RESEARCH OVERVIEW

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Here I give an overview of the various strands in my research, which concerns rigorous statistical mechanics usually through the lens of probability theory. The order of topics is roughly made so that those of greatest present active interest to me are earlier. Citations of the form [$\alpha 2$] or [$\delta 1$] are to the bibliography of others' work and of the form [A3] or [E1] to the appended publication list of mine.

A: Phase boundary fluctuation and randomly growing interfaces

This topic concerns the fluctuation exponents and scaling limits of static and dynamically defined random models of interfaces. In such systems, local Gaussian fluctuation competes with global curvature constraints to determine longitudinal and radial fluctuation exponents of $2/3$ and $1/3$. I have studied static models, partly in collaboration with Yuval Peres, and more recently dynamic ones, in large part with Ivan Corwin.

Phase separation: To set the scene for explaining the static results first, consider the ferrogmagnetic Ising model, in which spins of two types, positive and negative, are assigned to the integer-lattice sites in a domain such as a large box, with signs of opposite type in nearest-neighbour pairs being penalized according to a fixed parameter. The two populations of spins each prefer their own type, and, at supercritical inverse temperature, a dominant phase forms, with one type in the majority. If we condition at positive magnetization on an excess of signs of the opposite type, these spins collect together in a droplet. The droplet boundary is the object of study of phase separation. By duality considerations and the Fortuin-Kasteleyn representation, a close surrogate for this problem is the following one. For any planar lattice circuit Γ, write $\mathrm{INT}(\Gamma)$ for the region enclosed by that circuit. Let Γ_0 denote the outermost open circuit Γ such that $0 \in \mathrm{INT}(\Gamma)$. Condition a subcritical planar random cluster model P on the area constraint $\big|\mathrm{INT}(\Gamma_0)\big| = n^2$, with n high, and study the fluctuations present in Γ_0.

Three years ago, I addressed this last problem in the papers [B2,B3,B4]. The conditioned circuit Γ_0 contains area n^2 and has a diameter of order n. Local fluctuations have been defined [$\alpha 1$] by considering how the circuit deviates from the boundary $\partial\mathrm{conv}\big(\Gamma_0\big)$ of its convex hull. The maximum local roughness $\mathrm{MLR}\big(\Gamma_0\big)$ is defined to be the greatest distance of a point in the circuit from $\partial\mathrm{conv}\big(\Gamma_0\big)$. This radial notion of local deviation has a longitudinal counterpart in the form of maximum facet length $\mathrm{MFL}\big(\Gamma_0\big)$, which is defined to be the length of the longest line segment of which the polygon $\partial\mathrm{conv}\big(\Gamma_0\big)$ is comprised. These local definitions of roughness are of particular interest, because they are concerned with circuit behaviour on a scale at which the competition between local Gaussian fluctuation and its global constraint by curvature is manifest. In 2001, K. Alexander proved that an averaged version of local roughness scales as $n^{1/3}$. It has been an open problem to provide uniform control, to verify that $\mathrm{MLR}\big(\Gamma_0\big)$ and $\mathrm{MFL}\big(\Gamma_0\big)$ scale as $n^{1/3}$ and $n^{2/3}$. The principal conclusion of my investigation is that there exist constants $0 < c < C < \infty$ such that

$$P\bigg(c \leq \frac{\mathrm{MLR}\big(\Gamma_0\big)}{n^{1/3}\big(\log n\big)^{2/3}} \leq C \,\bigg|\, \big|\mathrm{INT}(\Gamma_0)\big| = n^2 \bigg) \to 1, \qquad \text{as } n \to \infty,$$

and that the same conclusion holds for the normalized quantity $n^{-2/3}\big(\log n\big)^{-1/3}\mathrm{MFL}\big(\Gamma_0\big)$. That is, the conjectured exponents for the power-laws in radial and longitudinal local deviation are derived, as well as exponents for the logarithmic correction for these quantities. These techniques are an extensive application of surgeries on the circuit, first to establish regeneration structure in Γ_0, and, then, to show that long boundary facets may be surgically reconstructed into provably unusual excesses in captured area. The logarithmic corrections identified are probably shared by a broad range of radially defined stochastic interface models including the ferrogmagnetic Ising example above.

Randomly growing interfaces and KPZ: Many mathematical models of interfaces which grow in time

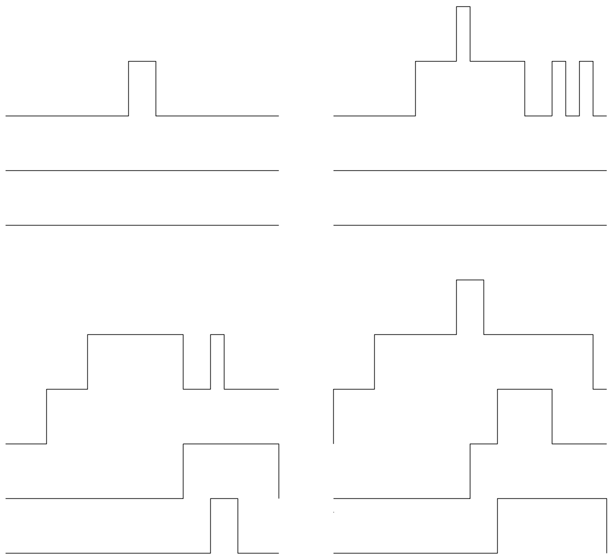

Figure 1: Four snapshots of an evolving multi-line PNG model. Nucleations on the top interface result in horizontally growing towers whose collision causes nucleations in the second line; over time, the nucleations percolate downwards so that growth occurs in many levels.

and are subject to forces of surface tension and randomness are predicted at late time to fall in the Kardar-Parisi-Zhang universality class. Members of this class of models are all expected to share the scaling characteristics already mentioned: when the global length is of order t, longitudinal and radial fluctuations will scale as $t^{2/3}$ and $t^{1/3}$.

In fact, the Gibbs resampling ideas which were so central to my investigation in 2009–10 of boundary fluctuation in phase separation have a conceptual counterpart in the realm of KPZ scaling limits. Ivan Corwin and I are engaged in a programme of research of studying the Gibbs properties of several canonical KPZ-style scaling limits and the consequence of these properties.

To explain how resampling is valuable in a dynamic setting, consider first an example, the polynuclear growth model, which was introduced in [$\alpha 3$]. Picture the real line as a terrain on which raindrops fall according to a space-time Poission process of unit intensity. At time zero, the *fertile* interval consists of the set $\{0\}$; as time evolves, this interval grows to the left and right at unit rate, taking the form $[-t, t]$ at time $t \geq 0$. Whenever a raindrop falls on the fertile interval, a line segment "tower" of unit height whose base is the location of impact is formed. Any given tower grows to both the left and the right at unit speed; when the sides of two towers touch, these two sides disappear, and the towers merge. Raindrops may fall also on the flat tops of existing towers, so that new towers are built on old ones.

In this way, the profile of tower tops is a interface exhibiting a local growth mechanism with smoothing, slope-dependent growth rates and space-time independent disorder. In fact, the model has a natural extension as a collection of growing interfaces (the multi-line PNG model), of which the tower top profile is the top curve: as illustrated in the figure, collision of tower sides in one layer causes nucleation of a new tower in the level below, so that over time, a sequence of collisions may propagate downwards causing several layers to begin growing.

It is a consequence of the Robinson-Schensted-Knuth correspondence that, at any given time $t \geq 0$, there is a simple rule for the conditional distribution of any given collection of curves on a given interval in the multi-line PNG model, given all the information about the curves elsewhere: the resampling rule involves several continuous time random walks conditioned on a natural mutual avoidance requirement.

This resampling rule is representative of a powerful analogy between static phase separation models and dynamic KPZ universality. If the time-width-height of the time-t multi-line PNG model is scaled by the classic KPZ scalings of $t/t^{2/3}/t^{1/3}$, one of the canonical limit objects of KPZ universality, the multi-line

Airy process, is obtained. In this scaling, it may be expected that the discrete resampling rule enjoyed by the multi-line PNG passes to a continuum counterpart that we call the Brownian-Gibbs property, under which several Brownian bridges are conditioned on mutual avoidance. In [B5], we prove that this is so, introducing a probabilistic perspective to KPZ universality which complements existing techniques such as asymptotic analysis of determinantal formulae. The new approach is used to augment convergence of certain discrete models to the multi-line Airy process, to provide L^∞ control on scaling limit interfaces and thus to prove a conjecture of Johansson [$\alpha 2$] regarding longitudinal deviation limits of such models as the top PNG curve.

As we will discuss in forthcoming work, the resampling approach may also be used to analyse the Hopf-Cole solution of the KPZ equation. This is because the solution of this equation at given time may be represented as the top curve in a system of many interfaces which satisfy a resampling property where Brownian bridges are sampled and an energetic penalty is applied should the curves go out of order (so that the new resampling rule is a soft version of the hard avoidance condition in the Airy case); this perspective may be invoked to prove strong regularity statements for the KPZ equation, and yields universal scaling exponents for the KPZ equation begun from very general initial data.

B: Biased motion in disordered media: ballisticity and trapping

When an external field to applied to a particle in open space, the particle will be set in motion. The Einstein relation predicts a linear relation between the velocity of a particle in the external field and the diffusivity of the particle without such a field. However, if the random medium in which the particle resides is disordered enough, and the external force is large, it has been expected physically that this linear response may be disrupted, and replaced by a phenomenon known as aging. Under aging, a random system becomes trapped in deep wells, spending a time there proportional to the system age. It has been predicted [$\beta 3$] that aging describes the dynamics of low temperature spin-glasses, and it has been proved [$\beta 1$] to arise in effective models, in which random walker jumps are time-changed at a random rate associated to walker location. The natural next step is to prove aging in systems where trapping occurs not by being built into the model's definition, but arises naturally in long-time dynamics.

In work initiated with Gérard Ben Arous at NYU, and pursued more recently with Alexander Fribergh, we have undertaken a detailed examination of the phenomenon of trapping and ballisticity of a biased particle in certain models of disordered systems. This programme of research may be roughly divided into three phases, the first two completed and the third in progress, as I now describe.

Part I. Biased walk on supercritical Galton-Watson trees.

In this problem, a random walk has a bias $\beta > 1$ away from the root in a random environment given by a supercritical Galton-Watson tree with leaves. This choice of environment is convenient for a study of ballisticity and trapping, because it splits natually into two pieces: a backbone, comprising of vertices from which emanates an infinite forward-going path to infinity, and a collection of finite traps, each of which hangs off the backbone. On the backbone, our biased walk will move linearly, while it is liable to be waylaid by visits to any one of the traps. This effect of trapping becomes more pronounced as the bias β increases, and indeed there exists a critical value β_c for the bias above which the walk moves at asymptotically zero speed. In this sub-ballistic regime, there is an explicitly computed exponent describing the sub-linear displacement made by the walk at late time. To understand trapping, it is natural to pose the question of whether we may scale particle distance and time in a manner that gives rise to a scaling limit object, a random Cantor-like function whose constant intervals correspond to trap sojourns.

In fact, in this problem, trap sojourn traps tend to concentrate around powers of the bias β. This creates a persistent periodic inhomogeneity, a discrete effect which does not diminish under late time rescaling, and which prevents the existence of any scaling limit. In [C1], we study subsequential scaling limits and in effect describe the trapping mechanism in detail for this model.

By modifying the bias $\beta > 1$ so that it is no longer deterministic but is chosen randomly and independently for each edge according to a law satisfying a certain non-lattice condition, this discrete periodic effect is disrupted, raising the prospect that scaling limits for walk displacement exist and are given by stable laws. We prove that this is indeed the case in [C2] and [C3], in a study which entails understanding the implications on long-term walk displacement made by the limiting random environments seen near the top, and the base, of the traps encountered at late time by the walk.

Part II. Biased random walk on the supercritical percolation cluster in $\mathbb{Z}^d$: exponents.

A more physically realistic model of biased motion impeded by obstacles is provided by choosing as the environment supercritical percolation on the Euclidean lattice $\mathbb{Z}^d$, for some $d \geq 2$. Conditionally on the origin belonging to the unique infinite component, a particle begins there, making random jumps only to neighbouring open sites, and favouring a certain direction with a bias $\beta > 1$. Predicted since the eighties to exhibit trapping [$\beta 4$], the model was the subject of works in 2003 by Sznitman [$\beta 5$], and Berger, Gantert and Peres [$\beta 2$], showing that small values of the bias yield ballistic motion and high values sub-ballistic motion.

Pursuing this line of inquiry, Alex Fribergh and I have shown in [C4] that the transition from ballisticity to trapping in this model is sharp: for any dimension $d \geq 2$ and supercritical percolation parameter $p > p_c$, there is a critical point β_c for the bias above which asymptotic walk speed is zero and below which it is positive. Intimately related to finding this critical value is an investigation of the sub-ballistic exponent for walk displacement in the zero speed case and of the geometry of the traps near the base of which the walk resides at late time. A novel technique of modifying the walk to skip trap sojourns is combined with electrical resistance theory to prove stronger estimates than previously known to the effect that long-term walk displacement occurs in ther direction of the drift.

Part III. Biased random walk on the supercritical percolation cluster in $\mathbb{Z}^d$: scaling limits.

The sharp phase transition to zero speed and the exponents for sub-ballistic displacement having been identified, a third phase of research naturally presents itself: to rescale time and space and prove convergence of walk displacement to a random scaling limit, a task completed now for biased walk on Galton-Watson trees. This task is the subject of an ongoing joint project with Alex Fribergh. There is significant geometric input arising from the nature of the traps into each which the walk may fall at late time: the width of their top and the means by which the walk approaches them, and the shape of the base and how this modifies by constant order the time spent in them. A striking dichotomy between the two and higher dimensional cases for this trap geometry occurs, with the traps having uniformly bounded cross sections only in the latter case.

In the tree case, constantly biased walk produced concentration of trap sojourn times on all scales that prevents the existence of walk displacement scaling limits, and non-lattice edge-bias randomization causes this effect to disappear. Interestingly, in the physical setting of $\mathbb{Z}^d$, the two effects appear to reside simultaneuously, the discrete inhomogeneity being present for rational choices of bias direction, and stable limits existing for generic directions due to a non-lattice condition on bias components being available in this case. Rigorously explaining these phenomena form an important part of the third phase of this inquir.

C: Self-avoiding walk

The self-avoiding walk is a celebrated model in probability, famous both for the simplicty of its definition and the difficulty of obtaining rigorously non-trivial information about its properties. Fixing any dimension $d \geq 2$, a self-avoiding walk γ of length n is an injective map $\gamma : \{0, \ldots, n\} \to \mathbb{Z}^d$ begun at the origin, $\gamma(0) = 0$, each of whose consecutive elements (γ_i, γ_{i+1}) are nearest-neighbors. That γ is injective – no vertex is visited twice – forces the walk to be self-avoiding. The self-avoiding walk (with length n) is then the uniform measure μ_{SAW_n} on the finite set of such walks.

It is anticipated by physicists that the typical endpoint displacement, given by the standard deviation of the Euclidean norm of γ_n, of the walk γ under μ_{SAW_n}, scales as n^ϕ, where ϕ is a dimension-dependent exponent. For example, when $d \geq 4$, it is anticipated that $\phi = 1/2$, reflecting the similarity between self-avoiding walk and Brownian motion in higher dimensions. In dimensions two and three, it is expected that $\phi \in (1/2, 1)$, with $\phi = 3/4$ when $d = 2$ predicted by conformal field theory and SLE.

Rigorous analysis of the behaviour of μ_{SAW_n} has been undertaken in dimension $d \geq 5$ by Brydges and Slade [γ1]. In dimensions two and three, however, there are very few rigorous results, and there has been none concerning the behaviour of the endpoint displacement.

In the direction of showing that $\phi < 1$, Hugo Duminil-Copin and I have proved in [D1] the following theorem, which shows that self-avoiding walk in any dimension $d \geq 2$ is sub-ballistic.

Theorem 0.1 *Let $v > 0$. There exists $\epsilon > 0$ such that, for each $n \in \mathbb{N}$,*

$$\mu_{\mathsf{SAW}_n}\Big(\max \big\{ ||\gamma_k|| : 0 \leq k \leq n \big\} \geq vn \Big) \leq e^{-\epsilon n}.$$

Central to ruling out ballisticity is showing that ballistic walks typically have a positive proportion of sites which are at renewal levels – these are points visited only once by say the vertical coordinate of the walk. Our proof uses unfolding surgery to manufacture walks with more renewal levels. We show that such surgery is efficient by an argument which owes something to the proof of Kesten's pattern theorem; this classical result [γ2] asserts that a finite subpath which is capable of appearing in the walk's interior actually appears in the walk on a uniformly positive fraction of occasions with overwhelming probability.

D: Noise sensitivity and dynamical percolation

In 1960, Kesten [δ1] proved that the critical point of bond percolation on $\mathbb{Z}^2$ is $\frac{1}{2}$, and that critical percolation in this model has no infinite cluster. As the percolation parameter increases through the critical point, large-scale, but not infinite, structure emerges at this value. In the physics literature, the term infinite incipient cluster has been applied to describe the large-scale clusters typically present at the critical point; it has been given mathematical meaning by Kesten, who defined the IIC to be the weak limit of the cluster of the origin at the critical point conditionally on this cluster having high radius.

Critical dynamical percolation, under which the open and closed bits constituting an instance of critical percolation are independently updated at rate one, is a canonical means of coupling together an uncountable collection of instances of critical percolation. Schramm and Steif proved in [δ2] that, on a measure zero set of times, the cluster of the origin is infinite under the process. In this sense, dynamical percolation offers a means of witnessing rare instances of inifinte structure in dynamical percolation, and it is natural to enquire how the IIC may be obtained from dynamical percolation.

In joint work [E2] with Gábor Pete and Oded Schramm, we have answered this question, first by showing that the natural first guess that the configuration at the first exceptional time under dynamical percolation differs from the IIC by being thinner than the latter measure, and by providing an alternative and rather explicit means of selecting a uniform exceptional time at which the configuration does have this property. Our work harnesses moment bounds on the length of absence of exceptional times available in [E1], in which we formulate theorems on exit-time tails from given sets in reversible Markov chains.

E: Spatial random permutations and the random stirring model

Background: In 1953, R. P. Feynman [ϵ1] used the path-formulation of quantum mechanics that he had recently introduced to propose a microscopic theory explaining a remarkable phase transition that occurs in liquid helium at extremely low temperatures. At such temperatures, a fraction of the helium becomes a superfluid, an extraordinary substance of zero viscosity, which creeps around the side of any container in which it is held, to form a mono-particle layer. Using what is now known as the Feynman-Kac representation, Feynman proposed a vivid interpretation of Bose-Einstein condensation, which is a phenomenon closely related to superfluidity, in terms of a system of repulsive diffusing particles.

Viewed mathematically, Feynman's model is a beautiful example of a random permutation whose law is induced by a natural spatial structure. The model has two parameters: $\lambda \in (0, \infty)$, the particle density, and $\beta \in (0, \infty)$, an inverse temperature, or, as we shall see, an index for a period of time. In the model, a dimension $d \geq 2$ is set, and a large box, of volume $N \gg 0$ (and sidelength $N^{1/d}$) is given. Initially, a total of λN particles are scattered independently, and uniformly at random, into the box. The particles then evolve as independent Brownian motions for a period β of time, with the box having periodic boundary conditions. These two randomnesses, of placement and movement, are then jointly conditioned by the requirement that the set of locations occupied by the particles at time β be the same as it was at time 0. This requirement permits the particles to exchange positions, so that, for each realization, a permutation on the set of particle indices is induced, in which the trajectory of each particle is followed for time β, with the particle moving from one element of the set of initial locations to another. As described, the model is a representation of an ideal bosonic gas. To model a weakly interacting gas, such as liquid helium, the model is adapted in the following way, in which we describe the case of a hard-core interaction for the sake of ease of exposition. Particles are scattered and evolve as before. In addition to conditioning on the ensemble of particle locations coinciding at times 0 and β, we further condition that, at no moment of time $t \in [0, \beta]$ is it the case that the members of any pair of particles are at distance less than one from each another. A permutation is associated to each realization as in the ideal case, and both models induce a measure on the symmetric group S_n of n elements.

The phenomena of superfluidity and Bose-Einstein condensation have long been expected [ϵ3] to correspond to the presence of certain phase transitions in the behaviour of this permutation measure as the inverse temperature β is increased at fixed density λ, with the transition emerging as the system size N is taken to infinity. For both the ideal and weakly interacting gas, it has been conjectured that, in dimension $d \geq 3$, there exists $\beta_c \in (0, \infty)$ such that, for $\beta > \beta_c$, the probability that the random permutation contains a cycle with a macroscopic fraction of elements (i.e., at least $\epsilon \in (0, 1)$) tends to 1 (in the sense that $\epsilon \searrow 0$ after the limit $N \to \infty$ has been taken); and that, for $\beta \in (0, \beta_c)$, the cycles remain finite, in the sense that the distribution of the cycle of containing a given point is tight in N. This phase transition is very widely believed by physicists to correspond to the occurrence of Bose-Einstein condensation (at sufficiently low temperature). In the case of an ideal gas, this conjecture has been verified, and indeed, the critical value explicitly identified [ϵ4], by an insightful analysis in Fourier variables that is, however, unavailable in the interacting case.

The Feynman-Kac representation of the repulsively interacting Bose gas is physically of great importance and has an obvious mathematical appeal. In the mathematical literature, beyond the ideal case, proofs that infinite cycles arise may have limited. For a natural related model whose physical interest has been highlighted by Balint Tóth [ϵ5], the random stirring model, Omer Angel [ϵ1] has proved the existence of such cycles for certain parameter values on regular trees.

In recent work, [F1] and F2], that extends Angel's, I have established that the transition from finite cycles to infinite ones occurs at a single critical point for a class of infinite trees including regular ones satisfying a lower bound on degree.

F: Kinetic limits of coagulating diffusive systems

An important aim in rigorous statistical mechanics is to explain how it is that all of the microscopic data consisting of the dynamical states of the particles composing a certain body may be reduced to an effective description which involves far fewer parameters but which is accurate for practical purposes. For example, a complete description of the molecules comprising the air in a room might specify the mass, location and momentum of all the constituent particles, while a practically accurate reduced description would record merely the values at macroscopic locations of a limited number of thermodynamic quantities, such as pressure and temperature, and explain how these quantities evolve. In this case, the mass of random data will typically reduce to a deterministic macroscopic description modelled by partial differential equations accounting for evolution of temperature or pressure as a function of macroscopic spatial variables.

The Smoluchowski diffusion-coagulation equation is a system of partial differential equations which one may expect to arise as such a limiting description when individual mass-bearing particles diffuse and are subject to an interaction of coagulation in pairs at close range. The functions $f_n : \mathbb{R}^d \times [0,\infty) \to [0,\infty)$, $n \in \mathbb{N}$ solve the PDE if, for each $n \in \mathbb{N}$ and $(x,t) \in \mathbb{R}^d \times [0,\infty)$,

$$\frac{\partial f_n}{\partial t}(x,t) = d(n)\Delta f_n(x,t) + Q_1^n(f)(x,t) - Q_2^n(f)(x,t)\,.$$

The final two terms are interaction terms, a gain term given by

$$Q_1^n(f)(x,t) = \frac{1}{2}\sum_{m=1}^{n-1} \beta(m,n-m) f_m(x,t) f_{n-m}(x,t)\,,$$

and a loss term by

$$Q_2^n(f) = f_n(x,t)\sum_{m=1}^{\infty} \beta(m,n) f_m(x,t)\,.$$

The quantity $f_n(x,t)$ has the interpretation of the density of mass n particles close to macroscopic location x at time t. I was introduced to the problem of deriving this system from a suitable microscopic model by James Norris, and undertook with Fraydoun Rezakhanlou and others an analysis which included such a derivation, in dimensions $d \geq 3$ [A2], $d = 2$ [A1], in the case of continuous mass variable [A4], and which further established conditions for mass conservation and uniqueness of solutions of the Smoluchowski PDE [A3]. Our kinetic derivation extends earlier analysis for the case of constant diffusion rates [$\zeta 1$] and for particles that annihilate rather than coagulate on collision [$\zeta 2$].

Note that the equations have two sets of parameters, diffusion rates $d : \mathbb{N} \to (0,\infty)$ and coagulation propensities $\beta : \mathbb{N}^2 \to (0,\infty)$. In the underlying microscopic model, a large number N of particles of varying masses are scattered initially, diffuse at the mass-determined rates $d(n)$, and are liable to coagulate in pairs inside an interaction range $\epsilon = \epsilon_N$; ϵ_N is chosen so that a typical particle will experience a bounded rate of interaction per unit time, uniformly in a high N limit. One of the challenges of deriving the Smoluchowski PDE system in such a high N limit is to determine how β depends on the microscopic parameters, that is, on the diffusion rates and on the details of pairwise collision inside the interaction range. A key element in the derivation is the proof of the Stosszahlansatz, the "collision number" ansatz, which asserts a pairwise independence of particle densities at all times. This result is a counterpart to one which was invoked by Boltzmann in his heuristic derivation of the eponymous equation for particle collision in a hard-core gas. Integral to proving the Stosszahlansatz, and to correctly deriving a formula for the macroscopic coagulation propensities β in terms of microscopic parameters, is the task of quantifying a short-range repulsive tendency in the model, wherein the microscopic neighbourhood of any given particle is less liable than average to contain another particle, due to the vulnerability of particles at close range to coagulation.

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PUBLICATION LIST

A: Phase boundary fluctuation and randomly growing interfaces

- *Fluctuation of planar Brownian loop capturing large area.* With Yuval Peres. Trans. Amer. Math. Soc. 360, no. 12, 6197–6230 (2008).
- *Phase separation in random cluster models I: uniform upper bounds on local deviation.* Comm. Math Phys. 310 , no. 2, 455–509 (2012).
- *Phase separation in random cluster models II: the droplet at equilibrium, and local deviation lower bounds.* Ann. Probab. 40 , no. 3, 921–978 (2012).
- *Phase separation in random cluster models III: circuit regularity.* J. Stat. Phys. 142, no. 2, 229–276 (2011).
- *Brownian Gibbs property for Airy line ensembles.* With Ivan Corwin. Submitted.

B: Biased motion in disordered media: ballisticity and trapping

- *Biased random walks on Galton-Watson trees with leaves.* with Gérard Ben Arous, Alexander Fribergh and Nina Gantert. Ann. Probab. 40, no. 1, 280–338 (2012).
- *Randomly biased walks on subcritical trees.* With Gérard Ben Arous. Comm. Pure Appl. Math. 65 , no. 11, 1481–1527 (2012).
- *Stable limit laws for randomly biased walks on supercritical trees.* Ann. Probab., to appear.
- *Phase transition for the speed of the biased random walk on the supercritical percolation cluster.* With Alex Fribergh. Comm. Pure Appl. Math., to appear.

C: Self-avoiding walk

- *Self-avoiding walk is sub-ballistic.* With Hugo Duminil-Copin. Submitted.

D: Noise sensitivity and dynamical percolation

- *Exit time tails from pairwise decorrelation in hidden Markov chains, with applications to dynamical percolation.* With Elchanan Mossel and Gábor Pete. Electron. J. Probab. 17 , article 68, 1–16 (2012).
- *Local time on the exceptional set of dynamical percolation, and the Incipient Infinite Cluster.* With Gábor Pete and Oded Schramm. Preprint.

E: Spatial random permutations and the random stirring model

- *Infinite cycles in the random stirring model on trees.* Submitted.
- *Sharp phase transition in the random stirring model on trees.* Submitted.

F: Coagulation and diffusion: the Smoluchowski PDE

- *Kinetic limit for a system of coagulating planar Brownian particles.* With Fraydoun Rezakhanlou. J. Stat. Phys. 124, 997–1040 (2006).
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G: Further topics

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