(v, V_B \setminus \{r\})\). (For later use: if \(V_-(v)\) has only one element, this element is labelled \(v_-\).)

1.5 Main results

We will study the leisurely game on root-reward trees. Here is our first principal conclusion.

**Theorem 1.8.** Let \(T = (V, E, 1_r)\) be a root-reward tree, and let \(K \subset (0, \infty)\) be compact. There exists \(\epsilon_0 = \epsilon_0(K) \in (0, 1)\) such that, when \(\epsilon \in (0, \epsilon_0)\) and \(\lambda \in K\), the following hold for each \(v \in V_O\).

1. The value of \(\text{Game}(\epsilon, \lambda, v)\) exists and equals \(h(\lambda, v)\).

2. The game \(\text{Game}(\epsilon, \lambda, v)\) has a Nash equilibrium. Any Nash equilibrium is a pair of pure strategies that lead the players to stake non-random amounts that maintain the fortune ratio: that is, \(\lambda_i = \lambda\) for \(i \in [1, F - 1]\) in the gameplay that arises at the Nash equilibrium.

Theorem 1.8(2) indicates that, under jointly optimal play, a shared and non-random proportion of each player’s present reserves will be stated. The question is obvious: what is this proportion? Our second principal result, Theorem 1.12, offers an answer. It is contingent on the usage of root-reward trees and the resulting property of differentiability that we now state.

**Proposition 1.9.** Let \(T = (V, E, 1_r)\) be a root-reward tree. For \(v \in V_O\), the function \((0, \infty) : \lambda \mapsto h(\lambda, v)\) is differentiable.
Definition 1.10. Set $\Delta : [0, \infty) \times V_O \to [0, 1]$ according to
$$\Delta(\lambda, v) = \max_{w \sim v} h(\lambda, w) - \min_{w \sim v} h(\lambda, w).$$

Proposition 1.9 permits us to define the stake function $\text{Stake} : (0, 1] \times [0, \infty) \times V_O \to [0, \infty)$ so that
$$\text{Stake}(\epsilon, \lambda, v) = \epsilon \Delta(\lambda, v) \frac{1}{(\lambda + 1)^2} \left| \frac{\partial}{\partial \lambda} h(\lambda, v) \right|. \quad (5)$$

Definition 1.11. Consider a strategy pair $(S_-, S_+) \in S_- \times S_+$ in Game$(\epsilon, \lambda, v)$ and the gameplay $X : [0, F] \to V$ governed by this strategy pair in this game. Let $i \in \mathbb{N}_+.$

1. Maxine conforms at the $i$th turn (under this gameplay) if at this turn she stakes the quantity $\lambda_{i-1} \text{Stake}(\epsilon, \lambda_{i-1}, X(i-1))$ and nominates the move $(X(i))_+$. The strategy $S_+$ is said to be conforming against $S_-$ if Maxine almost surely conforms at every turn.

2. Mina conforms at the $i$th turn (under the same gameplay) if she stakes $\text{Stake}(\epsilon, \lambda_{i-1}, X(i-1))$ and nominates a move to an element of $V_-(X(i-1))$ at this turn. The strategy $S_-$ is said to be conforming against $S_+$ if Mina almost surely conforms at every turn.

The strategy pair $(S_-, S_+)$ is said to be conforming if each component is conforming against the other.

Here then is the promised result that elucidates the form of Nash equilibria from Theorem 1.8(2).

Theorem 1.12. Let $(V, E, 1_r)$ be a root-reward tree. With $\epsilon_0 \in (0, 1)$ specified in Theorem 1.8(2), the condition $\epsilon \in (0, \epsilon_0)$ implies that each Nash equilibrium in Game$(\epsilon, \lambda, v)$ is a conforming strategy pair.

In the stake formula (5) a battle between space and money is evident in the relative sizes of numerator and denominator. This is also a battle between the short and long terms, as we will see shortly, in Proposition 1.16. A little notation is needed, so that the stake formula denominator may be identified as the mean time remaining in a suitably calibrated clock. Definition 1.7 specifies sets $V_-(v)$ for each $v \in V_O$. When the counter lies at a vertex $v \in V_O$ for which $|V_-(v)| \geq 2$, Mina may choose among the members of $V_-(v)$ when she nominates her move with the counter at $v$ while retaining a conforming strategy. We now specify a collection of gameplay processes that arise from different strategies for Mina compatible with such choices. Proposition 1.16 shows that in fact it makes no difference, at least for the purpose of measuring the mean calibrated time remaining in the game, which of these choices Mina makes; this result thus provides an alternative formula for the stake function $\text{Stake}(\epsilon, \lambda, v)$.

Definition 1.13. Let $(V, E, 1_r)$ be a root-reward tree. Let $\Theta$ denote the set of mappings $\theta : (0, \infty) \times V_O \times \mathbb{N} \to V$ such that $\theta(\lambda, v, i) \in V_-(v)$ for each $v \in V_O$.

(We write $\mathbb{N} = \mathbb{N}_+ \cup \{0\}.$)

Definition 1.14. For $\epsilon \in (0, 1], \theta \in \Theta$ and $\lambda \in (0, \infty)$, let $X_{\theta}(\epsilon, \lambda, \cdot) : [0, F_\theta] \to V$ denote $(1 - \epsilon)$-lazy $\lambda$-biased walk on $V$ (with index $\theta$). This is the Markov process such that $X_{\theta}(\epsilon, \lambda, 0) = v \in V_O$.
and, whenever \(X_\theta(\epsilon, \lambda, k) \in V_O\) for \(k \in \mathbb{N}\),

\[
X_\theta(\epsilon, \lambda, k + 1) = \begin{cases} 
X_\theta(\epsilon, \lambda, k) \text{ with probability } 1 - \epsilon, \\
(\lambda X_\theta(\epsilon, \lambda, k) + 1) \text{ with probability } \frac{\lambda}{1 + \lambda}, \\
\theta(\lambda, X_\theta(\epsilon, \lambda, k), k) \text{ with probability } \frac{1}{1 + \lambda}.
\end{cases}
\]

The process is stopped on arrival at \(V_B\), so that \(F_\theta = \min \{j \geq 0 : X_\theta(\epsilon, \lambda, j) \in V_B\}\).

The first two arguments will usually be \(\epsilon\) and \(\lambda\), and we will often use the shorthand \(X_\theta(\cdot) = X_\theta(\epsilon, \lambda, \cdot)\).

The next result expresses the conclusion of Theorem 1.12 in terms of counter evolution.

**Corollary 1.15.** Under the hypotheses of Theorem 1.12, the set of gameplay processes governed by Nash equilibria in Game(\(\epsilon, \lambda, v\)) is given by \(\{X_\theta: \theta \in \Theta\}\).

Set \(\text{TotVar}(\epsilon, \lambda, \theta, v) = \sum_{i=0}^{F_\theta-1} \Delta(\lambda, X_\theta(i))\) where \(X_\theta(0) = v\).

**Proposition 1.16.** Let \(T = (V, E, 1_r)\) be a root-reward tree. For \(\lambda \in [0, \infty)\), the value of the mean \(E_{\text{TotVar}}(\epsilon, \lambda, \theta, v)\) is independent of \(\theta \in \Theta\), and, in an abuse of notation, we denote it by \(E_{\text{TotVar}}(\epsilon, \lambda, v)\). Then \(E_{\text{TotVar}}(\epsilon, \lambda, v) = \epsilon^{-1}(\lambda + 1)^2 \frac{\partial}{\partial \lambda} h(\lambda, v)\) for \(\epsilon \in (0, 1]\). As such, we obtain an alternative formula for the stake function \(\text{(5)}\): for \(\epsilon \in (0, 1]\),

\[
\text{Stake}(\epsilon, \lambda, v) = \frac{\Delta(\lambda, v)}{E_{\text{TotVar}}(\epsilon, \lambda, v)} ,
\]

and indeed

\[
\text{Stake}(\epsilon, \lambda, v) = \frac{\epsilon \Delta(\lambda, v)}{E_{\text{TotVar}}(1, \lambda, v)} .
\]

A further main conclusion is an explicit formula for \(\text{Stake}(1, \lambda, v)\), \((\lambda, v) \in (0, \infty) \times V_O\), on a root-reward tree \((V, E, 1_r)\). The formula is expressed in terms of the Peres-Sunić decomposition of the tree, which is presented in Section 3 and it appears as Theorem 3.21 at the end of that section.

We end the introduction by stating two consequences of Theorem 3.21 that do not require new notation. The first is an explicit formula for the special case where \(V\) is an integer interval. The second states instances of the general formula where simple but interesting forms appear: these are the cases when \(\lambda\) is low; when it is one; and when it is high.

Let \(n \in \mathbb{N}_+\). The line graph \(L_n\) has vertex set \([0, n]\); its edges connect consecutive vertices. By choosing its root to be \(n\), and taking \(p = 1_n\), we obtain a root-reward graph \((L_n, \sim, 1_n)\) (where the edge-set is abusively denoted by \(\sim\)).

**Proposition 1.17.** For \(n \in \mathbb{N}_+\), consider the triple \((L_n, \sim, 1_n)\). Let \(i \in \llbracket 1, n - 1 \rrbracket\).

1. For \(\lambda \in (0, \infty) \setminus \{1\}\),

\[
\text{Stake}(\epsilon, \lambda, i) = \epsilon \cdot \left(\frac{\lambda + 1}{\lambda - 1} \cdot i \cdot \left(1 - \frac{n(\lambda - 1)}{n(\lambda_v - 1)}\right)^{-1}\right).
\]

2. For \(\lambda \in (0, \infty) \setminus \{1\}\), \(\text{Stake}(\epsilon, \lambda, i) = \text{Stake}(\epsilon, \lambda^{-1}, n - i)\).
3. We further have that Stake(\(\epsilon, 1, i\)) = \(\epsilon \cdot \frac{1}{(n-i)}\).

4. The preceding results are equally valid when the payment function \(1_n\) is replaced by any choice of \(p : \{0, n\} \rightarrow [0, \infty)\) for which \(p(0) < p(n)\).

In the next definition, and throughout, a path in a graph \((V, E)\) may not visit any element in \(V\) more than once.

**Definition 1.18.** Let \((V, E, 1, r)\) be a root-reward tree, and let \(v \in V\). Let \(P\) be any path in \((V, E)\) of minimum length that starts at \(r\); visits \(v\); and ends at a leaf in \(V\) other than \(r\) (which leaf we label \(z\)). Writing \(d : V \times V \rightarrow \mathbb{N}\) for graphical distance in \((V, E)\), set \(d_+(v) = d(v, r)\) and \(d_-(v) = d(v, z)\). In other words, \(d_+(v)\) is the distance along \(P\) from \(r\) to \(v\), and \(d_-(v)\) is the distance along this path from \(v\) to \(z\).

**Proposition 1.19.** Let \((V, E, 1, r)\) be a root-reward tree, and let \(v \in V\). Writing Stake\((\lambda, v)\) in place of Stake\((1, \lambda, v)\), we have that

\[
\lim_{\lambda \searrow 0} \text{Stake}(\lambda, v) = d_+(v)^{-1}, \quad \text{Stake}(1, v) = d_-(v)^{-1}d_+(v)^{-1}
\]

and

\[
\lim_{\lambda \nearrow \infty} \lambda^{d_-(v) - d_\text{min}(v)} \cdot \text{Stake}(\lambda, v) = \left(|J| \cdot d_\text{min}(v)\right)^{-1},
\]

where \(d_\text{min}(v)\) equals the minimum of \(d_-(w)\) as \(w\) ranges over vertices in the path \(Q\) from \(r\) to \(v\), and \(J\) is the set of vertices \(w\) on \(Q\) such that \(d_-(w) = d_\text{min}(v)\).

When \(v \in [1, n-1]\) lies on the line graph \(L_n\), \(\lim_{\lambda \nearrow \infty} \text{Stake}(\lambda, v)\) equals \(d_-(v)^{-1}\), and there is a certain symmetry between the players in the preceding result. But in general Stake\((\lambda, v)\) may tend rapidly to zero in the limit of high \(\lambda\), even as this function remains bounded away from zero as \(\lambda\) tends to zero.

### 1.5.1 Acknowledgments

The first author thanks Judit Zádor for help in preparing Figures 1, 2 and 3. He is supported by NSF grants DMS-1855550 and DMS-2153359 and a 2022 Simons Fellowship in Mathematics. The second author is supported by the ERC Consolidator Grant 772466 “NOISE”.

## 2 The big picture: the global saddle hope and the Poisson game

Our main assertions, Theorems 1.8 and 1.12, may be said to solve stake-governed tug-of-war for the applicable graphs and parameters, since these results determine game value and all practically important properties of Nash equilibria. The results are proved only for leisurely games on root-reward trees, however. In order to offer a simple means of seeing how the \(\lambda\)-biased infinity harmonic function may arise as game value in stake-governed tug-of-war, we state (and shortly prove) the next result.

For \(\lambda \in (0, \infty)\), the \(\lambda\)-idle zone of a boundary-payment graph \((V, E, p)\) is the set of \(v \in V_\Omega\) such that \(h(w, \lambda) = h(\lambda, v)\) for every neighbour \(w \in V\) of \(v\). We will see in Proposition 3.12(2) that the \(\lambda\)-idle zone of any root-reward tree is empty for \(\lambda \in (0, \infty)\).
Proposition 2.1. Let \((V, E, p)\) be a boundary-payment graph, and let \(\epsilon \in (0, 1]\). For \((\lambda, v) \in [0, \infty) \times V_O\), suppose that the \(\lambda\)-idle zone of \((V, E, p)\) is empty and that there exists a pure Nash equilibrium in \(\text{Game}(\epsilon, \lambda, v)\). Then \(\text{Val}(\epsilon, \lambda, v)\) exists and equals \(h(\lambda, v)\).

This is in essence our only result proved in a generality beyond root-reward trees. (The assumption of idle zone emptiness is a minor convenience that will facilitate the proof.) The proposition is however conditional and is not really one of our principal conclusions. We prove it next, in Section 2.1, by a simple argument of strategy mimicry. We present the result and its derivation because they offer a hypothesis for how to play stake-governed tug-of-war which the article at large examines; later in Section 2, we will interrogate the likely validity of the hypothesis, thus motivating the formulation of Theorems 1.8 and 1.12. The hypothesis in question, which may be contemplated for any boundary-payment graph, is that the Nash equilibrium existence assumption of Proposition 2.1 indeed holds, and that the joint stake at the hypothesised pure Nash equilibrium in \(\text{Game}(\epsilon, \lambda, v)\) takes the form \((\lambda, 1) \cdot S\). (This hypothesis is natural, given the proposition, because a constant fortune ratio of \(\lambda\) is compatible with the \(\lambda\)-biased form \(h(\lambda, v)\) for game value.) Proposition 2.1 offers no clue as to the form of the non-random stake function \(S\), but Theorem 1.12 asserts that, when this theorem’s hypotheses are in force, this function equals \(\text{Stake}(\epsilon, \lambda, v)\) as given in \((5)\). In Section 2.2, we present a heuristic argument that purports to identify \(S\) for the regular game as being equal to the right-hand side of \((5)\) with \(\epsilon = 1\). We will then draw attention to some difficulties that arise in trying to make this argument rigorous; and, in Section 2.3, we will challenge the purported conclusion for the regular game by analysing aspects of this game on three simple graphs. From these examples, we will learn that it is unrealistic to seek to prove Theorem 1.12 for the regular game without significant new restrictions. A particular problem that we will find is that one or other player may gain by deviating from the proposed equilibrium by going for broke, staking everything in an effort to win the next turn and end the game at that time. The leisurely game undermines the efficacy of go-for-broke staking. In Section 2.4, we analyse a Poisson game, which may be viewed as a game \(\text{Game}(0^+)\) of infinite leisure. In the Poisson game, time is continuous, and moves occur when Poisson clocks ring. We do not study this game rigorously, because significant questions arise in formulating and analysing it. Rather, we study it formally, where it acts as an \(\epsilon = 0^+\) idealization of the leisurely game. Indeed, we will see how formal calculations involving second derivatives of value for a constrained version of the Poisson game indicate that Theorem 1.12 may be expected to hold for this game. By the end of Section 2, we hope to have indicated clearly our motivations for studying the leisurely game and the rough form of the principal challenges that lie ahead as we prepare to prove Theorem 1.12. The section thus concludes with an overview of the structure of the remainder of the paper.

2.1 Strategy mimicry: proving Proposition 2.1

Lemma 2.2. Consider \(\text{Game}(\epsilon, \lambda, v)\) for \(\epsilon \in (0, 1]\), \(\lambda \in (0, \infty)\) and \(v \in V_O\). Suppose that the game has a pure Nash equilibrium. Suppose further that, for all \(P_- \in P_-\), there exists \(P_+ \in P_+\) such that \(M(P_-, P_+) \geq h(\lambda, v)\); and that, for all \(P_+ \in P_+\), there exists \(P_- \in P_-\) such that \(M(P_-, P_+) \leq h(\lambda, v)\). Then \(\text{Val}(\epsilon, \lambda, v)\) exists and equals \(h(\lambda, v)\).

Proof. Denote the hypothesised pure Nash equilibrium by \((P_0^+, P_0^-) \in P_+ \times P_-\). Write \(m = M(P_0^+, P_0^-)\). It follows directly from Definitions 1.4 and 1.5 that \(m\) equals \(\text{Val}(\epsilon, \lambda, v)\).

By assumption, there exists \(P_+ \in P_+\) such that \(M(P_0^+, P_+) \geq h(\lambda, v)\). But \(m \geq M(P_0^-, P_+)\) since...
\((P_0^0, P_0^1)\) is a Nash equilibrium. Hence \(m \geq h(\lambda, v)\). Similarly, there exists \(P^- \in \mathcal{P}_-\) such that \(M(P^-, P_0^+ \uparrow) \leq h(\lambda, v)\). Since \(m \leq M(P^-, P_0^+ \uparrow)\), we find that \(m \leq h(\lambda, v)\). Hence \(m = h(\lambda, v)\), and the lemma is proved.

**Proof of Proposition 2.1.** In view of Lemma 2.2 two assertions will suffice. First, for any pure strategy \(P^-\) for Mina in Game\((\epsilon)\), there exists a pure strategy \(P^+\) for Maxine in this game such that \(M(P^-, P^+) \geq h(\lambda, v)\). Second, for any given pure strategy \(P^+\) for Maxine in Game\((\epsilon)\), there exists a pure strategy \(P^-\) for Mina in this game such that \(M(P^-, P^+) \leq h(\lambda, v)\).

The two assertions have very similar proofs and we establish only the second of them. Suppose then given a pure strategy \(P^+\) for Maxine in Game\((\epsilon)\). Let \(\theta : V_\lambda \to V\) be such that \(\theta(v)\) is an \(h(\lambda, \cdot)\)-minimising neighbour of \(v\) for each \(v \in V_\lambda\). Let \(P^-\) denote the strategy for Mina in which she stakes the \(\lambda^{-1}\)-multiple of the stake offered by Maxine at any given turn and proposes the move \(\theta(v)\) when State\(\text{OfPlay} = (\lambda, v)\). If \(X\) denotes the resulting counter evolution, then \(h(\lambda, X)\) is a supermartingale, because

\[
E \left[ h(X_{i+1}) | X_i \right] \leq (1 - \epsilon) h(\lambda, X_i) + \epsilon \frac{1}{1+\lambda} \min_{u \sim X_i} h(\lambda, u) + \epsilon \frac{\lambda}{1+\lambda} \max_{u \sim X_i} h(\lambda, u) = h(\lambda, X_i).
\]

Since \(h\) takes values in \([0, 1]\), the supermartingale converges to a limiting value. Provided that the game ends in finite time almost surely, this limiting value is almost surely equal to Pay. To verify that the game ends almost surely, note that, when Mina wins the right to move, the value of \(h(\lambda, \cdot)\) evaluated at the counter location strictly decreases, because the \(\lambda\)-idle zone of \((V, E, p)\) is empty. Let \(d\) be the maximum length of a path in \(V_\lambda\). We see then that, if Mina wins the right to move on \(d\) consecutive occasions at which a move takes place, then the game will not endure beyond the last of these moves, because vertices cannot be revisited. Since the probability of Mina enjoying such a string of successes from a given move is positive (it is \((\lambda + 1)^{-d}\)), we confirm that the game will indeed end almost surely. Thus the payment that Mina makes is indeed almost surely \(\lim_n h(\lambda, X_n)\), whose mean is by Fatou’s lemma at most \(\liminf E h(\lambda, X_n)\), which is at most \(E h(\lambda, X_0) = h(\lambda, v)\) because \(h(\lambda, X_0)\) is a supermartingale. This confirms the second of the two assertions whose proof we promised in deriving Proposition 2.1.

The next result is in essence a consequence of a simplification of the preceding derivation. We omit the proof but discuss differences between the derivations. Mixed strategies in the stake game do not necessarily maintain the fortune ratio, so Proposition 2.1 requires the hypothesis that a pure Nash equilibrium exists so that its proof may merely address pure strategies in view of Lemma 2.2. For classical tug-of-war, this difficulty with mixed strategies is absent, and indeed the classical fact [VN28] that any two-person zero-sum game with finite strategy spaces has value may play the role of Lemma 2.2, thus, the value of classical biased tug-of-war (determined over mixed strategies) is unconditionally identified in the next result.

**Corollary 2.3.** Let \((V, E, p)\) be a boundary-payment graph, and let \(\lambda \in (0, \infty)\) and \(v \in V_\lambda\). Suppose that the \(\lambda\)-idle zone of \((V, E, p)\) is empty. The value of the game TugOfWar\((\lambda \frac{1}{1+\lambda}, v)\) exists and equals \(h(\lambda, v)\).

The idle-zone emptiness assumption merely permits the adaptation of the proof of Proposition 2.1. Corollary 2.3 may be readily obtained without this assumption from the study of biased tug-of-war in [PPS10] with suitable adaptations to handle the context of graphs.
2.2 The stake formula argued via perturbation

It is perhaps tempting given the proof of Proposition 2.1 to give credence to the discussed hypothesis, namely that the regular game has a pure Nash equilibrium. In more precise verse, we may make the assumption that

there exists a pure Nash equilibrium for Game(1, λ, v) (8)

and an open interval I containing λ for which,
under gameplay governed by this equilibrium,
the players stake
at each turn
at whose start
the relative fortune lies in I
a deterministic and shared proportion of their respective fortunes.

Suppose that this assumption holds, and call the proportion \( S(\mu, w) : V_O \rightarrow [0, 1] \), \((\mu, w) \in (0, \infty) \times V_O\), for Game(1, λ, v). Under gameplay governed by the hypothesised pure Nash equilibrium, the relative fortune of the players will remain constant, at its initial value λ; and the counter will move according to λ-biased infinity harmonic walk.

If this view is correct, then the natural task is to identify the form of the stake \( S(\lambda, \cdot) : V_O \rightarrow [0, 1] \) as a function of the vertices in open play. We will present a heuristic argument that identifies \( S(\lambda, v) \) as the right-hand side of the alternative stake formula (9), (with \( \epsilon = 1 \), since we consider the regular game). The principal assumption guiding the derivation is (8); in our informal presentation, we will indicate further assumptions as the need for them arises. Before we give the derivation, note that the formula’s right-hand side is a ratio of terms. In the numerator is \( \Delta(\lambda, v) \), namely the difference in mean payment according to whether Maxine wins or loses the first turn. In the denominator is \( \mathbb{E}_{\text{TotVar}}(1, \lambda, v) \), which is the mean value of the sum of such differences as the counter evolves during gameplay. The derivation will develop a theme mooted in Section 1.5: the ratio may be interpreted as a short-term gain divided by a long-term cost.

Set \( S = S(\lambda, v) \). Maxine and Mina will play according to the Nash equilibrium from (8). With StateOfPlay = \((\lambda, v)\), Mina will stake \( S \) and Maxine, \( \lambda S \). They will nominate moves to minimize or maximize the value of \( h(\lambda, \cdot) \) among neighbours of \( v \). To find the form of \( S \), we consider the possibility that, for the first turn only, Maxine perturbs her \( S \)-value by a small positive amount \( \eta \), so that she instead stakes \( \lambda(S + \eta) \). After the first turn, she adheres to the Nash equilibrium strategy. How will she be affected by this change? The net effect on mean payment will be a difference \( G - L \) of two positive terms, a gain \( G \) and a loss \( L \). The gain \( G \) is short term: Maxine may benefit by winning the first turn due to increased expenditure. The loss \( L \) is the price that she pays due to depleted resources as the second term begins.

We want to find the forms of \( G \) and \( L \), of course. To compute \( G \), note first that the increase in Maxine’s first turn win probability due to her alteration is

\[
\frac{\lambda(S + \eta)}{\lambda(S + \eta) + S} - \frac{\lambda}{\lambda + 1}
\]

which equals \( \eta S^{-1} \frac{\lambda}{(1 + \lambda)^2} \).
where we will treat \( \eta \) as an infinitesimal, so that \( \eta^2 = 0 \). If Maxine converts a loss into a win in this way, then her expected mean payment increases by

\[
\max_{u \sim v} h(\lambda, u) - \min_{u \sim v} h(\lambda, u)
\]

which equals \( \Delta(\lambda, v) \). The gain term \( G \) is the product of the two preceding displays:

\[
G = \eta S^{-1} \frac{\lambda}{(1+\lambda)^2} \Delta(\lambda, v) .
\] (9)

What of the loss term \( L \)? Maxine enters the second turn poorer than she would have been. Indeed, if \( \lambda_{alt} \) denotes the relative fortune at the second turn (and later) when Maxine alters her stake at the first turn, we have \( \lambda_{alt} = \lambda - \lambda(1 - S)^{-1} \eta \).

The counter evolves from the second turn as a \( \lambda_{alt} \)-biased walk, because the players adhere to the Nash equilibrium in (8) in light of the infinitesimal form of the perturbation to the relative fortune. The difference in the probability that Mina wins one of the later turns in the altered gameplay (when \( \eta > 0 \)) but not in the original one (when \( \eta = 0 \)) equals the positive quantity

\[
\lambda/(1 + \lambda) - \lambda_{alt}/(1 + \lambda_{alt}) = (1 - S)^{-1} \lambda(1 + \lambda)^{-2} \eta ,
\] (10)

where a brief calculation shows the displayed equality. What price will Maxine pay for any lost opportunities to move at these later turns? An argument in the style of the derivation of Russo’s formula for percolation will tell us the answer. Since \( \eta \) is infinitesimal, we may neglect the possibility that Maxine loses out twice. If she loses out when the counter is at \( w \in V_O \), the mean payment at the end of the game will drop by \( \Delta(\lambda_{alt}, w) \). This equals \( \Delta(\lambda, w) \) for our purpose because such a loss is incurred only with probability \( \eta^2 = 0 \), provided at least that \( \Delta(\cdot, w) \) is differentiable at \( \lambda \). By averaging over the counter trajectory \( X : [0, F] \rightarrow V \), we see that the loss in mean payment after the first turn equals

\[
L = \eta \cdot (1 - S)^{-1} \lambda(1 + \lambda)^{-2} \cdot \mathbb{E} \sum_{i=2}^{F} \Delta(\lambda, X_{i-1}) ,
\] (11)

where note that the turn index \( i \) runs from two; also recall that \( F \) is the time at which \( X \) reaches \( V_B \). (In fact, neglecting the possibility that Maxine loses out twice requires some assumption on the tail decay of \( F \), such as that \( \mathbb{E}F^2 < \infty \).)

When the two players adhere to a pure Nash equilibrium, Maxine will indeed stake \( \lambda S \) at the first turn. This means that \( G \) equals \( L \): if \( G > L \), Maxine could choose a small \( \eta > 0 \); if \( G < L \), a small \( \eta < 0 \); and in either case, she would gain. Thus, (9) equals (11), so that

\[
S^{-1} \Delta(\lambda, v) = (1 - S)^{-1} \mathbb{E} \sum_{i=2}^{F} \Delta(\lambda, X_{i-1}) .
\]

Rearranging, we find that

\[
S = \frac{\Delta(\lambda, v)}{\mathbb{E} \sum_{i=1}^{F} \Delta(\lambda, X_{i-1})} ,
\] (12)
where the summation (or turn) index $i$ now begins at one because $X_0 = v$. Since this denominator equals $E \text{TotVar}(1, \lambda, v)$ as it is specified before Proposition 1.16, we have thus completed a heuristic argument for the formula for the stake function $S = S(\lambda, v)$ given in (6) for $\epsilon = 1$.

We will note two criticisms of this heuristic or of the conclusions that we may be tempted to draw from it. The next definition is useful for expressing the first criticism and we will also use it later.

**Definition 2.4.** Let $I$ and $J$ be two intervals in $\mathbb{R}$ and suppose given $F : I \times J \rightarrow \mathbb{R}$. A point $(x, y) \in I \times J$ is a global saddle point in $I \times J$ if $F(x^*, y) \geq F(x, y) \geq F(x, y^*)$ for all $(x^*, y^*) \in I \times J$. Such a point is a local saddle point if these bounds hold for $(x^*, y^*) \in I^* \times J^*$ where $I^* \subseteq I$ and $J^* \subseteq J$ are intervals whose interiors respectively contain $x$ and $y$. The word ‘minimax’ is a synonym of ‘saddle’ for these definitions.

### 2.2.1 First criticism: the predicted saddle point may merely be local

The above argument is perturbative, and it claims only that the point $(\lambda S, S)$ is a local saddle point of the map that associates to each $(a, b) \in [0, \lambda] \times [0, 1]$ the value of the game resulting from joint stakes of $(a, b)$ at the first turn. (We will introduce notation for this value very soon.) The prospect that the saddle point is global needs further examination.

### 2.2.2 Second criticism: the saddle point formula may be badly defined

One or other player may have a choice of move nomination when $\text{StateOfPlay} = (\lambda, v)$. One choice may typically lead $\text{Game}(\epsilon)$ to end more quickly than another. This means that the denominator in (6) may be badly defined, because different gameplay processes $X$ that specify $\text{TotVar}(\epsilon, \lambda, v)$ may lead to different values of $E \text{TotVar}(\epsilon, \lambda, v)$. In other words, the perturbative argument shows that, under its assumptions, this denominator is well-defined; but it may not be.

### 2.3 The regular game: local saddle point and global interruption

If the identified saddle point is well-defined and global, we will informally say that *global saddle hope* is realized; in this case, the above two criticisms are not valid. We now interrogate the prospects for this hope by examining the regular game on three simple graphs. We will analyse a version of the game constrained at the first turn and verify whether a suitable saddle point emerges at the predicted location as the parameters governing the constraint in this game are varied.

Let $(V, E, p)$ be a boundary-payment graph and let $(\lambda, v) \in (0, \infty) \times V_O$. For $a \in [0, \lambda)$ and $b \in [0, 1)$, consider the constrained version $\text{Game}(1, \lambda, v, a, b)$ of $\text{Game}(1, \lambda, v)$ in which Maxine must stake $a$, and Mina $b$, at the first turn. (The appearance of five parameters in this order will be characteristic when first-turn-constrained games are considered.) Denote the value of the constrained game by $\text{Val}(1, \lambda, v, a, b)$. For the three graphs we will investigate, we will carry out what we may call a *GloshConch*: a *Global saddle hope Consistency check*. To perform the GloshConch for a given graph, we make the assumption that is naively prompted by Proposition 2.1:

**Definition 2.4.** Let $I$ and $J$ be two intervals in $\mathbb{R}$ and suppose given $F : I \times J \rightarrow \mathbb{R}$. A point $(x, y) \in I \times J$ is a global saddle point in $I \times J$ if $F(x^*, y) \geq F(x, y) \geq F(x, y^*)$ for all $(x^*, y^*) \in I \times J$. Such a point is a local saddle point if these bounds hold for $(x^*, y^*) \in I^* \times J^*$ where $I^* \subseteq I$ and $J^* \subseteq J$ are intervals whose interiors respectively contain $x$ and $y$. The word ‘minimax’ is a synonym of ‘saddle’ for these definitions.

for any given $\lambda \in (0, \infty)$ and $v \in V_O$, the value of $\text{Game}(1, \lambda, v)$ exists and equals $h(\lambda, v)$. (13)

Operating with (13), we will be able to compute constrained-game values $\text{Val}(1, \lambda, v, a, b)$ because, after the first turn, these games reduce to copies of the unconstrained game $\text{Game}(1)$. We will be in a position to determine whether the map $[0, \lambda] \times [0, 1) \rightarrow [0, \infty) : (a, b) \mapsto \text{Val}(1, \lambda, v, a, b)$ has
a saddle point at $(\lambda S, S)$, where $S = \text{Stake}(1, \lambda, v)$ is specified in (12). The global saddle hope promises that, whatever the value of $(\lambda, v) \in (0, \infty) \times V_O$, this point is a saddle, and indeed that the saddle is global. If this promise is delivered for all such parameters, we say that the GloshConch is passed; otherwise, it fails. Analysing a graph and finding that the GloshConch fails may entail identifying interesting structure for the maps of constrained value. We cannot however make detailed inferences from the GloshConch’s failure: we are in effect working by proof-by-contradiction, and we learn from the failure merely that either the global saddle hope is unfulfilled or (13) is false; in the latter case, we may infer that no pure Nash equilibria exist in Game$(1, \lambda, v)$, at least for some $(\lambda, v) \in (0, \infty) \times V_O$. (This inference is due to Proposition 2.1 at least for graphs without idle zones.) Further work would be needed to elucidate the structure of Nash equilibria and find the value of games for which the consistency check fails.

2.3.1 Normal form contour plots

A simple graphical representation will allow us to perform the GloshConch in three simple graphs. For a given boundary-payment graph, $\lambda \in (0, \infty)$ and $v \in V_O$, the function $[0, \lambda] \times [0, 1] \rightarrow [0, 1] : (a, b) \mapsto \text{Val}(1, \lambda, v, a, b)$ may be depicted as a contour plot in the $[a, b]$-rectangle. Mark points $(a_0, b_0)$ in $[0, \lambda] \times [0, 1]$ as red if they are maximizers of $[0, \lambda] \rightarrow (0, \infty) : a \mapsto \text{Val}(1, \lambda, v, a, b_0)$. Mark points $(a_0, b_0)$ in $[0, \lambda] \times [0, 1]$ as blue if they are minimizers of $[0, 1] \rightarrow (0, \infty) : b \mapsto \text{Val}(1, \lambda, v, a_0, b)$. We refer to the resulting sketches as normal form contour plots: see Figures 2 and 3 for several examples that we will shortly discuss. The resulting red and blue curves (which, as the sketches show, sometimes have discontinuities) are Maxine and Mina’s respective best responses to first-turn stakes offered by Mina and Maxine at given vertical and horizontal coordinates. A global saddle point for the plotted function occurs when the red and blue curves meet, provided that the function is continuous at the intersection point. (We will see shortly that the curves may however meet at a point where this continuity is lacking.)

2.3.2 Example I: The line graph $(L_2, \sim, 1_2)$

The first of the three graphs has only one vertex in open play. From 1, the game will end in one turn. If Maxine wins, the counter moves to 2, and Pay = 1. If she loses, the counter moves to 0 with no payment made. Note that $\text{Val}(1, \lambda, 1, a, b)$ equals $\frac{a}{a+b}$, with a global minimax achieved at $(\lambda, 1)$. This corresponds to the trivial conclusion that there is a unique Nash equilibrium where both players go for broke: what else could they do in a game with one turn?

2.3.3 Example II: The line graph $(L_3, \sim, 1_3)$

For this graph, the vertices 1 and 2 are in open play. Mina must play left and Maxine right, so move nomination is trivial. But stake decisions are less evident than they were in the preceding example. The $\lambda$-biased infinity harmonic values are

$$h(\lambda, 1) = \lambda^2(\lambda^2 + \lambda + 1)^{-1} \quad \text{and} \quad h(\lambda, 2) = \lambda(\lambda + 1)(\lambda^2 + \lambda + 1)^{-1}.$$  

For $a \in [0, \lambda)$ and $b \in [0, 1)$, $\text{Val}(1, \lambda, 1, a, b) = \frac{a}{a+b}h(\lambda_1, 2)$ and $\text{Val}(1, \lambda, 2, a, b) = \frac{a}{a+b} + \frac{b}{a+b}h(\lambda_1, 1)$, where $\lambda_1 = \frac{a}{a+b}$ is the relative fortune after the first turn in the constrained game. The predicted Nash equilibrium point is $(\lambda S, S)$ with $S$ given in (6) equal to $(1 + \lambda + \lambda^2)(2\lambda + 1)^{-1}(\lambda + 1)^{-1}$. (It is understood throughout this discussion of regular-game examples that $\epsilon = 1$ in (6). See Figure 1.)
for three normal form contour plots for the constrained game at vertex 2 on this graph. While the yellow cross $(\lambda S, S)$ marks a local saddle point for each $\lambda \in (0, \infty)$, it is only when the condition $\lambda \geq 1$ (such as in the middle and right plots) is met that this saddle point is global. Indeed, when $\lambda < 1$ (as in the left plot) and Mina stakes $S$ with StateOfPlay $= (\lambda, 2)$, Maxine prefers $\lambda$, the go-for-broke choice, to $\lambda S$ as her response. In an example as simple as $L_3$, the GloshConch fails—the global saddle hope is disappointed—for many values of $\lambda$. The first criticism, offered in Subsection 2.2.1 is valid.

![Figure 1: Normal form contour plots of $[0, \lambda] \times [0, 1] \rightarrow [0, 1] : (a, b) \mapsto \text{Val}(1, \lambda, v, a, b)$ for vertex $v = 2$ in the line graph $(L_3, \sim, 1_3)$, drawn under the assumptions of the GloshConch. The $\lambda$-values are $1/2, 1$ and $3$ from left to right.]

### 2.3.4 Example III: The $T$ graph

In the third example, move nomination as well as stake decisions are non-trivial for at least one player. Take copies of the line graphs $L_2$ and $L_3$ and identify vertex 1 in $L_2$ with vertex 3 in $L_3$. The result is the $T$ graph. It has two vertices in open play, which we call the north vertex $N$ and the south vertex $S$, with $N$ being the just identified vertex. We label the three leaves in the $T$ graph $0, 1$ and $2$, where $0$ is adjacent to $S$ and $1$ and $2$ are adjacent to $N$. We will consider the payment function given by the identity map on the leaves of $T$ with the just indicated notational convention: we name the leaves by the value of Pay that they offer.

Consider Game$(1, \lambda, N)$ on this boundary-payment graph. Mina has a choice between $1$ and $S$ when she nominates a move. This is a choice between a short and a long game, since the former nomination will end the game should she win the resulting move and the latter will keep the counter in open play, at $S$.

To compute the form of $h(\lambda, N)$ and $h(\lambda, S)$ subject to the boundary condition that $h(\lambda, i) = i$ for $i \in \{0, 2\}$, note that the $T$ graph may either be viewed as a copy $[1, N, 2]$ of $L_2$ to which the path $[0, S, N]$ has been adjoined, or as a copy $[0, S, N, 2]$ of $L_3$ to which $[1, N]$ has been adjoined. With the former view, we may specify $h_1(\lambda, N)$ as the value $\frac{2\lambda + 1}{\lambda + 1}$ of $\lambda$-biased infinity harmonic value at 1 on $L_2$ with boundary values 1 and 2, and then take $h_1(\lambda, S) = \frac{1}{\lambda + 1} h_1(\lambda, N)$. With the latter view, we set $h_2(\lambda, N) = 2\lambda(\lambda + 1)(\lambda^2 + \lambda + 1)^{-1}$ and $h_2(\lambda, S) = 2\lambda^2(\lambda^2 + \lambda + 1)^{-1}$ to be the $\lambda$-biased
values at 2 and 1 on \( L_3 \) with boundary data 0 and 2. On the \( T \) graph, for \( v \in \{N,S\} \),

\[
h(\lambda, v) = \begin{cases} h_1(\lambda, v) & \text{when } h_2(\lambda, S) \geq 1, \\ h_2(\lambda, v) & \text{when } h_2(\lambda, S) < 1. \end{cases}
\]

There is thus a critical value \( \lambda_c \), which equals the golden ratio \( \frac{\sqrt{5}+1}{2} \), such that \( h_1 \)-values are used when \( \lambda \geq \lambda_c \), and \( h_2 \)-values are used in the opposing case.

For \( a \in [0, \lambda) \) and \( b \in [0, 1) \), we will compute \( \text{Val}(1, \lambda, N, a, b) \) by use of (13). To do so, we need to understand whether Mina will choose to nominate 1 or \( S \) at the first turn of \( \text{Game}(1, \lambda, N) \). Writing \( \lambda_1 = \frac{\lambda-a}{1-b} \), she will nominate 1 if \( \lambda_1 > \lambda_c \), because \( h(\lambda_1, 1) = 1 \leq h(\lambda_1, S) \), and she will nominate \( S \) if \( \lambda_1 < \lambda_c \), because then \( h(\lambda_1, 1) = 1 > h(\lambda_1, S) \). (If \( \lambda_1 = \lambda_c \), she may nominate either of the two moves. Note also that she is able to make the indicated choices, because the updated relative fortune \( \lambda_1 \) has been encoded in \( \text{StateOfPlay} \) by the time she makes her move, as we emphasised after Definition 1.3.) Writing \( \omega = \frac{a}{a+b} \), we see then that

\[
\text{Val}(1, \lambda, N, a, b) = \begin{cases} 2\omega + 1 - \omega & \text{when } \lambda_1 > \lambda_c, \\ 2\omega + (1-\omega)\text{Val}(1, \lambda, S) & \text{when } \lambda_1 < \lambda_c. \end{cases}
\]

By (13), \( \text{Val}(1, \lambda_1, S) = h(\lambda_1, S) \). Thus, when \( \lambda_1 < \lambda_c \), \( \text{Val}(1, \lambda_1, S) \) equals \( h_2(\lambda_1, S) \). We find then that

\[
\text{Val}(1, \lambda, N, a, b) = \begin{cases} \omega + 1 & \text{when } \lambda_1 > \lambda_c, \\ 2\omega + 2(1-\omega)\frac{\lambda^2}{1+\lambda_1^2+\lambda^2} & \text{when } \lambda_1 < \lambda_c, \end{cases}
\]

with the right-hand formulas coinciding to specify \( \text{Val}(1, \lambda, N, a, b) \) in the critical case \( \lambda_1 = \lambda_c \), since

\[
\frac{2\lambda^2}{1+\lambda_c^2+\lambda^2} = 1.
\]

The saddle point \((\lambda S, S)\) in \((a, b) \mapsto \text{Val}(1, \lambda, N, a, b)\) predicted in (6) is not uniquely defined, because the denominator of the right-hand side differs according to whether Mina nominates 1 or \( S \) as her move. (This means that the second criticism, in Subsection 2.2.2, is valid. In fact, the criticism is valid in a strong sense, because, as we will see, Mina is motivated to nominate a move that leads to counter evolutions that are not \( \lambda \)-biased infinity harmonic walks in the sense of Definition 1.4.) This broadens the ambiguity in the definition of the denominator in (6).) The two saddle point predictions for the \( T \) graph are \((\lambda S, s)\) with

\[
s = (1+\lambda+\lambda^2)(2(\lambda+1)^{-1}(\lambda+1)^{-1}
\]

and \((\lambda, 1)\), corresponding to Mina’s nominations of \( S \) and 1. The latter point is the go-for-broke location, where each player bids it all. Five normal form contour plots for the \( T \) graph appear in Figure 2 with the former of the two predicted saddle points appearing as a yellow cross in each sketch.

The GloshConch fails in all five sketches, in the sense that neither of the predicted saddles is global in any case. (The first criticism voiced above is thus valid.) We discuss the lower-left plot, with \( \lambda = \lambda_c - 0.1 \), though several properties are shared with the other cases. The saddle \((\lambda S, s)\) is the yellow cross in the middle of the sketch. This saddle point is local but not global. Mina prefers the go-for-broke stake of one to the saddle-specified stake of \( s \) when Maxine stakes \( \lambda S \). Should Mina stake 1, Maxine also prefers the go-for-broke \( \lambda \). But if Maxine stakes this, Mina has a response that Maxine lacks: she may stake the maximum 1 but hold an infinitesimal amount \( dx \) in reserve, and then nominate the move \( S \). If she wins the turn, she will automatically win the next turn also (since

21
Figure 2: The $T$ graph and five normal form contour plots of $[0, \lambda] \times [0, 1] \rightarrow [0, 2] : (a, b) \mapsto \text{Val}(1, \lambda, N, a, b)$, where $N$ is the north vertex and the GloshConch’s assumptons are in force. The values of $\lambda$ in the upper middle and right are 0.9 and 1; on the lower left, middle and right, they are $\lambda_c - 1/10$, $\lambda_c$ and $\lambda_c + 1/10$. 
$dx/0 = \infty$), and the game will end with the counter at vertex 0. Indeed, there is a discontinuity in $[0, \lambda] \times [0, 1] \rightarrow [0, 2] : (a, b) \mapsto \text{Val}(1, \lambda, N, a, b)$ at the northeast corner $(\lambda, 1)$ which Mina may exploit by playing the long game in this way. Thus the predicted saddle at $(\lambda, 1)$ is not even a local saddle. Maxine reacts to a $1 - dx$ stake by Mina by staking an amount qualitatively of the form $1 - (dx)^{1/2}$, and, if the players alternate in best responses, the cycle of increasing withholdings from maximum stakes would continue until the withheld amounts are of unit order; eventually, Mina once again goes for broke. In this example, then, Mina’s luxury of move choice disrupts the predicted saddle at $(\lambda, 1)$. There is no pure Nash equilibrium. (A parallel role for vertex $N$ in the $T$ graph is mentioned in Kleinberg and Tardos’ work [KT08] on network-bargaining problems, where it is noted that a player at $N$ who negotiates with neighbours over splitting rewards on intervening edges is in a strong position. Experimental designs with negotiating participants have been set up and performed [CE01, CEGYS3] to test the strength of negotiators in such positions.)

2.4 The Poisson game

We have seen that go-for-broke is a principal mechanism that disrupts the global saddle hope, which is often unrealized in the regular game. The leisurely game is a variant that is designed largely in order to frustrate the efficacy of the go-for-broke strategy. Who would bet his life savings on the next turn when a shortly impending coin flip may dictate that no move will even take place? We now introduce an idealized low-$\epsilon$ limit of the leisurely game and carry out formal calculations that caricature the upcoming proof of Theorem 1.12 in which the global saddle hope will be demonstrated for the leisurely game.

The Poisson game on a boundary-payment graph $G = (V, E)$ takes place in continuous time. Let $\mathcal{P}$ denote a unit intensity Poisson process on $[0, \infty)$. With StateOfPlay$_0 = (\lambda, v) \in (0, \infty) \times V_O$, Maxine’s starting fortune is $\lambda_0 = \lambda$ and the starting counter location is $X_0 = v$. Maxine and Mina stake at time $t \in (0, \infty)$ at respective rates $a(t)$ and $b(t)$, where each of these processes is adapted with respect to the history of gameplay before time $t$. With instantaneous currency revaluation at time $t + dt$ to ensure that Mina holds one unit at this time, we see then that Maxine’s time $t + dt$ fortune equals

$$\lambda(t + dt) = \frac{\lambda(t) - a(t)dt}{1 - b(t)dt} = \lambda(t) - (a(t) - \lambda(t)b(t))dt,$$

where we work formally with an infinitesimal calculus for which $(dt)^2$ equals zero.

A move happens when a Poisson clock rings: when $t$ reaches an element of $\mathcal{P}$. Indeed, this is the game $\text{Game}(0^+)$ of infinite leisure; except it has been speeded up so that moves occur at a unit-order rate. If a move happens at a given time $t$, Maxine wins with probability $\frac{a(t)}{a(t) + b(t)}$; otherwise, Mina does. Thus it is the present stake rates that dictate the move outcome. It is intuitively plausible that the strategy mimicry argument for Proposition 2.1 in Section 2.1 would show that the game
value $\text{Val}(\lambda, v) = \text{Val}(0^+, \lambda, v)$ is given by the biased infinity harmonic value in Definition 1.6:

$$\text{Val}(\lambda, v) = h(\lambda, v);$$

and that, in jointly optimal play, the stake rates will satisfy

$$a(t) = \lambda b(t).$$

We do not seek to make rigorous these assertions—there are challenges in formulating strategy spaces and resulting outcomes in continuous-time games—but rather focus on the prospects that the Poisson game may realize the global saddle hope.

Let $a, b \in [0, \infty)$. Consider a constrained game $\text{Game}(0^+, \lambda, v, a, b)$, where Maxine and Mina are obliged during $[0, dt]$ to submit respective stake rates $a$ and $b$. They regain freedom at time $dt$. Write $\text{Val}(\lambda, v, a, b)$ for the value of $\text{Game}(0^+, \lambda, v, a, b)$. (We again omit a first argument $0^+$ in the notation for value.) Then we claim heuristically that

$$\text{Val}(\lambda, v, a, b) = \text{Val}(\lambda, v) + \Phi(\lambda, v, a, b)dt, \tag{16}$$

where $\Phi : (0, \infty) \times V_0 \times (0, \infty)^2 \rightarrow \mathbb{R}$ is given by

$$\Phi(\lambda, v, a, b) = -\text{Val}(\lambda, v) - (a - b\lambda)\text{Val}'(\lambda, v) + \frac{a}{a+b}\text{Val}(\lambda, v_+) + \frac{b}{a+b}\text{Val}(\lambda, v_-) \tag{17}$$

with $\text{Val}'(\lambda, v) = \frac{\partial}{\partial \lambda}\text{Val}(\lambda, v)$, and where $v_+ = v_+(\lambda)$ and $v_- = v_-(\lambda)$ are neighbours of $v$ that maximize, or minimize, $\text{Val}(\lambda, \cdot) = h(\lambda, \cdot)$. (These neighbours may not be unique, but we only consider such quantities as $h(\lambda, v_+)$. They may depend on $\lambda$, and indeed this may pose problems, but we prefer to suppress this dependence in the notation, and elide this difficulty, in this heuristic discussion. A related assumption that we make is that the derivative $\text{Val}'(\lambda, v)$ exists.) To explain why (16) holds, consider the partition of the state space offered by the events of absence of a Poisson point in $[0, dt]$; of the presence of such a point accompanied by a win for Maxine; and of such a presence alongside a win for Mina. By adding the values of the resulting subgames, we see that

$$\text{Val}(\lambda, v, a, b) = (1 - dt)\text{Val}(\lambda(dt), v) + dt \frac{a}{a+b}\text{Val}(\lambda(dt), v_+) + dt \frac{b}{a+b}\text{Val}(\lambda(dt), v_-).$$

Using $(dt)^2 = 0$, we may make three right-hand replacements $\lambda(dt) \rightarrow \lambda(0) = \lambda$. Also using $\lambda(dt) = \lambda - (a - \lambda b)dt$ (which is due to (14)), we obtain (16).

The function $(0, \infty)^2 \rightarrow \mathbb{R} : (a, b) \mapsto \Phi(\lambda, v, a, b)$ describes the infinitesimal reward in mean value for Maxine when the players jointly commit to $(a, b)$ rates for an instant from time zero. Game value, being an expected later payment, will be a martingale under jointly optimal play. With shorthand $\Phi(a, b) = \Phi(\lambda, v, a, b)$, we thus expect that $\Phi(a_0, b_0) = 0$ when $(a_0, b_0)$ denotes the stake pair to which the players infinitesimally commit under jointly optimal play. Moreover, the point $(a_0, b_0)$

\footnote{It may seem that the strategy mimicry argument in question would adapt to the Poisson case to show that the game value $\text{Val}(\lambda, v)$ equals $h(\lambda, v)$ only if players are restricted to the use of pure strategies. We elide the distinction between value with mixed or merely pure strategies in this heuristic discussion. In fact, even the question of how to define mixed strategy in the Poisson game may have several reasonable answers. To attempt nonetheless a rough summary: the instantaneous monitoring of an opponent’s strategy that the continuous time evolution in the Poisson game permits would make it impossible for a player to take advantage of the element of surprise that is a signature advantage of the use of mixed strategies. Tentatively, then, we may believe that the quantity $\text{Val}(\lambda, v)$ in the Poisson game does not change according to whether mixed strategies are permitted or prohibited.}
must be a global minimax for $\Phi$ if play is occurring at a Nash equilibrium (which is an interpretation of the informal phrase ‘jointly optimal play’ that we have been using). We also expect $a_0 = b_0\lambda$ in view of (15). To check these claims, note that

$$\frac{\partial}{\partial a} \Phi(a, b) = -\text{Val}'(\lambda, v) + \frac{b}{(a+b)^2} \left( \text{Val}(\lambda, v_+) - \text{Val}(\lambda, v_-) \right),$$

and

$$\frac{\partial}{\partial b} \Phi(a, b) = \lambda \text{Val}'(\lambda, v) - \frac{a}{(a+b)^2} \left( \text{Val}(\lambda, v_+) - \text{Val}(\lambda, v_-) \right).$$

At any saddle point $(a_0, b_0)$, the preceding two displays equal zero, so that

$$a_0 \text{Val}'(\lambda, v) = a_0 b_0 \frac{\lambda}{2} \left( \text{Val}(\lambda, v_+) - \text{Val}(\lambda, v_-) \right),$$

whence

$$b_0 = \frac{\text{Val}(\lambda, v_+) - \text{Val}(\lambda, v_-)}{(\lambda + 1)^2 \text{Val}'(\lambda, v)}. \tag{19}$$

Since $a_0$ equals $\lambda b_0$, we have heuristically argued that Maxine and Mina will adhere under jointly optimal play to stake rates $(\lambda, 1) \cdot b_0$, with the just derived form for $b_0$. With $\epsilon = dt$, we have derived a form of Theorem 1.12 for the Poisson game $\text{Game}(0^+)$, since $\text{Val}(\lambda, v)$ equals the biased infinity harmonic value $h(\lambda, v)$. In Section 2.2, we offered the global saddle hope, that a global minimax for first-turn stake constrained value exists for which the stake function equals (6). We have shown that a version of this hope is realized, with the stake function given by (5), at least heuristically and for $\epsilon$ infinitesimal.

### 2.5 The Poisson game by example

We reexamine in the Poisson case the three examples treated for the regular game in Section 2.3.
2.5.1 Example I: \((L_2, \sim, 1_2)\)

For the Poisson game on \((L_2, \sim, 1_2)\) with StateOfPlay = \((\lambda, 1)\), we have \(\text{Val}(\lambda, 1) = \frac{\lambda}{\lambda + 1}\) and that the global minimax \((a_0, b_0)\) equals \((\lambda, 1)\), since \((19)\) holds alongside \(a_0 = \lambda b_0\). Since there is only one move on this graph, this choice of stake rates may be the closest the Poisson game comes to go-for-broke: the uncertainty of when the next Poisson clock will ring limits the players’ tendency to stake at high rates.

2.5.2 Example II: \((L_3, \sim, 1_3)\)

We have that \(\frac{\lambda}{1 + \lambda} \cdot \text{Val}(\lambda, 1) = \text{Val}(\lambda, 2) = \frac{2\lambda^2}{1 + \lambda + \lambda^2}\). Thus, by \((17)\),

\[
\Phi(\lambda, 2, a, b) = \frac{\lambda(\lambda + 1)}{1 + \lambda + \lambda^2} \left( \frac{a}{a + b} - \frac{\lambda}{1 + \lambda} - \frac{a - b\lambda}{(1 + \lambda)^2} \left(2 + \frac{\lambda(1 - \lambda)}{1 + \lambda + \lambda^2}\right) \right).
\]  

(20)

The saddle point for \((20)\) is global. It equals \((\lambda b_0, b_0)\), where

\[
b_0^{-1} = 2 + \frac{\lambda(1 - \lambda)}{1 + \lambda + \lambda^2}.
\]

2.5.3 Example III: the \(T\) graph

Consider the graph \(T\) with the same boundary data as before. If \(\lambda \neq \lambda_c\), then the updated value \(\lambda(dt)\) in \((14)\) lies on the same side of \(\lambda_c\) as does \(\lambda\). This permits the \((a, b)\)-constrained value to be analysed by computing the relative values of \(\lambda\) and \(\lambda_c\) (provided that these are unequal), rather than by the relative values of \(\lambda_1\) and \(\lambda_c\) that were seen in the regular case.

When \(\lambda < \lambda_c\), Mina at \(N\) nominates a move to \(S\), and gameplay takes place on the \(L_3\) copy \([0, S, N, 2]\). Thus, by \((20)\),

\[
\Phi(\lambda, N, a, b) = \frac{\lambda(\lambda + 1)}{1 + \lambda + \lambda^2} \left( \frac{a}{a + b} - \frac{\lambda}{1 + \lambda} - \frac{a - b\lambda}{(1 + \lambda)^2} \left(2 + \frac{\lambda(1 - \lambda)}{1 + \lambda + \lambda^2}\right) \right).
\]

When \(\lambda > \lambda_c\), Mina at \(N\) nominates a move to \(1\), seeking to end the game at the present turn, and the \(L_2\) copy \([1, N, 2]\) dictates the outcome. The formula for \(\Phi(\lambda, N, a, b)\) now reads

\[
\frac{a}{a + b} - \frac{\lambda}{1 + \lambda} + \frac{a - b\lambda}{(1 + \lambda)^2}.
\]

(21)

When \(\lambda = \lambda_c\), the formula for \(\Phi(\lambda, N, a, b)\) is the minimum of the two preceding forms. This is because it is Mina’s prerogative at \(N\) to nominate \(1\) or \(S\), and she does so to minimize subgame value.

2.5.4 Contour plots and features seen in the Poisson examples

We have noted that the \(L_2\) saddle point \((a_0, b_0)\) equals \((\lambda, 1)\).

Three normal form contour plots of \((0, \infty)^2 \longrightarrow \mathbb{R} : (a, b) \mapsto \Phi(\lambda, N, a, b)\) appear in Figure 3. We indicate two basic features.

Don’t bet the house. Note how play has a compact character: a player will cut her losses, staking at rate zero, if her opponent spends big.
Figure 3: Plotting the Poisson game: normal form contour plots of \( [0, \infty)^2 \rightarrow \mathbb{R} : (a, b) \mapsto \Phi(\lambda, N, a, b) \) for the \( T \) graph. From left to right, the \( \lambda \)-values are \( 3/2, \lambda_c = 1.1618 \ldots, \) and 2. The left plot equally depicts \( \Phi(3/2, 2, a, b) \) for the line graph \( L_3 \), and the right one depicts \( \Phi(2, 1, a, b) \) for \( L_2 \). The global minimax point \((\lambda b_0, b_0)\), with \( b_0 \) specified in (19), is marked with a yellow cross in the left and right plots. The middle plot depicts the minimum of \( \Phi(\lambda_c, 2, a, b) \) for \( L_3 \) and \( \Phi(\lambda_c, 1, a, b) \) for \( L_2 \), and the two yellow crosses mark the predicted saddle point \((\lambda b_0, b_0)\) for these two functions. The minimum reflects Mina’s capacity to decide between nominating vertex \( S \) or vertex 1.

Don’t sit the next one out. The red and blue best response curves meet the origin tangentially to the coordinate axes in all three sketches. If the opponent offers no stake, the best response in each of the three cases is to offer a positive but infinitesimal stake.

The complexity of the regular game plots in Figures 1 and 2 has largely vanished in the off-critical plots, on the left and right in Figure 3. The trace that remains is in the middle, critical, plot. Maxine’s red response play runs the ridge between the lower-left \( L_3 \) yellow cross (the long game) to the upper-right \( L_2 \) yellow cross (the short game). We do not rigorously formulate the Poisson game in this article, but its study is well motivated, and we moot prospects for a rigorous inquiry in the final Section 8. To summarise our formal computations with and examples of this game: the global saddle hope appears to be realized under the assumptions (which we effectively supposed) that the set of maximizers or minimizers of \( h(\lambda, \cdot) \) among the neighbours of any given vertex is independent of \( \lambda \in (0, \infty) \); and that the partial derivative of \( h(\lambda, \cdot) \) in \( \lambda \) exists. The prospects of the global saddle hope remain untested if one or other of these assumptions fails. Line graphs meet the assumptions but the \( T \) graph does not, and further work would be needed to resolve these questions even in a graph as simple as the latter.

2.6 A simple case where the stake formula is valid in the regular game

We have seen that the global saddle hope fails for the regular game for several simple graphs. Here we present an example where the stake formula (5) is in fact valid for this game.

Let \( n \in \mathbb{N}_+ \). The half-ladder \((H_n, \sim, 1_0)\) is the root-reward tree formed from the line graph \( L_n \) and a collection of points \( i^*, 1 \leq i^* \leq n \), by attaching edges \( i \sim i^*, i \in [1, n] \). The reward vertex equals 0 and the field of open play is \([1, n]\). Consider Game(1, \( \lambda, n \)). The game has \( n \) turns. Maxine must win at each turn so that the counter evolution \( n \rightarrow n - 1 \rightarrow \cdots \rightarrow 1 \rightarrow 0 \) ends at 0. A victory
for Mina at the $i$th turn will lead her to end the game with the move $n - i + 1 \rightarrow (n - i + 1)^*$ provided that the game is on at the start of this turn, whatever the value of $i \in [1, n]$. It is natural to suspend currency revaluation in analysing the game and instead take the view that the stake strategies are indexed by possibly random vectors $\tilde{x} = (x_i : i \in [1, n])$ (for Mina) and $\tilde{y} = (y_i : i \in [1, n])$ (for Maxine) of non-negative entries that respectively sum to one and $\lambda$. At the start of the game, Mina places $x_i$ units and Maxine $y_i$ units against the vertex $i$ for each $i \in [1, n]$. These deposits are the stakes to be played at vertex $i$ at the start of a turn. For $d \geq 0$, write $\tilde{d}$ for the constant $n$-vector with sum $d$, so that each entry is $d/n$. Let $\tilde{y}$ be a non-random strategy for Maxine. Suppose that she plays against Mina’s constant strategy $\hat{1}$. Note that the mean payment $M(\hat{1}, \tilde{y})$ equals $\prod_{i=1}^{n} \frac{y_i}{1/n + y_i}$. It follows readily from the strict concavity of $(0, \infty) \longrightarrow (0, \infty) : z \mapsto -\log(1 + z)^{-1}$ that the maximum over $\tilde{y}$ of $M(\hat{1}, \tilde{y})$ equals $\lambda^n(1 + \lambda)^{-n}$ and is attained uniquely by $\tilde{y} = \lambda$. This is equally true if $\tilde{y}$ is permitted to vary over random strategies for Maxine. Now suppose instead that Mina plays a strategy $\tilde{x}$ against Maxine’s constant $\lambda$. We have $M(\tilde{x}, \lambda) = \prod_{i=1}^{n} \frac{\lambda/n}{\lambda n + x_i}$. Permit $\tilde{x}$ to vary over non-random strategies for Mina. By the strict convexity of $z \mapsto -\log(1 + z)$, this expression is minimized with value $\lambda^n(1 + \lambda)^{-n}$ and a unique minimizer $\tilde{x} = \hat{1}$. The conclusion holds equally when variation is permitted over random choices of $\tilde{x}$. These are the essentials of the proof of the next result.

**Proposition 2.5.** Let $n \in \mathbb{N}_+$. The value of the regular game $Game(1, \lambda, n)$ on the half-ladder $H_n$ equals $\lambda^n(1 + \lambda)^{-n}$. The game has a unique Nash equilibrium. Let $i \in [1, n]$. Should play under the Nash equilibrium continue to the start of the $i$th turn, the counter will lie at $n + 1 - i$ at this time. The proportion of her remaining reserves that each player will then stake is equal to $(n + 1 - i)^{-1}$. The index of the final turn of the game has the law $G \land n$, where $G \geq 1$ is a geometric random variable of parameter $\mathbb{P}(G = 1) = (1 + \lambda)^{-1}$.

Note that the stake formula in its guise (6) is satisfied for $Game(1, \lambda, n)$ because (6) states that $\text{Stake}(1, \lambda, n) = \frac{\lambda^{n-1}(1+\lambda)^{-n}}{\sum_{i=1}^{n} x_i^{n-1}(1+\lambda)^{-n}} = 1/n$. The summand is the product of the probability that play continues to the $i$th turn and the value $\Delta(\lambda, n + 1 - i)$.

### 2.7 Structure of the remainder of the article

The $T$-graph at the point $\lambda = \lambda_c$ has illustrated that, if the set of move nominations that a player may make without surrendering value changes as $\lambda$ rises through a certain point, complications arise for analysis, even in the idealized Poisson game. In working with root-reward trees, we are choosing to sidestep this difficulty, by using a framework in which such special values of $\lambda$ do not exist. In Section 3 we review the Peres–Sunić algorithm from [PS19] for finding biased infinity harmonic functions on graphs. In Theorem 3.3 we will establish that a related decomposition of the graph into paths (which has already been suggested in our discussion of the $T$ graph) has no dependence on $\lambda \in (0, \infty)$ in the case of root-reward trees. We will prove the $h$-differentiability Proposition 1.9 and several other needed facts about $h$ on root-reward trees.

In Section 4 we will introduce finite time-horizon games $Game_n(\epsilon, \lambda, v)$ whose gameplay is in essence the truncation of that in $Game(\epsilon, \lambda, v)$ to the first $n$ turns. Our finite-horizon games will be set up so that each has value $h(\lambda, v)$, with no error needed to account for the finiteness of $n$: see Lemma 4.6 for the case of classical tug-of-war; (later, Proposition 6.1(1) will assert this for the stake-governed version). As such, finite-horizon classical tug-of-war becomes a tool for deriving the
stake function formulas in Proposition 1.16. Indeed, we prove Proposition 1.16 in Section 5, by an argument concerning tug-of-war that is inspired by the heuristic perturbative identification of the stake function in the stake-governed version of the game seen in Section 2.2.

Section 6 presents the backbone of our proofs in the article. It proves a counterpart Proposition 6.1 for the finite-horizon games Game_n(\epsilon, \lambda, v) of Theorems 1.8 and 1.12. This proposition shows that the global saddle hope is realized, and finds the location of the global saddle point, in the finite games. The result will be proved by the fundamental technique of backwards induction, for which we need to consider finite-horizon games; in this proof, the formal computations in Section 2.4 for the Poisson game at \epsilon = 0^+ are perturbed so that \epsilon is permitted to be small but positive. By taking the horizon to infinity, we will derive Theorems 1.8 and 1.12 in Section 7. We conclude in Section 8 by reviewing some proof aspects and indicating several directions for developing theory and applications of stake-governed games.

3 The Peres-Šunić decomposition and root-reward trees

Here we recall or derive the information that we will need about biased infinity harmonic functions.

3.1 Peres and Šunić’s algorithm for finding \lambda-biased infinity harmonic functions

Peres and Šunić [PS19] present an algorithm for computing biased infinity harmonic functions on a finite graph. We recall it in the context of a boundary-payment graph (V, E, p) and then discuss important aspects of the resulting decomposition in the case of root-reward trees. Each pair of vertices \(x, y \in V\) is connected by a path that we label \([x, y]\) and whose length (or number of edges) we denote by \(\ell[x, y]\). (The path may not be unique, but Peres and Šunić find the notational abuse convenient. Most of our attention will be offered to root-reward trees for which the path is unique.) The vertices \(x\) and \(y\) are the endpoints of this path; further vertices on the path are called internal.

Let \(a, b \in \mathbb{R}\) with \(a \leq b\) be given. If \([x, y]\) has consecutive vertices \(x = z_0, z_1, \ldots, z_n = y\), the function \(h\) on \([x, y]\) equals

\[h(z_i) = A + B\rho^i \text{ for } i \in [0, n]\]  

with \(\rho = \lambda^{-1}\) and

\[A = \frac{b - \rho^n a}{1 - \rho^n} \text{ and } B = -\frac{b - a}{1 - \rho^n};\]  

when \(\lambda = 1\), we take the linear interpolation of the boundary data. This function is the \(\lambda\)-biased infinity harmonic function \(h\) on \([x, y]\) given boundary data \(h(x) = a\) and \(h(y) = b\).

Let \(g : V' \to \mathbb{R}\) for some \(V' \subseteq V\) and let \(\rho \in (0, \infty)\). For \(x, y \in V'\), the \(\rho\)-slope of \(g\) on \([x, y]\) equals

\[
g(y) - g(x) - \rho^{\ell[x, y]} g(x) \frac{1}{1 + \rho + \cdots + \rho^{\ell[x, y]-1}}.
\]

Note that 1-slope equals the ordinary slope \(\frac{g(y) - g(x)}{y - x}\). When \(g = 1_r\) for a root-reward tree \((V, E, 1_r)\), we will speak of “the \(\rho\)-slope of \([x, y]\)” in place of “the \(\rho\)-slope of \(g\) on \([x, y]\)”.

Definition 3.1. A decomposition is a sequence—we will say ‘list’—of edge-disjoint paths in \((V, E)\) such that every edge in \(E\) lies in a path on this list, and the endpoints of a given path on the list lie in the union of \(V_B\) and the set of internal vertices of the preceding paths in the list. Any
decomposition determines a $\rho$-extension of the payment function $p : V_B \rightarrow [0, \infty)$, this being the function $f : V \rightarrow [0, \infty)$ which is iteratively specified by setting the internal vertex values along each path in the decomposition according to the formula (22) where the endpoint values $a$ and $b$ are the already assigned values of $f$ (which is $p$ on $V_B$). A decomposition is $\rho$-valid if the $\rho$-slopes of the successive paths for the decomposition form a non-increasing sequence. A decomposition is $\rho$-perfect if, at each stage of the iteration, the $\rho$-slope of the next path in the list is the largest among all available paths.

The next result is proved in [PS19, Section 3]. We will reprove it, together with a refinement of it, in the specific case of root-reward trees, in the next subsection.

**Proposition 3.2.** Consider a boundary-payment graph $G = (V, E, p)$ and $\lambda \in (0, \infty)$. The $\lambda^{-1}$-extension determined by any $\lambda^{-1}$-perfect decomposition is the unique $\lambda$-biased infinity harmonic function that coincides with $p$ on $V_B$.

For $n \in \mathbb{N}_+$, the line graph $L_n$ has a unique decomposition: this is a list with one item, $L_n$ itself. A simple graph with non-trivial decompositions is the $T$ graph, introduced in Subsection 2.3.4. Recall that the condition $h(\lambda, S) = 1$ is satisfied by a unique value $\lambda_c = \frac{\sqrt{5}+1}{2}$ of $\lambda \geq 0$. When $\lambda \in [0, \lambda_c)$, the $\lambda^{-1}$-perfect decomposition is unique, taking the form $\{[0, 2], [1, N]\}$. When $\lambda \in (\lambda_c, \infty)$, the decomposition is also unique and now takes the form $\{[1, 2], [0, N]\}$. And when $\lambda = \lambda_c$, there are two $\lambda^{-1}$-perfect decompositions, these being the two just stated.

### 3.2 The decomposition for root-reward trees

Recall Definition 1.7 of root-reward trees.

**Theorem 3.3.** On any root-reward tree, the $\lambda^{-1}$-extension determined by any $\lambda^{-1}$-valid decomposition is the unique $\lambda$-biased infinity harmonic extension of the boundary data. Also, there is a $\lambda^{-1}$-valid decomposition that is independent of $\lambda \in (0, \infty)$.

**Proof.** Let $u \in V_B$ with $u \neq r$. The $\rho$-slope of the path $[u, r]$ equals $(\rho - 1)(\rho^{\ell[u, r]} - 1)^{-1}$ when $\rho \neq 1$; when $\rho = 1$, it equals $\ell[u, r]^{-1}$. We see then that, whatever the value of $\rho = \lambda^{-1} \in (0, \infty)$, any first element in a $\lambda^{-1}$-valid decomposition will take the form $[u, w]$, where $\ell[u, r]$ attains the minimum value for $u \in V_B$ with $u \neq w$.

Suppose that an initial sequence of paths in a $\lambda^{-1}$-valid decomposition has been constructed, and that the function $g$ has been defined on the set of vertices lying on these paths. Let $u$ be a vertex in one of the paths in the initial sequence. A path with upper end $u$ takes the form $[z, u]$ where $z \in V_B$, $z \neq u$, and $u$ is the only vertex in the path that may be an internal vertex of one of the paths in the initial sequence. Such a path $[z, u]$ is called admissible if the value of $\ell[z, u]$ is minimal among all paths with upper end $u$. We claim that the next path in any $\lambda^{-1}$-valid decomposition is admissible, for some such $u$. Indeed, for any path $[z, u]$ with upper end at $u$, the $\rho = \lambda^{-1}$-slope of the path equals $g(u)(\rho - 1)(\rho^{\ell[z, u]} - 1)^{-1}$ when $\rho \neq 1$; when $\rho = 1$, it equals $g(u)\ell[z, u]^{-1}$. Thus, such a path can continue the decomposition only if it is admissible.

From the claim, we see that a decomposition is $\lambda^{-1}$-valid only if each term is admissible given its list of predecessors. While a decomposition could satisfy the italicized condition without being

---

"We adopt throughout the usage ‘$\lambda$-biased’ for the infinity harmonic functions specified in Definition 1.6; however, in the present section, we employ the Peres-Sumić usage of $\lambda^{-1}$ in naming other objects from their work."
\(\lambda^{-1}\)-valid, the \(\lambda^{-1}\)-extension determined by any such decomposition coincides with that of a \(\lambda^{-1}\)-valid decomposition. Indeed, since \((V,E)\) is a tree, it makes no difference to assigned values in what order admissible paths are admitted to the decomposition, because these admissions occur in disjoint subtrees of \((V,E)\). Specifically, a \(\lambda^{-1}\)-valid decomposition may be obtained by admitting as the next term given an initial sequence the admissible path that is minimal for some arbitrary ordering of paths, such as that arising from lexicographical ordering given an alphabetical labelling of \(V\). Since this decomposition does not depend on \(\lambda \in (0,\infty)\), we have completed the proof of Theorem 3.3.

In the preceding proof, we saw that there may be several \(\lambda^{-1}\)-valid decompositions of a root-reward tree (even several \(\lambda^{-1}\)-perfect ones). For example, there may be several paths that attain the minimum length between the non-reward boundary \(V_B \setminus \{r\}\) and the root \(r\); to obtain a perfect decomposition, we may place any of these paths first, include suitable fragments of the others, and then include the rest of the tree by a suitably specified inductive means. To reduce this ambiguity, we introduce a decomposition into subtrees instead of paths.

**Definition 3.4.** A root-reward tree \((V,E,1_r)\) is called basic if the distance \(d(u,r)\) is independent of the leaf \(u \in V_B\) with \(u \neq r\). The shared distance will be called the span.

We now decompose a root-reward tree \(T\) into basic subtrees.

**Definition 3.5.** Let \((V,E,1_r)\) be a root-reward tree. The basic partition of this tree is a collection \(C\) of induced basic subtrees such that each element in \(E\) belongs to one subtree, specified by induction on the value of \(|V|\) in the following way.

Let \((V_0,E_0)\) denote the subtree of \((V,E)\) induced by the set of vertices that lie on paths of the form \([u,r]\), where \(u\) ranges over elements of \(V_B \setminus \{r\}\) such that \(d(u,r)\) attains the minimum value among such elements. The tree \((V_0,E_0,1_r)\) is basic. It is an element in the basic partition of \((V,E,1_r)\). The remaining elements are found by use of the inductive hypothesis. Consider the post-removal subgraph \((V,E) \setminus (V_0,E_0)\). This is the graph \((V',E')\) with edge-set \(E' = E \setminus E_0\) whose set of vertices \(V'\) equals the set of vertices in \(V\) to which an edge in \(E'\) is incident. The graph \((V',E')\) is a forest each of whose components \((V_1,E_1)\) has a unique vertex \(r_1\) in \(V_0\). For each component, \((V_1,E_1,1_{r_1})\) is a root-reward tree. Applying the inductive hypothesis, we obtain the basic partition of this tree; we denote it by \(C_1\). The basic partition of \((V,E,1_r)\) is set equal to \(C\), where \(C\) is the union of the singleton with element \((V_0,E_0,1_r)\) and the sets \(C_1\) indexed by the above components \((V_1,E_1)\).

**Definition 3.6.** Let \(v \in V\) with \(v \neq r\). A junction of \(v\) is a vertex \(w\) on the path \([r,v]\) from the root to the parent of \(v\) that is the root of a basic tree \(B\) of \((V,E,1_r)\) that has an edge traversed by the path \([r,v]\). It is easy to see (from the span-minimizing property of the basic trees in our construction), that the root \(w\) is actually the first vertex of \(B\) encountered by the path \([r,v]\). Let \(\{j_i : i \in [0,k]\}\) denote a list of the junctions of \(v\) in the order that they are encountered along \([r,v]\). Thus, \(j_0 = r\). We further set \(j_{k+1} = v\). For \(i \in [0,k]\), let \(T_i\) denote the basic tree that contains the path \([j_i,j_{i+1}]\); thus, \(j_i\) is the root of \(T_i\). Set \(d_i = d(j_i,j_{i+1})\), where \(d(\cdot,\cdot)\) denotes graphical distance. Set \(s_i\) equal to the span of \(T_i\). The list \(\{(s_i,d_i) : i \in [0,k]\}\) is called the journey data of \(v\) in \((V,E,1_r)\). (In the case of the root \(r\), we set \(k = 0\), and take the journey data to be \((s_0,0)\), where \(s_0\) is the distance between \(r\) and \(V_B \setminus \{r\}\).)
Definition 3.7. For \( k \in \mathbb{N}_+ \) and \( \ell \in [0, k] \), set
\[
H(\lambda, k, \ell) = \begin{cases} 
\frac{1-\lambda^{-(k-\ell)}}{1-\lambda^{-k}} & \text{for } \lambda \in (0, \infty) \setminus \{1\}, \\
\ell/k & \text{for } \lambda = 1.
\end{cases}
\]

Lemma 3.8. For such \( k \) and \( \ell \) as above, \( H(\lambda, k, \ell) \) is equal to \( h(\lambda, \ell) \) for the line graph \( L_k \) with root 0.

Proof. Consider (22) with \([x, y]\) equal the right-to-left path \([k, 0]\) and with \( a = 0 \) and \( b = 1 \).

Lemma 3.9. Let \((V, E, 1_r)\) be a root-reward tree. The \( \lambda \)-biased infinity harmonic function \( h(\lambda, \cdot) : V \to [0, 1] \) specified by (4) with \( p = 1_r \) takes the form
\[
h(\lambda, v) = \prod_{i=0}^{k} H(\lambda, s_i, d_i),
\]
(25)
where \( \{(s_i, d_i) : i \in [0, k]\} \) is the journey data of \( v \in V \).

Corollary 3.10. Let \((V, E, 1_r)\) be a root-reward tree. For \( v \in V \), \( h(\cdot, v) : [0, \infty) \to [0, 1] \) is increasing.

Proof. This follows directly from Lemma 3.9.

Proof of Lemma 3.9. An induction on \( k \in \mathbb{N} \). When \( k = 0 \), \( v \) lies on a path \([r, w]\) of minimum length between \( r \) and another leaf \( w \) in \( V \). As we saw at the start of the proof of Theorem 3.3, this path forms the first element in a \( \lambda^{-1} \)-valid decomposition. We see then that (25) holds in view of Proposition 3.2 and Lemma 3.8 applied to the line graph \([w, r]\). To prove the case of general \( k \in \mathbb{N}_+ \), denote the right-hand side of (25) by \( f(v) \), and note that
\[
f(v) = f(j_k)H(\lambda, s_k, d_k) = h(\lambda, j_k)H(\lambda, s_k, d_k)
\]
where the latter equality is due to the inductive hypothesis at index \( k - 1 \). We complete the induction by arguing that the right-hand expression equals \( h(\lambda, v) \). Indeed, when one constructs a \( \lambda^{-1} \)-valid decomposition and the corresponding \( \lambda^{-1} \)-extension from the basic partition, then the path containing \( v \) will have \( j_k \) as its upper end, with the already constructed value \( h(\lambda, j_k) \), and a non-reward leaf as its lower end, with value zero. The product formula for \( h(\lambda, v) \) readily follows.

We next state another simple consequence of the decomposition into basic trees. Recall the notation \( V_- (v) \) from Definition 1.7.

Definition 3.11. For \( v \in V_O \), let \( V_- (\lambda, v) \) denote the set of neighbours \( u \in V \) of \( v \) that attain the minimum of \( h(\lambda, \cdot) \) among these neighbours.

Proposition 3.12. Let \((V, E, 1_r)\) denote a root-reward tree. Fix \( \lambda \in (0, \infty) \) and let \( v \in V_O \).

1. The sets \( V_- (v) \) and \( V_- (\lambda, v) \) are equal.

2. Furthermore, \( h(\lambda, v_+) > h(\lambda, v) \).

Proof. In the basic partition of \( T = (V, E, 1_r) \), let \( B \) be the unique basic tree that contains \( v \) as a non-root vertex. Denote the root of \( B \) by \( j \), and let \( T_v \) denote the descendent tree of \( v \) (not
Consider the graph homomorphism Proposition 3.14. The following proposition is the main point of this construction.

By the construction of the $\lambda^{-1}$-extension using the basic partition (or computationally, by Lemma 3.9 and the fact that $H(\lambda, 1, d)$ is increasing in $d$), it is clear that the vertices in $\mathcal{V}_-(\lambda, v)$ are exactly the children of $v$ (all of them in $T_v$) that lead to the leaves of $B$ that lie in $T_v$. Furthermore, $v_+$ is the unique neighbour of $v$ with $h(\lambda, v_+) > h(\lambda, v)$. On the other hand, by definition, the vertices in $\mathcal{V}_-(v)$ are exactly the children of $v$ leading to $\mathcal{V}_-\infty(v)$. Since these two sets of leaves were proved to be equal in the previous paragraph, we also have equality of the sets of children: $\mathcal{V}_-(v) = \mathcal{V}_-(\lambda, v)$.

We will now represent the information contained in the basic partition and the journey data using a simplified tree. Figure 4 illustrates the forthcoming definition.

Definition 3.13. A basic tree $(V, E, 1_r)$ is called linear if $V_B \setminus \{r\}$ is a singleton. A linear basic tree is a copy of the line graph $L_s$, where $s$ is the span of the tree. A root-reward tree is essential if every tree $C$ in its basic partition is linear.

For any basic tree $(V, E, 1_r)$ with span $s$, there is a natural graph homomorphism $\phi$ onto the linear tree $(L_s, \sim, 1_s)$, given by $\phi(v) := s - d(r, v)$. We call this $\phi$ the linearization of the basic tree.

If $(V_k, E_k, 1_{rk})$ is an element in the basic partition of a root-reward tree $(V, E, 1_r)$, then note that its linearization $\phi_k$ naturally extends to a graph homomorphism $\overline{\phi_k}$ from $(V, E, 1_r)$, by acting as the identity on the complement of $(V_k, E_k, 1_{rk})$. Furthermore, $\overline{\phi_k}$ also establishes a natural bijection between the elements of the basic partition of $(V, E, 1_r)$ and those of $\overline{\phi_k}((V, E, 1_r))$: even though some roots may get identified under $\overline{\phi_k}$, no edges outside of $E_k$ are identified, and this bijection of edges induces the bijection between subtrees.

The essence tree $(V_{ess}, E_{ess}, 1_{res})$ of $(V, E, 1_r)$ is the root-reward tree given by taking the linearization $\phi_k$ of each basic tree in its basic partition, then composing all the extensions $\overline{\phi_k}$ in an arbitrary order. (By the identification of basic subtrees under each $\overline{\phi_k}$ explained in the previous paragraph, this composition is obviously commutative.) The final graph homomorphism from $(V, E, 1_r)$ to $(V_{ess}, E_{ess}, 1_{res})$ will be denoted by $\Phi$.

The following proposition is the main point of this construction.

Proposition 3.14. Consider the graph homomorphism $\Phi : (V, E, 1_r) \rightarrow (V_{ess}, E_{ess}, 1_{res})$ of Definition 3.13. Write $h_{ess}(\lambda, \cdot) : V_{ess} \rightarrow [0, 1]$ for the $\lambda$-biased infinity-harmonic function on the essence root-reward tree, for any $\lambda \in (0, \infty)$. Then, for every $v \in V$,

$$h(\lambda, v) = h_{ess}(\lambda, \Phi(v)).$$

Proof. By construction, the journey data for $v$ in $(V, E, 1_r)$ equals the journey data for $\Phi(v)$ in $(V_{ess}, E_{ess}, 1_{res})$. Thus the result follows from Lemma 3.9.

We are going to strengthen this connection between the $\lambda$-biased infinity-harmonic functions into a connection between the $\lambda$-biased random walks of Definition 1.14. We start with a couple of simple lemmas.
Lemma 3.15. Suppose that a root-reward tree \((V, E, 1_r)\) contains a vertex \(u \in V\) and leaves \(v, w \in V_B \setminus \{r\}\) for which there exist edge-disjoint paths \([u, v]\) and \([u, w]\) in the descendent tree of \(u\) whose lengths equal the distance between \(u\) and \(V_B \setminus \{r\}\). Then \((V, E, 1_r)\) is not an essential tree.

Proof. Consider the basic tree \(B\) in \((V, E, 1_r)\) that contains the first edge in \([u, v]\). By the iterative construction of the basic partition of \((V, E, 1_r)\) in Definition 3.1, \(B\) has root \(z \in V\) such that \([z, v]\) and \([z, w]\) are paths that contain \(u\) and whose lengths equal the distance between \(z\) and \(V_z \cap (V_B \setminus \{r\})\) in the descendent tree \(T_z = (V_z, E_z)\) of \(z\). Thus \(B\) contains the paths \([u, v]\) and \([u, w]\) and is non-linear. \(\square\)

Recall from Definition 1.7 that \(v_+\) is the parent of \(v \in V, v \neq r\), and that \(V_-(v)\) denotes the set of children of \(v\) whose distance to the non-reward boundary \(V_B \setminus \{r\}\) attains the minimum among these children.

Lemma 3.16. Let \(v \in V_O\).

1. We have that \((\Phi(v))_+ = \Phi(v_+).\)

2. For \(w \in V_{\text{ess}},\) the set \(V_-(w)\) is a singleton.

3. Denote the unique element of \(V_-(w)\) by \(w_-\). The intersection of the preimage of \((\Phi(v))_-\) under \(\Phi\) and the set of children of \(v\) equals \(V_-(v)\).

Proof: (1). The relation between parent and child is preserved at each linearization step in the construction of \(\Phi\) in Definition 3.13.

(2). The essence tree is essential, so Lemma 3.15 implies this statement.
(3). A child $x$ of $v$ such that $Φ(x) = (Φ(v))_-$ shares its journey data with $(Φ(v))_-$. Thus, such $x$ minimize the distance to the non-reward boundary among the children of $v$: that is, $x ∈ V_-(v)$. □

Recall from Definition 1.14 the $(1-ε)$-lazy $λ$-biased random walks $X_θ : [0, F] → V$ on root-reward trees, indexed by $θ ∈ Θ$. In the essence tree, $V_-(v)$ is a singleton $\{v_\}$ for each $v ∈ V_O$ by Lemma 3.16(2). The space $Θ$ is thus also a singleton in this case. We may denote the process $X_θ$, where $Θ = \{θ\}$, by $X_{ess}$. To be explicit, $X_{ess} : [0, F_{ess}] → V_{ess}$ is the Markov process such that $X_{ess}(0) = v ∈ V_{ess}$ and, for $k ∈ N$ such that $X_{ess}(k) ∈ V_O$,

$$X_{ess}(k+1) = \begin{cases} X_{ess}(k) & \text{with probability } 1-ε, \\ (X_{ess}(k))_+ & \text{with probability } ε \frac{λ}{1+λ}, \\ (X_{ess}(k))_- & \text{with probability } ε \frac{1}{1+λ}. \end{cases}$$

**Proposition 3.17.** Let $(V, E, 1_r)$ be a root-reward tree.

1. The processes $X_θ$, $X_θ(0) = v ∈ V_O$, are such that the $V_{ess}$-valued processes $Φ ∘ X_θ : [0, F_θ] → V_{ess}$ are each equal in law to $X_{ess} : [0, F_{ess}] → V_{ess}$, $X_{ess}(0) = Φ(v)$.

2. The value

$$E \text{TotVar}(ε, θ, λ, v) = E \sum_{i=0}^{F_θ-1} \left( h(λ, (X_θ(i))_+) - h(λ, θ(λ, X_θ(i), i)) \right)$$

is independent of $θ ∈ Θ$.

Proposition 3.17(2) is an important step to resolving the second criticism of the heuristic perturbation argument, which was voiced in Subsection 2.2.2; the denominator in (6) has a well-defined mean.

**Proof:** (1). We prove by induction on $k ∈ N$ that the restricted process $Φ ∘ X_θ : [0, k ∧ F_θ] → V_{ess}$ coincides in law with $X_{ess} : [0, k ∧ F_θ] → V_{ess}$, $X_{ess}(0) = Φ(v)$. The base case $k = 0$ is trivial. Suppose the inductive hypothesis at index $k$. Given that $X_θ(k) = w ∈ V_O$, the value $X_θ(k+1)$ equals $w_+$, $w$ or some element of $V_-(w)$ with respective probabilities $ε \frac{λ}{1+λ}$, $1-ε$ and $ε \frac{1}{1+λ}$. Given that $X_{ess}(k) = Φ(w)$, the value $X(k+1)$ equals $(Φ(w))_+$, $Φ(w)$ or $(Φ(w))_-$ with the same respective probabilities. Thus Lemma 3.16 shows that the processes $Φ ∘ X_θ$ and $X_{ess}$ share the conditional distribution of their value at time $k+1$ whenever these processes are conditioned to assume any given value in $V_O$ at time $k$. Thus we obtain the inductive hypothesis at index $k+1$.

(2). Let $θ ∈ Θ$. By the preceding part, we may find a coupling so that $Φ ∘ X_θ$ equals $X_{ess}$ almost surely. By Proposition 3.14 and Lemma 3.16(1), we see then that, almost surely,

$$h(λ, (X_θ(i))_+) = h(λ, (X_{ess}(i))_+).$$

By Proposition 3.14, Lemma 3.16(3) and Proposition 3.12 we also have that

$$h(λ, θ(λ, X_θ(i), i)) = h(λ, (X_{ess}(i))_-)$$

almost surely. Thus,

$$E \text{TotVar}(ε, θ, λ, v) = E \sum_{i=0}^{F_θ-1} \left( h(λ, (X_{ess}(i))_+) - h(λ, (X_{ess}(i))_-) \right)$$

35
Thus, for moves made by $\gamma$, the set of paths $(J, \lambda, 0)$.

Next we state and prove bounds on the $\lambda$-derivatives. We have not tried to optimize the exponents of $|V|$; what will be important for us is that there are bounds that depend only on the tree, not on $\lambda$.

**Lemma 3.18** (Value derivative bounds). There exists a constant $C > 0$, which is dependent neither on the root-reward tree $(V, E, 1_r)$ nor on $\lambda \in (0, \infty)$, such that

1. $0 \leq \frac{\partial}{\partial x} h(\lambda, v) \leq C|V|^2$, and
2. $|\frac{\partial^2}{\partial x^2} h(\lambda, v)| \leq C|V|^4$.

Let $J$ denote a compact subset of $(0, \infty)$. There exists a constant $c > 0$, which may depend on the tree $(V, E, 1_r)$ as well as on $J$, such that, for any $\lambda \in J$,

3. $\frac{\partial}{\partial x} h(\lambda, v) \geq c$.

**Proof:** (1,2). Consider first the case of the line graph $L_n$. Let $i \in [1, n - 1]$. Let $\Gamma_k(i)$ denote the set of paths $\gamma : [0, k] \rightarrow [0, n]$ such that $\gamma_0 = i, \gamma_k = n$ and $\gamma_{\ell} \notin \{0, n\}$ if $\ell \in [1, k - 1]$. For $\gamma \in \Gamma_k(i)$, set $p_\gamma(\lambda) = \lambda^j(\lambda + 1)^{-k}$, where $j = (k + n - i)/2 \in [0, k]$ is the number of rightward moves made by $\gamma$. Note that

$$h(\lambda, i) = \sum_{k=1}^{\infty} q_k(\lambda, i)$$

where $q_k(\lambda, i)$ equals the probability $P_i(X_F = n, F = k)$ that $\lambda$-biased walk $X : [0, F] \rightarrow [0, n]$, $X(0) = i$, on $L_n$ finishes at $n$ at time $k$. Note that $q_k(\lambda, i) = \sum_{\gamma \in \Gamma_k(i)} p_\gamma(\lambda) = |\Gamma_k(i)| \lambda^j(\lambda + 1)^{-k}$.

Thus, we have that

$$\frac{\partial}{\partial x} q_k(\lambda, i) = \left( \frac{j}{\lambda^2} - \frac{k}{\lambda + 1} \right) q_k(\lambda, i)$$

and

$$\frac{\partial^2}{\partial x^2} q_k(\lambda, i) = \left( \frac{j(j-1)}{\lambda^4} + \frac{k(k+1)}{(1+\lambda)^2} - \frac{2jk}{\lambda(1+\lambda)} \right) q_k(\lambda, i).$$

Thus, for $\lambda \geq 1$,

$$\left| \frac{\partial}{\partial x} q_k(\lambda, i) \right| \leq 2kq_k(\lambda, i)$$

and

$$\left| \frac{\partial^2}{\partial x^2} q_k(\lambda, i) \right| \leq \left( k(k-1) + k(k+1) + 2k^2 \right) q_k(\lambda) = 4k^2 q_k(\lambda, i).$$

Furthermore, we have

$$\sum_{\ell=1}^{\infty} q_\ell(\lambda, i) \leq P(F \geq k) \leq C \exp\{-ck/n^2\}.$$
starting position in \([1, n - 1]\), there is a uniformly positive probability for the walk to hit \(n\) (using \(\lambda \geq 1\) again).

The preceding displayed bounds together show that \(\frac{\partial}{\partial \lambda} q_k(\lambda, i)\) and \(\frac{\partial^2}{\partial \lambda^2} q_k(\lambda, i)\) are summable in \(k\), uniformly in \(\lambda \geq 1\). Therefore, by [Tao16, Corollary 3.7.3], we find that \(h(\lambda, i)\) is twice differentiable in \(\lambda\), with

\[
0 \leq \frac{\partial}{\partial \lambda} h(\lambda, i) \leq 2 \sum_{k=1}^{\infty} k q_k(\lambda, i) = 2 \sum_{k=1}^{\infty} \sum_{\ell=k}^{\infty} q_\ell(\lambda, i) \leq 2 \sum_{k=1}^{\infty} C \exp\{-c k/n^2\} \leq C' n^2
\]

and

\[
\left| \frac{\partial^2}{\partial \lambda^2} h(\lambda, i) \right| \leq 4 \sum_{k=1}^{\infty} k^2 q_k(\lambda, i) = 4 \sum_{k=0}^{\infty} (2k + 1) \sum_{\ell=k+1}^{\infty} q_\ell(\lambda, i) \leq \sum_{k=0}^{\infty} C k \exp\{-c k/n^2\} \leq C' n^4,
\]

where the identities in the middle use summation by parts, while the lower bound on \(\frac{\partial}{\partial \lambda} h(\lambda, i)\) follows from Corollary [3.10]. Since \(|V| = n + 1\) for the graph \(L_n\), we have obtained Lemma [3.18(1,2)] for line graphs when \(\lambda \geq 1\). The case for line graphs when \(\lambda \in (0, 1)\) is treated by noting that \(h(\lambda, i) = 1 - h(\lambda - 1, n - i)\).

We now consider the general case of a root-reward tree \((V, E, r)\). Let \(v \in V_0\). In view of Lemma [3.9], we may make use of the product formula (25) for \(h(\lambda, v)\). Applying the just established assertion for line graphs, and that the factors \(H(\lambda, \cdot)\) in (25) are at most one, we find that

\[
0 \leq \frac{\partial}{\partial \lambda} h(\lambda, i) \leq C' \sum_{i=0}^{k} s_i^2
\]

and

\[
\left| \frac{\partial^2}{\partial \lambda^2} h(\lambda, i) \right| \leq C' \sum_{i,j=0}^{k} s_i^2 s_j^2,
\]

where note that the journey data \(\{(s_i, d_i) : i \in [0, k]\}\) of \(v\) satisfies \(\sum_{i=0}^{k} s_i \leq |V|\). The two right-hand sides are maximized by the choice \(k = 0\) and \(s_0 = |V|\). In this way, we obtain Lemma [3.18(1,2)] in the general case.

(3). This result follows from Lemma [3.9] and a computation of the derivative of \(H(\lambda, k, \ell)\) from Definition [3.7].

### 3.4 Further lemmas concerning game value

We record two simple results that we will need.

**Lemma 3.19.** Let \(K \subset (0, \infty)\) be compact. There exists a positive constant \(c = c(K, |V|)\) such that \(h(\lambda, v) - h(\lambda, v) \geq c\) and \(h(\lambda, v) - h(\lambda, v) \geq c\) for all \(v \in V_0\) and \(\lambda \in K\). (Here, \(v\) denotes any child of \(v\).)

**Proof.** Since \(h(\lambda, v) > h(\lambda, v)\) by Proposition [3.12(2)], it suffices to prove that \(h(\lambda, v) - h(\lambda, v)\) is uniformly positive under the hypotheses of the lemma. This difference is positive by Proposition [3.12(2)]. It is also continuous in \(\lambda\) (and even differentiable—see Lemma [3.18]), so the compactness of \(K\) yields the claim.
Lemma 3.20. Let $\epsilon \in (0,1]$. Set $\lambda(\epsilon) = \frac{\lambda - \epsilon a}{1 - \epsilon b}$. Suppose that $\lambda/2 \geq \epsilon a \geq 0$ and $1/2 \geq \epsilon b \geq 0$.

1. We have that $\lambda/2 \leq \lambda(\epsilon) \leq 2\lambda$.
2. Let $K > k > 0$. If $\lambda \in [k, K]$, then $\lambda(\epsilon) \in [k/2, 2K]$.

Proof: (1). Note that $\lambda(\epsilon) \geq \lambda - \epsilon a \geq \lambda/2$ and $\lambda(\epsilon) \leq \lambda - \epsilon b \leq 2\lambda$.

(2). This follows directly from the preceding part.

3.5 The stake function computed explicitly

Here is the explicit formula for the stake function that we promised at the end of the introduction.

Theorem 3.21. For $\lambda \in (0, \infty) \setminus \{1\}$ and $\ell \in \mathbb{N}$, set $\Psi(\lambda, \ell) = \frac{\lambda - \ell}{1 - \lambda \ell}$. Let $(V, E, 1_r)$ be a root-reward tree. Let $v \in V$ have journey data $\{ (s_i, d_i) : i \in [0, k] \}$ for $k \in \mathbb{N}$.

1. When $\lambda \in (0, \infty) \setminus \{1\}$, we have that

$$\text{Stake}(1, \lambda, v) = \frac{(\lambda - 1)\Psi(\lambda, s_k - d_k)}{(\lambda + 1) \sum_{i=0}^{k} ((s_i - d_i)\Psi(\lambda, s_i - d_i) - s_i\Psi(\lambda, s_i))}.$$ 

2. When $\lambda = 1$,

$$\text{Stake}(1, 1, v) = \left( (s_k - d_k) \sum_{i=0}^{k} d_i \right)^{-1}.$$

Before attempting the proof, we derive the stake function on line graphs.

Proof of Proposition 1.17. Here, $h(\lambda, i)$ is evaluated for the line graph $(L_n, \sim, 1_n)$ at vertex $i \in [1, n-1]$.

(1). We have that $h(\lambda, i)$ equals $H(\lambda, n-i, n)$ from Definition 3.7, where the reversed position $n-i$ is considered because, in the proposition, the root of $L_n$ is at $n$, whereas this root is at zero in Lemma 3.8. The sought formula follows from this lemma and by a computation of (5).

(2). This follows by computing the right-hand side of the formula in the first part. A game-theoretic interpretation is available, given Theorem 1.12. If Maxine and Mina switch roles, the root-reward tree $L_n$ is reflected about $n/2$, and the initial fortune $\lambda$ is replaced by $\lambda^{-1}$, then the payment made or received by either player is unchanged in law. Thus the sought formula reduces to that in the preceding part.

(3). The function $(0, \infty) \rightarrow [0,1] : \lambda \mapsto h(\lambda, i)$ is seen to be continuously differentiable in view of Lemma 3.8 when $\lambda = 1$, $\frac{\partial}{\partial \lambda} h(\lambda, i)$ equals $\frac{i(n-i)}{2n}$, as we find by computing $\lim_{\lambda \downarrow 1} \frac{\partial}{\partial \lambda} h(\lambda, i)$. The claimed stake formula thus results by computing the limit as $\lambda \downarrow 1$ of the formula obtained in the proposition’s first part.

(4). In computing (5), a factor of $p(n) - p(0)$ appears in numerator and denominator, so that the formula coincides with the special case already treated. Alternatively, a game-theoretic interpretation, which is contingent on Theorem 1.12, if we deduct the constant $p(0)$ from Pay, and then revalue currency so that Pay $\rightarrow$ Pay/$(p(n) - p(0))$, we do not affect strategy. 

38
Proof of Theorem 3.21 (1). When $\lambda \in (0, \infty)$ is not equal to one, we compute terms in the expression $\text{Stake}(1, \lambda, v)$ as specified in [5] by means of the formula for $h(\lambda, v)$ in Lemma 3.9. Whatever the value of $\lambda \in (0, \infty)$, the numerator in the formula satisfies

$$\Delta(\lambda, v) = (H(\lambda, s_k, d_k - 1) - H(\lambda, s_k, d_k + 1)) \Pi_{i=0}^{k-1} H(\lambda, s_i, d_i);$$

(26)

when $\lambda \neq 1$, it thus equals

$$\frac{\lambda^{-(s_k-d_k)}}{1-\lambda^{-s_k}} (\lambda - \lambda^{-1}) \Pi_{i=0}^{k-1} H(\lambda, s_i, d_i) = \frac{\lambda^{-(s_k-d_k)}}{1-\lambda^{-s_k}} (\lambda - \lambda^{-1}) h(\lambda, v).$$

That is, $\Delta(\lambda, v) = \Psi(\lambda, s_k - d_k)(\lambda - \lambda^{-1}) h(\lambda, v)$. To compute the denominator of (26), note that

$$\frac{\partial}{\partial \lambda} h(\lambda, v) = h(\lambda, v) \sum_{i=0}^{k} \frac{\partial}{\partial \lambda} \log H(\lambda, s_i, d_i) = h(\lambda, v) \lambda^{-1} \sum_{i=0}^{k} ((s_i - d_i) \Psi(\lambda, s_i - d_i) - s_i \Psi(\lambda, s_i))^2.$$

Dividing the obtained expression for $\Delta(\lambda, v)$ by the product of $(\lambda + 1)^2$ and the last display, we obtain the formula for $\text{Stake}(1, \lambda, v)$ asserted by Theorem 3.21 (1).

(2). Now $\lambda$ equals one. By (26) with $\lambda = 1$, we find that $\Delta(\lambda, v) = 2 s_k^{-1} \Pi_{i=0}^{k-1} H(1, s_i, d_i)$. Since $H(1, s_k, d_k)$ equals $d_k / s_k$, we obtain $\Delta(\lambda, v) = 2(s_k - d_k)^{-1} h(1, v)$.

We showed in the proof of Proposition 1.17(3) that $\frac{\partial}{\partial \lambda} H(\lambda, s, d)|_{\lambda=1} = \frac{d(s-d)}{2s}$ for $s \in \mathbb{N}_+$ and $d \in [0, s]$. Since $H(1, s, d) = (s-d) / s$, $\frac{\partial}{\partial \lambda} \log H(\lambda, s, d)|_{\lambda=1} = d/2$. Thus,

$$\frac{\partial}{\partial \lambda} h(\lambda, v)|_{\lambda=1} = h(\lambda, v) \sum_{i=0}^{k} \frac{\partial}{\partial \lambda} \log H(\lambda, s_i, d_i)|_{\lambda=1} = 2^{-1} h(1, v) \sum_{i=0}^{k} d_i.$$

From these inputs, we obtain Theorem 3.21(2). \qed

We may now derive the second consequence of Theorem 3.21 stated in the introduction.

Proof of Proposition 1.19. We may write the quantities $d_-(v)$ and $d_+(v)$ from Definition 1.18 in terms of the journey data $\{(s_i, d_i) : i \in [0, k]\}$ of $v \in V$. Indeed, it is readily verified that $d_-(v) = s_k - d_k$ and $d_+(v) = \sum_{i=0}^{k} d_i$. For $l \in \mathbb{N}_+$, $\lim_{\lambda \to 0} \Psi(\lambda, \ell) = -1$. The first two assertions in Proposition 1.19 arise from Theorem 3.21(1) in view of these facts. The third assertion is implied by the upcoming corollary. \qed

The next result treats the high $\lambda$ asymptotic for the stake function.

Corollary 3.22. Let $(V, E, 1_r)$ be a root-reward tree, and let $v \in V$ have journey data $\{(s_i, d_i) : i \in [0, k]\}$. Let $J$ denote the set of $j \in [0, k]$ such that $s_j - d_j$ is minimal. As $\lambda \to \infty$,

$$\text{Stake}(1, \lambda, v) = \frac{1}{|J|(s_i - d_i)} \cdot \lambda^{s_i - d_i - (s_k - d_k)} (1 + O(\lambda^{-1})),

where $i$ is any element of $J$.

Proof. This is due to Theorem 3.21(1); $d_i > 0$ for each $i \in [0, k]$; and $\lim_{\lambda \to \infty} \Psi(\lambda, \ell) \lambda^\ell = 1$ for $\ell \in \mathbb{N}_+$.

A further consequence of Theorem 3.21 will be needed.
Proposition 1.19. Since the function is strictly positive, it is bounded away from zero uniformly on $\lambda(0, L)$. 

Proof. Since Stake($\lambda, \lambda, v$), in which Stake($\lambda, \lambda, v$) is continuous: this follows directly from Theorem 3.21(1) when $\lambda \neq 1$, and from a short computation involving both parts of this theorem in the remaining case. The function may be extended to a continuous function on $[0, \infty)$ by the first assertion of Proposition 1.19. Since the function is strictly positive, it is bounded away from zero uniformly on $[0, L]$ for any $L > 0$. \hfill \Box

4 Finite-horizon games: the basics

Let $n \in \mathbb{N}_+$. In Game$_{\epsilon}(\lambda, \lambda, v)$, the rules of Game($\lambda, \lambda, v$) are followed, but the game is forcibly ended at the end of the $n^{th}$ turn if the counter has then yet to arrive in $V_B$ (so that $X_n \in V_O$). If the game ends for this reason, the value of Pay equals $h(\lambda, X_n)$. A given payment—say zero or one—would seem to be a more basic choice, but our choice will in essence permit us to view Game$_{\epsilon}(\lambda, \lambda, v)$ as a truncation of Game($\lambda, \lambda, v$) to the first $n$ turns without the truncation introducing any mean error.

To modify the definition of the mixed and pure strategy spaces $S_{\pm}$ and $P_{\pm}$ in treating the finite game Game$_{\epsilon}(n)$ for $n \in \mathbb{N}_+$, we merely replace the turn index set $\mathbb{N}_+$ by $[1, n]$ in Definition 1.3 calling the new spaces $S_{\pm}(n)$ and $P_{\pm}(n)$. With this change made, the value Val$_n(\epsilon, \lambda, v)$ of Game$_{\epsilon}(n, \lambda, v)$ remains specified by Definition 1.4.

Definition 4.1. For $\epsilon \in (0, 1]$ and $n \in \mathbb{N}$, let $h_n(\epsilon, \lambda, v) : (0, \infty) \times V \rightarrow (0, \infty)$ be iteratively specified by $h_0(\epsilon, \lambda, v) = h(\lambda, v)$ for $v \in V$ and, for $n \in \mathbb{N}_+$,

$$h_n(\epsilon, \lambda, v) = \begin{cases} p(v) & \text{for } v \in V_B, \\
\frac{\epsilon \lambda}{\lambda + 1} \max_{u \sim v} h_{n-1}(\epsilon, \lambda, u) + \frac{\epsilon}{\lambda + 1} \min_{u \sim v} h_{n-1}(\epsilon, \lambda, u) + (1 - \epsilon) h_{n-1}(\epsilon, \lambda, v) & \text{for } v \in V_O.\end{cases}$$

Lemma 4.2. Let $(V, E, p)$ be a boundary-payment graph. For $\epsilon \in (0, 1]$, $\lambda \in [0, \infty)$ and $v \in V$, $h_n(\epsilon, \lambda, v)$ equals $h(\lambda, v)$.

Proof. An induction on $n \in \mathbb{N}$. The base case $n = 0$ holds by definition. Let $n \in \mathbb{N}_+$. The inductive hypothesis at index $n - 1$ and Definition 1.4 imply that

$$h_n(\epsilon, \lambda, v) = \begin{cases} 0 & \text{for } v \in V_B, \\
\frac{\epsilon \lambda}{\lambda + 1} \max_{u \sim v} h(\lambda, u) + \frac{\epsilon}{\lambda + 1} \min_{u \sim v} h(\lambda, u) - \epsilon h(\lambda, v) & \text{for } v \in V_O.\end{cases}$$

Thus, the inductive hypothesis at index $n$ holds by 1.4. \hfill \Box

Definition 4.3. For $v \in V_O$, let $V_+(\lambda, v)$ denote the set of neighbours in $V$ of $v$ that attain the maximum of $h(\lambda, \cdot)$ among these neighbours. Let $V_-(\lambda, v)$ be the counterpart where the minimum is instead considered.

In root-reward trees, $V_+(\lambda, v)$ is a singleton whose element is the parent $v_+$, for any $v \in V$. This property transmitted to the specification of lazy biased walks that we saw in Definition 1.14. We are led to a redefinition of these walks in the more general context of boundary-payment graphs (which, briefly at least, we now consider), in which $V_+(\lambda, v)$ may have several elements; we also work now with a finite horizon $n$. 40
Definition 4.4. Let $\Theta_n(\lambda)$ denote the set of mappings $(\theta_-, \theta_+) : V_O \times [0, n - 1] \longrightarrow V \times V$ such that $\theta_-(v, i) \in V_-(\lambda, v)$ and $\theta_+(v, i) \in V_+(\lambda, v)$ for each $v \in V_O$. For $\theta = (\theta_-, \theta_+) \in \Theta_n(\lambda)$, let $X_\theta : [0, F] \longrightarrow V$ denote the Markov process such that $X_\theta(0) = v \in V_O$ and, for $k \in \mathbb{N}_+$,

$$X_\theta(k + 1) = \begin{cases} X_\theta(k) \text{ with probability } 1 - \epsilon, \\ \theta_+(X_\theta(k), k) \text{ with probability } \epsilon \frac{\lambda}{1 + \lambda}, \\ \theta_-(X_\theta(k), k) \text{ with probability } \epsilon \frac{1}{1 + \lambda}. \end{cases}$$

The process is stopped either at time $n$ or on arrival at $V_B$, so that $F$ is the minimum of $n$ and $\min \{ j \geq 0 : X_\theta(j) \in V_B \}$.

Definition 4.5. Let $(V, E, p)$ be a boundary-payment graph. Let $\epsilon \in (0, 1], q \in [0, 1]$ and $v \in V_O$. The leisurely version $\text{TugOfWar}(\epsilon, q, v)$ of tug-of-war may be specified by adapting Definition 1.2. At any given turn, a move takes place with probability $\epsilon$, the decision taken independently of other randomness; if a move does take place, the existing rules apply.

Lemma 4.6. Let $(V, E, p)$ be a boundary-payment graph. A pure Nash equilibrium exists for the game $\text{TugOfWar}_n(\epsilon, \frac{\lambda}{1 + \lambda}, v)$, the finite horizon version of the one in Definition 1.5. Under any pure Nash equilibrium, the gameplay process has the law of $X_\theta : [0, F] \longrightarrow V$ for some $\theta \in \Theta_n(\lambda)$. The mean payment at any such equilibrium equals $h(\lambda, v)$.

Proof. We prove this by induction on $n$. If Maxine and Mina propose respective random moves $v_+$ and $v_-$ at the first turn in $\text{TugOfWar}_n(\epsilon, \frac{\lambda}{1 + \lambda}, v)$, and then adhere to strategies in a Nash equilibrium, then, by the inductive hypothesis for index $n - 1$, the mean payment in the resulting gameplay will equal the mean of

$$\epsilon \frac{\lambda}{1 + \lambda} h(\lambda, v_+) + \epsilon \frac{1}{1 + \lambda} h(\lambda, v_-) + (1 - \epsilon) h(\lambda, v),$$

where the mean is taken over the randomness in their move choices. We see then that it is necessary and sufficient for a pure strategy pair to be a Nash equilibrium that $v_+ \in V_+(\lambda, v)$ and $v_- \in V_-(\lambda, v)$ hold alongside the players adhering jointly to play at a pure Nash equilibrium in the subgame copy of $\text{TugOfWar}_{n-1}(\epsilon)$ that takes place after the first turn. By taking $v_+$ and $v_-$ to be deterministic elements of the respective sets, we obtain the inductive hypothesis’s claim about the form of Nash equilibria; in view of (4), the mean payment formula then results from the displayed equation for the above such choices of $v_+$ and $v_-$. \qed

5 The lambda-derivative of game value: proving Proposition 1.16

Here we use probabilistic reasoning for classical tug-of-war inspired by the perturbative argument presented heuristically for the stake-governed version in Section 2.2 in order to prove that $\frac{\partial h}{\partial \lambda}(\lambda, v)$ equals $\lambda^{-2} \mathbb{E} \text{TotVar}(\epsilon, \lambda, \theta, v)$. In order to prove Proposition 1.16 of which the above statement forms part, we will first argue that $\text{Stake}(\epsilon, \lambda, v)$ from (5) has an alternative, $n$-dependent, form $\text{Stake}_n(\epsilon, \lambda, v)$. We thus have cause to introduce some notation that relates the infinite-horizon game with its finite-horizon counterpart. The usage of the notation $\cdot | n$ will be characteristic when truncation of an aspect of infinite gameplay to the first $n$ turns is concerned.

Definition 5.1. Let $\theta$ be an element of the set $\Theta$ specified in Definition 1.14. For $\lambda \in (0, \infty)$, $v \in V_O$ and $n \in \mathbb{N}_+$, let $X_{\theta|n}(\lambda, \cdot) : [0, F_{\theta|n}] \longrightarrow V$, $X_{\theta|n}(0) = v$, be given by $X_{\theta|n}(\lambda, k) = X_{\theta}(\lambda, k)$.
for \( k \in [0, F_{\theta|n}] \). Here the truncated finish time \( F_{\theta|n} \) equals \( F_{\theta} \land n \). If the counter is in open play at time \( n \), so that \( X_{\theta}(n) \in V_O \), formally set \( F_{\theta|n} \) equal to \( n + 1 \).

Extending the shorthand in Definition 1.14, we will write \( X_{\theta|n}(\cdot) = X_{\theta|n}(\lambda, \cdot) \).

To specify the quantity Stake\(_n\)(\( \epsilon, \lambda, v \)), let \( \theta \in \Theta \). For \( \lambda \in (0, \infty) \), set  

\[
\text{TotVar}(\epsilon, \lambda, \theta|n, v) = \sum_{i=0}^{F_{\theta|n}-1} \Delta(\lambda, X_{\theta|n}(i)),
\]

where recall that \( X_{\theta|n}(0) = v \), and \( \Delta(\lambda, u) = h(\lambda, u_+) - h(\lambda, u_-) \) with \( u_- \in \mathcal{V}_-(u) \). A device in fact permits simpler notation. We specify \( X_{\theta}(i) \) to equal its terminal value \( X_{\theta}(F_{\theta}) \) whenever \( i > F_{\theta} \); thus, \( X_{\theta} \) is now a random map from \( \mathbb{N} \) to \( V \). And we set \( \Delta(\lambda, v) = 0 \) for \( v \in V_B \), so that \( \Delta(\lambda, \cdot) \) now maps \( V \) to \([0, 1]\). Under these conventions,

\[
\text{TotVar}(\epsilon, \lambda, \theta|n, v) = \sum_{i=0}^{n-1} \Delta(\lambda, X_{\theta}(i)).
\]

(27)

This quantity is a finite-horizon counterpart to \( \text{TotVar}(\epsilon, \lambda, \theta, v) \), which is defined before Proposition 1.16. We set

\[
\text{Stake}_n(\epsilon, \lambda, v) = \frac{\Delta(\lambda, v)}{\text{TotVar}(\epsilon, \lambda, \theta|n, v) + \epsilon^{-1}(\lambda + 1)^2 \sum_{w \in V_O} \frac{\partial}{\partial \lambda} h(\lambda, v) \cdot \mu_n(v, w)},
\]

where \( \mu_n(v, w) = \mathbb{P}(X_{\theta}(n) = w) \), so that \( \{\mu_n(v, w) : w \in V_O\} \) is the sub-probability measure of the counter’s location in the event that the game is unfinished at time \( n \).

(28)

(It is perhaps worth noting that, when \( n \) is high, Stake\(_n\)(\( \epsilon, \lambda, v \)) is equal to the product of \( \epsilon \) and a unit-order quantity. Indeed, \( \text{TotVar}(\epsilon, \lambda, \theta|n, v) \) typically reports \( \Theta(\epsilon^{-1}) \) terms for each jump that \( X_{\theta} \) makes, because this process is a \((1 - \epsilon)\)-lazy walk. The implied unit-order quantity is apparent in \((5)\).)

The next result shows that the new stake formula \((28)\) equals the original one \((5)\). Recall from Proposition 1.9 that the function \((0, \infty) \rightarrow (0, 1) : \lambda \mapsto h(\lambda, v) \) is differentiable for \( v \in V_O \).

**Proposition 5.2.**

1. For \( n \in \mathbb{N}_+ \),

\[
\frac{\partial}{\partial \lambda} h(\lambda, v) = \frac{\epsilon}{(\lambda + 1)^2} \cdot \mathbb{E} \text{TotVar}(\epsilon, \lambda, \theta|n, v) + \sum_{w \in V_O} \frac{\partial}{\partial \lambda} h(\lambda, w) \cdot \mu_n(v, w).
\]

2. Let \( \theta \in \Theta \). The expression \( \mathbb{E} \sum_{i=0}^{\infty} \Delta(\lambda, X_{\theta}(\epsilon, \lambda, i)) \) is finite. For each \( n \in \mathbb{N}_+ \), it equals

\[
\mathbb{E} \text{TotVar}(\epsilon, \lambda, \theta|n, v) + \epsilon^{-1}(\lambda + 1)^2 \sum_{w \in V_O} \frac{\partial}{\partial \lambda} h(\lambda, w) \cdot \mu_n(v, w).
\]

(29)

3. This expression also equals \( \epsilon^{-1}(\lambda + 1)^2 \frac{\partial}{\partial \lambda} h(\lambda, v) \).

42
4. The stake function (28) is independent of $n \in \mathbb{N}_+$ and equals \text{Stake}(\epsilon, \lambda, v)$ as specified in (5).

**Proof of Proposition 1.16.** Recall that $\mathbb{E}_{\text{TotVar}}(\epsilon, \lambda, v) = \mathbb{E} \sum_{i=0}^{\infty} \Delta(\lambda, X_\theta(i))$. By Proposition 5.2(3), this expression equals $\epsilon^{-1}(\lambda + 1)^2 \frac{\partial}{\partial \lambda} h(\lambda, v)$. Naturally then the alternative stake formulas (6) and (7) are obtained from the original formula (5).

**Proof of Proposition 5.2:** (1). By Lemma 4.2, $h(\lambda, v)$ equals $h_n(\epsilon, \lambda, v)$. The latter quantity is the value of TugOfWar$_n(\epsilon, \frac{\lambda}{1+\lambda}, v)$ by Lemma 4.6. It is this interpretation that we exploit in this proof. Let $\theta \in \Theta$. By Lemma 4.6 there exists a Nash equilibrium in TugOfWar$_n(\epsilon, \frac{\lambda}{1+\lambda}, v)$ such that the counter evolution has the law $X_{\theta|n} : [0, F_{\theta|n}] \rightarrow V$, $X_{\theta|n}(0) = v$. (The notational convention leading to (27) will however be in force, so that the domain of this process may formally be taken to be $[0, n]$.) We provide a coupling of these counter evolutions that permits them to be realized simultaneously for differing values of $\lambda$. To this end, let \{$U_i : i \in [0, n-1]$\} denote an independent sequence of random variables, each having the uniform law on $[0, 1]$. Let $i \in [1, n]$. If a move takes place at the $i$th turn in the coupled copy of TugOfWar$_n(\epsilon, \frac{\lambda}{1+\lambda}, v)$, then the condition that ensures that it is Maxine who wins the right to move is that $U_{i-1} \leq \frac{\lambda}{1+\lambda}$.

Let $\phi > 0$. In order to study the difference $h_n(\epsilon, \lambda + \phi, v) - h_n(\epsilon, \lambda, v)$, we call the coupled copy of the $\lambda$-biased game the original game; and the $(\lambda + \phi)$-biased game, the alternative game. In addition, we specify the intermediate game. To do so, let the disaccord set $D \subseteq [1, n]$ denote the set of indices of turns at which a move occurs that Maxine wins in the alternative game, but not in the original one. Then let $\sigma \in [1, n]$ denote the minimum of $D$. That is, $\sigma$ is the smallest $i \in [1, n]$ such that $U_i \in (\frac{\lambda}{1+\lambda}, \frac{\lambda+\phi}{1+\phi+1}]$. (Which is to say, in essence: $\sigma$ is the first turn that Maxine wins in the alternative game but not in the original one. Note however that $\sigma$ is defined even if the game has already finished by the time that its value is reached. This device will permit a convenient independence property.) If $D$ is empty, set $\sigma = n + 1$. Gameplay in the intermediate game follows that of the alternative game until move $\sigma$ when, at least if $\sigma \leq n$, this gameplay diverges from that in the original game; after this, the gameplay coincides with that in the original game. In other words, the cutoff $\chi$ in the condition $U_{i-1} \leq \chi$ takes the value $\chi = \frac{\lambda+\phi}{\lambda+\phi+1}$ until move $i = \sigma$ and then changes to $\chi = \frac{\lambda}{\lambda+1}$ for higher-indexed moves.

With the indices 0, 1 and 2 denoting the original, intermediate and altered games, write $P_i$, $0 \leq i \leq 2$, for the mean value of Pay under the corresponding game. As we have noted, $h(\lambda, v) = h_n(\lambda, v)$ is the value of TugOfWar$_n(\epsilon, \frac{\lambda}{1+\lambda}, v)$; thus,

$$P_0 = h(\lambda, v) \quad \text{and} \quad P_2 = h(\lambda + \phi, v).$$

Let $j \in [1, n]$. The occurrence of $\sigma = j$ imparts no information regarding the trajectory $X_\theta$ for any value of $\lambda$, because $\sigma$ takes the form $G \land (n + 1)$ where $G \geq 1$ is a geometric random variable of parameter $\frac{\lambda+\phi}{\lambda+\phi+1} - \frac{\lambda}{\lambda+1}$ that is independent of other randomness. We see then that

$$\mathbb{E} \left[(P_1 - P_0) 1_{\sigma = j}\right] = \mathbb{E} \left[(P_1 - P_0) \mid \sigma = j\right] \mathbb{P}(\sigma = j) = \mathbb{E} \Delta(\lambda, X_\theta(j-1)) \mathbb{P}(\sigma = j),$$

where the just noted independence was used in the latter equality to write $\mathbb{E} \left[(P_1 - P_0) \mid \sigma = j\right]$ as the mean $\mathbb{E} \Delta(\lambda, X_\theta(j-1))$ of the difference in terminal payment of $X_\theta$ with parameter $\lambda$ according to whether Maxine or Mina wins at turn $j$, the mean taken over the location $X_\theta(j-1)$ as specified in the original dynamics (and thus with parameter $\lambda$). Recalling $F_{\theta|n}$ from Definition 5.1, note then
that

\[ P_1 - P_0 = \mathbb{E} \sum_{i=0}^{n-1} \Delta(\lambda, X_0(i)) \zeta_i + \mathbb{P}(\sigma = n+1) \mathbb{E} \left[ (h(\lambda + \phi, X_0(n)) - h(\lambda, X_0(n))) 1_{F_{\theta, n+1}} \right], \]

where \( \zeta_i = \mathbb{P}(\sigma = i+1) \). We claim that

\[ \zeta_i = \epsilon \left( \frac{\phi}{(\lambda+\epsilon)^2} + i\epsilon^2 \theta(1) \right). \]

To verify this, note that \( \zeta_i = (1-\epsilon)^i \alpha \), where \( \alpha > 0 \) is the probability that, at a given turn, a move takes place that Maxine wins in the altered game but not in the original one. This quantity satisfies

\[ \alpha = \epsilon \cdot \left( \frac{\lambda+\phi}{\lambda+\epsilon + 1} \right) = \epsilon \frac{\phi}{(\lambda+\epsilon)^2} + \epsilon \frac{\lambda}{\lambda+\epsilon} \theta(1). \]

From this, the claim readily follows. The claim yields the consequence that

\[ \mathbb{P}(\sigma = n+1) = 1 - n\epsilon \frac{\phi}{(\lambda+\epsilon)^2} + n^2 \epsilon^2 \theta(1). \]

Applying the claim and its consequence leads to the formula

\[ P_1 - P_0 = \epsilon \left( \frac{\phi}{(\lambda+\epsilon)^2} + n\epsilon \theta(1) \right) \cdot \sum_{i=0}^{n-1} \Delta(\lambda, X_0(i)) \]

\[ + \left( 1 - n\epsilon \frac{\phi}{(\lambda+\epsilon)^2} + n^2 \epsilon^2 \theta(1) \right) \sum_{w \in V_0} (h(\lambda + \phi, w) - h(\lambda, w)) \cdot \mu_n(v, w). \]

Note also that

\[ P_2 - P_1 \leq \mathbb{P}(|D| \geq 2), \]

whose right-hand side is at most \( n^2 \alpha^2 = n^2 \epsilon^2 \theta(2) \).

Since \( P_2 - P_0 \) equals \( h(\lambda + \phi, v) - h(\lambda, v) \) by (30), the two preceding displays yield Proposition 5.2(1) when a limit \( \phi \searrow 0 \) is taken.

(2). Proposition 5.2(1) implies that the expressions (29) are equal to \( \epsilon^{-1}(\lambda+1)^2 \frac{\partial}{\partial \lambda} h(\lambda, v) \), whatever the value of \( n \in \mathbb{N}_+ \). This common value is finite by Proposition 1.9. The sequence \( \mathbb{E} \sum_{i=0}^{n-1} \Delta(\lambda, X_0(i)) \) converges pointwise in \( \lambda \in (0, \infty) \) as \( n \to \infty \) to \( \mathbb{E} \sum_{i=0}^{\infty} \Delta(\lambda, X_0(i)) \). Since the \( n \)-indexed sequence \( \epsilon^{-1}(\lambda+1)^2 \sum_{w \in V_0} \frac{\partial}{\partial \lambda} h(\lambda, w) \cdot \mu_n(v, w) \) is non-negative, we find that \( \mathbb{E} \sum_{i=0}^{\infty} \Delta(\lambda, X_0(i)) \) is finite. Note that this expression at least \( q(\lambda) \mathbb{E} F_0 \), where \( q(\lambda) = \min_{u \in V_0} \Delta(\lambda, u) \) is positive by Lemma 3.19. Thus, \( \mathbb{E} F_0 \) is finite, so that

\[ \sum_{w \in V_0} \mu_n(v, w) = \mathbb{P}(F_0 > n) \to 0 \text{ as } n \to \infty, \tag{31} \]

and

\[ \sum_{w \in V_0} \frac{\partial}{\partial \lambda} h(\lambda, w) \cdot \mu_n(v, w) \leq \sup_{w \in V_0} \frac{\partial}{\partial \lambda} h(\lambda, w) \cdot \sum_{w \in V_0} \mu_n(v, w) \to 0 \text{ as } n \to \infty. \]

This implies that the common value of the expressions (29) is \( \mathbb{E} \sum_{i=0}^{\infty} \Delta(\lambda, X_0(i)) \).

(3). This follows from Proposition 5.2(1,2).

(4). This is due to Lemma 4.2 and the proposition’s first part. \( \square \)

An alternative characterization of the stake function will be useful.
Lemma 5.3. The unique solution $b$ of the equation
\[(1 - b) \frac{\partial}{\partial x} h(\lambda, v) = \frac{\lambda}{x+1} \frac{\partial}{\partial x} h(\lambda, v_+) + \frac{1}{x+1} \frac{\partial}{\partial x} h(\lambda, v_-)\] is $b = \text{Stake}(1, \lambda, v)$.

Proof. For $v \in V$ and $i \in \mathbb{N}$, set $W(\lambda, v, i) = \mathbb{E} \sum_{j=i}^{\infty} \Delta(\lambda, X_\theta(1, \lambda, i))$, where $X_\theta(1, \lambda, 0) = v$. (Note that $W(\lambda, v, 0)$ equals $\mathbb{E} \text{TotVar}(1, \lambda, v)$.) By Proposition 5.2(3), $W(\lambda, v, 0) = (\lambda + 1)^2 \frac{\partial}{\partial x} h(\lambda, v)$. By multiplying (32) by $(\lambda + 1)^2$, we thus find that
\[(1 - b)W(\lambda, v, 0) = W(\lambda, v, 1)\]
This implies that $b = \frac{\Delta(\lambda, v)}{W(\lambda, v, 0)}$. Thus, $b$ equals Stake($1, \lambda, v$) as it is specified in the alternative formula (7) in Proposition 1.16. This completes the proof. \[\square\]

6 The leisurely game on root-reward trees

We now derive a counterpart to our main results for games of finite horizon, for which the fundamental technique of backwards induction is available. The principal result, Proposition 6.1 indicates that the global saddle hope offered in Section 2 is realized for finite horizon leisurely games. This proposition asserts the existence of a non-random stake proportion that the players share when they adhere to any Nash equilibrium. The result moreover offers a formula for this shared proportion.

Proposition 6.1. Let $T = (V, E, 1_v)$ be a root-reward tree, and let $I \subset (0, \infty)$ be compact. There exists $\epsilon_0 = \epsilon_0(I) \in (0, 1)$ such that, when $\epsilon \in (0, \epsilon_0), \lambda \in I$ and $n \in \mathbb{N}_+$, the following hold for each $v \in V_0$.

1. The value of Game$_n(\epsilon, \lambda, v)$ exists and equals $h(\lambda, v)$.

2. The game Game$_n(\epsilon, \lambda, v)$ has a Nash equilibrium. Under play governed by any Nash equilibrium, each player stakes a common non-random proportion $S_n(\epsilon, \lambda, v)$ of her reserves. The set of gameplay processes governed by Nash equilibria is given by $\{X_{\theta|n} : \theta \in \Theta\}$. The shared proportion $S_n(\epsilon, \lambda, v)$ equals the stake function Stake($\epsilon, \lambda, v$) specified in (5).

The derivation of Proposition 6.1 follows the template offered by the Poisson game in Section 2.4; we will inductively locate a global minimax in a first-turn-constrained variant of Game$_n(\epsilon)$ via computations that perturb the Poisson case by raising the move probability $\epsilon$ from $0^+$ to small positive values. (In this way, we overcome the first criticism, voiced in Subsection 2.2.1, that the saddle point identified in the heuristic perturbation argument may merely be local.)

Proof of Proposition 6.1. The result will be proved by induction on $n$. We write Hyp($n, i$), $n \in \mathbb{N}_+$ and $i \in \{1, 2\}$, to refer to Proposition 6.1(i) with index value $n$. The base case may be viewed as $n = 0$: Game$_0(\epsilon, \lambda, v)$ is moveless, with Pay = $h(\lambda, v)$, so that Hyp($0, 1$) holds trivially; we may formally specify Hyp($0, 2$) to be the vacuous statement.

In Section 2.4 we discussed a constrained version Game($0^+, \lambda, v, a, b$) of the Poisson game in which the players are obliged, for an infinitesimal duration, to offer respective stake rates $a$ and $b$. In the leisurely analogue, Game$_n(\epsilon, \lambda, v, a, b)$ is a constrained version of the finite-horizon game
Game$_n(\epsilon, \lambda, v)$, in which Maxine is obliged to stake $ae$, and Mina to stake $be$, at the first turn. The parameters $a$ and $b$ are non-negative real numbers, and the form of the stakes is chosen so that the Poisson game may be viewed (after time change!) as a low-$\epsilon$ limit of the leisurely games Game($\epsilon$). In the next lemma, we see that, roughly put, the derivative expression (17) forms the dominant term in the difference of constrained value and actual value in the leisurely setup. The result uses the notation Val$_{n-1}$ and Val$_n$ to emphasise the inductively developed relationship between game values as the horizon rises, but it may be worth bearing in mind that Val$_{n-1}(\epsilon, \lambda(\epsilon), v)$ is nothing other than $h(\lambda(\epsilon), v)$, by Hyp($n-1, 1$), so that the upcoming $\Psi_n$ is actually independent of $n$.

**Lemma 6.2.** Let $n \in \mathbb{N}_+$ and suppose Hyp($n-1, 1$) and Hyp($n-1, 2$). Let $(a, b) \in (0, \lambda \epsilon^{-1}) \times (0, \epsilon^{-1})$. Let Game$_n(\epsilon, \lambda, v, a, b)$ denote Game$_n(\epsilon, \lambda, v)$ subject to the constraint that Maxine and Mina stake $(ae, be)$ at the first turn.

1. The game Game$_n(\epsilon, \lambda, v, a, b)$ has value when $\epsilon \in (0, 1)$, with this value Val$_n(\epsilon, \lambda, v, a, b)$ given by

$$\text{Val}_n(\epsilon, \lambda, v, a, b) = \epsilon \Psi_n(\epsilon, a, b) + h(\lambda, v),$$

where

$$\Psi_n(\epsilon, a, b) = \epsilon^{-1}\left(\text{Val}_{n-1}(\epsilon, \lambda(\epsilon), v) - h(\lambda, v)\right) - \text{Val}_{n-1}(\epsilon, \lambda(\epsilon), v)
+ \frac{a}{a+b}\text{Val}_{n-1}(\epsilon, \lambda(\epsilon), v_+) + \frac{b}{a+b}\text{Val}_{n-1}(\epsilon, \lambda(\epsilon), v_-),$$

with $\lambda(\epsilon) = \frac{\lambda-a\epsilon}{1-b\epsilon}$. By $v_-$ is denoted any element of the set $\mathcal{V}_-(v)$ specified in Definition 1.7. Note that $\text{Val}_{n-1}(\epsilon, \lambda(\epsilon), v_-)$ is well defined, since this quantity equals $h(\lambda, v_-)$ in view of Hyp($n-1, 1$) and is thus independent of the choice of $v_-$ by Proposition 3.12.

2. Let $(y, z) \in V \times V$ be a pair of neighbours of $v$. A Nash equilibrium for Game$_n(\epsilon, \lambda, v, a, b)$ exists for which the move nominations at the first turn for Maxine and Mina are the respective elements of $(y, z)$ if and only if $y = v_+$ and $z \in V_-(v)$.

**Proof.** At the start of the second move of Game$_n(\epsilon, \lambda, v, a, b)$, a copy of Game$_{n-1}(\epsilon)$ remains to be played. It is Hyp($n-1, 2$) that furnishes Nash equilibria for the two parties to play this subgame. (The value of $\lambda$ has been updated to be $\lambda(\epsilon)$, but Hyp($n-1, 2$) is robust enough to handle this, because any value of $\lambda \in (0, \infty)$ is viable for the application of this hypothesis; it is here that our usage of root-reward trees is needed.)

Let $P_n^+(a, b)$ denote a strategy for Maxine in Game$_n(\epsilon, \lambda, v, a, b)$ at whose first turn she nominates $v_+$ (and, of course, stakes $ae$) and whose dictates for later turns adhere to a Nash equilibrium offered by Hyp($n-1, 2$). Let $P_n^-(a, b)$ denote a strategy for Mina in Game$_n(\epsilon, \lambda, v, a, b)$ at whose first turn she nominates a given element $v_-$ of $\mathcal{V}_-(v)$ (and stakes $be$) and whose later turns again adhere to a Nash equilibrium offered by Hyp($n-1, 2$). We will argue that

$$M\left(P_n^-(a, b), P_n^+(a, b)\right) = \epsilon \Psi_n(\epsilon, a, b) + h(\lambda, v)$$

and that $(P_n^-(a, b), P_n^+(a, b))$ is a Nash equilibrium for Game$_n(\epsilon, \lambda, v, a, b)$. To confirm these claims, note that

$$M\left(P_n^-(a, b), P_n^+(a, b)\right) = (1-\epsilon)\text{Val}_{n-1}(\epsilon, \lambda(\epsilon), v) + \epsilon \frac{a}{a+b}\text{Val}_{n-1}(\epsilon, \lambda(\epsilon), v_+) + \epsilon \frac{b}{a+b}\text{Val}_{n-1}(\epsilon, \lambda(\epsilon), v_-),$$

(36)
where recall that $\lambda(\epsilon) = \frac{-\lambda\epsilon}{1-\epsilon}$. The three right-hand terms are the mean of the final payment $\text{Pay}$ that Mina makes to Maxine when multiplied by indicator functions on the absence of a move at the first turn; on a move which Maxine wins; and on a move which Mina does. The equalities are valid because the players are maintaining their sides of a Nash equilibrium from the second turn onwards. By Hyp$(n - 1, 1)$, the right-hand side of (36) equals $\epsilon \Psi_n(\epsilon, a, b) + h(\lambda, v)$, as desired to derive (35). If $P_-$ is a strategy for Mina by which she nominates $u \not\in \mathcal{V}_-(v)$ at the first move, then

$$M(P_-, P^+_{\epsilon}(a, b)) \geq (1 - \epsilon) \text{Val}_{n-1}(\epsilon, \lambda, v) + \epsilon \frac{a}{a+b} \text{Val}_{n-1}(\epsilon, \lambda, v+) + \epsilon \frac{b}{a+b} \text{Val}_{n-1}(\epsilon, \lambda, v),$$

as we see from noting that $P^+_{\epsilon}(a, b)$ adheres from the second turn to Maxine’s component of a Nash equilibrium for Game$_{n-1}(\epsilon)$. The right-hand side of (37) exceeds that of (36). Thus, $P_-$ cannot be Mina’s component in a Nash equilibrium. Similar reasoning shows that a strategy for Maxine under which a move to a vertex other than $v_+$ is nominated at the first turn cannot be Maxine’s component in any Nash equilibrium. This completes the proof of Lemma 6.2.

The leisurely $\Psi_n$ is counterpart to the Poisson $\Phi$. Next we compute first and second derivatives for $\Psi_n$ and search for a global saddle point, as we did for $\Phi$ later in Section 2.4. Lemma 6.3(3) shows that the saddle is global in an arbitrary compact region in $(a, b)$ if $\epsilon$ is suitably small; it remains to treat other choices of $(a, b)$.

**Lemma 6.3.** Let $n \in \mathbb{N}_+$ and suppose that Hyp$(n - 1, 1)$ holds.

1. We have that

$$\frac{\partial^2}{\partial \epsilon^2} \Psi_n(\epsilon, a, b) = -\frac{2b}{(a+b)^2} \left(h(\lambda(\epsilon), v_+) - h(\lambda(\epsilon), v_-)\right) + \epsilon \cdot \left(\frac{b}{(a+b)^2} + 1\right) |V|^4 \cdot O(1),$$

where the factor implied by the big-O notation is bounded above in absolute value uniformly in $v \in \mathcal{V}_\epsilon$, $\epsilon \in (0,1/2]$ and $(\lambda, a, b) \in (0, \infty) \times (0, \lambda/\epsilon) \times (0, 1/\epsilon)$ subject to be $\leq 1/2$.

2. And also that

$$\frac{\partial^2}{\partial \epsilon^2} \Psi_n(\epsilon, a, b) = \frac{2a}{(a+b)^2} \left(h(\lambda(\epsilon), v_+) - h(\lambda(\epsilon), v_-)\right) + \epsilon \cdot \left(\frac{b}{(a+b)^2} + 1\right) \max\{\lambda^2, 1\} |V|^4 \cdot O(1),$$

with the same interpretation of big-O.

3. Let $\Lambda \subset (0, \infty)^3$ take the form of a product of a compact subset of $(0, \infty)$ (in the first coordinate); a bounded subset of $(0, \infty)$ (in the second); and a compact subset of $(0, \infty)$ (in the final coordinate). There exists $\epsilon_0 = \epsilon_0(\Lambda, |V|) > 0$ such that

$$\sup \frac{\partial^2}{\partial \epsilon^2} \Psi_n(\epsilon, a, b) < 0$$

where the supremum is taken over $\epsilon \in (0, \epsilon_0)$ and $(\lambda, a, b) \in \Lambda$.

Now suppose instead that $\Lambda \subset (0, \infty)^3$ takes the form of a product of a compact subset of $(0, \infty)$, a compact subset of $(0, \infty)$, and a bounded subset of $(0, \infty)$, in the respective coordinates. Then there exists $\epsilon_0 = \epsilon_0(\Lambda, |V|) > 0$ such that

$$\inf \frac{\partial^2}{\partial \epsilon^2} \Psi_n(\epsilon, a, b) > 0,$$

where the infimum is taken over $\epsilon \in (0, \epsilon_0)$ and $(\lambda, a, b) \in \Lambda$. 47
Proof: (1). By Hyp($n - 1, 1$), $\text{Val}_{n-1}(\epsilon, \lambda, w) = h(\lambda, w)$, so that Proposition 6.3.1 implies that $(0, \infty) \to [0, 1]: \lambda \mapsto \text{Val}_{n-1}(\epsilon, \lambda, w)$ is differentiable for $w \in V_O$. We may thus compute from (34) that

$$\frac{\partial}{\partial a} \Psi_n(\epsilon, a, b) = \frac{b}{(a+b)^2} \left( h(\lambda(\epsilon), v_+) - h(\lambda(\epsilon), v_-) \right) - \frac{\epsilon}{1-\epsilon} \frac{1}{a+b} \left( ah'(\lambda(\epsilon), v_+) + bh'(\lambda(\epsilon), v_-) \right),$$

where the prime indicates the derivative with respect to the first coordinate. We further find that

$$\frac{\partial^2}{\partial a^2} \Psi_n(\epsilon, a, b) = -\frac{2b}{(a+b)^3} \left( h(\lambda(\epsilon), v_+) - h(\lambda(\epsilon), v_-) \right) - \frac{2\epsilon}{(1-\epsilon)(a+b)^2} \cdot \frac{b}{(a+b)^2} \left( h'(\lambda(\epsilon), v_+) - h'(\lambda(\epsilon), v_-) \right) + \frac{\epsilon(1-\epsilon)}{(1-\epsilon)(a+b)^2} \left( ah''(\lambda(\epsilon), v_+) + bh''(\lambda(\epsilon), v_-) \right).$$

By Lemma 3.18, we obtain

$$\left| \frac{\partial}{\partial a} \Psi_n(\epsilon, a, b) + \frac{2b}{(a+b)^3} \left( h(\lambda(\epsilon), v_+) - h(\lambda(\epsilon), v_-) \right) \right| \leq 4\epsilon \frac{b}{(a+b)^3} C|V|^2 + 8\epsilon C|V|^4 + 8\epsilon^2 C|V|^2,$$

since $be \leq 1/2$ and $\epsilon \leq 1/2$. Thus Lemma 6.3.1.

(2). Similarly, we find that

$$\frac{\partial}{\partial b} \Psi_n(\epsilon, a, b) = -\frac{a}{(a+b)^2} \left( h(\lambda(\epsilon), v_+) - h(\lambda(\epsilon), v_-) \right) + \frac{1-\epsilon}{1-\epsilon} \frac{1}{a+b} \left( ah'(\lambda(\epsilon), v_+) + bh'(\lambda(\epsilon), v_-) \right),$$

and that $\frac{\partial^2}{\partial a^2} \Psi_n(\epsilon, a, b)$ equals

$$-\frac{2a}{(a+b)^3} \left( h(\lambda(\epsilon), v_+) - h(\lambda(\epsilon), v_-) \right) - \frac{2\lambda}{(1-\epsilon)(a+b)^2} \cdot \frac{a}{(a+b)^2} \left( h'(\lambda(\epsilon), v_+) - h'(\lambda(\epsilon), v_-) \right) + \frac{\epsilon(1-\epsilon)}{(1-\epsilon)(a+b)^2} \lambda \left( 2(1-\epsilon)h'(\lambda(\epsilon), v) + \lambda h''(\lambda(\epsilon), v) \right) + \frac{2\epsilon^2}{(1-\epsilon)(a+b)^2} \left( ah'(\lambda(\epsilon), v_+) + bh'(\lambda(\epsilon), v_-) \right) + \frac{\epsilon^2 \lambda}{(1-\epsilon)(a+b)^2} \left( ah''(\lambda(\epsilon), v_+) + bh''(\lambda(\epsilon), v_-) \right).$$

Thus, Lemma 3.18 $be \leq 1/2$ and $\epsilon \leq 1/2$ imply that $\frac{\partial^2}{\partial a^2} \Psi_n(\epsilon, a, b) - \frac{2b}{(a+b)^3} \left( h(\lambda(\epsilon), v_+) - h(\lambda(\epsilon), v_-) \right)$ is in absolute value at most $16\epsilon \lambda C|V|^2 \left( \frac{a}{(a+b)^2} + 4 |V|^2 \right) + 32\epsilon^2 \lambda C|V|^2 (1 + \lambda C|V|^2)$. Lemma 6.3.2 follows.

(3). Next, note that, when $(\lambda, b)$ lies in a compact subset of $(0, \infty)$, and $a$ lies in a bounded subset of $(0, \infty)$ (so that $a$ may approach zero, but $\lambda$ and $b$ may not), there exists $e_0 \in (0, 1)$ such that, when $\epsilon \in (0, e_0)$, $h(\lambda(\epsilon), v_+) - h(\lambda(\epsilon), v_-)$ exceeds a positive constant. Indeed, Lemma 3.20 offers control on $\lambda(\epsilon)$ so that Lemma 3.19 may be applied to reach this conclusion. We then obtain the first assertion of Lemma 6.3.1 from Lemma 6.3.1,2; the latter assertion has a similar proof.

Recall the compact set $I \subset (0, \infty)$ in the hypothesis of Proposition 6.1. Let $c$ and $K$ be positive, with

$$\text{Stake}(1, \lambda, v) \in (c, K).$$  

(38)
The function \( \text{Stake}(1, \lambda, v) \) from (5) is bounded uniformly away from zero and infinity as \( \lambda \) ranges over \( I \): to see this, note that the numerator in (5) is bounded above due to \( h \leq 1 \); it is bounded below by Lemma 3.19, while the denominator is bounded above by Lemma 3.18(1); and it is bounded below by Lemma 3.18(3). Thus, such choices of \( c \) and \( K \) may indeed be made. The values of these parameters will be further specified shortly, in the proof of Lemma 6.4(3).

Let \( \Lambda_0 = \{ (\lambda, a, b) \in (0, \infty)^3 : \lambda \in I, a \in \lambda \cdot [c, K], b \in [c, K] \} \). We make a choice of a set \( \Lambda \) that contains \( \Lambda_0 \) for which both parts of Lemma 6.3(3) are applicable; we may take \( \Lambda \) equal to \( I \times [c \cdot \inf I, K \cdot \sup I] \times [c, K] \). Applying this result furnishes a value \( \epsilon_0 = \epsilon_0(\Lambda) \in (0, 1) \), which is independent of the inductive parameter \( n \), for which the condition \( \epsilon \in (0, \epsilon_0) \) implies that the bounds \( \frac{\partial^2}{\partial a^2} \Psi_n(\epsilon, a, b) < 0 \) and \( \frac{\partial^2}{\partial b^2} \Psi_n(\epsilon, a, b) > 0 \) hold for \((a, b) \in \lambda \cdot [c, K] \times [c, K] \) whenever \( \lambda \in I \).

This implies the existence of a unique global minimax point for the function \( \lambda \cdot [c, K] \times [c, K] \) \( \rightarrow \mathbb{R} : (a, b) \mapsto \Psi_n(\epsilon, a, b) \) whenever \( \epsilon \in (0, \epsilon_0) \) and \( \lambda \in I \). In principle, this point may depend on \( \epsilon \) or lie on the boundary of \( \lambda \cdot [c, K] \times [c, K] \).

**Lemma 6.4.** Let \((a_0, b_0)\) be the just identified global minimax point in the rectangle \( \lambda [c, K] \times [c, K] \). For a choice of \( c > 0 \) that is suitably low, and of \( K < \infty \) that is suitably high—these choices made independently of \( \lambda \) in the theorem hold.

1. We have that \( a_0 = \lambda b_0 \).

2. The value of \((a_0, b_0)\) is independent of \( \epsilon \in (0, \epsilon_0) \). It equals \((\lambda S, S)\), where \( S = \text{Stake}(1, \lambda, v) \).

3. The function \( [0, \lambda \epsilon^{-1}] \times [0, \epsilon^{-1}] \) \( \rightarrow \mathbb{R} : (a, b) \mapsto \text{Val}_n(\epsilon, \lambda, v, a, b) \) has a global minimax at \((a_0, b_0)\).

**Proof:** (1). First we claim that \( \Psi_n(\epsilon, a, b) \) is constant on the line \( b = \lambda^{-1} a \), for \( b \in [0, \epsilon^{-1}] \). Indeed, \( \lambda(\epsilon) \) equals \( \lambda \) for any \((a, b)\) on this line with \( b \in [0, \epsilon^{-1}] \); the same in fact remains true even when \( b \in \{0, \epsilon^{-1}\} \), by the two reset rules recorded in the paragraphs that precede Definition 1.3. Thus, (5) shows what we claim. Next define \((a'_0, b'_0) \in (0, \infty)^2 \) such that \((a'_0, b_0)\) and \((a_0, b'_0)\) lie on the line \( b = \lambda^{-1} a \). To confirm that \( a_0 = \lambda b_0 \), we must check that \((a'_0, b'_0) = (a_0, b_0)\). Note that

\[
\Psi_n(\epsilon, a'_0, b_0) \leq \Psi_n(\epsilon, a_0, b_0) \leq \Psi_n(\epsilon, a_0, b'_0)
\]

with the respective bounds strict if \( a'_0 \neq a_0 \) or \( b'_0 \neq b_0 \). (This inference uses the membership of the rectangle \( \lambda \cdot [c, K] \times [c, K] \) by \((a'_0, b'_0)\).) The first and the third displayed terms are however equal, because the \((a, b)\)-argument pairs lie on the slope-\( \lambda^{-1} \) line. Thus, \((a'_0, b'_0) = (a_0, b_0)\) as we sought to show. We have found that \( a_0 = \lambda b_0 \).

(2). The point \((a_0, b_0)\) is known to take the form \((\lambda b_0, b_0)\), and our task is to identify \( b_0 \). We do so by finding the value of \( b \in (0, \infty) \) for which \( \frac{\partial}{\partial b} \Psi_n(\epsilon, a, b) \) evaluated at \((a, b) = (\lambda b, b)\) equals zero. This evaluation is

\[
\left( b^{-1}(1 + \lambda) - h(\lambda, v_+) - h(\lambda, v_-) \right) - \frac{\partial}{\partial a} h(\lambda, v) + \left( \frac{1}{1 + \lambda^2} \frac{\partial}{\partial \lambda} h(\lambda, v) \right) \left( \frac{1}{1 + \lambda^2} \frac{\partial}{\partial \lambda} h(\lambda, v) - h(\lambda, v) \right)
\]

This expression equals zero for a unique value of \( b \). This solution is \( b = \text{Stake}(1, \lambda, v) \). Indeed, taking \( b \) equal to this value causes the expression in both lines of the display to equal zero. For the first line, this is due to (5) with \( \epsilon = 1 \); while the second line vanishes when \( b \) satisfies the condition
in Lemma 5.3 so this result shows this vanishing. By now, we know that the minimax point \((a_0, b_0)\) in the rectangle \(R = \lambda \cdot [c, K] \times [c, K]\) exists and satisfies \(a_0 = \lambda b_0\); and, if it lies in the interior of \(R\), it must be \((\lambda b, b)\) for the just identified solution \(b\). To demonstrate Lemma 6.4(2), it suffices then to argue that neither of the corners \((\lambda c, c)\) and \((\lambda K, K)\) can be the minimax in \(R\). Because \(b \in (c, K)\) (this due to \(b = \text{Stake}(1, \lambda, v)\) and (38)), and the last displayed partial-\(a\) derivative is decreasing in \(b\), this display is positive at \(b = c\) and negative at \(b = K\). Thus \(\Psi_n(\epsilon, a, b)\) increases in rising \(a\) about \((\lambda c, c)\) and in falling \(a\) about \((\lambda K, K)\). The minimax in \(R\) can only be at \((\lambda b, b)\), as we sought to show.

(3). Our task is to show that

\[
\Psi_n(\epsilon, a_0, b) > \Psi_n(\epsilon, a_0, b_0)
\]

when \(b \in (0, \infty) \setminus \{b_0\}\), and

\[
\Psi_n(\epsilon, a, b_0) < \Psi_n(\epsilon, a_0, b_0)
\]

when \(a \in (0, \infty) \setminus \{a_0\}\). For such \((a, b)\) that lie in \(\lambda \cdot [c, K] \times [c, K]\), these bounds arise from Lemma 6.3(3).

We will next obtain (39) in the remaining case that \(b \in [0, c] \cup [K, \infty)\). Lemma 6.4(3) will be obtained once this is shown alongside (40) in its remaining case, when \(a \in \lambda \cdot \left([0, c] \cup [K, \infty)\right)\). We do not offer the proof of (40) in this case because to do so would be repetitive.

To address the remaining case for (39), we will invoke the next presented Lemma 6.5 to gain control of \(\Psi\)-differences under large variations of an argument. First we argue that (39) holds for \(b \in (0, b_0)\); since \(c < \text{Stake}(1, \lambda, v) = b_0\) by (38), this handles low \(b\)-values in the remaining case. To obtain this bound, consider what Lemma 6.3(3) has to say about \(\frac{\partial^2}{\partial b^2} \Psi_n(\epsilon, a, b_0)\) when we take \(\Lambda\) equal to \{\(\lambda\) \times \{a_0\}\} \times (0, b_0]\); this parameter choice is admissible, because this choice of \(\Lambda\) is independent of \(\epsilon \in (0, \epsilon_0)\) for \(\epsilon_0 > 0\) small enough, by Lemma 6.4(2). Alongside the vanishing of the first derivative \(\frac{\partial}{\partial b} \Psi_n(\epsilon, a, b_0)\) at \(b = b_0\), Lemma 6.3(3) then implies that \(\Psi_n(\epsilon, a_0, b) - \Psi_n(\epsilon, a_0, b_0)\) is positive when \(b \in (0, b_0)\), provided that the condition \(\epsilon \in (0, \epsilon_0)\) is imposed for a suitably small \(\epsilon_0 > 0\); thus, we confirm that (39) holds for \(b \in (0, b_0)\).

We now seek to obtain (39) for large choices of \(b\); indeed, in order to obtain (39), it suffices to derive this bound when \(b \geq K\). It is here that we specify the value of \(K\). Let \(\eta > 0\) satisfy \(2\eta < \Psi_n(\epsilon, a_0, b_0 + 1) - \Psi_n(\epsilon, a_0, b_0)\), where Lemma 6.3(3) shows that the right-hand side is positive. We set \(K = a_0(\eta^{-1} - 1)\) and reduce \(\eta > 0\) if need be in order to ensure that \(K \geq b_0 + 1\) (as well as that the lower bound on \(K\) demanded by (38) holds). Let \(b \geq K\). By applying Lemma 6.5 with its parameter pair \((b, \beta)\) set equal to the presently specified \((b, K)\), we find that

\[
\Psi_n(\epsilon, a_0, b) - \Psi_n(\epsilon, a_0, K) \geq -2\eta.
\]

Since \(K \geq b_0 + 1\), Lemma 6.3(3) implies that \(\Psi_n(\epsilon, a_0, K) - \Psi_n(\epsilon, a_0, b_0) \geq \Psi_n(\epsilon, a_0, b_0 + 1) - \Psi_n(\epsilon, a_0, b_0)\), whose right-hand side is at least \(2\eta\). Alongside the preceding display, we thus obtain (39) for \(b \geq K\). This completes the verification of (39) and thus the proof of Lemma 6.4(3).

The lemma that was just used demonstrates the limited benefit that arises from increasing a high stake. It gives crude but rigorous expression of one of two features that were illustrated by example in Section 2.5.
Lemma 6.5 (Don’t bet the house). Let $a \in (0, \infty)$ and $\eta \in (0, 1)$. Suppose that $\epsilon^{-1} > b \geq \beta \geq a(\eta^{-1} - 1)$. Then
\[ \Psi_n(\epsilon, a, \beta) - \Psi_n(\epsilon, a, b) \leq 2\eta. \]

For use in the proof, we set $\lambda_x = \frac{\lambda - ax}{1 - xe}$ for $x \in [0, \epsilon^{-1})$.

Proof of Lemma 6.5. By Hyp($n - 1, 1$),
\[ \Psi_n(\epsilon, a, \beta) - \Psi_n(\epsilon, a, b) = (\epsilon^{-1} - 1)(h(\lambda, \epsilon) - h(\lambda, \beta)) + \frac{a}{a+\beta}h(\lambda, v_+) - \frac{a}{a+b}h(\lambda, v_-) + \frac{b}{a+b}h(\lambda, v_+) - \frac{b}{a+b}h(\lambda, v_-). \]

Since $\frac{a}{a+\beta} \leq \eta$ and $\frac{b}{a+b} \geq 1 - \eta$, we find that
\[ \Psi_n(\epsilon, a, \beta) - \Psi_n(\epsilon, a, b) \leq \eta + h(\lambda, v_-) - h(\lambda, v_-) + \eta h(\lambda, v_-). \]

To finish the proof, we wish to confirm that the right-hand side is at most $2\eta$. Note that $\beta \leq b < \epsilon^{-1}$ implies that $\lambda_b \leq \lambda_\beta$, so that $h(\lambda, v_-) \leq h(\lambda, v_-)$ by Corollary 3.10. The sought bound then follows from $h \leq 1$. This completes the proof of Lemma 6.5.

The value $\text{Val}_n(\epsilon, \lambda, v)$ of the game $\text{Game}_n(\epsilon, \lambda, v)$ equals $\text{Val}_n(\epsilon, \lambda, v, a_0, b_0)$ for the minimax $(a_0, b_0)$ identified in Lemma 6.4(3) because this minimax is indeed global.

By Lemma 6.4 and Hyp($n - 1, 1$), we have that
\[ \text{Val}_n(\epsilon, \lambda, v) = (1 - \epsilon)h(\lambda, v) + \epsilon \frac{\lambda}{1+\lambda}h(\lambda, v_+) + \epsilon \frac{1}{1+\lambda}h(\lambda, v_-). \]

This right-hand side equals $h(\lambda, v)$, which was specified in (4). Thus Hyp($n, 1$) is obtained. By Lemma 6.4(3), the function $(0, \infty)^2 \to \mathbb{R} : (a, b) \mapsto \text{Val}_n(\epsilon, \lambda, a, b, v)$ has been shown to have a global minimax at $(a_0, b_0)$ with $a_0 = \lambda b_0$; move nominations are addressed by Lemma 6.2(2). We see then that a Nash equilibrium is obtained when, at the first turn, Maxine and Mina stake $a_0\epsilon$ and $b_0\epsilon$, and nominate moves to $v_+$ and to an arbitrary element of $\mathcal{V}_-(v)$, and adhere in later play to the strategy components of the Nash equilibrium furnished by Hyp($n - 1, 2$). Since the global minimax is strict, it must be unique, so no deviation in first-turn stakes is compatible with a Nash equilibrium; nor is later deviation tolerable, because Hyp($n - 1, 2$) determines these later stakes. Thus, we confirm Hyp($n, 2$). We have derived Hyp($n, i$) for $i \in \{1, 2\}$ and completed the derivation of the inductive step and thus of Proposition 6.1.

7 Proving the main results from their finite horizon counterparts

Here we prove Theorems 1.8 and 1.12 and Corollary 1.15.

Definition 1.11 concerns strategy pairs that result in gameplay where both players conform at every turn with probability one. It is useful to consider explicit strategies that satisfy the stronger condition that they demand that the player conform at every turn regardless of the strategy adopted by her opponent. These are the strongly conforming strategies that we now specify.

Definition 7.1. The strongly conforming strategy for Maxine in $\text{Game}(\epsilon, \lambda, v)$ demands that, when $\text{StateOfPlay}$ equals an arbitrary $(\mu, w) \in (0, \infty) \times \mathcal{V}_0$, Maxine stakes $\mu \cdot \text{Stake}(\epsilon, \mu, w)$ and nominates the move $w_+$, regardless of the index of the turn in question. If $(\mu, w) \in \{\infty\} \times \mathcal{V}_0$ (so that Mina is
bankrupt), Maxine stakes a given positive but finite sum and nominates \( w_+ \). Strongly conforming
growth strategies for Mina are indexed by \( \theta \) in the index set \( \Theta \) specified in Definition 1.13. The strategy so
indexed by \( \theta \) demands that, when StateOfPlay = \((\mu, w) \in (0, \infty) \times V_O\) and the turn index is \( i \in \mathbb{N}_+ \),
Mina stakes Stake(\( \epsilon, \mu, w \)) and nominates the move \( \theta(\mu, w, i - 1) \). If \((\mu, w) \in \{0\} \times V_O\), Mina stakes
a given quantity on \((0, 1)\)—one-half, say—and nominates a definite move away from the root—let’s say \( \theta(1, w, i - 1) \).

The next result shows that it is a mistake to deviate from conforming play if one’s opponent
conforms.

**Proposition 7.2.** Let \( P_{\text{conf}}^- \) and \( P_{\text{conf}}^+ \) denote strongly conforming strategies for Mina and Maxine
in Game(\( \epsilon, \lambda, v \)). Let \( S_- \in S_- \) be a strategy for Mina that is not conforming against \( P_{\text{conf}}^+ \). Then
\[
M(S_-, P_{\text{conf}}^+) > h(\lambda, v).
\]

And likewise let \( S_+ \in S_+ \) be a strategy for Maxine that is not conforming against \( P_{\text{conf}}^- \). Then
\[
M(P_{\text{conf}}^-, S_+) < h(\lambda, v).
\]

This result is the principal element in the passage from finite to infinite horizon that we are un-
dertaking. Recall from Section 1.3 that the finish time of Game(\( \epsilon, \lambda, v \)) is denoted by \( F \). A critical
element in making this passage is gaining an understanding that under suitable strategy pairs in
the infinite-horizon game, play is certain to end at a finite time: in other words, that \( F < \infty \), or
\( X_i \in V_B \) for some \( i \in \mathbb{N}_+ \), almost surely. If both players conform, this is easy enough: the counter
evolution is a lazy walk \( X_\theta \) for some \( \theta \in \Theta \), so that (31) in the proof of Proposition 5.2(2) bounds
the tail of the finish time. Indeed, it would not be hard to obtain an exponentially decaying bound
on the tail in this case. When one player conforms but the other may not, proving the finiteness
of the finish time is more delicate. Next is a result that does so, with Mina the conforming party.
There is an additional hypothesis that regularises Maxine’s play when her fortunes are good.

**Definition 7.3.** Let \( D > 0 \). A strategy \( S_+ \in S \) for Maxine in Game(\( \epsilon, \lambda, v \)) is said to be strongly
conforming above fortune \( D \) if it meets the demand set in the above specification of the strategy
that is strongly conforming for Maxine whenever \((\mu, w) \in (0, \infty) \times V_O\) satisfies \( \mu \geq D \).

**Lemma 7.4.** Let \( D > 0 \). Let \( S_+ \in S_+ \) be strongly conforming above fortune \( D \). In the game
Game(\( \epsilon, \lambda, v \)), \( F \) is finite almost surely under the strategy pair \((P_{\text{conf}}^-, S_+)\).

**Proof.** Consider the gameplay governed by the strategy pair \((P_{\text{conf}}^-, S_+)\).

An index \( i \in \mathbb{N}_+ \) is called Mina if the game is unfinished at the start of the \( i^{\text{th}} \) turn; at the start of
the turn, StateOfPlay = \((\mu, w) \in [0, \infty] \times V_O\), where the fortune \( \mu \) is less than \( D \); a move is selected
to take place at the \( i^{\text{th}} \) turn; and Mina wins the right to make the resulting move.

An index \( i \in \mathbb{N}_+ \) is called high if the game is unfinished at the start of the \( i^{\text{th}} \) turn; and, at the start
of this turn, StateOfPlay = \((\mu, w) \in [0, \infty] \times V_O\), where \( \mu \geq D \).

Set
\[
c = \inf \left\{ \mu^{-1} \text{Stake}(\epsilon, \mu, w) : \mu \in [0, D), w \in V_O \right\}.
\]

By Corollary 3.23 \( c > 0 \). We claim that \( c/(1 + c) \) is a lower bound on the probability that Mina
wins any given turn at whose start the fortune is less than \( D \) and at which a move is selected to take

52
place. Indeed, if StateOfPlay = (µ, w) for µ < D, then Mina will stake Stake(ε, µ, w) and Maxine will stake at most µ, so that Mina’s win probability given a move is at least \[ \frac{\text{Stake}(ε, µ, w)}{µ + \text{Stake}(ε, µ, w)} \geq \frac{1}{e-1+1}; \]
whence the claim.

Let \( d \) denote the number of vertices on the longest path in \( V_0 \). Since \( P_{\text{conf}} \) is strongly conforming, a string of consecutive Mina indices may have at most \( d \) elements, because the game will finish at or before the move corresponding to the final index in the string.

Suppose that \( F \geq j \) for \( j \in \mathbb{N}_+ \). It follows from the above claim that, whatever the status of turn indices on \([0, j - 1]\), there is conditional probability at least \( εc/(c+1) \) that \( j \) is Mina or high or \( F = j \). In any block \([i-1)d + 1, id]\), and given that \( F \geq (i-1)d + 1 \), there is thus conditional probability at least \( (εc/(c+1))^d \) that one of the following occurs: every index in the block is Mina; at least one index in the block is high; or \( F \leq id \). By the preceding paragraph, the first alternative implies the third. And if any index is high, \( F < ∞ \), because both players conform from a turn with a high index and this ensures a finite finish as we noted in the paragraph preceding Definition 7.3. (Note that Definition 7.1 treats opponent bankruptcy in a way that ensures a finite finish time in a high index and this ensures a finite finish as we noted in the paragraph preceding Definition 7.3.) Thus, \( F < ∞ \) almost surely.

We extend the truncation notation \( \cdot|_n \) in Definition 5.1 to strategies.

**Definition 7.5.** Let \( S \in S_− \). The horizon-\( n \) truncation of \( S \), denoted by \( S|_n \), is the element in \( S_(n) \) formed by restricting \( S \) to the first \( n \) turns of Game(\( ϵ, λ, v \)). We may equally use this notation when \( S \in S_+ \).

**Lemma 7.6.** Let \( D > 0 \).

1. Let \( S_− \in S_− \). Then
\[
M(S_−, P_{conf}^+) \geq \limsup_n M(S_−|_n, P_{conf}^+|_n).
\]

2. Now let \( S_+ \in S_+ \) be strongly conforming above fortune \( D \). We have that
\[
M(P_{conf}^−, S_+) \leq \liminf_n M(P_{conf}^−|_n, S_+|_n).
\]

**Proof:** (1). Note that
\[
M(S_−, P_{conf}^+) = E_{(S_−, P_{conf}^+)}[\text{Pay} \cdot 1_{F=∞}] + E_{(S_−, P_{conf}^+)}[\text{Pay} \cdot 1_{F<∞}].
\]
Recall that the rules of the infinite-horizon game declare that, should the game be unfinished in any finite time—that is, should \( F \) be equal to \( ∞ \)—then the terminal payment \( \text{Pay} \) equals one. We invoke this rule now, to find the first right-hand term in (41) equals \( P(F = ∞) \). We may also speak of the finish time \( F \) in the finite-horizon game Game\(_n(\epsilon, \lambda, v)\); when we do so, \( F \) will be at most \( n \) for gameplay that finish by the counter leaving open play, and we will adopt the convention of Definition 5.1 by recording the event of an unfinished game in the form \( F = n + 1 \).

Note further that \( M(S_−|_n, P_{conf}^+|_n) = \zeta_1(n) + \zeta_2(n) \), where \( \zeta_1(n) = E_{(S_−|_n, P_{conf}^+|_n)}[\text{Pay} \cdot 1_{F=∞}] \) and \( \zeta_2(n) = E_{(S_−|_n, P_{conf}^+|_n)}[\text{Pay} \cdot 1_{F<n}] \). Since the infinite-horizon gameplay never finishes precisely when all of its truncations are also unfinished, we find that \( \limsup_n \zeta_1(n) \leq P(F = ∞) \). We also have that \( \zeta_2(n) \) increases in \( n \) to the limiting value \( E_{(S_−, P_{conf}^+)}[\text{Pay} \cdot 1_{F<∞}] \), which is the second right-hand term in (41). Thus do we prove Lemma 7.6(1).
Since $F < \infty$ under $(P_{\text{conf}}^-, S_\infty)$ by Lemma 7.4, we have that
\[ M(P_{\text{conf}}^-, S_\infty) = \mathbb{E}(P_{\text{conf}}^-, S_\infty)[\text{Pay} \cdot 1_{F < \infty}] . \] (42)

Note that
\[ M(P_{\text{conf}}^-, S_\infty) = \mathbb{E}(P_{\text{conf}}^-, S_\infty)[\text{Pay} \cdot 1_{F = n + 1}] + \mathbb{E}(P_{\text{conf}}^-, S_\infty)[\text{Pay} \cdot 1_{F \leq n}] . \]

The former right-hand term is non-negative, and the latter increases to $\mathbb{E}(P_{\text{conf}}^-, S_\infty)[\text{Pay} \cdot 1_{F < \infty}]$.

From (42), we find then that
\[ \lim_{n} \inf M(P_{\text{conf}}^-, S_\infty) \geq M(P_{\text{conf}}^-, S_\infty), \]

as we sought to show.

An asymmetry under good fortune was noted between Mina and Maxine after Proposition 1.19. It is in fact this asymmetry that has led us to treat the two players differently in the preceding proof. While Maxine’s strategy was circumscribed at high fortunes by a definition in Lemma 7.6(2), it was the ‘Mina pays one’ rule for unfinished games that enabled the proof of Lemma 7.6(1). Indeed, with a rule of the form ‘Mina pays less than one’, we cannot hope to obtain Lemma 7.6(1) by circumscribing Mina’s strategy at low fortunes similarly as we did Maxine’s: see Section 8.2.

Definition 7.7. Let $S_- \in S_-$, $S_+ \in S_+$ and $D \in (0, \infty)$. Let $S_+(D)$ denote the element of $S_+$ that is strongly conforming above fortune $D$ for which stakes and move nominations offered when StateOfPlay = $(\mu, w) \in (0, \infty) \times V_0$ for $\mu \in [0, D)$ are governed by $S_+$.

Recall that $\lambda = \lambda_0$ is the initial fortune in Game($\epsilon, \lambda, v$).

Lemma 7.8. Let $S_- \in S_-$ be a strategy for Mina that is non-conforming against $P_{\text{conf}}^+$. Then $\lim_{n} \inf M(S_-|n, P_{\text{conf}}^+|n) > h(\lambda, v)$.

Let $S_+ \in S_+$ be a strategy for Maxine that is non-conforming against $P_{\text{conf}}^-$. There exists $\delta > 0$ such that $D > \lambda$ implies that
\[ \lim_{n} \sup M(P_{\text{conf}}^-, S_+(D)|n) < h(\lambda, v) - \delta, \]

where $S_+(D)$ is specified in Definition 7.7.

Proof: (1). This proof is omitted because it is in essence a slightly simplified version of the proof of the second part.

(2). We extend the notation of Definition 7.3 so that $S_+(\infty)$ denotes $S_+$. Consider gameplay under the strategy pair $(P_{\text{conf}}^-, S_+(\infty))$. Let the deviation set $D$ denote the set of indices $i \in \mathbb{N}_+$ such that there is positive probability that Maxine does not conform at the $i$th turn in the sense of Definition 1.11. Since $S_+(\infty)$ is by assumption non-conforming, $D$ is non-empty. Let $j \in \mathbb{N}_+$ denote the least element in $D$.

To derive Lemma 7.8(2), we will now argue that it suffices to prove the bound
\[ \lim_{n} \sup M(P_{\text{conf}}^-, S_+(\infty)|n) < h(\lambda, v). \] (43)
For $D \in (0, \infty)$, let $S^j_\pm(D)$ denote the element of $S_\pm$ obtained by modifying $S_\pm(D)$ so that play by Maxine at every turn with index at least $j + 1$ is conforming. We now make two claims, which will prove that deriving the bound (43) is indeed sufficient for our purpose. First, 

$$M(P_{\text{conf}}^\pm|_n, S^j_+(D)|_n) \geq M(P_{\text{conf}}^\pm|_n, S_+(D)|_n) \text{ for } D \in (0, \infty].$$

Second, 

$$M(P_{\text{conf}}^\pm|_n, S^j_+(D)|_n) = M(P_{\text{conf}}^\pm|_n, S^j_+(\infty)|_n) \text{ for } D > \lambda.$$ 

Let $D > \lambda$. Applying (44), (45) and (43), we obtain Lemma 7.6.3.

We now prove (44). Proposition 6.1(2) shows that Maxine’s play under $P_{\text{conf}}^\pm|_n, S^j_+(D)|_n$ forms a Nash equilibrium in Game$_n(\epsilon, \lambda, v)$. So in fact the two gameplays coincide throughout the lifetime of the game. This proves (44).

The two claims justified, it remains, in order to obtain Lemma 7.8.2, to prove (43). Set $S = S^j_+(\infty)$. Our task is to show that $\limsup_n M(P_{\text{conf}}^\pm|_n, S^j_+|_n) < h(\lambda, v)$. We thus consider gameplay under $(P_{\text{conf}}^\pm|_n, S^j_+|_n)$. The deviation set $D$ associated to this gameplay has a unique element, $j$. Let $E_j$ denote the event that, at the $j$th turn, Maxine’s stake does not conform. Let $F_j$ denote the counterpart event where ‘move nomination’ replaces ‘stake’. Note that $P(E_j) + P(F_j) > 0$. Definition 1.11 indicates the form of these two events: on $E_j$, Maxine’s stake at the $j$th turn is not equal to $\lambda_{j-1} \cdot \text{Stake}(\epsilon, \lambda_{j-1}, X_{j-1})$; while $F_j$ occurs when Maxine nominates a move at this turn that lies outside of $V_+(X_{j-1})$.

If the strategy $S$ is modified at the $j$th turn so that Maxine conforms against $P_{\text{conf}}^\pm$ at this turn, a strategy in Game($\epsilon, \lambda, v$) that is conforming against $P_{\text{conf}}^\pm$ results, in the sense of Definition 1.11. Denote this modified strategy by $P_{\text{conf}}^\pm$ and its index by $\theta \in \Theta$. Note then that 

$$M(P_{\text{conf}}^\pm|_n, S^j_+|_n) = \sum_{i=1}^4 \gamma_i(A, n) \text{ and } M(P_{\text{conf}}^\pm|_n, P_{\text{conf}}^\pm|_n) = \sum_{i=1}^4 \gamma_i(B, n),$$

where $\gamma_i(A, n), i \in [1, 4]$, are the expected values of Pay $\cdot 1_{E_j \cap F_j}$, Pay $\cdot 1_{F_j \cap E_j}$, Pay $\cdot 1_{E_j \cap F_j}$ and Pay $\cdot 1_{F_j \cap F_j}$ under the strategy pair $(P_{\text{conf}}^\pm|_n, S^j_+|_n)$, and $\gamma_i(B, n), i \in [1, 4]$, are the counterpart quantities under $(P_{\text{conf}}^\pm|_n, P_{\text{conf}}^\pm|_n)$.

We claim that 

$$\gamma_i(A, n) - \gamma_i(B, n) \text{ is negative and independent of } n \geq j \text{ for } i \in [1, 3],$$

and also that $\gamma_4(A, n) = \gamma_4(B, n)$. The latter statement has the simplest proof: the discrepancy between the two concerned strategies for Maxine is manifest only at the $j$th turn and this discrepancy
does not affect gameplay when $E_j^c \cap F_j^c$ occurs. We now prove in turn the three assertions made in [47]. First, note that

$$\gamma_1(A, n) - \gamma_1(B, n) = \sum_{w \in \mathcal{V}_O} \mathbb{P}(X_\theta(j - 1) = w, E_j \cap F_j) \int \alpha_n(x) \, d\mu_{j,w}(x),$$

where recall that $\theta \in \Theta$ is the index of Maxine’s strongly conforming strategy $P_+^{\text{conf}}$; the quantity $\alpha_n(x)$ is specified in the notation of Lemma 6.2 by

$$\alpha_n(x) = \text{Val}_{n+1-j}(\epsilon, \lambda_{j-1}, w, \lambda_{j-1}s, x) - \text{Val}_{n+1-j}(\epsilon, \lambda_{j-1}, w, \lambda_{j-1}s, s); \quad (48)$$

and the law $\mu_{j,w}$ is the conditional distribution of the stake offered by Maxine at the $j^{\text{th}}$ turn given that $E_j$, $F_j^c$ and $X_\theta(j) = w$ occur. The quantity $s$ equals Stake($\epsilon, \lambda_{j-1}, w$) from [5], so that $(\lambda_{j-1}s, s)$ is the global minimax for $(0, \infty)^2 \rightarrow [0, 1] : (a, b) \mapsto \text{Val}_{n+1-j}(\epsilon, \lambda_{j-1}, w, a, b)$ identified in Lemma 6.3 (3) and by Proposition 6.1 (2). Note that the law $\mu_{j,w}$ assigns zero mass to the point $s$. Note then that, for $x \in (0, \infty)$, $x \neq \lambda_{j-1}s$,

$$\text{the quantity } \alpha(x) = \alpha_n(x) \text{ is negative, and independent of } n \geq j. \quad (49)$$

Indeed, the second derivative in the variable $a$ that appears in Lemma 6.3 (3) is negative, and it is independent of $n$ because so $\Psi_n$ is, as we noted before Lemma 6.2. Thus we obtain (47) for $i = 1$.

Note next that, by Lemma 4.2, $\gamma_2(A, n) - \gamma_2(B, n)$ equals

$$\sum_{w \in \mathcal{V}_O} \mathbb{P}(X_\theta(j - 1) = w, F_j \cap E_j) \epsilon_{\lambda_{j-1}}(1 + \lambda_{j-1})^{-1} \int (h(\lambda_{j-1}, z) - h(\lambda_{j-1}, w_+)) \, d\zeta_{j,\lambda_{j-1}}(z),$$

where $\zeta_{j,\lambda_{j-1}}(\cdot)$ is the discrete law charging neighbours of $w$ that is given by the conditional distribution of Maxine’s move nomination at the $j^{\text{th}}$ turn given that $F_j$, $E_j^c$ and $X_\theta(j - 1) = w$ occur and the fortune is $\lambda_j$. Indeed, if a move takes place at the turn that Maxine wins at which she nominates $z$, she suffers a conditional mean change of $h(\lambda_j, z) - h(\lambda_{j-1}, w_+)$ in terminal payment; the respective probabilities attached to the italicized terms are $\epsilon$, $\lambda_{j-1}(1 + \lambda_{j-1})^{-1}$ and $\zeta_{j,\lambda_{j-1}}(z)$. The law $\zeta_{j,\lambda_{j-1}}(\cdot)$ does not charge the vertex $w_+$ because Maxine’s $j^{\text{th}}$ move nomination is non-conforming when $F_j$ occurs. Thus $h(\lambda_{j-1}, z) - h(\lambda_{j-1}, w_+)$ is seen to be negative (and independent of $n$) when $z$ is in the support of $\zeta_{j,\lambda_{j-1}}(\cdot)$, by Proposition 6.12. Thus, (47) holds for $i = 2$.

Next we consider $i = 3$. In this case, Maxine makes a mistake at the $j^{\text{th}}$ turn both in her stake and in her move nomination. Let $\psi_{j,w}$ denote the conditional distribution of the stake and move nomination pair offered by Maxine at the $j^{\text{th}}$ turn given that $E_j \cap F_j$ and $X_\theta(j) = w$ occur. (We could write $\psi_{j,\lambda_{j-1},\lambda_{j-1},w}$: the stake is decided knowing $\lambda_{j-1}$; the move is nominated knowing $\lambda_j$.) The respective marginals of $\psi_{j,w}$ charge neither the point $\lambda_{j-1}s$ nor the vertex $w_+$. We have that

$$\gamma_3(A, n) - \gamma_3(B, n) = \sum_{w \in \mathcal{V}_O} \mathbb{P}(X_\theta(j - 1) = w, E_j \cap F_j) \int (\alpha(x) + \beta_n(x, z)) \, d\psi_{j,w}(x, z),$$

---

This fortune is $\lambda_j$, not $\lambda_{j-1}$, because players learn the revised fortune before nominating a move: see the update to StateOfPlay in the first step in Section 1.3.
where $\alpha(x)$ is specified in \ref{3.9}, and
\[ \beta_n(x, z) = \epsilon \frac{x}{x+s} \left( \text{Val}_{n-j} \left( \frac{\lambda_{j-1}-x}{1-s}, z \right) - \text{Val}_{n-j} \left( \frac{\lambda_{j-1}-x}{1-s}, w_- \right) \right). \]

As above, $\alpha(x)$ is the conditional mean change in payment resulting from Maxine’s faulty decision to stake $x$ at the $j$th turn. The corresponding change caused by her move nomination of $z$, given the stake pair $(\lambda_{j-1}-x, x)$, is $\beta_n(x, z)$; indeed, with probability $\epsilon$, a move takes place at the $j$th turn; with probability $\frac{x}{x+s}$, Maxine wins the right to make the resulting move; and, if she does so, she incurs a change in conditional mean payment of $\text{Val}_{n-j} \left( \frac{\lambda_{j-1}-x}{1-s}, z \right) - \text{Val}_{n-j} \left( \frac{\lambda_{j-1}-x}{1-s}, w_- \right)$ by nominating $z$ instead of the optimal choice $w_+$. The quantity $\beta(z, a) = \beta_n(x, z)$ is negative, and it is independent of $n \geq j$ in view of Corollary 3.10 and (50). Since $\rho$, the strategy pair $\left( \text{P}_{\text{conf}}^{n-j}, \text{P}_{+}^{n-j} \right)$ denotes the event that Maxine’s fortune is at least $D$ before some turn in Game$(\epsilon, \lambda, v)$, we return to (46) to learn that
\[ M \left( S_{n}^{+}, \text{P}_{-}^{\text{conf}} \right) - M \left( \text{P}_{-}^{\text{conf}}, S_{+}^{\text{conf}} \right) = \sum_{i=1}^{3} \left( \gamma_{i}(A) - \gamma_{i}(B) \right) \]
is negative and independent of $n \geq j$. Since the strategy pair $\left( \text{P}_{\text{conf}}^{n-j}, \text{P}_{+}^{\text{conf}} \right)$ is a Nash equilibrium in Game$n(\epsilon, \lambda, v)$ by Proposition 6.1(2), $M \left( \text{P}_{\text{conf}}^{n-j}, \text{P}_{+}^{\text{conf}} \right)$ equals $h(\lambda, v)$ by Proposition 6.1(1) and Lemma 4.2. Recalling the shorthand $S = S_{+}^{\text{conf}}$, we obtain (43) and thus complete the proof of Lemma 7.8(2).

Recall Definition 7.3.

**Lemma 7.9.** For $\eta > 0$, any sufficiently high $D > 0$ is such that
\[ M \left( \text{P}_{\text{conf}}^{n-j}, S_{+}^{\text{conf}} \right) \leq M \left( \text{P}_{-}^{\text{conf}}, S_{+}^{\text{conf}}(D) \right) + \eta. \]

**Proof.** Lemma 3.9 implies that $\lim_{\lambda \to \infty} h(\lambda, v) = 1$ for $v \in V$. Thus any high enough $D > 0$ satisfies
\[ h(D, v) \geq 1 - \eta \quad \text{for} \quad v \in V. \quad (50) \]

Let $\text{HighFortune}_{+}(D, \infty)$ denote the event that Maxine’s fortune is at least $D$ before some turn in Game$(\epsilon, \lambda, v)$. Note that
\[ M \left( \text{P}_{-}^{\text{conf}}, S_{+}^{\text{conf}} \right) = \mathbb{E}_{\left( \text{P}_{-}^{\text{conf}}, S_{+}^{\text{conf}} \right)} \left[ \text{Pay} \cdot \mathbf{1}_{\text{HighFortune}_{+}(D, \infty)} \right] + \mathbb{E}_{\left( \text{P}_{-}^{\text{conf}}, S_{+}^{\text{conf}} \right)} \left[ \text{Pay} \cdot \mathbf{1}_{\text{HighFortune}_{+}(D, \infty)} \right]. \]

The first right-hand term is at most $\rho := \mathbb{P}_{\left( \text{P}_{-}^{\text{conf}}, S_{+}^{\text{conf}} \right)} \left( \text{HighFortune}_{+}(D, \infty) \right)$, and the second coincides with its counterpart in the next equality:
\[ M \left( \text{P}_{-}^{\text{conf}}, S_{+}^{\text{conf}}(D) \right) = \mathbb{E}_{\left( \text{P}_{-}^{\text{conf}}, S_{+}^{\text{conf}}(D) \right)} \left[ \text{Pay} \cdot \mathbf{1}_{\text{HighFortune}_{+}(D, \infty)} \right] + \mathbb{E}_{\left( \text{P}_{-}^{\text{conf}}, S_{+}^{\text{conf}}(D) \right)} \left[ \text{Pay} \cdot \mathbf{1}_{\text{HighFortune}_{+}(D, \infty)} \right]. \]

The first right-hand term in the last display satisfies
\[ \rho \cdot \mathbb{E}_{\left( \text{P}_{-}^{\text{conf}}, S_{+}^{\text{conf}}(D) \right)} \left[ \text{Pay} | \text{HighFortune}_{+}(D, \infty) \right] \geq \rho(1 - \eta), \]
where the inequality is due to Corollary 3.10 and (50). Since $\rho \leq 1$, the lemma has been proved. \qed
Proof of Proposition 7.2. The first assertion follows from Lemma 7.6(1) and Lemma 7.8(1). To prove the second, let \( D > \lambda \). By Lemma 7.6(2) with \( S_+ \) there taken equal to the present \( S_+(D) \), and Lemma 7.8(2), we find that \( M(P_{-\text{conf}}, S_+(D)) < h(\lambda, v) - \delta \). By increasing \( D > 0 \) if need be, we may apply Lemma 7.9 with \( \eta = \delta/2 \) to obtain \( M(P_{-\text{conf}}, S_+) < h(\lambda, v) - \delta/2 \). Whence the proposition’s second assertion.

Proofs of Theorems 1.8 and 1.12. Continuing to use our notation for strongly conforming strategies, we claim that

\[
M(P_{-\text{conf}}, P_{+\text{conf}}) = h(\lambda, v). \tag{51}
\]

Indeed, the counter evolution under gameplay governed by the strategy pair \((P_{-\text{conf}}, P_{+\text{conf}})\) takes the form \( X_\theta : [0, F_\theta] \rightarrow V \), \( X_\theta(0) = v \). The process \([0, F_\theta] \rightarrow [0, 1] : i \mapsto h(\lambda, X_\theta(i))\) is a martingale. Moreover, and as we noted after Proposition 7.2, it implies that \( F_\theta \) is finite almost surely. Thus the mean terminal payment as a function of the starting location satisfies the system (4). Hence, we obtain (51).

Proposition 7.2 thus demonstrates that any strategy pair whose components are strongly conforming is a Nash equilibrium; we thus obtain the assertions in Theorem 1.8(1) that the value of \( \text{Game}(\epsilon, \lambda, v) \) equals \( h(\lambda, v) \) and in Theorem 1.8(2) that a Nash equilibrium exists in \( \text{Game}(\epsilon, \lambda, v) \).

Suppose now that \((S_-, S_+)\) is a Nash equilibrium at least one of whose components is non-conforming against the opposing element, in the sense that, under gameplay governed by \((S_-, S_+)\), there is a positive probability that Maxine or Mina does not conform at some move. Let \( \ell_+ \in \mathbb{N}_+ \) be the minimum turn index \( i \) at which there is a positive probability that Maxine does not conform at the \( i \)-th turn; we take \( \ell_- = \infty \) if all such probabilities are zero. Let \( \ell_- \) denote the counterpart quantity for Mina. Suppose that \( \ell_- \leq \ell_+ \); we have then that \( \ell_- \) is finite. Let \( P_{+\text{conf}} \) denote the strategy for Maxine that coincides with \( S_+ \) at turns with index at most \( \ell_- - 1 \), and that adheres to Maxine’s strongly conforming strategy at turns with index at least \( \ell_- \). (Our notation would be inconsistent if \( P_{+\text{conf}} \) could fail to be strongly conforming. And in fact this difficulty may occur. But this is due merely to non-conforming choices that Maxine may make in states of play that are almost surely inaccessible under the gameplay governed by \((S_-, P_{+\text{conf}})\). We may harmlessly correct such deviations of Maxine, so that \( P_{+\text{conf}} \) may indeed be supposed to be strongly conforming.) Mina’s play under the strategy pair \((S_-, P_{+\text{conf}})\) is non-conforming, because it is with positive probability that she does not conform at the turn with index \( \ell_- \). We have then that

\[
h(\lambda, v) = M(S_-, S_+) \geq M(S_-, P_{+\text{conf}}) > M(P_{-\text{conf}}, P_{+\text{conf}}) = h(\lambda, v),
\]

where the first equality, and the first inequality, are due to \((S_-, S_+)\) being a Nash equilibrium; the strict inequality is by appeal to Proposition 7.2 in view of the strong conformity of \( P_{+\text{conf}} \) and the just noted non-conformity of the component \( S_- \) in the strategy pair \((S_-, P_{+\text{conf}})\); and the latter equality is due to \((P_{-\text{conf}}, P_{+\text{conf}})\) being a Nash equilibrium (a fact shown earlier in this proof). The displayed contradiction shows that \((S_-, S_+)\) is not a Nash equilibrium, at least under the assumption that \( \ell_- \leq \ell_+ \); but there was no loss of generality in making this assumption, because the opposing case has a similar proof. In this way, we complete the proof of Theorem 1.8(2) and obtain Theorem 1.12.

Proof of Corollary 1.15. Elements of the pair \((P_{-\text{conf}}, P_{+\text{conf}})\) considered in the preceding proofs are strongly conforming, and the index \( \theta \in \Theta \) of \( P_{+\text{conf}} \) is arbitrary. The gameplay process \( X_\theta \) results
from the choice of $\theta \in \Theta$, and thus all of the claimed processes occur as gameplay processes governed by Nash equilibria in Game($\epsilon, \lambda, v$).

In order to show that gameplay can be no other process, we develop Definition 1.11(2). Let $(S_-, S_+) \in S_- \times S_+$, $\theta \in \Theta$ and $i \in \mathbb{N}_+$. Mina conforms with index $\theta$ at the $i^{th}$ turn under the gameplay governed by $(S_-, S_+)$ if she stakes Stake($\epsilon, \lambda_{i-1}, X(i-1)$) and nominates the move $\theta(\lambda_{i-1}, X(i-1), i-1)$ at this turn when this given strategy pair is adopted. The strategy $S_-$ is said to be conforming with index $\theta$ against $S_+$ if Mina almost surely conforms with index $\theta$ at every turn. (To establish the relation to Definition 1.11, note that $S_-$ conforms against $S_+$ if and only if $S_-$ conforms with index $\theta$ against $S_+$ for some $\theta \in \Theta$.)

Theorem 1.12 asserts that any Nash equilibrium in Game($\epsilon, \lambda, v$) is a conforming strategy pair. Maxine’s hand is thus forced at every turn. Mina’s choice at a given turn is restricted to the selection of an element of $\mathcal{V}_-(w)$ for her move nomination, where $w \in V_{\Omega}$ is the present counter location. By Definition 1.3, she may only take account of the values of StateOfPlay and the turn index in making her choice. This forces her play to be conforming with index $\theta$ against Maxine’s chosen strategy, for some $\theta \in \Theta$. This completes the proof of Corollary 1.15.

8 Directions and open problems

We offer an overview of some prospects for developments of the concepts and proofs in the article in several subsections that begin with more specific and technical aspects and end with broader themes.

8.1 Strategy spaces and reset rules

We have not hesitated to posit limitations on game design and strategy spaces in the interests of proof simplicity provided that such assumptions change nothing essential about the anticipated game values and equilibria. These assumptions can be reviewed. The pair of reset rules recorded in Section 1.3 provides easy means of settling questions in proofs that arise from joint-zero or joint-total stakes at a given turn. But in fact players are anyway tempted away from zero or total stakes, as comments under the pair of slogans in Subsection 2.5.4 indicate and further rigorous argument may be expected to imply.

Legal strategies in Definition 1.3 permit only the present values of Maxine’s fortune and counter location, alongside the turn index, to inform players’ choices at a given turn. These definitions could be broadened so that the game history is available, though we do not anticipate that the basic structure of value and equilibria to shift in response to this broadening. Players have been permitted to randomize stakes, but not moves, and a similar comment may be made in this regard. Choices that are mixed for both move and stake would render the mixed strategy spaces convex, which may be an attractive feature should fixed point theorems be brought to bear in efforts to prove the existence of equilibria in variants of the game: we turn to this topic in the next paragraph but one.

8.2 The payment when the game is unfinished

In Section 1.3, we specified that Mina will pay one unit when Game($\epsilon, \lambda, v$) fails to finish. This choice is needed to permit our proof of Lemma 7.6(1). Indeed, if Pay is a given value less than one when the
finish time is infinite, Mina can sometimes oppose conforming play from Maxine and achieve a lower mean payment than she would obtain were she to conform. To see this, consider the graph formed by attaching a leaf $z$ along an edge to the vertex $n - 1$ in the root-reward line graph $([0, n], \sim, 1_n)$. This root-reward tree has root $n$, and two further leaves, 0 and $z$. The vertex $n - 2$ has journey data $\{(2, 1), (n - 1, 1)\}$. The game played from $n - 2$ is a contest of two rounds: if Maxine is to win, the counter must first reach $n - 1$; and then $n$. Suppose that $\lambda \gg 1$. In the first round, little is at stake; but the second is more contested. Indeed, if both players conform, then Corollary 3.22 shows that the stake offered in Game$(1, \lambda, n - 2)$, namely Stake$(1, \lambda, n - 2)$, takes the form $\lambda^{3-n}(1 + o(1))$ as $\lambda \to \infty$. Suppose that Mina modifies conforming play only at vertex $n - 2$, where she stakes one-half (and plays left). From $n - 2$, the counter typically moves left; then it returns to $n - 2$, usually after one further turn, with the value of $\lambda$ almost doubling since the last visit to $n - 2$. Except with probability of order $\lambda^{3-n}$, the game will never end. Conditionally on this non-finishing event, the gameplay empirical process $h(\lambda_i, X_i)$ will converge to one almost surely. But Mina’s terminal payment will be a given constant less than one, by fiat. Thus we see that the rule for unfinished games, even when it makes the seemingly mundane demand that the terminal payment be a given constant on $[0, 1]$, is not merely a technical detail to ensure well-specified play: if the constant is less than one, Maxine must do something to deviate from conforming play to overcome this unending filing of extensions from Mina. From this example, we see that the ‘Mina pays one’ rule should be viewed, alongside the use of root-reward trees and a small move probability $\epsilon$, as a hypothesis that substantially enables our proofs. It would be of interest to inquire whether, admitting these other assumptions, the rule ‘Mina pays $p$’, for $p \in [0, 1)$ given, leads to the biased-infinity game value found under the $p = 1$ rule.

8.3 Abstract existence results for Nash equilibria

We have constructed Nash equilibria by concrete means rather than by seeking to use abstract results. Von Neumann [VN28] proved the existence of value in two-person zero-sum games with finite strategy spaces via his minimax theorem, and continuum strategy spaces have been treated by Glicksberg [Gli51], provided that the mean payoff is continuous. When it exists, the value $\text{Val}(1, \lambda, v, a, b)$ of the first-turn-constrained regular game is not in most cases continuous at $(a, b) = (\lambda, 1)$: the final paragraph of Section 2.3 addresses this point (in a special case). Existence results for equilibria where the continuity hypothesis is weakened have been obtained by Simon [Sim87], and it would be of interest to examine the applicability of such theory to stake-governed games.

8.4 Prospects for specifying and analysing the Poisson game

A formal analysis of the Poisson game in Section 2.4 provided (what we hoped to be!) a simple and attractive point of departure for our proofs concerning the leisurely game. One may wish to treat the Poisson game rigorously. Significant conceptual questions must be answered to make rigorous sense of this continuous-time game, however. One could say that each player must adhere to a strategy that is adapted to the strict past history of gameplay. With such instantaneous communication, strategies such as ‘I’ll stake twice what she just staked’ can however lead to a folie-à-deux. Presumably, they should be banned by suitable constraints on strategy spaces. Simon and Stinchcombe [SS89] have developed a framework for addressing such problems which may be applicable for stake-governed games.

For many boundary-payment graphs, it may be that the global saddle hope is realized for the
Poisson game at $\lambda$-values outside a finite set at which the Peres-Šunić decomposition changes. It is conceivable, however, that a player may force the running $\lambda$-value onto the special finite set for a positive measure of times, so that the more complex behaviour apparent in Figure 3 (middle) becomes germane even if the initial value of $\lambda$ is generic. This is highly speculative, but such possibilities should be borne in mind when the Poisson game is analysed.

8.5 Stake-governed tug-of-war beyond root-reward trees

Theorem 3.3 attests that the Peres-Šunić decomposition is independent of $\lambda \in (0, \infty)$ for root-reward trees. This condition is fundamental to our analysis, because it disables the capacity (seen in Subsection 2.3.4) of a player to bamboozle an opponent by uncertainties over whether she will play a short or long game. Indeed, the global saddle hope appears to be false in many cases for the regular game Game(1, $\lambda$, $v$). A challenge is to make sense of the complexities of behaviour apparent in such contour plots as those in Figures 1 and 2 and to resolve questions about the existence and structure of Nash equilibria for stake-governed tug-of-war when the global saddle hope fails.

8.6 Predictions for the regular game

In Section 2, we saw that the ‘big picture’ prediction of optimal play governed by the stake formula (5) is inaccurate in the regular game (with $\epsilon = 1$) for at least some values of $\lambda$ in graphs as simple as the $T$ graph. But we also saw that the prediction is correct in one case at least, that of the half-ladder $H_n$ for $n \in \mathbb{N}_+$. It would be interesting to find further examples where the prediction is valid and to seek to characterize the graphs for which it is.

8.7 A one-parameter family of games

In a ‘Poorman’ variant of the Richman games that is analysed in [LLP+99], the higher staking player at a given turn wins the right to move, with the stake of this player surrendered to a third party. We may vary this game to bring it closer to stake-governed tug-of-war if we instead insist that the stakes of both players are thus surrendered. Call this the Poormen game. In classical tug-of-war, the right to move is allocated according to a fair coin flip, independently of the stakes (which are thus unnecessary). The Poormen game and tug-of-war may be viewed as $p = \infty$ and $p = 0$ endpoints of a one-parameter family of games. Indeed, let $p \in [0, \infty]$. If Maxine stakes $a$ and Mina $b$ at a given turn, then Maxine’s win probability is $a^p / (a^p + b^p)$; the rules of stake-governed tug-of-war are otherwise in force. It would be of interest to study this continuum of games, along which stake-governed tug-of-war occupies the position $p = 1$.

8.8 Continuous-game versions and PDE

Tug-of-war attracted great interest in PDE from its inception because of its capacity to prove properties and develop intuitions about such famously subtle problems as the uniqueness and regularity theory of the infinity Laplacian on domains in Euclidean space. It is very natural to seek to take the same path for the stake-governed version. Mina and Maxine will play on a domain in $\mathbb{R}^d$, moving a counter a given small distance when one or other wins the right to do so. Does the stake formula (5) take a counterpart form in the limit of small step size? Does the alternative stake formula have a counterpart whose denominator is expressed in terms of a limiting stochastic process for continuum gameplay—an $\infty$-Brownian motion? Which PDE does the continuum stake function satisfy?
These questions may also be posed for the case of the \( p \)-Laplacian, which arises in tug-of-war with interjections of noise at each turn \[\text{PS08, Lew20},\] because we may naturally specify a noisy version of stake-governed tug-of-war in the Euclidean setting. This setting may be more tractable than the \( p = \infty \) case because the concerned objects—functions and stochastic processes—are at least somewhat more regular and, presumably, more straightforward to define.

### 8.9 Stake-governed selection games

Study the same questions—existence of pure Nash equilibria, the value of the stake function, and so on—for the stake-governed versions of random-turn selection games based on monotone Boolean functions, such as iterated majority, AND/OR trees, or critical planar percolation \[\text{PSSW07}].\] Of interest is to investigate the possible interplay between changes in the bias \( \lambda \) and phase transition phenomena \[\text{OSSS05, DCRT19},\] especially in light of our formula \( [9] \) connecting optimal stake and expected length of the game, and of the role of low revealment algorithms in the sharpness of phase transitions.

### 8.10 Dynamic strategies for political advertising: stake games played in parallel

Mina wears a red hat and Maxine a blue one. On each of fifty boundary-payment graphs, a counter is placed at some vertex. In each graph, the payment function \( f : V_B \rightarrow [0, \infty) \) takes the form \( f = \kappa 1_D \) for some \( D \subseteq V_B \) and constant \( \kappa \in \mathbb{N}_+ \) determined by the graph. For some graphs, such as CA and NY, \( \kappa \) is around forty, and \( V_B \setminus D \) comprises a few isolated sites; for others, such as SD and WY, \( \kappa \) is about five, and it is \( D \) that is a small and isolated set; while on such graphs as FL and PA, neither \( D \) nor \( V_B \setminus D \) is evidently more exposed in a suitable harmonic sense. The sum of \( \kappa \)-values over the graphs is 538. Maxine and Mina have respective budgets of \( \lambda \) and one at the outset. At each turn, each must allocate a stake in each graph from her remaining fortune; a move takes place in each graph at each turn, and the right to move is allocated to a given player in each graph with probability equal to the ratio of that player’s stake, and the combined stakes, offered for that graph at that turn. Maxine’s objective is to maximize the probability of receiving at least 270 units; or, if we exercise the liberty to choose our assumptions, simply to maximize the mean terminal payment.

### References


**ALAN HAMMOND**
DEPARTMENTS OF MATHEMATICS AND STATISTICS, U.C. BERKELEY
899 EVANS HALL, BERKELEY, CA, 94720-3840, U.S.A.
*Email:* alanmh@berkeley.edu

**GÁBOR PETE**
ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS
REÁLTANODA U. 13-15., BUDAPEST 1053 HUNGARY
AND
MATHEMATICAL INSTITUTE, BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS
MÜEGYETEM RKP. 3., BUDAPEST 1111 HUNGARY
*Email:* gabor.pete@renyi.hu