

# EXPONENTS GOVERNING THE RARITY OF DISJOINT POLYMERS IN BROWNIAN LAST PASSAGE PERCOLATION

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ABSTRACT. In last passage percolation models lying in the KPZ universality class, long maximizing paths have a typical deviation from the linear interpolation of their endpoints governed by the two-thirds power of the interpolating distance. This two-thirds power dictates a choice of scaled coordinates, in which these maximizers, now called polymers, cross unit distances with unit-order fluctuations. In this article, we consider Brownian last passage percolation in these scaled coordinates, and prove that the probability of the presence of  $k$  disjoint polymers crossing a unit-order region while beginning and ending within a short distance  $\epsilon$  of each other is bounded above by  $\epsilon^{k^2-1+o(1)}$ . This result, which we conjecture to be sharp, yields understanding of the uniform nature of the coalescence structure of polymers, and plays a foundational role in [Ham17c] in proving comparison on unit-order scales to Brownian motion for polymer weight profiles from general initial data. The present paper also contains an on-scale articulation of the two-thirds power law for polymer geometry: polymers fluctuate by  $\epsilon^{2/3}$  on short scales  $\epsilon$ .

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### 1. INTRODUCTION

**1.1. KPZ universality, last passage percolation models, and scaled coordinates.** The  $1 + 1$  dimensional Kardar-Parisi-Zhang (KPZ) universality class includes a wide range of random interface models suspended over a one-dimensional domain, in which growth in a direction normal to the surface competes with a smoothening surface tension in the presence of a local randomizing force that roughens the surface. These characteristic features are evinced by many last passage percolation models. Such an LPP model comes equipped with a planar random environment, which is independent in disjoint regions. Directed paths, that are permitted say to move only in a direction in the first quadrant, are then assigned energy via this randomness, by say integrating the environment's value along the path. For a given pair of planar points, the path attaining the maximum weight over directed paths with such endpoints is called a geodesic.

For LPP models lying in the KPZ class, a geodesic that crosses a large distance  $n$  (in say a north-easterly direction) has an energy that grows linearly in  $n$  with a standard deviation of order  $n^{1/3}$ . If the lower geodesic endpoint is held fixed, and the higher one is permitted to vary horizontally, then the geodesic energy as a function of the variable endpoint plays the role of the random interface mentioned at the outset. (With the first endpoint thus fixed, the energy profile goes by the name 'narrow wedge'.) Non-trivial correlations in the geodesic energy are witnessed when this horizontal variation has order  $n^{2/3}$ . These assertions have been rigorously demonstrated for only a few LPP models, each of which enjoys an integrable structure: the seminal work of Baik, Deift and Johansson [BDJ99] rigorously established the one-third exponent, and the GUE Tracy-Widom

distributional limit, for the case of Poissonian last passage percolation, while the two-thirds power law for maximal transversal fluctuation was derived for this model by Johansson [Joh00].

In seeking to understand the canonical structures of KPZ universality, we are led, in view of these facts, to represent the field of geodesics in a scaled system of coordinates, under which a northeasterly displacement of order  $n$  becomes a vertical displacement of one unit, and a horizontal displacement of order  $n^{2/3}$  becomes a unit horizontal displacement. Moreover, the accompanying system of energies also transfers to scaled coordinates, with the scaled geodesic energy being specified by centring about the mean value and division by the typical scale of  $n^{1/3}$ .

**1.2. Probabilistic proof techniques for KPZ, and scaled Brownian LPP.** The theory of KPZ universality has advanced through physical insights, numerical analysis, and several techniques of integrable or algebraic origin. We will not hazard a summary of literature to support this one-sentence history, but refer to the reader to [Cor12] for a KPZ survey from 2012; in fact, integrable and analytic approaches to KPZ have attracted great interest around and since that time. Now, it is hardly deep or controversial to say that many problems and models in KPZ are intrinsically random. This fact may suggest that it would be valuable to approach the problems of KPZ universality from a predominately probabilistic perspective.

An important illustration is offered by Brownian last passage percolation. This LPP model has very attractive probabilistic features: for example, in the narrow wedge case, the scaled geodesic energy profile may be embedded as the uppermost curve in a Dyson diffusion, namely a system of one-dimensional Brownian motions conditioned on mutual avoidance (with a suitable boundary condition). May probabilistic analysis be brought to bear on Brownian LPP depicted in the scaled coordinates that we have described, with light thus shed on universal KPZ structures?

The present article forms part of a four-paper answer to this question. The companion papers are [Ham17a], [Ham17b] and [Ham17c]. Our focus is on scaled coordinates, and we will adopt the terms *polymer* and *weight* to refer to scaled geodesics and their scaled energy. The reader may glance ahead to the right sketch in Figure 2 for a depiction of a polymer, bearing in mind that, when the scaling parameter  $n$  is high, the microscopic backtracking apparent in the illustration becomes negligible, so that the lifetime  $[t_1, t_2]$  of a polymer may be viewed as a vertical interval, with the polymer being a random real-valued function defined on that interval; the range of the polymer thus crosses the planar strip  $\mathbb{R} \times [t_1, t_2]$  from the lower to the upper side.

**1.3. Principal conclusions and themes in overview.** In this article, we will reach three principal conclusions concerning scaled Brownian LPP. As we informally summarise them now, we omit mention of the scaling parameter  $n \in \mathbb{N}$ : roughly, our assertions should be understood uniformly in high choices of this parameter.

- Consider the event that there exist  $k$  polymers of unit lifetime  $[0, 1]$ , where each begins and ends in a given interval of some given *small* scaled length  $\epsilon$ . Theorem 1.1, states an upper bound of  $\epsilon^{k^2-1+o(1)}$  on the probability of this event. The left sketch of the upcoming Figure 3 depicts the event with  $k = 3$ .
- Derived as a consequence, Theorem 1.4, asserts that the probability that  $m$  polymers coexist disjointly in a unit-order region has a superpolynomial decay in  $m$ .
- We have mentioned that the maximal geodesic fluctuation in Poissonian last passage percolation has been shown in [Joh00] to be governed by an exponent of two-thirds. Here is a further expression of this two-thirds power law, showing that this exponent also governs

local behaviour of the scaled geodesic: a polymer fluctuates by more than  $\epsilon^{2/3}r$  on a short duration  $\epsilon$  with probability at most  $\exp\{-O(r^\alpha)\}$  (for a broad range of values of  $r$ ). In Theorem 1.5, we will prove a Brownian LPP version of this assertion, with  $\alpha = 3/4$ .

It is the first of these results, namely the exponent bound of the form  $k^2 - 1 + o(1)$  on the rarity of  $k$  disjoint polymers with nearby endpoints, which we regard as the most fundamental. Four reasons, discussed in turn next, will help to explain why.

*1. Applications.* Theorem 1.1 is very useful. In fact, Theorem 1.4 will be derived as a rather direct consequence. More fundamentally, Theorem 1.1 plays a key role in the principal highway of implications in our four-paper study. Narrow wedge polymer weight profiles resemble Brownian motion on a unit scale: indeed, Corollary 2.12 and Proposition 2.5 of [Ham17a] demonstrate that, after an affine adjustment, narrow wedge profiles withstand a very demanding comparison to Brownian bridge. In this narrow wedge case, random growth is initiated from a point. Such growth may also begin from much more general initial data. The main conclusion of [Ham17c], that article's Theorem 1.2, asserts that the corresponding polymer weight profiles enjoy in a uniform sense a strong resemblance to Brownian motion on unit scales. Theorem 1.1 is an engine that drives the derivation of this conclusion. Figure 1 gives a very impressionistic account of why the theorem is valuable for this purpose.

We anticipate that Theorem 1.1 will have a central role in an expanding inquiry into KPZ universality via probabilistic proof methods. For suitable LPP models, the profile weight profile in the upper-left sketch of Figure 1 converges distributionally [PS02, Joh03] in the high scaling parameter limit  $n \rightarrow \infty$  to the  $\text{Airy}_2$  process after the subtraction of a parabola. This process, which has finite-dimensional distributions specified by Fredholm determinants, is a central object in KPZ universality. Conjecturally, however, the KPZ universality class contains richer objects, such as the Airy sheet [CQR15], which is expected to encode simultaneously the collection of scaled weights between any pair of planar endpoint locations. Many copies of the  $\text{Airy}_2$  process are coupled together in the yet-to-be-constructed Airy sheet. It is already interesting to consider how two such processes rooted at distinct planar points may be coupled: the random process sending  $y \in \mathbb{R}$  to the *difference* in scaled weight between polymers beginning at  $(-1, 0)$  and at  $(1, 0)$  and ending together at  $(y, 1)$  is such an example, in which a natural attribute of KPZ universality which should be determined by the Airy sheet. The behaviour of this difference process is closely related to the presence of pairs of disjoint polymers with nearby endpoints, and we anticipate that Theorem 1.1 will have a key role in elucidating questions arising from the study of this process.

*2. The KPZ fixed point.* An important rigorous advance concerning KPZ universality beyond the narrow wedge profile is the recent construction [MQR17] of the KPZ fixed point. The narrow wedge polymer weight profile may be viewed as a time-one snapshot of a scaled random growth process initiated from the origin at time zero. Growth may be initiated from a much more general initial condition. In [MQR17], Matetski, Quastel and Remenik have utilized a biorthogonal ensemble representation found by [Sas05, BFPS07] associated to the totally asymmetric exclusion process in order to find Fredholm determinant formulas for the multi-point distribution of the height function of this growth process begun from an arbitrary initial condition. Using these formulas to take the KPZ scaling limit, the authors construct a scale invariant Markov process that lies at the heart of the KPZ universality class. The time-one evolution of this Markov process may be applied to very general initial data, and the result is the scaled profile begun from such data, which generalizes the  $\text{Airy}_2$  process seen in the narrow wedge case. It would be a valuable step to adapt [MQR17] to Brownian last passage percolation, in an effort to render the present article's theorems and consequences applicable to universal KPZ structure: this outcome should be rather direct in the case of the

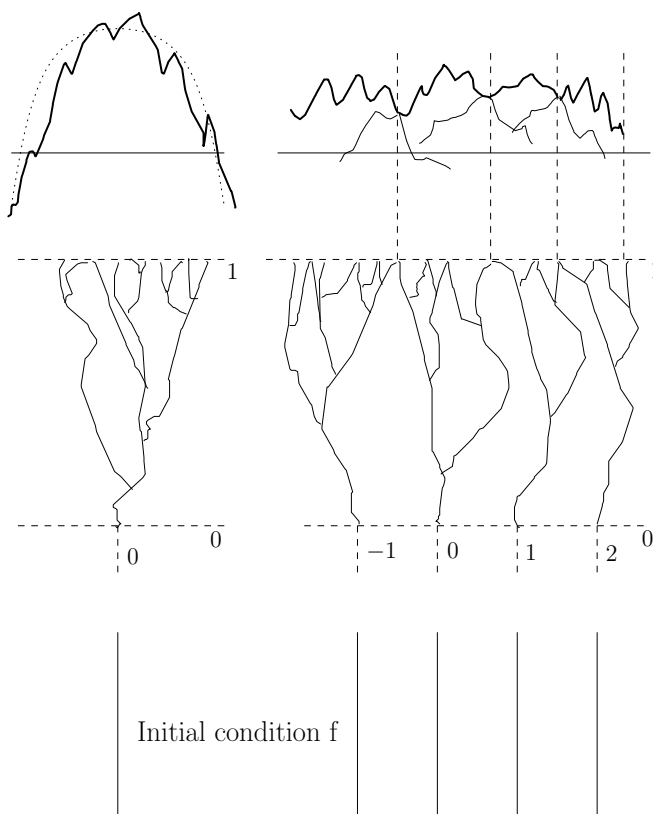


FIGURE 1. The left and right sketches each depict three aspects of scaled random growth in Brownian LPP, corresponding to two different initial conditions. These initial conditions  $f$  are depicted at the bottom. On the left, the narrow wedge case of growth from zero is depicted. The polymer weight profile, top left, is known to be locally Brownian (and globally parabolic). In the middle left, we see the associated system of polymers, forming a tree with root at the origin. On the right, growth is instead initiated from the set of integers. The field of polymers now forms a forest, rather than a tree, and the polymer weight profile, which is depicted in bold, is the maximum of the profiles associated to the roots of the various trees. Indeed, a general polymer weight follows the profile associated to the consecutive trees in the associated polymer forest. This profile may be understood to be locally Brownian as an inference from the Brownian resemblance enjoyed by the ‘narrow-wedge’ profiles associated to the individual trees. To make this idea work, we need to understand that there are not too many trees: their number per unit length must be shown to be tight in the scaling parameter  $n$ . It is here that Theorem 1.1, and Theorem 1.4, provide the necessary input.

consequence [Ham17c, Theorem 1.2] concerning the Brownian nature of general initial condition profiles, while it would be merely an intermediate step for this article’s results, since construction

of the Airy sheet (and the putatively associated sheet of polymers) remains an important open problem.

*3. Polymer surgery as a KPZ tool.* The proof of Theorem 1.1 has conceptual interest. Two principal probabilistic techniques are at work in the four-paper study, namely *Brownian Gibbs resamplings* and *polymer surgery*. The latter is introduced in order to prove Theorem 1.1. In order to explain how, it is useful first to discuss the former. This first, resampling, tool is employed to analyse the narrow wedge polymer weight profile by viewing it as the top curve in a Dyson diffusion. Allied with a simple observation from integrable probability, the Karlin-McGregor formula, the technique has been used in [Ham17a] to solve a cousin of the problem addressed in Theorem 1.1. The solution will later be presented as Theorem 5.1. Crudely, in place of considering, as Theorem 1.1 does,  $k$  disjoint polymers with nearby endpoints, we may instead consider the event that there exists a system of  $k$  scaled paths, crossing between two *given* points at a unit vertical displacement, the paths pairwise disjoint except for these shared endpoints, and with each path coming close to being a polymer, in the sense that each path weight is close to maximal for the given endpoints. Theorem 5.1 identifies in a sharp way the decay rate in the probability of this event as a natural parameter that measures this closeness tends to zero: the theorem identifies the exponent  $k^2 - 1$  as governing this rate.

How then will polymer surgery help to prove Theorem 1.1? Faced with  $k$  polymers with nearby endpoints, we seek to deform them surgically near their endpoints in order to typically achieve a shared-endpoint outcome whose probability is determined by Theorem 5.1: Figure 3 is a one-sketch summary. A probabilistic bridge is being built into an understanding of integrable origin, and the proof technique is thus emblematic of a broader aim of expanding KPZ horizons by probabilistic means.

*4. The exponent  $k^2 - 1$ .* In its key application in [Ham17c], Theorem 1.1 is applied when  $k = 3$ , for a triple of polymers. The form of the exponent is important in this application, dictating that a certain Radon-Nikodym derivative that describes the strength of Brownian comparison lies in  $L^{3-}$ . Merely knowing that the exponent is at least a positive constant, for example, would not be adequate even for deriving  $L^1$ -membership.

We believe that the  $k^2 - 1$  exponent in Theorem 1.1 is sharp and will present a conjecture to this effect in Subsection 1.7.1. It is rigorously known that, in Theorem 1.1's proof, nothing is lost in the exponent after the reduction to the shared endpoint Theorem 5.1 is made, because the latter result provides a sharp resolution of the cousin problem discussed above in the third point.

**1.4. A suggestion for further reading of overview.** In seeking to explain the progress made in this article in a few paragraphs, we have also offered a whistle-stop tour of ideas in a broader study of scaled Brownian LPP. Such a brief presentation may well prompt questions, and we refer the reader who wants a more detailed but still informal overview of the main results and ideas in the four-paper study to [Ham17a, Section 1.2].

For the reader who wishes to read the broader study, the present article would be read after [Ham17a] and [Ham17b] but before [Ham17c]. However, the present article has been written so that it may be read on its own. As such, consulting the other articles, including the further reading just suggested, is an optional extra.

**1.5. Connections: probabilistic tools, geodesics and coalescence.** The idea of taking limited integrable inputs, such as control of narrow wedge polymer profiles at given points, and exploiting them via probabilistic means to reach much stronger conclusions is central to this article and

the broader four-paper Brownian LPP study. A similar conceptual approach governs many ideas in [BSS16] and [BSS17b], where the slow bond conjecture for the totally asymmetric exclusion process, concerning the macroscopic effects of slightly attenuating passage of particles through the origin, is proved, and the associated invariant measures determined. Such a probabilistic technique is also encountered in [BSS17a]. Indeed, in [BSS17a, Theorem 2], an assertion similar to, and in fact slightly stronger than, Theorem 1.5 has been proved for exponential last passage percolation: in essence, the assertion in the third bullet point in the preceding discussion has been verified for  $\alpha = 1$ .

Roughly opposite to the phenomenon of geodesic disjointness is the process of geodesic coalescence. An interesting arena of study of this coalescence is the class of first passage percolation models, which are variants of last passage percolation that are not integrable. Although scaling exponents are not rigorously known in this setting, it has been heuristically appreciated for some time that certain circumstances should coincide in this context: the non-existence of bi-infinite geodesics; an exponent of one-third in the standard deviation of geodesic energy; and smoothness of the limiting shape of energy-per-unit-length as a function of angle. A summary of progress until 1995 regarding coalescence of geodesics is offered by Newman in [New95]; this progress included geodesic uniqueness for given direction, subject to a curvature assumption on the limiting shape. The theory of semi-infinite geodesics was developed by [FP05] and [Cou11]. Building on this theory, Pimentel adapted Newman's technique for uniqueness to the context of exponential LPP, in which the curvature assumption is known to be verified. In this work, it is also proved that, for two geodesics of given direction that begin at points separated by distance  $\ell$ , it is typical that coalescence of the geodesic occurs by a time of order  $\ell^{3/2}$ . Moreover, Pimentel conjectured that the probability that coalescence has failed to occur by time  $r\ell^{3/2}$  has tail  $r^{-2/3}$ . In the context of finite geodesics, [BSS17a] has proved an upper bound on this failure probability of the form  $r^{-c}$  for some positive  $c > 0$ , and there seems to be promise in these techniques to obtain a sharper form of the result. Hoffman [Hof08] initiated very fruitful progress on infinite geodesics in first passage percolation by using Busemann functions, with significant geometric information emerging in [DH14, AH16] and [DH17]. Busemann functions have also been studied in non-integrable last passage percolation contexts: see [GRAS15a] and [GRAS15b].

**1.6. Brownian last passage percolation.** In the two remaining introductory sections, we specify this model precisely, and set up notation for the use of scaled coordinates; and we state our main results.

**1.6.1. The model's definition.** This model was introduced by [GW91] and further studied in [OY02]; we will call it Brownian LPP. On a probability space carrying a law labelled  $\mathbb{P}$ , we let  $B : \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}$  denote an ensemble of independent two-sided standard Brownian motions  $B(k, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ ,  $k \in \mathbb{Z}$ .

Let  $i, j \in \mathbb{Z}$  with  $i \leq j$ . We denote the integer interval  $\{i, \dots, j\}$  by  $\llbracket i, j \rrbracket$ . Further let  $x, y \in \mathbb{R}$  with  $x \leq y$ . With these parameters given, we consider the collection of non-decreasing lists  $\{z_k : k \in \llbracket i+1, j \rrbracket\}$  of values  $z_k \in [x, y]$ . With the convention that  $z_i = x$  and  $z_{j+1} = y$ , we associate an energy to any such list, namely  $\sum_{k=i}^j (B(k, z_{k+1}) - B(k, z_k))$ . We may then define the maximum energy,  $M_{(x,i) \rightarrow (y,j)}^1$ , to be the supremum of the energies of all such lists.

**1.6.2. A geometric view: staircases.** In order to make a study of those lists that attain this maximum energy, we begin by noting that the lists are in bijection with certain subsets of  $[x, y] \times [i, j] \subset \mathbb{R}^2$  that we call *staircases*. Staircases offer a geometric perspective on Brownian LPP and perhaps help in visualizing the problems in question.

The staircase associated to the non-decreasing list  $\{z_k : k \in \llbracket i + 1, j \rrbracket\}$  is specified as the union of certain horizontal planar line segments, and certain vertical ones. The horizontal segments take the form  $[z_k, z_{k+1}] \times \{k\}$  for  $k \in \llbracket i, j \rrbracket$ . Here, the convention that  $z_i = x$  and  $z_{j+1} = y$  is again adopted. The right and left endpoints of each consecutive pair of horizontal segments are interpolated by a vertical planar line segment of unit length. It is this collection of vertical line segments that form the vertical segments of the staircase.

The resulting staircase may be depicted as the range of an alternately rightward and upward moving path from starting point  $(x, i)$  to ending point  $(y, j)$ . The set of staircases with these starting and ending points will be denoted by  $SC_{(x,i) \rightarrow (y,j)}$ . Such staircases are in bijection with the collection of non-decreasing lists considered above. Thus, any staircase  $\phi \in SC_{(x,i) \rightarrow (y,j)}$  is assigned an energy  $E(\phi) = \sum_{k=i}^j (B(k, z_{k+1}) - B(k, z_k))$  via the associated  $z$ -list.

1.6.3. *Energy maximizing staircases are called geodesics.* A staircase  $\phi \in SC_{(x,i) \rightarrow (y,j)}$  whose energy attains the maximum value  $M_{(x,i) \rightarrow (y,j)}^1$  is called a geodesic from  $(x, i)$  to  $(y, j)$ . It is a simple consequence of the continuity of the constituent Brownian paths  $B(k, \cdot)$  that such a geodesic exists for all choices of  $(x, y) \in \mathbb{R}^2$  with  $x \leq y$ . As we will later explain, in Lemma 3.1, the geodesic with given endpoints is almost surely unique.

1.6.4. *The scaling map.* For  $n \in \mathbb{N}$ , consider the  $n$ -indexed *scaling map*  $R_n : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by

$$R_n(v_1, v_2) = \left( 2^{-1} n^{-2/3} (v_1 - v_2), v_2/n \right).$$

The scaling map acts on subsets  $C$  of  $\mathbb{R}^2$  by setting  $R_n(C) = \{R_n(x) : x \in C\}$ .

(Clearly,  $n$  must be positive. In fact,  $\mathbb{N}$  will denote  $\{1, 2, \dots\}$  throughout.)

1.6.5. *Scaling transforms staircases to zigzags.* The image of any staircase under  $R_n$  will be called an  $n$ -zigzag. The starting and ending points of an  $n$ -zigzag  $Z$  are defined to be the image under  $R_n$  of such points for the staircase  $S$  for which  $Z = R_n(S)$ .

Note that the set of horizontal lines is invariant under  $R_n$ , while vertical lines are mapped to lines of gradient  $-2n^{-1/3}$ . As such, an  $n$ -zigzag is the range of a piecewise affine path from the starting point to the ending point which alternately moves rightwards along horizontal line segments and northwesterly along sloping line segments, where each sloping line segment has gradient  $-2n^{-1/3}$ ; the first and last segment in this journey may be either horizontal or sloping.

1.6.6. *Scaled geodesics are called polymers.* For  $n \in \mathbb{N}$ , the image of any geodesic under the scaling map  $R_n$  will be called an  $n$ -polymer, or often simply a polymer.

1.6.7. *Some basic notation.* For  $\ell \geq 1$ , we write  $\mathbb{R}_{\leq}^{\ell}$  for the subset of  $\mathbb{R}^{\ell}$  whose elements  $(z_1, \dots, z_{\ell})$  are non-decreasing sequences. When the sequences are increasing, we instead write  $\mathbb{R}_{<}^{\ell}$ . We also use the notation  $A_{\leq}^{\ell}$  and  $A_{<}^{\ell}$ . Here,  $A \subset \mathbb{R}$  and the sequence elements are supposed to belong to  $A$ . We will typically use this notation when  $\ell = 2$ .



1.6.8. *Compatible triples.* Let  $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}_{<}^2$ , which is to say that  $n \in \mathbb{N}$  and  $t_1, t_2 \in \mathbb{R}$  with  $t_1 < t_2$ . Taking  $x, y \in \mathbb{R}$ , does there exist an  $n$ -zigzag from  $(x, t_1)$  to  $(y, t_2)$ ? As far as the data  $(n, t_1, t_2)$  is concerned, such an  $n$ -zigzag may exist only if

$$t_1 \text{ and } t_2 \text{ are integer multiples of } n^{-1}. \tag{1}$$

We say that data  $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}_{<}^2$  is a *compatible triple* if it verifies the last condition.

An important piece of notation associated to a compatible triple is  $t_{1,2}$ , which we will use to denote the difference  $t_2 - t_1$ . The law of the underlying Brownian ensemble  $B : \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}$  is invariant under integer shifts in the first, curve indexing, coordinate. This translates to a distributional invariance of scaled objects under vertical shifts by multiples of  $n^{-1}$ , something that makes the parameter  $t_{1,2}$  of far greater significance than  $t_1$  or  $t_2$ .

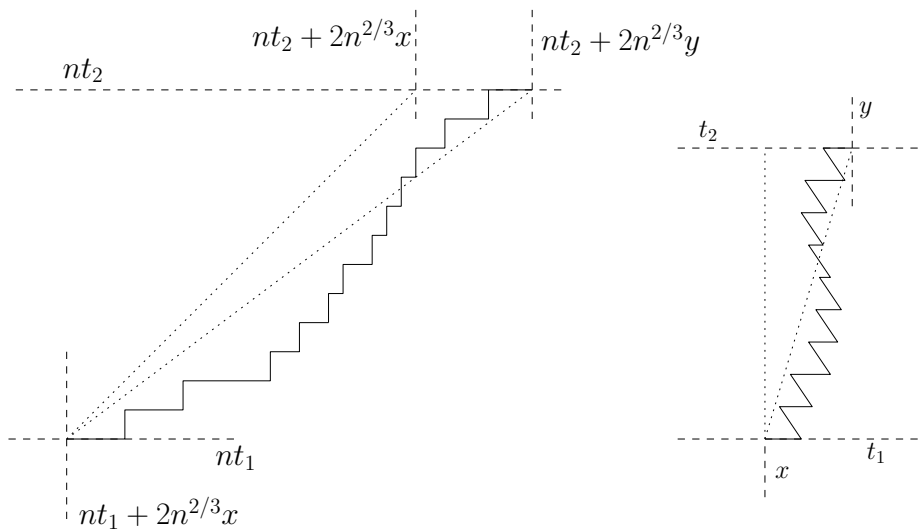


FIGURE 2. Let  $(n, t_1, t_2)$  be a compatible triple and let  $x, y \in \mathbb{R}$ . The endpoints of the geodesic in the left sketch have been selected so that, when the scaling map  $R_n$  is applied to produce the right sketch, the  $n$ -polymer  $\rho_{n;(x,t_1)}^{(y,t_2)}$  results.

Supposing now that  $(n, t_1, t_2)$  is indeed a compatible triple, the condition  $y \geq x - 2^{-1}n^{1/3}t_{1,2}$  ensures that the preimage of  $(y, t_2)$  under the scaling map  $R_n$  lies northeasterly of the preimage of  $(x, t_1)$ . Thus, an  $n$ -zigzag from  $(x, t_1)$  to  $(y, t_2)$  exists in this circumstance. We have mentioned that geodesics exist uniquely for given endpoints almost surely. Taking the image under scaling, this translates to the almost sure existence and uniqueness of the  $n$ -polymer from  $(x, t_1)$  to  $(y, t_2)$ . This polymer will be denoted  $\rho_{n;(x,t_1)}^{(y,t_2)}$ : see Figure 2. This notation has characteristics in common with several later examples, in which objects are described in scaled coordinates. Round bracketed expressions in the subscript or superscript will refer to a space-time pair, with the more advanced time in the superscript. Typically some aspect of the  $n$ -polymer from  $(x, t_1)$  to  $(y, t_2)$  is being described when this  $\text{TBA}_{n;(x,t_1)}^{(y,t_2)}$  notation is used.

**1.7. Main results, and a conjecture.** This article reaches conclusions in two main directions: it establishes upper bounds on the probability of the coexistence of disjoint polymers with nearby endpoints, and it gives expression to the two-thirds power law that dictates polymer geometry,

showing that polymers fluctuate by order  $\epsilon^{2/3}$  on a short scale  $\epsilon$ . In the next two subsections, the principal conclusions in these directions are respectively stated.

1.7.1. *The rarity of many disjoint polymers.* Let  $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}_{<}^2$  be a compatible triple, and let  $I, J \subset \mathbb{R}$  be intervals. Set  $\text{MaxDisjtPoly}_{n;(I,t_1)}^{(J,t_2)}$  equal to the maximum cardinality of a *pairwise disjoint* set of  $n$ -polymers each of whose starting and ending points have the respective forms  $(x, t_1)$  and  $(y, t_2)$  where  $x$  is some element of  $I$  and  $y$  is some element of  $J$ .

(We make two parathetical comments. First, our notation here is an extension of the usage described a moment ago. In this case, the first element in the space-time pairs  $(I, t_1)$  and  $(J, t_2)$  is an interval, rather than a point. Second, the condition of pairwise disjointness in this definition can be slightly weakened so that our assertions remain valid. We prefer to defer elaborating on this. A remark in Section 5.7 will offer an altered definition of  $\text{MaxDisjtPoly}_{n;(I,t_1)}^{(J,t_2)}$  whose use results in slightly stronger results.)

Our first main conclusion treats the problem of polymers of shared unit-order lifetimes that cross between intervals that are of a small length  $\epsilon$ , albeit one that is independent of the scaling parameter  $n$ . Theorem 1.1 provides an upper bound of  $\epsilon^{k^2-1+o(1)}$  on the probability that  $k$  such polymers may coexist disjointly; here,  $k \in \mathbb{N}$  is fixed, and the result is asserted uniformly in high  $n$ .

**Theorem 1.1.** *There exists a positive constant  $G$  such that the following holds. Let  $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}_{<}^2$  be a compatible triple with  $nt_{1,2}$  even. Let  $k \in \mathbb{N}$ ,  $\epsilon > 0$  and  $x, y \in \mathbb{R}$  satisfy the conditions that  $k \geq 2$ ,*

$$\epsilon \leq G^{-4k^2}, \quad nt_{1,2} \geq G^{k^2} (1 + |x - y|^{36} t_{1,2}^{-24}) \epsilon^{-G}$$

and  $t_{1,2}^{-2/3} |x - y| \leq \epsilon^{-1/2} (\log \epsilon^{-1})^{-2/3} G^{-k}$ .

Setting  $I = [x - t_{1,2}^{2/3} \epsilon, x + t_{1,2}^{2/3} \epsilon]$  and  $J = [y - t_{1,2}^{2/3} \epsilon, y + t_{1,2}^{2/3} \epsilon]$ , we have that

$$\mathbb{P}\left(\text{MaxDisjtPoly}_{n;(I,t_1)}^{(J,t_2)} \geq k\right) \leq \epsilon^{(k^2-1)/2} \cdot Q,$$

where  $Q$  is a positive correction term that is bounded above by  $G^{k^3} \exp\{G^k (\log \epsilon^{-1})^{5/6}\}$ .

Theorem 1.1 is a slightly simplified form of a more detailed result that will be presented later, Theorem 6.1. Despite this simpler form, it is worth mentioning that, since a two-thirds power governs the horizontal coordinate under the scaling transformation, there is no generality lost by considering the special case  $t_1 = 0$  and  $t_2 = 1$ , and thus  $t_{1,2} = 1$ , in Theorem 1.1. (This point is explained further when the *scaling principle* is discussed in Section 5.1.) The reader is thus encouraged to read this result with  $t_1 = 0$  and  $t_2 = 1$ , as well as with  $x$  and  $y$  both supposed to be at most one say. Indeed, doing so shows that the next result follows immediately.

**Corollary 1.2.** *For each  $k \in \mathbb{N}$ ,  $k \geq 2$ , we have that*

$$\liminf_{\epsilon \searrow 0} \liminf_n \frac{\log \mathbb{P}\left(\text{MaxDisjtPoly}_{n;([-\epsilon, \epsilon], 0)}^{([-\epsilon, \epsilon], 1)} \geq k\right)}{\log \epsilon} \geq k^2 - 1.$$

Note that, despite the  $\geq$  symbol, this result is an *upper bound* on the concerned probability. The formulation permits us to express a conjecture in a symmetric form which indicates a belief that the exponent  $k^2 - 1$  in Theorem 1.1 is sharp.

**Conjecture 1.3.** For each  $k \in \mathbb{N}$ ,  $k \geq 2$ , we have that

$$\limsup_{\epsilon \searrow 0} \limsup_n \frac{\log \mathbb{P}\left(\text{MaxDisjtPoly}_{n;[-\epsilon, \epsilon], 0}^{([-\epsilon, \epsilon], 1)} \geq k\right)}{\log \epsilon} \leq k^2 - 1.$$

If the conjecture is valid, then equality will hold in the corollary and the conjecture.

The second main conclusion is a rather direct consequence of the first. It asserts that, for polymers that cross between unit-length intervals separated by unit-order times, the maximum number of such disjoint polymers has a tail that decays super-polynomially.

**Theorem 1.4.** *There exist constants  $G \geq g > 0$  such that the following holds. Let  $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}_{<}^2$  be a compatible triple with  $nt_{1,2}$  even. Further let  $x, y \in \mathbb{R}$ ,  $h \in \mathbb{N}$  and  $m \in \mathbb{N}$ . Suppose that*

$$m \geq (Gh)^{2g^{-1}G} \vee (|x - y|t_{1,2}^{-2/3} + 2h)^3$$

and  $nt_{1,2} \geq \max\{1, (|x - y|t_{1,2}^{-2/3} + 2h)^{36}\}Gm^G$ . Then

$$\mathbb{P}\left(\text{MaxDisjtPoly}_{n;(I,t_1)}^{(J,t_2)} \geq m\right) \leq m^{-g(\log \log k)^2},$$

where  $I$  denotes the interval  $[x, x + ht_{1,2}^{2/3}]$  and  $J$  the interval  $[y, y + ht_{1,2}^{2/3}]$ .

Focus on the meaning of this theorem is brought by considering  $t_1 = 0$ ,  $t_2 = 1$ ,  $h = 1$ , with  $x$  and  $y$  chosen to lie in a unit interval about the origin. In this case, the demanded lower bound on  $m$  merely excludes an initial bounded interval, and the lower bound on  $n$  takes the form  $n \geq \Theta(m^G)$ . The latter condition is not demanding given that much of the interest in the use of scaled coordinates lies either in the high  $n$  limit or in statements made uniformly for high enough  $n$ . Theorem 1.4 is such a statement; as seen by these parameter choices, it asserts that there is at most probability  $m^{-g(\log \log m)^2}$  of  $m$  disjoint polymers crossing a bounded region, uniformly in high  $n$ .

**1.7.2. Polymer fluctuation.** Our principal conclusion in this regard, Theorem 1.5, will here be stated, after a few paragraphs in which we attend to setting up and explaining some necessary notation and concepts. The theorem is used to prove Theorem 1.1 and it is also needed in [Ham17c].

Let  $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}_{<}^2$  be a compatible triple, and let  $x, y \in \mathbb{R}$ . The polymer  $\rho_{n;(x,t_1)}^{(y,t_2)}$  has been defined to be a subset of  $\mathbb{R} \times [t_1, t_2]$  containing  $(x, t_1)$  and  $(y, t_2)$ , but really as  $n$  rises towards infinity, it becomes more natural to seek to view it as a random function that maps its lifetime  $[t_1, t_2]$  to the real line. In choosing to adopt this perspective, we will abuse notation: taking  $t \in [t_1, t_2]$ , we will speak of the value  $\rho_{n;(x,t_1)}^{(y,t_2)}(t) \in \mathbb{R}$ , as if the polymer were in fact a function of  $[t_1, t_2]$ . Some convention must be adopted to resolve certain microscopic ambiguities as we make use of this new notation, however. First, we will refer to  $\rho_{n;(x,t_1)}^{(y,t_2)}(t)$  only when  $t \in [t_1, t_2]$  satisfies  $nt \in \mathbb{Z}$ , a condition that ensures that the intersection of the set  $\rho_{n;(x,t_1)}^{(y,t_2)}$  with the line at height  $t$  takes place along a horizontal planar interval.

Second, we have to explain which among the points in this interval  $\rho_{n;(x,t_1)}^{(y,t_2)} \cap \{(\cdot, t) : \cdot \in \mathbb{R}\}$  we wish to denote by  $\rho_{n;(x,t_1)}^{(y,t_2)}(t)$ . To present and explain our convention in this regard, we let  $\ell_{(x,t_1)}^{(y,t_2)}$  denote the planar line segment whose endpoints are  $(x, t_1)$  and  $(y, t_2)$ . Adopting the same

perspective as for the polymer, we abuse notation to view  $\ell_{(x,t_1)}^{(y,t_2)}$  as a function from  $[t_1, t_2]$  to  $\mathbb{R}$ , so that  $\ell_{(x,t_1)}^{(y,t_2)}(t) = t_{1,2}^{-1}((t_2 - t)x + (t - t_1)y)$ .

Our convention will be to set  $\rho_{n;(x,t_1)}^{(y,t_2)}(t)$  equal to  $z$  where  $(z, t)$  is that point in the horizontal segment  $\rho_{n;(x,t_1)}^{(y,t_2)} \cap \{(\cdot, t) : \cdot \in \mathbb{R}\}$  whose distance from  $\ell_{(x,t_1)}^{(y,t_2)}(t)$  is maximal. (An arbitrary tie-breaking rule, say  $\rho_{n;(x,t_1)}^{(y,t_2)}(t) \geq \ell_{(x,t_1)}^{(y,t_2)}(t)$ , resolves the dispute if there are two such points.) The reason for this very particular convention is that our purpose in using it is to explore, in the soon-to-be-stated Theorem 1.5, upper bounds on the probability of large fluctuations between the polymer  $\rho_{n;(x,t_1)}^{(y,t_2)}$  and the line segment  $\ell_{(x,t_1)}^{(y,t_2)}$  that interpolates the polymer's endpoints. Our convention ensures that the form of the theorem would remain valid were any other convention instead adopted.

In order to study the intermediate time  $(1 - a)t_1 + at_2$  (in the role of  $t$  in the preceding), we now let  $a \in (0, 1)$  and impose that  $at_{1,2}n \in \mathbb{Z}$ : doing so ensures that, as desired,  $t \in n^{-1}\mathbb{Z}$ , where  $t = (1 - a)t_1 + at_2$ .

Consider also  $r > 0$ . Define the *polymer deviation regularity* event

$$\text{PolyDevReg}_{n;(x,t_1)}^{(y,t_2)}(a, r) = \left\{ \left| \rho_{n;(x,t_1)}^{(y,t_2)}((1-a)t_1 + at_2) - \ell_{(x,t_1)}^{(y,t_2)}((1-a)t_1 + at_2) \right| \leq rt_{1,2}^{2/3} (a \wedge (1-a))^{2/3} \right\}, \quad (2)$$

where  $\wedge$  denotes minimum. For example, if  $a \in (0, 1/2)$ , the polymer's deviation from the interpolating line segment, at height  $(1 - a)t_1 + at_2$  (when the polymer's journey has run for time  $at_{1,2}$ ), is measured in the natural time-to-the-two-thirds scaled units obtained by division by  $(at_{1,2})^{2/3}$ , and compared to the given value  $r > 0$ .

For intervals  $I, J \subset \mathbb{R}$ , we extend this definition by setting

$$\text{PolyDevReg}_{n;(I,t_1)}^{(J,t_2)}(a, r) = \bigcap_{x \in I, y \in J} \text{PolyDevReg}_{n;(x,t_1)}^{(y,t_2)}(a, r).$$

The perceptive reader may notice a problem with the last definition. The polymer  $\rho_{n;(x,t_1)}^{(y,t_2)}$  is well defined almost surely for given endpoints, but this property is no longer assured as the parameters vary over  $x \in I$  and  $y \in J$ . In the case of exceptional  $(x, y)$  where several  $n$ -polymers move from  $(x, t_1)$  to  $(y, t_2)$ , we interpret  $\rho_{n;(x,t_1)}^{(y,t_2)}$  as the union of all these polymers, for the purpose of defining  $\rho_{n;(x,t_1)}^{(y,t_2)}(t)$ . This convention permits us to identify worst case behaviour, so that the event  $\neg \text{PolyDevReg}_{n;(I,t_1)}^{(J,t_2)}(a, r)$  is triggered by a suitably large fluctuation on the part of any concerned polymer. Here, and later,  $\neg A$  denotes the complement of the event  $A$ .

We are ready to state our conclusion concerning polymer fluctuation. Bounds in this and many later results have been expressed explicitly up to two positive constants  $c$  and  $C$ . See Section 4.3 for an explanation of how the value of this pair of constants is fixed. We further set  $c_1 = 2^{-5/2}c \wedge 1/8$ .

**Theorem 1.5.** *Let  $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}_{<}^2$  be a compatible triple, with  $nt_{1,2}$  even, and let  $x, y \in \mathbb{R}$ .*

(1) *Let  $a \in [1 - 10^{-11}c_1^2, 1)$  satisfy  $at_{1,2} \in n^{-1}\mathbb{Z}$ . Suppose that  $n \in \mathbb{N}$  satisfies*

$$nt_{1,2} \geq \max \left\{ 10^{29}(1-a)^{-25}c^{-18}, 10^{24}c^{-18}(1-a)^{-25}|x-y|^{36}t_{1,2}^{-24} \right\}.$$

Let  $r > 0$  be a parameter that satisfies

$$r \geq \max \left\{ 10^9 c_1^{-4/5}, 15C^{1/2}, 87(1-a)^{1/3} t_{1,2}^{-2/3} |x-y| \right\}$$

and  $r \leq 3(1-a)^{25/9} n^{1/36} t_{1,2}^{1/36}$ .

Writing  $I = [x, x + t_{1,2}^{2/3}(1-a)^{2/3}r]$  and  $J = [y, y + t_{1,2}^{2/3}(1-a)^{2/3}r]$ , we have that

$$\mathbb{P} \left( \neg \text{PolyDevReg}_{n;(I,t_1)}^{(J,t_2)}(a, 2r) \right) \leq 44Cr \exp \left\{ -10^{-11} c_1 r^{3/4} \right\}.$$

(2) The same statement holds verbatim when appearances of  $a$  are replaced by  $1-a$ .

One aspect of our presentation may be apparent from the form of this theorem: we have chosen to be fairly explicit in recording hypothesis bounds in our results. This approach can provide quite lengthy formulas when hypotheses are recorded, and we encourage the reader not to be distracted by this from the essential meaning of results. In the present case, for example, we may set  $t_1 = 0$  and  $t_1 = 1$ , so that  $t_{1,2} = 1$  results. We may note that  $a$  must be a certain small distance from either zero or one, and that the condition  $a \in n^{-1}\mathbb{Z}$  is a negligible constraint, given that we are interested in high choices of  $n$ . Working in a case where  $x$  and  $y$  are bounded above by one, we are stipulating that  $n \geq n_0(a)$ . The parameter  $r$ , whose role is to gauge scaled fluctuation, is then constrained to lie between a universal constant and a multiple of  $n^{1/36}$ . We see then that the theorem is asserting that the maximum scaled fluctuation during the first, or last, duration  $a$  of time among all polymers beginning and ending in a given unit interval, separated at a unit duration, exceeds  $r \in [\Theta(1), \Theta(1)n^{1/36}]$  with probability at most  $\exp\{-O(1)r^{3/4}\}$ . Since we are interested in high  $n$ , and the bound so available when  $r = \Theta(1)n^{1/36}$  is rapidly decaying in  $n$ , the condition imposed on  $r$  is quite weak.

If the reader is ever disconcerted by the various conditions imposed in results, it may be helpful to consider the formal case where  $n = \infty$ , in which upper bounds, such as  $r \leq \Theta(1)n^{1/36}$  above, become obsolete.

As a matter of convenience for the upcoming proofs, we also state a version of Theorem 1.5 in which the intervals  $I$  and  $J$  are singleton sets.

**Proposition 1.6.** *Let  $(n, t_1, t_2) \in \mathbb{R}_{<}^2$  be a compatible triple, with  $nt_{1,2}$  even. Let  $x, y \in \mathbb{R}$  and let  $a \in [1 - 10^{-11}c_1^2, 1)$  satisfy  $at_{1,2} \in n^{-1}\mathbb{Z}$ . Suppose that*

$$nt_{1,2} \geq \max \left\{ 10^{29}(1-a)^{-25}c^{-18}, 10^{24}c^{-18}(1-a)^{-25}|x-y|^{36}t_{1,2}^{-24} \right\}.$$

Let  $r > 0$  be a parameter that satisfies

$$r \geq \max \left\{ 10^9 c_1^{-4/5}, 15C^{1/2}, 87(1-a)^{1/3} t_{1,2}^{-2/3} |x-y| \right\}$$

and  $r \leq 3(1-a)^{25/9} n^{1/36} t_{1,2}^{1/36}$ . Then

$$\mathbb{P} \left( \neg \text{PolyDevReg}_{n;(x,t_1)}^{(y,t_2)}(a, r) \right) \leq 22Cr \exp \left\{ -10^{-11} c_1 r^{3/4} \right\}.$$

*Remark.* The argument leading to the proposition will show that the result equally applies when it is instead supposed that  $a \in (0, 10^{-11}c_1^2]$ , provided that the instances of  $1-a$  in the hypothesis conditions are replaced by  $a$ .

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## 2. A ROAD MAP FOR PROVING THE RARIETY OF DISJOINT POLYMERS WITH CLOSE ENDPOINTS

In this section, we present a very coarse overview of the strategy of the proof of our result, Theorem 1.1, establishing the improbability of the event that several disjoint polymers begin and end in a common pair of short intervals. (Theorem 1.5's proof depends on different ideas and will appear towards the end of the paper.) Before we begin, we first present a key concept: the scaled energy, or *weight*, of an  $n$ -zigzag.

### 2.1. Polymer weights.

2.1.1. *Staircase energy scales to zigzag weight.* Let  $n \in \mathbb{N}$  and  $(i, j) \in \mathbb{N}_{>}^2$ . Any  $n$ -zigzag  $Z$  from  $(x, i/n)$  to  $(y, j/n)$  is ascribed a scaled energy, which we will refer to as its weight,  $\text{Wgt}(Z) = \text{Wgt}_n(Z)$ , given by

$$\text{Wgt}(Z) = 2^{-1/2}n^{-1/3} \left( E(S) - 2(j - i) - 2n^{2/3}(y - x) \right) \quad (3)$$

where  $Z$  is the image under  $R_n$  of the staircase  $S$ .

2.1.2. *Maximum weight.* Let  $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}_{>}^2$  be a compatible triple. Suppose that  $x, y \in \mathbb{R}$  satisfy  $y \geq x - 2^{-1}n^{1/3}t_{1,2}$ . Define

$$\text{Wgt}_{n;(x,t_1)}^{(y,t_2)} = 2^{-1/2}n^{-1/3} \left( M_{(nt_1+2n^{2/3}x, nt_1) \rightarrow (nt_2+2n^{2/3}y, nt_2)}^1 - 2nt_{1,2} - 2n^{2/3}(y - x) \right).$$

Thus,  $\text{Wgt}_{n;(x,t_1)}^{(y,t_2)}$  equals the maximum weight of any  $n$ -zigzag from  $(x, t_1)$  to  $(y, t_2)$ . When the polymer  $\rho_{n;(x,t_1)}^{(y,t_2)}$  is well defined, as it is almost surely, we have that  $\text{Wgt}_{n;(x,t_1)}^{(y,t_2)} = \text{Wgt}(\rho_{n;(x,t_1)}^{(y,t_2)})$ . It is also worth noting, however, that the system of weights  $\text{Wgt}_{n;(x,t_1)}^{(y,t_2)}$  is well defined almost surely, even as the five concerned parameters vary over admissible choices, despite the possible presence of exceptional parameter choices at which the corresponding polymer is not unique.

2.2. **The road map.** We begin by revisiting a theme from Subsection 1.7.2. When  $n$  is large, an  $n$ -polymer such as  $\rho_{n;(x,t_1)}^{(y,t_2)}$  is rather naturally viewed as a random function of its lifetime  $[t_1, t_2]$ , one that begins at  $x$  and ends at  $y$ . The microscopic ambiguities in the specification of this random function become vanishingly small in the limit of high  $n$ . Indeed, for the purposes of these paragraphs of overview, we yield to the temptation to set  $n = \infty$ , and discuss polymers as objects arising after a scaling limit has been taken. In this way,  $\rho_{\infty;(x,t_1)}^{(y,t_2)}$  is interpreted as the scaled limit polymer between space-time locations  $(x, t_1)$  and  $(y, t_2)$ . The polymer may be viewed as a subset of  $\mathbb{R}^2$ , living inside the strip  $\mathbb{R} \times [t_1, t_2]$ , or it may be viewed as a random, real-valued, function of the lifetime  $[t_1, t_2]$ . Speaking of such objects as  $\rho_{\infty;(x,t_1)}^{(y,t_2)}$  raises substantial questions about the uniqueness of a limiting description as  $n$  tends to infinity. In this non-rigorous overview, we have no intention of trying to address these questions. We simply use the framework of  $n = \infty$  as a convenient device for heuristic discussion, since this framework is unencumbered by microscopic details (such as the structure of zigzags). The term zigzag has become rather inapt in this context; it may be replaced by the term *path*, a path being any continuous real-valued function defined on a given interval  $[t_1, t_2]$ . Each

path has a weight and a path of maximum weight given its endpoints is a polymer. When these endpoints are  $(x, t_1)$  and  $(y, t_2)$ , this maximum weight is called  $\text{Wgt}_{\infty; (x, t_1)}^{(y, t_2)}$ .

Viewed through the prism of  $n = \infty$ , the event that Theorem 1.1 discusses is depicted in the left sketch of Figure 3. Here we take  $(n, t_1, t_2) = (\infty, 0, 1)$ . Depicted with  $k = 3$  is the event  $\text{MaxDisjtPoly}_{\infty; ([x-\epsilon, x+\epsilon], 0)}^{([y-\epsilon, y+\epsilon], 1)} \geq k$  that there exist  $k$  disjoint polymers with lifetime  $[0, 1]$  each of which begins at distance at most  $\epsilon$  from  $x \in \mathbb{R}$  and ends at such a distance from  $y \in \mathbb{R}$ .

It is valuable to bear in mind three general features of polymer geometry and weight that correspond to powers of  $2/3$ ,  $1/3$  and  $1/2$ . The first of these three principles has been articulated rigorously by Theorem 1.5. For the latter two, we will recall in Section 5 corresponding results for Brownian LPP from [Ham17b].

*A power of two-thirds dictates polymer geometry.* A polymer whose lifetime is  $[t_1, t_2]$ , and thus has duration  $t_{1,2}$ , fluctuates laterally, away from the planar line segment that interpolates its endpoints, by an order of  $t_{1,2}^{2/3}$ .

*A power of one-third dictates polymer weight.* A polymer whose lifetime is  $[t_1, t_2]$  has a weight of order  $t_{1,2}^{1/3}$ . Actually, this is only true if the polymer makes no significant lateral movement. For example,  $\rho_{\infty; (x, t_1)}^{(y, t_2)}$  may be expected to have a weight of order  $t_{1,2}^{1/3}$  provided that the endpoints verify  $|y - x| = O(t_{1,2}^{2/3})$ . The last condition indicates that the endpoint discrepancy is of the order of the polymer fluctuation, so that such a polymer is not deviating more than would be expected by a polymer whose starting and ending points coincide.

*A power of one-half dictates polymer weight differences.* Consider two polymers of unit duration both of whose endpoints differ by a small quantity  $\epsilon$ . Then the polymers' weights typically differ by an order of  $\epsilon^{1/2}$ . For example,  $\text{Wgt}_{\infty; (x+\epsilon, 0)}^{(y+\epsilon, 1)} - \text{Wgt}_{\infty; (x, 0)}^{(y, 1)}$  typically has order  $\epsilon^{1/2}$ , at least when  $|y - x| = 0(1)$ .

We now introduce an event that has much in common with the event that  $\text{MaxDisjtPoly}_{\infty; ([x-\epsilon, x+\epsilon], 0)}^{([y-\epsilon, y+\epsilon], 1)}$  is at least  $k$ . Let  $\eta > 0$ . The *near polymer* event  $\text{NearPoly}_{\infty, k; (x, 0)}^{(y, 1)}(\eta)$  is said to occur when there exist  $k$  paths mapping  $[0, 1]$  to  $\mathbb{R}$ , each beginning at  $x \in \mathbb{R}$  and ending at  $y \in \mathbb{R}$ , but which are pairwise disjoint except at the endpoint locations, such that the sum of the paths' weights exceeds  $k \cdot \text{Wgt}_{\infty; (x, 0)}^{(y, 1)} - \eta$ . To understand the meaning of this definition, note that, since each of these  $k$  paths has a weight bounded above by that of the polymer with these given endpoints, the sum of the paths' weights may be at most  $k \cdot \text{Wgt}_{\infty; (x, 0)}^{(y, 1)}$ . Since weights are of unit order, due to a unit duration being considered, we see that, when  $\eta > 0$  is fixed to be a small value, the event  $\text{NearPoly}_{\infty, k; (x, 0)}^{(y, 1)}(\eta)$  expresses the presence of a system of  $k$  near polymers, with shared endpoints but otherwise disjoint, whose sum weight misses the maximum value attainable in principle by a smaller than typical discrepancy of  $\eta$ .

The backbone of our derivation of Theorem 1.1 consists of arguing that the occurrence of the event  $\text{MaxDisjtPoly}_{\infty; ([x-\epsilon, x+\epsilon], 0)}^{([y-\epsilon, y+\epsilon], 1)} \geq k$  typically entails that  $\text{NearPoly}_{\infty, k; (x, 0)}^{(y, 1)}(\eta)$  also occurs, where here  $\eta$  is set to be of the order of  $\epsilon^{1/2}$ . Before we offer an overview of why this assertion may be expected to be true, we point out that the order of probability of the latter event is understood. Indeed, the use of shared endpoints in the definition of  $\text{NearPoly}_{\infty, k; (x, 0)}^{(y, 1)}(\eta)$  permits the use of integrable techniques arising from the Robinson-Schensted-Knuth correspondence: this event is shown in [Ham17a] to have

probability of order  $\eta^{k^2-1+o(1)}$ . The result in question will shortly be reviewed as Corollary 5.2. In reality, the result actually concerns the events  $\text{NearPoly}_{n,k;(x,0)}^{(y,1)}(\eta)$  when  $n \in \mathbb{N}$  is finite, and makes an assertion uniformly in high  $n$  about them. Anyway, since we are taking  $\eta = \Theta(\epsilon^{1/2})$ , we see that the form of Theorem 1.1 may be expected to follow from such a bound.

Now, how is it that we may argue that the  $k$  disjoint polymer event  $\text{MaxDisjtPoly}_{\infty;([x-\epsilon, x+\epsilon], 0)}^{([y-\epsilon, y+\epsilon], 1)} \geq k$  does indeed typically entail  $\text{NearPoly}_{\infty,k;(x,0)}^{(y,1)}(\eta)$  with  $\eta = \Theta(\epsilon^{1/2})$ ? The first event provides much of the structure of the  $k$  near polymers that would realize the second. But in the second, the near polymers begin and end at precisely the same points, rather than merely nearby one another. What is needed is surgery to alter the  $k$  polymers that begin and end in  $[x - \epsilon, x + \epsilon] \times \{0\}$  and  $[y - \epsilon, y + \epsilon] \times \{1\}$  so that their endpoints are actually equal.

Our device enabling this surgery to effect local correction of endpoint locations is called a *multi-polymer bouquet*. Let  $[t_1, t_2]$  be a short real interval, and let  $u \in \mathbb{R}$ . Fixing an increasing list of  $k$  real values,  $(v_1, \dots, v_k)$ , each of unit order, we may consider the problem of constructing a system of  $k$  paths, each with lifetime  $[t_1, t_2]$ , with each path beginning at  $(u, t_1)$ , and with the  $i^{\text{th}}$  path ending at  $(u + t_{1,2}^{2/3} v_i, t_2)$ ; crucially, we insist that the paths be disjoint, except for their shared starting location. Such a bouquet of paths is depicted in the middle sketch in Figure 3. These paths are moving on the natural two-thirds power fluctuation scale given their endpoints, so it is a natural belief that they may be selected so that the sum of their weights has order  $t_{1,2}^{1/3}$ , which is the natural polymer weight scale for polymers of duration  $t_{1,2}$ . It is one of the upcoming challenges for the proof of Theorem 1.1 to prove that the construction of such a path system may be undertaken. We will refer to this challenge as the problem of *bouquet construction*.

Accepting for now that such a bouquet of  $k$  paths, tied together at a common base point, may be constructed, it is of course also natural to believe that such a construction may be undertaken where instead it is at the endpoint time that the various paths share a common location. The two constructions may be called a forward and a backward bouquet.

The two bouquets may be used to tie together the nearby endpoints of the  $k$  polymers that arise in the event  $\text{MaxDisjtPoly}_{\infty;([x-\epsilon, x+\epsilon], 0)}^{([y-\epsilon, y+\epsilon], 1)} \geq k$ . Rather than altering the paths so that they begin and end at  $(x, 0)$  and  $(y, 1)$ , we will prefer to extend the duration of the paths a little in each direction. Because the spatial discrepancy of endpoint locations is of order  $\epsilon$ , the two-thirds power law for polymer geometry indicates that the natural time scale for our polymer bouquets will be  $\epsilon^{3/2}$ . A forward bouquet of  $k$  paths will be used at the start and a backward one at the end. The  $k$  paths in the forward bouquet will each begin at  $(x, -\epsilon^{3/2})$ ; their other endpoints, at time zero, will be the  $k$  starting points of the duration  $[0, 1]$  polymers. The backward bouquet will take the  $k$  ending points of the polymers and tie them together at the common location  $(y, 1 + \epsilon^{3/2})$ .

In this way, a system of  $k$  paths from  $(x, -\epsilon^{3/2})$  to  $(y, 1 + \epsilon^{3/2})$  will be obtained: see the right sketch in Figure 3. Each is split into three pieces: a short  $\epsilon^{3/2}$ -duration element of the forward bouquet, a long  $[0, 1]$ -time piece given by one of the polymers, and then a further short element of the backward bouquet. This system will be shown to typically realize  $\text{NearPoly}_{\infty,k;(x,-\epsilon^{3/2})}^{(y,1+\epsilon^{3/2})}(\eta)$  with  $\eta = \Theta(\epsilon^{1/2})$ . Note that the starting and ending times are  $-\epsilon^{3/2}$  and  $1 + \epsilon^{3/2}$ , rather than zero and one.

We now summarise the challenges involved in proving this. The value  $n$  is finite, with statements to be understood uniformly in high choices of this parameter.



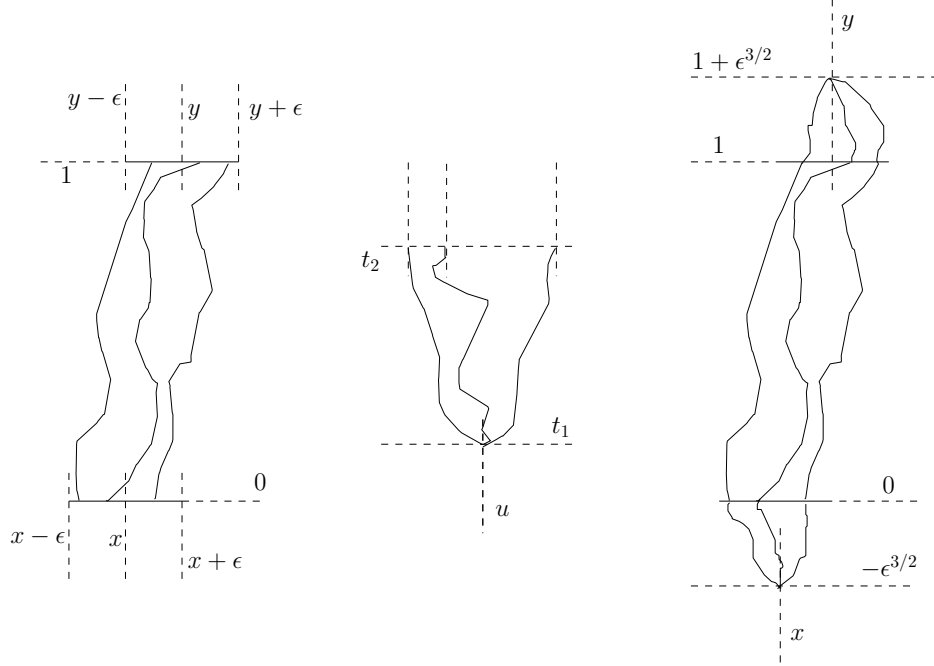


FIGURE 3. *Left:* The three polymers concerned in the formally specified event  $\text{MaxDisjtPoly}_{\infty;([x-\epsilon, x+\epsilon], 0)}^{([y-\epsilon, y+\epsilon], 1)} \geq 3$ . *Middle:* A bouquet of three paths of lifetime  $[t_1, t_2]$ . *Right:* Two bouquets, facing in opposite directions, are employed to extend the original three polymers into longer paths, each beginning and ending at  $(x, -\epsilon^{3/2})$  and  $(y, 1 + \epsilon^{3/2})$ .

- **Polymer weight similarity:** if the construction is to result in a collection of  $k$  paths from  $(x, -\epsilon^{3/2})$  to  $(y, 1 + \epsilon^{3/2})$  each of whose weights reaches within order  $\epsilon^{1/2}$  of the maximum achievable value  $\text{Wgt}_{n; (x, -\epsilon^{3/2})}^{(y, 1 + \epsilon^{3/2})}$ , then we will want to begin by confirming that the difference in weight between any pair of the  $k$  polymers moving between  $[x - \epsilon, x + \epsilon] \times \{0\}$  and  $[y - \epsilon, y + \epsilon] \times \{1\}$  itself has order  $\epsilon^{1/2}$ . This is an instance of the one-half power law principle enunciated above.
- **Bouquet construction:** the forward and backward bouquets, defined on the time intervals  $[-\epsilon^{3/2}, 0]$  and  $[1, 1 + \epsilon^{3/2}]$ , must be constructed so that each bouquet has cumulative weight of the desired order  $\epsilon^{1/2}$ .
- **Final polymer comparison:** even if our original polymers have similar weights, and they can be altered to run from  $(x, -\epsilon^{3/2})$  to  $(y, 1 + \epsilon^{3/2})$  with weight changes of order  $\epsilon^{1/2}$ , there could still be trouble in principle. We need to know that each of the extended paths has a weight suitably close to  $\text{Wgt}_{n; (x, -\epsilon^{3/2})}^{(y, 1 + \epsilon^{3/2})}$ . Thus, we need to verify that the polymer running from  $(x, -\epsilon^{3/2})$  to  $(y, 1 + \epsilon^{3/2})$  has a weight that exceeds the extended paths' weights by an order of at most  $\epsilon^{1/2}$ .

In a sense, the third of these challenges is simply a restatement of the entire problem. When we discuss it later, however, we will resolve the problem in light of the solutions of the first two difficulties.

This completes our first overview of the road map to the proof of Theorem 1.1. We have been left with three challenges to solve. In the next three sections, we will present definitions and tools and cite results that will be used to implement the road map, as well as to prove the polymer fluctuation Theorem 1.5. After these tools have been described, we will describe at the end of Section 5 the structure of the rest of the paper. In essence, we will then return to the road map, elaborate further in outline how to resolve its three challenges, and implement it rigorously.

### 3. TOOLS: STAIRCASE COLLECTIONS AND MULTI-POLYMERS

The definition and study of the event  $\text{NearPoly}_{n,k;(x,0)}^{(y,1)}(\eta)$  (for  $k$  fixed and  $n$ , although now finite, high in applications), and the problem of bouquet construction, will both involve the use of systems of staircases or zigzags. We begin this discussion of tools by setting up definitions in this regard.

**3.1. Staircase collections.** Two staircases are called horizontally separate if there is no planar horizontal interval of positive length that is a subset of a horizontal interval in both staircases.

We now introduce notation for collections of pairwise horizontally separate staircases. For  $k \in \mathbb{N}$ , let  $(x_i, s_i)$  and  $(y_i, f_i)$ ,  $i \in \llbracket 1, k \rrbracket$ , be a collection of pairs of elements of  $\mathbb{R} \times \mathbb{N}$ . (The symbols  $s$  and  $f$  are used in reference to the staircases' heights at the *start* and *finish*.)

Let  $SC_{(\bar{x}, \bar{s}) \rightarrow (\bar{y}, \bar{f})}^k$  denote the set of  $k$ -tuples  $(\phi_1, \dots, \phi_k)$ , where  $\phi_i$  is a staircase from  $(x_i, s_i)$  to  $(y_i, f_i)$  and each pair  $(\phi_i, \phi_j)$ ,  $i \neq j$ , is horizontally separate. (This set may be empty; in order that it be non-empty, it is necessary that  $y_i \geq x_i$  and  $f_i \geq s_i$  for  $i \in \llbracket 1, k \rrbracket$ .) Note also that  $SC_{(x_1, s_1) \rightarrow (y_1, f_1)}^1$  equals  $SC_{(x_1, s_1) \rightarrow (y_1, f_1)}$ .

We also associate an energy to each member of  $SC_{(\bar{x}, \bar{s}) \rightarrow (\bar{y}, \bar{f})}^k$ . Each of the  $k$  elements of any  $k$ -tuple in  $SC_{(\bar{x}, \bar{s}) \rightarrow (\bar{y}, \bar{f})}^k$  has an energy, as we described in Subsection 1.6.2. Define the energy  $E(\phi)$  of any  $\phi = (\phi_1, \dots, \phi_k) \in SC_{(\bar{x}, \bar{s}) \rightarrow (\bar{y}, \bar{f})}^k$  to be  $\sum_{j=1}^k E(\phi_j)$ .

When  $SC_{(\bar{x}, \bar{s}) \rightarrow (\bar{y}, \bar{f})}^k \neq \emptyset$ , we further define the maximum  $k$ -tuple energy

$$M_{(\bar{x}, \bar{s}) \rightarrow (\bar{y}, \bar{f})}^k = \sup \left\{ E(\phi) : \phi \in SC_{(\bar{x}, \bar{s}) \rightarrow (\bar{y}, \bar{f})}^k \right\}. \quad (4)$$

An  $k$ -tuple of staircases that attains this maximum may be called a multi-geodesic.

**3.2. Maximizer uniqueness.** Next is [Ham17c, Lemma A.1]. Since this result is drawn the final article in our four-paper study, it is worth mentioning that the result has a fairly short and self-contained proof in [Ham17c, Appendix A].

**Lemma 3.1.** *Let  $k \in \mathbb{N}$  and let  $(x_i, s_i)$  and  $(y_i, f_i)$ ,  $i \in \llbracket 1, k \rrbracket$ , be a collection of pairs of points in  $\mathbb{R} \times \mathbb{N}$  such that  $SC_{(\bar{x}, \bar{s}) \rightarrow (\bar{y}, \bar{f})}^k$  is non-empty. Then, except on a  $\mathbb{P}$ -null set, there is a unique element of  $SC_{(\bar{x}, \bar{s}) \rightarrow (\bar{y}, \bar{f})}^k$  whose energy attains  $M_{(\bar{x}, \bar{s}) \rightarrow (\bar{y}, \bar{f})}^k$ .*

When it exists, we denote the unique maximizer by

$$\left( P_{(\bar{x}, \bar{s}) \rightarrow (\bar{y}, \bar{f}); i}^k : i \in \llbracket 1, k \rrbracket \right) \in SC_{(\bar{x}, \bar{s}) \rightarrow (\bar{y}, \bar{f})}^k.$$

Consider the partial order on  $\mathbb{R}^2$  such that  $(x_1, x_2)$  is at most  $(y_1, y_2)$  if and only if  $x_1 \leq x_2$  and  $y_2 \leq y_1$ .

In the case that  $k = 1$ , and  $(x, s)$  and  $(y, f)$  are elements of  $\mathbb{R} \times \mathbb{N}$  with  $(y, f)$  exceeding  $(x, s)$  in the partial order, the set  $SC^1_{(x,s) \rightarrow (y,f)}$  is non-empty. The maximizer in Lemma 3.1 is in this special case denoted by  $P^1_{(x,s) \rightarrow (y,f)}$ .

**3.3. Maximum weights of zigzag systems.** Each  $n$ -zigzag is the image under the scaling map  $R_n$  of a staircase. Two such zigzags will be called horizontally separate if their staircase counterparts have this property. The weight of a collection of zigzags is the sum of the weights of those zigzags.

We wish to record notation for the maximum weight of a pairwise horizontally disjoint collection of  $[t_1, t_2]$ -lifetime zigzags with prescribed endpoints.

To this end, let  $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}_{\leq}^2$  be a compatible triple. Taking  $k \in \mathbb{N}$ , a parameter that will denote the number of zigzags, consider  $k$  pairs  $(x_i, y_i) \in \mathbb{R}^2$ ,  $i \in \llbracket 1, k \rrbracket$ , where we suppose that  $y_i \geq x_i - 2^{-1}n^{1/3}t_{1,2}$  for each  $i \in \llbracket 1, k \rrbracket$ . Further suppose that the vectors  $\bar{x} = (x_1, \dots, x_k)$  and  $\bar{y} = (y_1, \dots, y_k)$  are non-decreasing. We will write  $\text{Wgt}_{n,k;(\bar{x},t_1)}^{(\bar{y},t_2)}$  for the maximum weight associated to  $k$  horizontally separate zigzags moving consecutively between  $(x_i, t_1)$  to  $(y_i, t_2)$  for  $i \in \llbracket 1, k \rrbracket$ . Formally, we define

$$\text{Wgt}_{n,k;(\bar{x},t_1)}^{(\bar{y},t_2)} = 2^{-1/2}n^{-1/3} \left( M^k_{(nt_1\bar{1}+2n^{2/3}\bar{x},nt_1\bar{1}) \rightarrow (nt_2\bar{1}+2n^{2/3}\bar{y},nt_2\bar{1})} - 2knt_{1,2} - 2n^{2/3} \sum_{i=1}^k (y_i - x_i) \right), \quad (5)$$

where  $\bar{1} \in \mathbb{R}^k$  denotes the vector whose components equal 1.

**3.4. Multi-polymers and their weights.** Retaining these parameters, we now may offer a geometric view of this multi-weight. The maximum energy  $M^k$  seen on the right-hand side of (5) is attained by a certain multi-geodesic which, after the scaling map is applied, may be viewed as a multi-polymer; the weight specified in (5) is the sum of the weights of the zigzags that constitute this multi-polymer. To set up notation in this regard, note that Lemma 3.1 ensures the almost sure existence of the  $k$ -tuple

$$\left( P^k_{(nt_1\bar{1}+2n^{2/3}\bar{x},nt_1\bar{1}) \rightarrow (nt_2\bar{1}+2n^{2/3}\bar{y},nt_2\bar{1})}; i : i \in \llbracket 1, k \rrbracket \right) \in SC^k_{(nt_1\bar{1}+2n^{2/3}\bar{x},nt_1\bar{1}) \rightarrow (nt_2\bar{1}+2n^{2/3}\bar{y},nt_2\bar{1})}. \quad (6)$$

Indeed, the right-hand set is clearly non-empty under our hypotheses, permitting the use of the above lemma. We define the multi-polymer

$$\left( \rho_{n,k,i;(\bar{x},t_1)}^{(\bar{y},t_2)} : i \in \llbracket 1, k \rrbracket \right)$$

to be the  $k$ -tuple of  $n$ -zigzags whose elements are the respective images under  $R_n$  of the elements in the  $k$ -tuple in (6). The  $k$ -tuple itself will be denoted by  $\rho_{n,k;(\bar{x},t_1)}^{(\bar{y},t_2)}$ . In this way, the  $i^{\text{th}}$  component  $\rho_{n,k,i;(\bar{x},t_1)}^{(\bar{y},t_2)}$  is an  $n$ -zigzag with starting and ending points  $(x_i, t_1)$  and  $(y_i, t_2)$ . This zigzag is not necessarily a polymer.

Each of the  $k$  zigzag components of  $\rho_{n,k;(\bar{x},t_1)}^{(\bar{y},t_2)}$  has a weight via (3): the  $i^{\text{th}}$  weight is  $\text{Wgt}(Z) = 2^{-1/2}n^{-1/3} (E(S) - 2nt_{1,2} - 2n^{2/3}(y_i - x_i))$ , where  $S$  is the pre-image staircase in question. This  $i^{\text{th}}$  weight will be denoted by  $\text{Wgt}_{n,k,i;(\bar{x},t_1)}^{(\bar{y},t_2)}$ . The weight of the multi-polymer  $\rho_{n,k;(\bar{x},t_1)}^{(\bar{y},t_2)}$  equals the component sum  $\sum_{i=1}^k \text{Wgt}_{n,k,i;(\bar{x},t_1)}^{(\bar{y},t_2)}$ , and is seen to be equal to  $\text{Wgt}_{n,k;(\bar{x},t_1)}^{(\bar{y},t_2)}$ .

An important special case arises when  $\bar{x}$  and  $\bar{y}$  are both constant vectors, of the form  $\bar{x} = x\bar{1} \in \mathbb{R}^k$  and  $\bar{y} = y\bar{1} \in \mathbb{R}^k$ . The multi-polymer may be called a *multi-polymer watermelon* in this case. The

cases when merely one of  $\bar{x}$  and  $\bar{y}$  is a constant vector are also significant: these are the forward and backward bouquets that have been discussed in the road map.

#### 4. MULTI-WEIGHT ENCODING LINE ENSEMBLES: DEFINITIONS AND KEY PROPERTIES

As the road map has described, a key input drawing in part on integrable probability, Corollary 5.2, will assert that  $\text{NearPoly}_{n,k;(x,0)}^{(y,1)}(\eta)$  has probability  $\eta^{k^2-1+o(1)}$  uniformly in high  $n$ . This assertion about the weight of a multi-polymer watermelon falls within the realm of integrable probability because this weight is the maximum possible attained by a system of  $k$  zigzags, with given shared endpoints but otherwise disjoint, and, as such, it may be expressed as the value of the sum of the  $k$  uppermost curves in a line ensemble that is naturally associated to Brownian LPP (by means of the RSK correspondence). In this section, we set up the notation for these ensembles of random curves.

**4.1. Line ensembles that encode polymer weights.** Let  $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}_{\geq}^2$  be a compatible triple, and let  $x \in \mathbb{R}$ . We define the scaled forward line ensemble

$$\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2} : \llbracket 1, nt_{1,2} + 1 \rrbracket \times [x - 2^{-1}n^{1/3}t_{1,2}, \infty) \rightarrow \mathbb{R}$$

rooted at  $(x, t_1)$  with duration  $t_{1,2}$  by declaring that, for each  $k \in \llbracket 1, nt_{1,2} + 1 \rrbracket$  and  $y \geq x - 2^{-1}n^{1/3}t_{1,2}$ ,

$$\sum_{i=1}^k \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(i, y) = \text{Wgt}_{n,k;(x\bar{1},t_1)}^{(y\bar{1},t_2)}. \quad (7)$$

The ensemble is called ‘forward’, and the notation is adorned with the symbol  $\uparrow$ , because it is the spatial location  $y$  attached to the more advanced time  $t_2$  that is treated as the variable. We stand at  $(x, t_1)$  and look forward in time to  $t_2$  to witness behaviour as a function of location  $y$ . When  $k = 1$ , the lowest indexed ensemble curve  $y \rightarrow \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(1, y)$  records the polymer weight profile from  $(x, t_1)$  to  $(y, t_2)$ . The sum of the  $k$  lowest indexed curves records the multi-polymer watermelon weight between these locations.

Equally, we may stand at  $(y, t_2)$  and look backward in time to a variable location  $x \in \mathbb{R}$  at time  $t_1$ . Indeed, fixing  $y \in \mathbb{R}$ , we define the scaled backward line ensemble

$$\mathcal{L}_{n;t_1}^{\downarrow;(y,t_2)} : \llbracket 1, nt_{1,2} + 1 \rrbracket \times (-\infty, y + 2^{-1}n^{1/3}t_{1,2}] \rightarrow \mathbb{R}$$

rooted at  $(y, t_2)$  with duration  $t_{1,2}$  by declaring that, for each  $k \in \llbracket 1, nt_{1,2} + 1 \rrbracket$  and  $x \leq y + 2^{-1}n^{1/3}t_{1,2}$ ,

$$\sum_{i=1}^k \mathcal{L}_{n;t_1}^{\downarrow;(y,t_2)}(i, x) = \text{Wgt}_{n,k;(x\bar{1},t_1)}^{(y\bar{1},t_2)}.$$

It is valuable to hold a vivid picture of the two ensembles. Each is an ordered system of random continuous curves, with the lowest indexed curve uppermost. The ensemble curve of any given index locally resembles Brownian motion but globally follows the contour of the parabola  $-2^{-1/2}(y - x)^2 t_{1,2}^{-4/3}$ , as a function of  $y$  or  $x$  in the forward or backward case. Each ensemble adopts values of order  $t_{1,2}^{1/3}$  when  $x$  and  $y$  differ by an order of  $t_{1,2}^{2/3}$ . More negative values, dictated by parabolic curvature, are witnessed outside this region. This description holds sway in a region that expands from the origin as the parameter  $n$  rises.

Clearly, then, our ensembles have fundamental differences according to the value of  $t_{1,2}$ : sharply peaked ensemble curves when  $t_{1,2}$  is small, and much flatter curves when  $t_{1,2}$  is large. A simple

further parabolic transformation will serve to put the ensembles on a much more equal footing. Since the ensembles are already scaled objects, we will use the term ‘normalized’ to allude to the newly transformed counterparts. A scaled forward ensemble and its normalized counterpart are depicted in Figure 4.

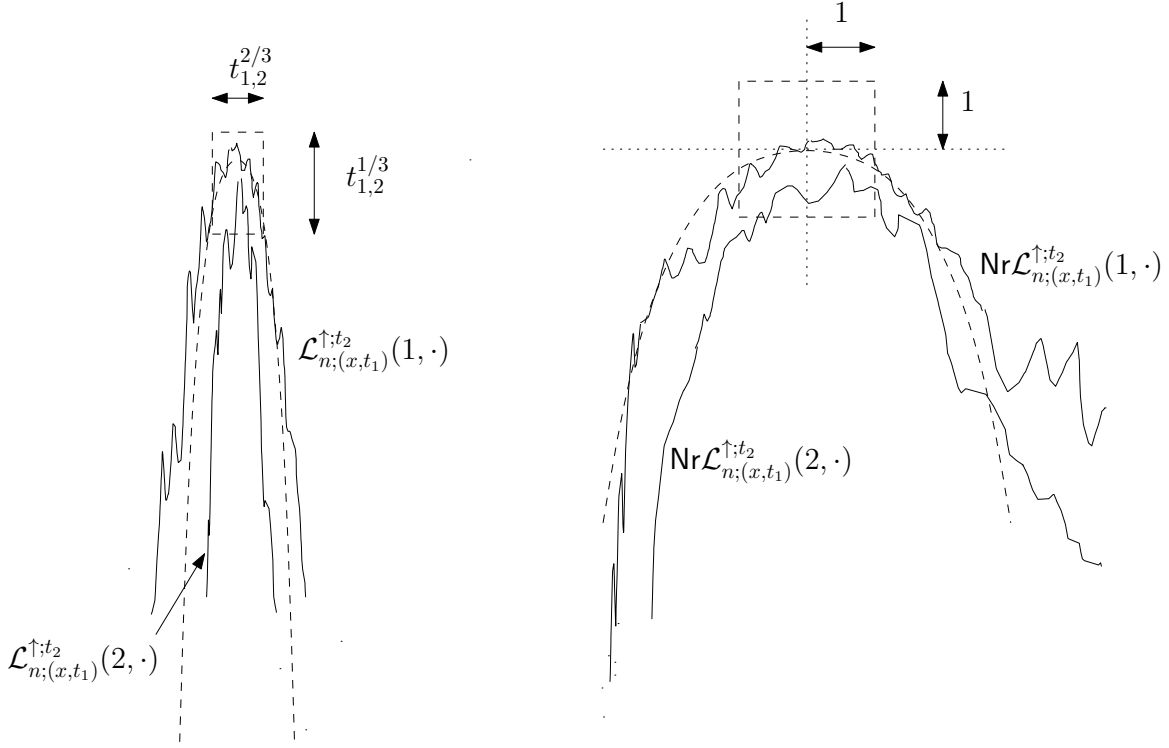


FIGURE 4. Let  $(n, t_1, t_2)$  be a compatible triple, with  $t_{1,2}$  some small fixed value, and  $n$  large. For  $x \in \mathbb{R}$  given, the two sketches show the highest two curves in the scaled, and normalized, forward line ensembles rooted at  $(x, t_1)$  with duration  $t_{1,2}$ .

That is, we define the *normalized* forward ensemble

$$\text{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2} : \llbracket 1, nt_{1,2} + 1 \rrbracket \times \left[ -2^{-1}(nt_{1,2})^{1/3}, \infty \right) \rightarrow \mathbb{R}, \quad (8)$$

setting

$$\text{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(k, z) = t_{1,2}^{-1/3} \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(k, x + t_{1,2}^{2/3} z) \quad (9)$$

for each  $k \in \llbracket 1, nt_{1,2} + 1 \rrbracket$  and  $z \geq -2^{-1}n^{1/3}t_{1,2}^{-2/3}$ . Similarly, the normalized backward ensemble

$$\text{Nr}\mathcal{L}_{n;t_1}^{\downarrow;(y,t_2)} : \llbracket 1, nt_{1,2} + 1 \rrbracket \times \left( -\infty, 2^{-1}(nt_{1,2})^{1/3} \right] \rightarrow \mathbb{R}, \quad (10)$$

is specified by setting

$$\text{Nr}\mathcal{L}_{n;t_1}^{\downarrow;(y,t_2)}(k, z) = t_{1,2}^{-1/3} \mathcal{L}_{n;t_1}^{\downarrow;(y,t_2)}(k, y + t_{1,2}^{2/3} z)$$

for each  $k \in \llbracket 1, nt_{1,2} + 1 \rrbracket$  and  $z \leq 2^{-1}n^{1/3}t_{1,2}^{-2/3}$ .

The curves in the new ensembles locally resemble Brownian motion as before, but they have been centred and squeezed so that now the parabola that dictates their overall shape is  $-2^{-1/2}z^2$ . This picture is accurate in a region that expands as the parameter  $nt_{1,2}$  rises.

**4.2. Brownian Gibbs line ensembles.** The notion of a Brownian Gibbs line ensemble was introduced in [CH14] to capture a system of ordered curves that arise by conditioning Brownian motions or bridges on mutual avoidance. The precise definition is not logically needed in this article, but we offer an informal summary next.

4.2.1. *An overview.* Let  $n \in \mathbb{N}$  and let  $I$  be a closed interval in the real line. A  $\llbracket 1, n \rrbracket$ -indexed line ensemble defined on  $I$  is a random collection of continuous curves  $\mathcal{L} : \llbracket 1, n \rrbracket \times I \rightarrow \mathbb{R}$  specified under a probability measure  $\mathbb{P}$ . The  $i^{\text{th}}$  curve is thus  $\mathcal{L}(i, \cdot) : I \rightarrow \mathbb{R}$ . (The adjective ‘line’ has been applied to these systems perhaps because of their origin in such models as Poissonian LPP, where the counterpart object has piecewise constant curves. We will omit it henceforth.) An ensemble is called *ordered* if  $\mathcal{L}(i, x) > \mathcal{L}(i+1, x)$  whenever  $i \in \llbracket 1, n-1 \rrbracket$  and  $x$  lies in the interior of  $I$ . The curves may thus assume a common value at any finite endpoint of  $I$ . We will consider ordered ensembles that satisfy a condition called the Brownian Gibbs property. Colloquially, we may say that an ordered ensemble is called Brownian Gibbs if it arises from a system of Brownian bridges or Brownian motions defined on  $I$  by conditioning on the mutual avoidance of the curves at all times in  $I$ .

4.2.2. *Defining  $(c, C)$ -regular ensembles.* We are interested in ensembles that are not merely Brownian Gibbs but that hew to the shape of a parabola and have one-point distributions for the uppermost curve that enjoy tightness properties. We will employ the next definition, which specifies a  $(\bar{\phi}, c, C)$ -regular ensemble from [Ham17a, Definition 2.4], in the special case where the vector  $\bar{\phi}$  equals  $(1/3, 1/9, 1/3)$ .

**Definition 4.1.** Consider a Brownian Gibbs ensemble of the form

$$\mathcal{L} : \llbracket 1, N \rrbracket \times [-z_{\mathcal{L}}, \infty) \rightarrow \mathbb{R},$$

and which is defined on a probability space under the law  $\mathbb{P}$ . The number  $N = N(\mathcal{L})$  of ensemble curves and the absolute value  $z_{\mathcal{L}}$  of the finite endpoint may take any values in  $\mathbb{N}$  and  $[0, \infty)$ .

Let  $C$  and  $c$  be two positive constants. The ensemble  $\mathcal{L}$  is said to be  $(c, C)$ -regular if the following conditions are satisfied.

- (1) **Endpoint escape.**  $z_{\mathcal{L}} \geq cN^{1/3}$ .
- (2) **One-point lower tail.** If  $z \in [-z_{\mathcal{L}}, \infty)$  satisfies  $|z| \leq cN^{1/9}$ , then

$$\mathbb{P}\left(\mathcal{L}(1, z) + 2^{-1/2}z^2 \leq -s\right) \leq C \exp\{-cs^{3/2}\}$$

for all  $s \in [1, N^{1/3}]$ .

- (3) **One-point upper tail.** If  $z \in [-z_{\mathcal{L}}, \infty)$  satisfies  $|z| \leq cN^{1/9}$ , then

$$\mathbb{P}\left(\mathcal{L}(1, z) + 2^{-1/2}z^2 \geq s\right) \leq C \exp\{-cs^{3/2}\}$$

for all  $s \in [1, \infty)$ .

A Brownian Gibbs ensemble of the form

$$\mathcal{L} : \llbracket 1, N \rrbracket \times (-\infty, z_{\mathcal{L}}] \rightarrow \mathbb{R}$$

is also said to be  $(c, C)$ -regular if the reflected ensemble  $\mathcal{L}(\cdot, -\cdot)$  is. This is equivalent to the above conditions when instances of  $[-z_{\mathcal{L}}, \infty)$  are replaced by  $(-\infty, z_{\mathcal{L}}]$ .

We will refer to these three regular ensemble conditions as Reg(1), Reg(2) and Reg(3).

4.2.3. *The normalized forward and backward ensembles are  $(c, C)$ -regular.* Our reason for invoking the theory of regular Brownian Gibbs ensembles is that the normalized Brownian LPP ensembles verify the definition. This assertion is made by the next result; as we explain shortly, in Section 5.8, its proof has in essence been provided in [Ham17a] and will formally be given in Appendix B.

**Proposition 4.2.** *Let  $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}_{<}^2$  be a compatible triple.*

(1) *Let  $x \in \mathbb{R}$ . The ensemble  $\mathcal{L}$  given by*

$$\text{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2} : \llbracket 1, nt_{1,2} + 1 \rrbracket \times \left[ -2^{-1}(nt_{1,2})^{1/3}, \infty \right) \rightarrow \mathbb{R},$$

*is Brownian Gibbs, where  $N(\mathcal{L}) = nt_{1,2} + 1$  and  $z_{\mathcal{L}} = 2^{-1}(nt_{1,2})^{1/3}$ .*

(2) *There exist positive constants  $C$  and  $c$ , which may be chosen independently of all such choices of the parameters  $t_1, t_2, x$  and  $n$ , such that the ensemble  $\mathcal{L}$  is  $(c, C)$ -regular.*

(3) *If, in place of  $x$ , we consider  $y \in \mathbb{R}$ , and the ensemble  $\mathcal{L}$  given by*

$$\text{Nr}\mathcal{L}_{n;t_1}^{\downarrow;(y,t_2)}(1, \cdot) : \llbracket 1, nt_{1,2} + 1 \rrbracket \times \left( -\infty, 2^{-1}(nt_{1,2})^{1/3} \right] \rightarrow \mathbb{R},$$

*then the two preceding assertions hold, the second now independently of the parameters  $t_1, t_2, y$  and  $n$ .*

**4.3. Some useful properties of regular ensembles.** We are about to state Proposition 4.3, whose four parts assert the various properties, beyond the regular sequence conditions Reg(1), Reg(2) and Reg(3), of the normalized ensembles that we will be employing.

We fix henceforth the values of the two positive constants  $C$  and  $c$ , specifying them by Proposition 4.2. Since bounding the constants would render hypotheses of our results to be explicit, we mention that they are determined in [Ham17a, Appendix A.1] via Ledoux [Led07, (5.16)] and Aubrun's [Aub05, Proposition 1] bounds on the lower and upper tail of the maximum eigenvalue of a matrix in the Gaussian unitary ensemble.

We now specify two sequences  $\{C_k : k \geq 1\}$  and  $\{c_k : k \geq 1\}$ , their values expressed in terms of  $C$  and  $c$ . This usage for  $C_k$  and  $c_k$  is retained throughout the paper. We set, for each  $k \geq 2$ ,

$$C_k = \max \left\{ 10 \cdot 20^{k-1} 5^{k/2} \left( \frac{10}{3 \cdot 2^{3/2}} \right)^{k(k-1)/2} C, e^{c/2} \right\} \quad (11)$$

as well as  $C_1 = 140C$ ; and

$$c_k = \left( (3 - 2^{3/2})^{3/2} 10^{-3/2} \right)^{k-1} c_1, \quad (12)$$

with  $c_1 = 2^{-5/2} c \wedge 1/8$ . Note that  $\limsup C_k^{1/k^2} < \infty$  and  $\liminf c_k^{1/k} > 0$ .

Let  $Q : \mathbb{R} \rightarrow \mathbb{R}$  denote the parabola  $Q(x) = 2^{-1/2} x^2$ .

**Proposition 4.3.** *Suppose that  $\mathcal{L} = \mathcal{L}_N$ , mapping either  $\llbracket 1, N \rrbracket \times [-z_{\mathcal{L}}, \infty)$  or  $\llbracket 1, N \rrbracket \times (-\infty, z_{\mathcal{L}}]$ , to  $\mathbb{R}$ , is a  $(c, C)$ -regular ensemble, where  $N \in \mathbb{N}$  and  $z_{\mathcal{L}} \geq 0$ .*

(1) *(Pointwise curve lower bound) Let  $k \in \mathbb{N}$  and  $z, s \in \mathbb{R}$ . Suppose that  $N \geq k \vee (c/3)^{-18} \vee 6^{36}$ ,  $|z| \leq 2^{-1} c N^{1/18}$  and  $s \in [0, 2N^{1/18}]$ . Then*

$$\mathbb{P} \left( \mathcal{L}_N(k, z) + Q(z) \leq -s \right) \leq C_k \exp \left\{ -c_k s^{3/2} \right\}.$$

- (2) (*Uniform curve lower bound*) For  $k \in \mathbb{N}$ , let  $E_k = 20^{k-1}2^{k(k-1)/2}E_1$  where  $E_1 = 10C$ . Set  $r_0 = 5(3 - 2^{3/2})^{-1}$ ,  $r_1 = 2^{3/2}$ , and  $r_k = \max\{5^3, r_0 r_{k-1}\}$  for  $k \geq 2$ .

Whenever  $k \in \mathbb{N}$  and  $(t, r, y) \in \mathbb{R}$  satisfy  $N \geq k \vee (c/3)^{-18} \vee 6^{36}$ ,  $t \in [0, N^{1/18}]$ ,  $r \in [r_k, 2N^{1/18}]$  and  $y \in c/2 \cdot [-N^{1/18}, N^{1/18}]$ ,

$$\mathbb{P}\left(\inf_{x \in [y-t, y+t]} (\mathcal{L}_N(k, x) + Q(x)) \leq -r\right) \leq \left(t \vee 5 \vee (3 - 2^{3/2})^{-1/2} r_{k-1}^{1/2}\right)^k \cdot E_k \exp\{-c_k r^{3/2}\}.$$

- (3) (*No Big Max*) For  $|y| \leq 2^{-1}cN^{1/9}$ ,  $r \in [0, 4^{-1}cN^{1/9}]$ ,  $t \in [2^{7/2}, 2N^{1/3}]$  and  $N \geq c^{-18}$ ,

$$\mathbb{P}\left(\sup_{x \in [y-r, y+r]} (\mathcal{L}_N(1, x) + 2^{-1/2}x^2) \geq t\right) \leq (r+1) \cdot 6C \exp\{-2^{-9/2}ct^{3/2}\}.$$

- (4) (*Collapse near infinity*) For  $\eta \in (0, c]$ , let  $\ell = \ell_\eta : \mathbb{R} \rightarrow \mathbb{R}$  denote the even function which is affine on  $[0, \infty)$  and has gradient  $-5 \cdot 2^{-3/2}\eta N^{1/9}$  on this interval, and which satisfies  $\ell(\eta N^{1/9}) = (-2^{-1/2} + 2^{-5/2})\eta^2 N^{2/9}$ . If  $N \geq 2^{45/4}c^{-9}$ , then

$$\begin{aligned} & \mathbb{P}\left(\mathcal{L}_N(1, z) > \ell(z) \text{ for some } z \in D \setminus [-\eta N^{1/9}, \eta N^{1/9}]\right) \\ & \leq 6C \exp\left\{-c\eta^3 2^{-15/4} N^{1/3}\right\}. \end{aligned}$$

The set  $D$  is the spatial domain of  $\mathcal{L}$ , either  $[-z_{\mathcal{L}}, \infty)$  or  $(-\infty, z_{\mathcal{L}}]$ .

These four assertions are proved in [Ham17a]. Respectively, they appear as the following results in that article: Proposition 2.7, Proposition A.2, Proposition 2.29, and Proposition 2.31.

A few words about the meaning of the four parts of this proposition: the first part asserts a lower bound for a curve in a regular ensemble of any given index. This holds by definition when the index is  $k = 1$  via Reg(2) but is non-trivial in the other cases. The second part strengthens this conclusion to speak of the minimum value of such a curve on a compact interval. The third similarly strengthens the one-point upper tail Reg(3). In regard to the fourth, note that Reg(2) and Reg(3) do not assert that curves hew to the parabola  $-2^{-1/2}z^2$  globally, but only in an expanding region about the origin, of width  $2cN^{1/9}$  centred at the origin, where  $N$  is the ensemble curve cardinality. Proposition 4.3(4) offers a substitute control on curves far from the origin, showing them to decay at a rapid but nonetheless linear rate in the region beyond scale  $N^{1/9}$ .

These four assertions from [Ham17a] are all consequences of the theory of Brownian Gibbs resampling introduced in [CH14] and developed in [Ham17a]. They are all rather simple consequences of this theory, with short proofs in [Ham17a]. It is in fact in the upcoming Theorem 5.1 that we cite a result from [Ham17a] that harnesses the Brownian Gibbs theory from [Ham17a] in a substantial way.

## 5. SOME FURTHER KEY INPUTS

**5.1. The scaling principle.** Fundamental to the theory of the KPZ fixed point is the triple  $(1/3, 2/3, 1)$  of exponents that reflects the scaling laws for weight, polymer geometry, and polymer lifetime. The triple manifests itself in the context of our use of scaled coordinates. It is a simple consequence of the definition of the scaling transformation  $R_n$  and of the energy-weight relationship (3) that the following useful fact holds true.



*The scaling principle.* Let  $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}_{<}^2$  be a compatible triple. Any statement concerning the system of  $n$ -zigzags, including weight information, is equivalent to the corresponding statement concerning the system of  $nt_{1,2}$ -zigzags, provided that the following changes are made:

- the index  $n$  is replaced by  $nt_{1,2}$ ;
- any time is multiplied by  $t_{1,2}^{-1}$ ;
- any weight is multiplied by  $t_{1,2}^{1/3}$ ;
- and any horizontal distance is multiplied by  $t_{1,2}^{-2/3}$ .

A little more explanation of the scaling principle appears in [Ham17b, Section 2.3].

**5.2. Several disjoint near polymers with shared endpoints.** Our road map offering a route to Theorem 1.1 involves specifying and bounding the probability of an event  $\text{NearPoly}$ . This work has been carried out in [Ham17a], and we here recall the needed result.

Define

$$\text{NearPoly}_{n,k;(x,t_1)}^{(y,t_2)}(\eta) = \left\{ \text{Wgt}_{n,k;(x\bar{1},t_1)}^{(y\bar{1},t_2)} \geq k \cdot \text{Wgt}_{n;(x,t_1)}^{(y,t_2)} - t_{1,2}^{1/3} \eta \right\}.$$

The presence of the  $t_{1,2}^{1/3}$  factor that multiplies  $\eta$  is consistent with the scaling principle. It means that  $\eta$  has the role of a measure of discrepancy from the maximum weight attainable in principle, independently of the value of  $t_{1,2}$ . This  $\text{NearPoly}$  may also be expressed using a scaled forward ensemble: namely,

$$\text{NearPoly}_{n,k;(x,t_1)}^{(y,t_2)}(\eta) = \left\{ \sum_{i=1}^k \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(i, y) \geq k \cdot \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(1, y) - t_{1,2}^{1/3} \eta \right\}.$$

The next result is perhaps the most consequential of this article's inputs. It is [Ham17a, Theorem 1.12] up to a relabelling of one parameter.

**Theorem 5.1.** *There exist constants  $K_0 \geq 1$ ,  $K_1 > 0$ ,  $a_0 \in (0, 1)$  and  $\eta_0 > 0$  and a positive sequence  $\{\beta_k : k \in \mathbb{N}\}$  with  $\limsup \beta_k^{1/k} < \infty$  such that, for  $n, k \in \mathbb{N}$ ,  $x \in \mathbb{R}$  and  $\eta \in (0, (\eta_0)^{k^2})$  satisfying  $k \geq 2$ ,  $n \geq k \vee (K_0)^{k^2} (\log \eta^{-1})^{K_0}$  and  $|x| \leq a_0 n^{1/9}$ ,*

$$\eta^{k^2-1} \cdot e^{-K_1 k^5} \leq \mathbb{P} \left( \text{NearPoly}_{n,k;(0,0)}^{(x,1)}(\eta) \right) \leq \eta^{k^2-1} \cdot \exp \left\{ \beta_k (\log \eta^{-1})^{5/6} \right\}.$$

The actual consequence that will be used when we implement the road map is now stated.

**Corollary 5.2.** *There exist constants  $K_0 \geq 1$ ,  $K_1 > 0$ ,  $a_0 \in (0, 1)$  and  $\eta_0 > 0$  and a positive sequence  $\{\beta_k : k \in \mathbb{N}\}$  with  $\limsup \beta_k^{1/k} < \infty$  such that, for  $n, k \in \mathbb{N}$ ,  $(t_1, t_2) \in \mathbb{R}_{<}^2$ ,  $x, y \in \mathbb{R}$  and  $\eta \in (0, (\eta_0)^{k^2})$  satisfying  $k \geq 2$ ,  $t_{1,2} n \geq k \vee (K_0)^{k^2} (\log \eta^{-1})^{K_0}$  and  $t_{1,2}^{-2/3} |y - x| \leq a_0 n^{1/9}$ ,*

$$\eta^{k^2-1} \cdot e^{-K_1 k^5} \leq \mathbb{P} \left( \text{NearPoly}_{n,k;(x,t_1)}^{(y,t_2)}(\eta) \right) \leq \eta^{k^2-1} \cdot \exp \left\{ \beta_k (\log \eta^{-1})^{5/6} \right\}.$$

**Proof.** This follows from Theorem 5.1 by the scaling principle. □

**5.3. A useful tool: tail behaviour for polymer weight suprema and infima.** Short scale rewiring of zigzags is a central part of the plan in the road map. The next definition and result will be used in order to prove that such rewiring occurs at a manageable cost.

For  $x, y \in \mathbb{R}$ ,  $w_1, w_2 \geq 0$  and  $r > 0$ , let  $\text{PolyWgtReg}_{n;([x, x+w_1], t_1)}^{([y, y+w_2], t_2)}(r)$  denote the event that, for all  $(u, v) \in [0, w_1] \times [0, w_2]$ ,

$$\left| t_{1,2}^{-1/3} \text{Wgt}_{n;(x+u, t_1)}^{(y+v, t_2)} + 2^{-1/2} t_{1,2}^{-4/3} (y + v - x - u)^2 \right| \leq r.$$

When  $w_1 = 0$ , so that the interval  $[x, x + w_1]$  is a singleton, we write  $x$  in place of  $\{x\}$  or  $[x, x]$  in using this notation.

The next result, [Ham17b, Corollary 2.1], expresses the one-third power law for polymer weight that was discussed in Section 2.2.

**Corollary 5.3.** *Let  $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}_{>}^2$  be a compatible triple. Suppose further that the integer  $nt_{1,2}$  is even and that it is at least  $10^{29} \vee 2(c/3)^{-18}$ . Let  $x, y \in \mathbb{R}$ , and let  $a, b \in \mathbb{N}$ , be such that  $|x - y| t_{1,2}^{-2/3} + \max\{a, b\} - 1 \leq 3^{-1} 2^{-2/3} c (nt_{1,2})^{1/18}$ . Let  $r \in [33, 4(nt_{1,2})^{1/18}]$ . Then*

$$\mathbb{P}\left(\neg \text{PolyWgtReg}_{n;([x, x+at_{1,2}^{2/3}], t_1)}^{([y, y+bt_{1,2}^{2/3}], t_2)}(r)\right) \leq ab \cdot 400C \exp\{-c_1 2^{-8} r^{3/2}\}.$$

**5.4. Polymer weight regularity.** Recall from the road map that a power of one-half dictates the Hölder continuity of polymer weights as the endpoints are varied horizontally. A rigorous version of this assertion has been developed in [Ham17b], and we recall it now, using the notation  $Q : \mathbb{R} \rightarrow \mathbb{R}$ ,  $Q(u) = 2^{-1/2} u^2$ , to denote the parabola that dictates the global shape of the weight profile.

**Theorem 5.4.** *Let  $n \in 2\mathbb{N}$  and  $x, y \in \mathbb{R}$  satisfy  $n \geq 10^{29} c^{-18}$  and  $|x - y| \leq 2^{-5/3} 3^{-1} c n^{1/18}$ . Let  $\epsilon \in (0, 2^{-4}]$  and  $R \in [10^4, 10^3 n^{1/18}]$ . Then*

$$\mathbb{P}\left(\sup_{\substack{u_1, u_2 \in [x, x+\epsilon] \\ v_1, v_2 \in [y, y+\epsilon]}} \left| \text{Wgt}_{n;(u_2, 0)}^{(v_2, 1)} + Q(v_2 - u_2) - \text{Wgt}_{n;(u_1, 0)}^{(v_1, 1)} - Q(v_1 - u_1) \right| \geq \epsilon^{1/2} R\right)$$

is at most  $10032 C \exp\{-c_1 2^{-19} R^{3/2}\}$ .

It is also convenient to record a version of this result in which the parabolic curvature term is absent. Let  $I$  and  $J$  denote two closed intervals in the real line, each of length  $\epsilon$ . Define the *local weight regularity* event

$$\text{LocWgtReg}_{n;(I, 0)}^{(J, 1)}(\epsilon, r) = \left\{ \sup_{\substack{x_1, x_2 \in I \\ y_1, y_2 \in J}} \left| \text{Wgt}_{n;(x_2, 0)}^{(y_2, 1)} - \text{Wgt}_{n;(x_1, 0)}^{(y_1, 1)} \right| \leq r \epsilon^{1/2} \right\}.$$

Next is [Ham17b, Corollary 6.3]. The new hypothesis  $|x - y| \leq \epsilon^{-1/2}$  controls parabolic curvature and permits comparison of polymer weights without parabolic adjustment.

**Corollary 5.5.** *Let  $n \in 2\mathbb{N}$  and  $x, y \in \mathbb{R}$  satisfy  $n \geq 10^{29} c^{-18}$  and  $|x - y| \leq \epsilon^{-1/2} \wedge 2^{-5/3} 3^{-1} c n^{1/18}$ . Let  $\epsilon \in (0, 2^{-4}]$  and  $R \in [2 \cdot 10^4, 10^3 n^{1/18}]$ . Then*

$$\mathbb{P}\left(\neg \text{LocWgtReg}_{n;([x, x+\epsilon], 0)}^{([y, y+\epsilon], 1)}(\epsilon, R)\right) \leq 10032 C \exp\{-c_1 2^{-20-1/2} R^{3/2}\}. \quad (13)$$

**5.5. Polymer ordering.** An important challenge that lies ahead is bouquet construction, and a key difficulty here will be to ensure the disjointness, except at the shared endpoint, of the concerned zigzags. Some natural monotonicity properties of multi-polymers will be needed in the proof. Lemma 5.6 is our result in this regard. It will be proved in Appendix A.

Let  $(n, t_1, t_2)$  be a compatible triple. We introduce two ordering relations,  $\prec$  and  $\preceq$ , on the space of  $n$ -zigzags with lifetime  $[t_1, t_2]$ . To define the relations, let  $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$  and consider a zigzag  $Z_1$  from  $(x_1, t_1)$  to  $(y_1, t_2)$  and another  $Z_2$  from  $(x_2, t_1)$  to  $(y_2, t_2)$ .

*The  $\prec$  relation.* We declare that  $Z_1 \prec Z_2$  if  $x_1 \leq y_1$ ,  $x_2 \leq y_2$ , and the two polymers are horizontally separate.

*The  $\preceq$  relation.* Consider again  $Z_1$  and  $Z_2$ . We declare that  $Z_1 \preceq Z_2$  if ‘ $Z_2$  lies on or to the right of  $Z_1$ ’: formally, if  $Z_2$  is contained in the union of the closed horizontal planar line segments whose left endpoints lie in  $Z_1$ .

**Lemma 5.6.** *Let  $(n, t_1, t_2)$  be a compatible triple.*

- (1) *Let  $(x_1, x_2), (y_1, y_2) \in \mathbb{R}_{\leq}^2$  and  $k \in \mathbb{N}$ . Then the multi-polymer components satisfy*

$$\mathbb{P}\left(\rho_{n,k,i;(x_1\bar{1}),t_1}^{(y_1\bar{1},t_2)} \preceq \rho_{n,k,i;(x_2\bar{1}),t_1}^{(y_2\bar{1},t_2)} \quad \forall i \in \llbracket 1, k \rrbracket\right) = 1.$$

- (2) *Let  $Z_1, Z_2$  and  $Z_3$  be  $n$ -zigzags of lifetime  $[t_1, t_2]$  that verify  $Z_1 \prec Z_2$  and  $Z_2 \preceq Z_3$ . Then  $Z_1 \prec Z_3$ .*

A rather simple sandwiching fact about polymers will also be needed.

**Lemma 5.7.** *Let  $(n, t_1, t_2)$  be a compatible triple, and let  $(x_1, x_2), (y_1, y_2) \in \mathbb{R}_{\leq}^2$ . Suppose that there is a unique  $n$ -polymer from  $(x_i, t_1)$  to  $(y_i, t_2)$ , both when  $i = 1$  and  $i = 2$ . (This circumstance occurs almost surely, and the resulting polymers have been labelled  $\rho_{n;(x_1,t_1)}^{(y_1,t_2)}$  and  $\rho_{n;(x_2,t_1)}^{(y_2,t_2)}$ .) Now let  $\rho$  denote any  $n$ -polymer that begins in  $[x_1, x_2] \times \{t_1\}$  and ends in  $[y_1, y_2] \times \{t_2\}$ . Then  $\rho_{n;(x_1,t_1)}^{(y_1,t_2)} \preceq \rho \preceq \rho_{n;(x_2,t_1)}^{(y_2,t_2)}$ .*

**Proof.** When two  $n$ -zigzags  $Z_1$  and  $Z_2$  share their starting and ending heights, it is easy enough to define associated minimum and maximum zigzags  $Z_1 \wedge Z_2$  and  $Z_1 \vee Z_2$ . The minimum begins and ends at the leftmost of the starting and ending points of  $Z_1$  and  $Z_2$ ; during its lifetime, it follows one or other of the two zigzags, always keeping as far to the left as possible. It is a similar story for the maximum. A formal definition is made for staircases (but this hardly changes for zigzags) in the opening paragraphs of Appendix A. The polymer  $\rho$  begins and ends its journey to the right of  $\rho_{n;(x_1,t_1)}^{(y_1,t_2)}$ . If the condition  $\rho_{n;(x_1,t_1)}^{(y_1,t_2)} \preceq \rho$  is to be violated,  $\rho$  must pass a sojourn in its lifetime to the left of the other polymer. This would make the minimum  $\rho \wedge \rho_{n;(x_1,t_1)}^{(y_1,t_2)}$  distinct from  $\rho_{n;(x_1,t_1)}^{(y_1,t_2)}$ . A moment’s thought shows, however, that this minimum is itself a polymer between  $(x_1, t_1)$  and  $(y_1, t_2)$ . This is a contradiction to polymer uniqueness for these endpoints, which is known as a direct consequence of Lemma 3.1 with  $\ell = 1$ . Thus,  $\rho_{n;(x_1,t_1)}^{(y_1,t_2)} \preceq \rho$ . The second claimed ordering has a similar derivation.  $\square$

**5.6. Operations on polymers: splitting and concatenation.** Two very natural operations are now discussed.

A polymer may be split into two pieces. Let  $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}_{\leq}^2$  is a compatible triple, and let  $(x, y) \in \mathbb{R}^2$  satisfy  $y \geq x - 2^{-1}n^{1/3}t_{1,2}$ . Let  $t \in (t_1, t_2) \cap n^{-1}\mathbb{Z}$ . Suppose that the almost sure

event that  $\rho_{n;(x,t_1)}^{(y,t_2)}$  is well defined occurs. Select any element  $(z,t) \in \rho_{n;(x,t_1)}^{(y,t_2)}$ . The removal of  $(z,t)$  from  $\rho_{n;(x,t_1)}^{(y,t_2)}$  creates two connected components. Taking the closure of each of these amounts to adding the point  $(z,t)$  to each of them. The resulting sets are  $n$ -zigzags from  $(x,t_1)$  to  $(z,t)$ , and from  $(z,t)$  to  $(y,t_2)$ ; indeed, it is straightforward to see that these are the unique  $n$ -polymers given their endpoints. We use a concatenation notation  $\circ$  to represent this splitting. In summary,  $\rho_{n;(x,t_1)}^{(y,t_2)} = \rho_{n;(x,t_1)}^{(z,t)} \circ \rho_{n;(z,t)}^{(y,t_2)}$ . Naturally, we also have  $\text{Wgt}_{n;(x,t_1)}^{(y,t_2)} = \text{Wgt}_{n;(x,t_1)}^{(z,t)} + \text{Wgt}_{n;(z,t)}^{(y,t_2)}$ . Indeed, the concatenation operation may be applied to any two  $n$ -zigzags for which the ending point of the first equals the starting point of the second. Since the zigzags are subsets of  $\mathbb{R}^2$ , it is nothing other than the operation of union. The weight is additive under the operation.

**5.7. A closure property for a space of several disjoint polymers.** A rather technical point about stability under closure of systems of disjoint polymers is now addressed.

Let  $(n, t_1, t_2)$  be a compatible triple, let  $I$  and  $J$  be two closed real intervals, and let  $k \in \mathbb{N}$ . We now define two random subsets of  $I^k \times J^k \subset \mathbb{R}^{2k}$ . Set  $\text{DisjtIndex}_{n,k;(I,t_1)}^{(J,t_2)}$  to be the collection of vectors  $(x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k) \in I^k \times J^k$  such that there exists a collection of  $k$  pairwise disjoint  $n$ -polymers which consecutively move from  $(x_i, t_1)$  to  $(y_i, t_2)$ . This set is non-empty precisely when  $\text{MaxDisjtPoly}_{n;(I,t_1)}^{(J,t_2)} \geq k$ . We further define  $\text{HorSepIndex}_{n,k;(I,t_1)}^{(J,t_2)}$ , by replacing the condition of pairwise disjoint by that of pairwise horizontally separate in the above.

**Lemma 5.8.** *The closure of  $\text{DisjtIndex}_{n,k;(I,t_1)}^{(J,t_2)}$ , when viewed as a subset of  $\mathbb{R}^{2k}$ , is contained in  $\text{HorSepIndex}_{n,k;(I,t_1)}^{(J,t_2)}$ .*

**Proof.** Let  $(\bar{x}^i, \bar{y}^i)$ ,  $i \in \mathbb{N}$ , be elements of  $\text{DisjtIndex}_{n,k;(I,t_1)}^{(J,t_2)}$  that converge to  $(\bar{x}, \bar{y}) \in \mathbb{R}^k \times \mathbb{R}^k$ . Associated to index  $i$  is a system of disjoint polymers from  $(\bar{x}, t_1)$  to  $(\bar{y}, t_2)$ . It is a simple matter to extract a subsequence of these indices such that all the endpoint locations of the horizontal intervals in each of the  $k$  polymers converge pointwise. By considering the  $k$   $n$ -zigzags with these locations given by the limiting values, we may note that we are dealing with a collection of polymers, due to the continuity of the underlying Brownian motions  $B(\ell, \cdot)$ ,  $\ell \in \mathbb{Z}$ , in Brownian LPP. Moreover, these  $k$   $n$ -zigzags are pairwise horizontally separate: indeed, we are equipping the space of  $n$ -zigzags of lifetime  $[t_1, t_2]$  with the Hausdorff topology, so that it is enough to note that the property of a  $k$ -tuple of zigzags having a pair that is not horizontally separate is an open condition for the  $k$ -wise product topology on such tuples of zigzags.  $\square$

*Remark.* Theorems 1.1 and 1.4 make assertions about the random variable  $\text{MaxDisjtPoly}_{n;(I,t_1)}^{(J,t_2)}$ . The value of this random variable cannot decrease if in its definition in Subsection 1.7.1 we replace the italicized *pairwise disjoint* by *pairwise horizontally separate*. If we were to redefine the random variable with this change, then the new versions of the results would imply the old ones. In fact, all proofs in this article are valid for the altered definition, and we take the liberty of adopting it henceforth.

**5.8. Organization of the remainder of the paper.** Our principal results are the disjoint polymer estimates Theorems 1.1 and 1.4, and the polymer fluctuation bound Theorem 1.5. A road map has been outlined for the proof of Theorem 1.1, and, in order to communicate this conceptually central idea with suitable emphasis, the next section, Section 6, is devoted to rigorously formulating it. We mention here that, in fact, this proof will invoke Theorem 1.5: this aspect of the road

map has yet to be explained, but the resolution of the third challenge in the road map will involve polymer fluctuation bounds. Theorem 1.4 is a rather straightforward consequence of Theorem 1.1, and its proof appears at the end of Section 6.

The proof of Theorem 1.5 has a rather different flavour. We present the proof next, in Section 7. An outline of the proof is offered at the beginning of the section.

The paper ends with three appendices. The first two of these present proofs for two remaining tools that we have described.

The multi-polymer ordering Lemma 5.6 is proved in Appendix A. In our presentation of results, and in the overall perspective of this article, we have been eager to focus attention on scaled coordinates. But of course Lemma 5.6 is really simply the scaled counterpart to a result that concerns multi-geodesics. This Lemma A.1 appears late in the paper, in accordance with our wish to encourage the reader to focus on the scaled coordinate interpretation of the subject. It is, however, a basic result that may have an independent interest.

Proposition 4.2, which asserts that our normalized ensembles (8) and (10) are regular, has in essence been proved already in [Ham17a], on the basis of a fundamental result of O’Connell and Yor describing a counterpart ensemble of curves in unscaled coordinates. Appendix B explains this connection and provides the proof of this proposition, which involves a mundane argument to reconcile a slight difference in ensemble notation between [Ham17a] and the present work.

We close this section by discussing two conventions that will govern the presentation of upcoming proofs. The second of these comments also explains what is contained in Appendix C.

5.8.1. *Boldface notation for quoted results.* During the upcoming proofs, we will naturally be making use of the various tools that we have recalled: the Reg conditions and the regular ensemble properties Proposition 4.3, and results presented earlier in this section. The statements of such results involve several parameters, in several cases including  $(n, t_1, t_2)$ , spatial locations  $x$  and  $y$ , and positive real parameters such as  $r$ . We will employ a device that will permit us to disregard notational conflict between the use of such parameters in the contexts of the ongoing proof in question and the statements of quoted results. When specifying the parameter settings of a particular application, we will allude to the parameters of the result being applied in boldface, and thus permit the reuse of the concerned symbols.

5.8.2. *The role of hypotheses invoked during proofs.* When we quote results in order to apply them, we will take care, in addition to specifying the parameters according to the just described convention, to indicate explicitly what the conditions on these parameters are that will permit the quoted result in question to be applied. Of course, it is necessary that the hypotheses of the result being proved imply all such conditions. The task of verifying that the hypotheses of a given result are adequate for the purpose of obtaining all conditions needed to invoke the various results used during its proof may be called the calculational derivation of the result in question. This derivation is necessary, but also in some cases lengthy and unenlightening: a succession of trivial steps. We have chosen to separate the calculational derivations of most of our results from their proofs. These derivations may be found in Appendix C at the end of the paper. This appendix also contains the proofs of four lemmas that are invoked in the proof of Theorem 1.5. These proofs are exclusively calculational matters.

## 6. RARIETY OF MANY DISJOINT POLYMERS: PROOFS OF THEOREMS 1.1 AND 1.4

Theorems 1.1 and 1.4 are slightly simplified versions of the next two results, in which hypotheses on parameters are stated in a more explicit form. The results make reference to  $\eta_0$ ,  $K_0$  and  $a_0$ . These are positive constants that are fixed by Theorem 5.1. The sequence of positive constants  $\{\beta_i : i \in \mathbb{N}\}$ , which verifies  $\limsup_{i \in \mathbb{N}} \beta_i^{1/i} < \infty$ , is also supplied by this theorem.

**Theorem 6.1.** *Let  $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}_{<}^2$  be a compatible triple with  $nt_{1,2}$  even. Let  $k \in \mathbb{N}$ ,  $\epsilon > 0$  and  $x, y \in \mathbb{R}$  satisfy the conditions that  $k \geq 2$ ,*

$$\epsilon \leq \min \left\{ (\eta_0)^{4k^2}, 10^{-616} c_k^{22} k^{-115}, \exp \{ -C^{3/8} \} \right\}, \quad (14)$$

$$nt_{1,2} \geq \max \left\{ 2(K_0)^{k^2} (\log \epsilon^{-1})^{K_0}, 10^{584} c_k^{-48} k^{240} c^{-36} \epsilon^{-222} \max \{ 1, |x-y|^{36} t_{1,2}^{-24} \}, a_0^{-9} |y-x|^9 t_{1,2}^{-6} \right\}, \quad (15)$$

as well as  $|y-x|t_{1,2}^{-2/3} \leq \epsilon^{-1/2} (\log \epsilon^{-1})^{-2/3} \cdot 10^{-8} c_k^{2/3} k^{-10/3}$ . Then

$$\begin{aligned} & \mathbb{P} \left( \text{MaxDisjtPoly}_{n; ([x-t_{1,2}^{2/3}\epsilon, x+t_{1,2}^{2/3}\epsilon], t_1)}^{([y-t_{1,2}^{2/3}\epsilon, y+t_{1,2}^{2/3}\epsilon], t_2)} \geq k \right) \\ & \leq \epsilon^{(k^2-1)/2} \cdot 10^{32k^2} k^{15k^2} c_k^{-3k^2} C_k (\log \epsilon^{-1})^{4k^2} \exp \{ \beta_k (\log \epsilon^{-1})^{5/6} \}. \end{aligned}$$

**Theorem 6.2.** *There exists a positive constant  $m_0$ , and a sequence of positive constants  $\{H_i : i \in \mathbb{N}\}$  for which  $\sup_{i \in \mathbb{N}} H_i \exp \{ -2(\log i)^{11/12} \}$  is finite, such that the following holds. Let  $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}_{<}^2$  be a compatible triple with  $nt_{1,2}$  even. Let  $x, y \in \mathbb{R}$ ,  $a, b \in \mathbb{N}$  and  $m \in \mathbb{N}$ . Write  $h = a \vee b$ . Suppose that*

$$m \geq m_0 \vee (|x-y|t_{1,2}^{-2/3} + 2h)^3$$

and

$$nt_{1,2} \geq \max \left\{ 2(K_0)^{(12)^{-2}(\log \log m)^2} (\log m)^{K_0}, a_0^{-9} (|y-x|t_{1,2}^{-2/3} + 2h)^9, \right. \\ \left. 10^{325} c^{-36} m^{465} \max \{ 1, (|y-x|t_{1,2}^{-2/3} + 2h)^{36} \} \right\}. \quad (16)$$

Then

$$\mathbb{P} \left( \text{MaxDisjtPoly}_{n; ([x, x+at_{1,2}^{2/3}], t_1)}^{([y, y+bt_{1,2}^{2/3}], t_2)} \geq m \right) \leq m^{-(145)^{-1}(\log \beta)^{-2}(0 \vee \log \log m)^2} \cdot h^{(\log \beta)^{-2}(0 \vee \log \log m)^2/288+3/2} H_m.$$

Here,  $\beta$  is specified to be  $e \vee \limsup_{i \in \mathbb{N}} \beta_i^{1/i}$ .

**Proof of Theorems 1.1 and 1.4.** These results are direct consequences of their just stated counterparts.  $\square$

In this section, then, our job is to prove Theorems 6.1 and 6.2. Our principal task is to prove Theorem 6.1. This we do first, by implementing the road map rigorously. Theorem 6.2 then emerges, at the end of Section 6, as a fairly straightforward consequence: after all, many disjoint polymers running between unit intervals entail several disjoint polymers running between short intervals.

**6.1. Bouquet construction.** As we noted at the end of Section 2.2, there are three main challenges involved in implementing the road map, beyond the **NearPoly** probability estimate Corollary 5.2. The technical input concerning the first of the three challenges, polymer weight similarity, has been cited already, as Corollary 5.3.

We now address the second challenge: the construction of forward and backward bouquets of suitable weight, dictated by the exponent of one-third.

In Section 3.3, we introduced the maximum weight  $\text{Wgt}_{n,k;(\bar{x},t_1)}^{(\bar{y},t_2)}$  of a system of  $k$  pairwise horizontally separate zigzags that make the journey from  $(x_i, t_1)$  to  $(y_i, t_2)$  for  $i \in \llbracket 1, k \rrbracket$ . In forward bouquet construction, we specialise to  $\bar{x} = x\bar{\mathbf{1}}$  for some  $x \in \mathbb{R}$ ; in backward bouquet construction, we instead choose  $\bar{y} = y\bar{\mathbf{1}}$  for a given  $y \in \mathbb{R}$ .

The aim of bouquet construction will be achieved in the guise of Corollary 6.4, which shows that the cumulative weight of the maximum weight bouquet, when measured in the suitable one-third power units, is tight, uniformly in the scaling parameter  $n$ . This result is a direct consequence of Proposition 6.3, which asserts an upper and a lower bound on the maximum bouquet weight. The first of these bounds is easy, because polymer weights offer a control from above on bouquet weights. The lower bound is more delicate: horizontal separateness of zigzags must be ensured without damaging weight significantly. Our technique will be a diagonal argument, with the ordering Lemma 5.6 playing an important role in establishing the necessary form of disjointness.

**Proposition 6.3.** *Let  $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}_{<}^2$  be a compatible triple. Let  $k \in \mathbb{N}$ ,  $\bar{u} \in \mathbb{R}_{\leq}^k$  and  $r > 0$ .*

(1) *Suppose that  $nt_{1,2}$  is at least  $t_{1,2}^{-6}|u_i - x|^9 c^{-9}$  for  $i \in \llbracket 1, k \rrbracket$ , and that  $r \geq k$ . Then*

$$\mathbb{P}\left(t_{1,2}^{-1/3} \cdot \text{Wgt}_{n,k;(x\bar{\mathbf{1}},t_1)}^{(\bar{u},t_2)} \geq -2^{-1/2} \sum_{i=1}^k t_{1,2}^{-4/3} (u_i - x)^2 + r\right) \leq k \cdot C \exp\left\{-ck^{-3/2}r^{3/2}\right\}.$$

(2) *Suppose that  $nt_{1,2}$  is at least  $\max\{k, 3^{18}c^{-18}, 6^{36}\}$  and is also bounded below for each  $i \in \llbracket 1, k \rrbracket$  by  $2^{18}c^{-18}t_{1,2}^{-12}|u_i - x|^{18}$ . Suppose further that  $r \in [4k^2, 9k^2(nt_{1,2})^{1/3}]$ . Then*

$$\mathbb{P}\left(t_{1,2}^{-1/3} \cdot \text{Wgt}_{n,k;(x\bar{\mathbf{1}},t_1)}^{(\bar{u},t_2)} \leq -2^{-1/2} \sum_{i=1}^k t_{1,2}^{-4/3} (u_i - x)^2 - r\right) \leq 3k^2 C_k \exp\left\{-2^{-3}k^{-3}c_k r^{3/2}\right\}.$$

As we prepare to state our bouquet construction tool, Corollary 6.4, we mention a further aspect of this construction. It would be natural to suppose, on the basis of the ideas presented in Section 2.2, that, on the event  $\text{MaxDisjtPoly}_{(n;[x-\epsilon, x+\epsilon], 0)}^{([y-\epsilon, y+\epsilon], 1)} \geq k$ , if the  $k$  disjoint polymers that move between  $[x - \epsilon, x + \epsilon] \times \{0\}$  and  $[y - \epsilon, y + \epsilon] \times \{1\}$  have endpoints  $(u_i, 0)$  and  $(v_i, 1)$ , for  $i \in \llbracket 1, k \rrbracket$ , then the forward and backward bouquets will be chosen to be the multi-polymers whose weights are  $\text{Wgt}_{n,k;(x\bar{\mathbf{1}}, -\epsilon^{3/2})}^{(\bar{u}, 0)}$  and  $\text{Wgt}_{n,k;(y\bar{\mathbf{1}}, 1+\epsilon^{3/2})}^{(\bar{v}, 1)}$ . A microscopic detail must be addressed, however. The first of these multi-polymers is measurable with respect to the randomness in the region  $\mathbb{R} \times (-\infty, 0]$  and the second to the randomness in  $\mathbb{R} \times [1, \infty)$ . Our purpose in the upcoming surgery will be better served were a choice of the bouquet pair to be made for which this assertion is valid with the two *open* regions  $\mathbb{R} \times (-\infty, 0)$  and  $\mathbb{R} \times (1, \infty)$  instead. What is needed is a modified definition in which no use is made of the randomness in the horizontal line with the height of the fixed endpoint: height zero or height one for the forward and backward bouquets.

In the modification, the weight function  $\text{Wgt}$  will be replaced by a proper weight function  $\text{PropWgt}$ , in a sense we now specify. Recall that if  $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}_{<}^2$  is a compatible triple and  $Z$  is an  $n$ -zigzag with starting point  $(x, t_1)$  and ending point  $(y, t_2)$ , then  $Z$  begins with a planar line segment abutting  $(x, t_1)$ , which is either horizontal or sloping, and likewise ends with such a segment abutting  $(y, t_2)$ . We will call  $Z$  *backward proper* if the line segment that abuts  $(x, t_1)$  is sloping, and *forward proper* if the line segments that abuts  $(y, t_2)$  is sloping.

We now define the *proper weight*  $\text{PropWgt}_{n,k;(x\bar{1},t_1)}^{(\bar{u},t_2)}$  to equal the maximum weight associated to  $k$  horizontally separate *forward proper* zigzags moving consecutively between  $(x, t_1)$  to  $(u_i, t_2)$  for  $i \in \llbracket 1, k \rrbracket$ . The  $k$ -tuple maximizer will be denoted by  $\rho_{n,k;(x\bar{1},t_1)}^{\text{prop};(\bar{u},t_2)}$ . (Almost sure uniqueness of the maximizer follows in essence from Lemma 3.1 but is not needed for our purpose.) The proper weight definition modifies  $\text{Wgt}_{n,k;(x\bar{1},t_1)}^{(\bar{u},t_2)}$  by introducing the insistence that the concerned zigzags be forward proper. Similarly, we define  $\text{PropWgt}_{n,k;(\bar{v},t_1)}^{(y\bar{1},t_2)}$  to equal the maximum weight associated to  $k$  horizontally separate *backward proper* zigzags moving consecutively between  $(v_i, t_1)$  to  $(y, t_2)$  for  $i \in \llbracket 1, k \rrbracket$ . The  $k$ -tuple maximizer is  $\rho_{n,k;(\bar{v},t_1)}^{\text{prop};(y\bar{1},t_2)}$ .

Define the *forward bouquet regularity event*  $\text{ForBouqReg}_{n,k;(x,t_1)}^{(\bar{u},t_2)}(r)$  to equal

$$\left\{ \left| (t_{1,2} - n^{-1})^{-1/3} \cdot \text{PropWgt}_{n,k;(x\bar{1},t_1)}^{(\bar{u},t_2)} + 2^{-1/2} \sum_{i=1}^k (t_{1,2} - n^{-1})^{-4/3} (u_i + 2^{-1}n^{-2/3} - x)^2 \right| \leq r \right\}$$

and the *backward bouquet regularity event*  $\text{BackBouqReg}_{n,k;(\bar{v},t_1)}^{(y,t_2)}(r)$  to equal

$$\left\{ \left| (t_{1,2} - n^{-1})^{-1/3} \cdot \text{PropWgt}_{n,k;(\bar{v},t_1)}^{(y\bar{1},t_2)} + 2^{-1/2} \sum_{i=1}^k (t_{1,2} - n^{-1})^{-4/3} (v_i - 2^{-1}n^{-2/3} - y)^2 \right| \leq r \right\}.$$

**Corollary 6.4.** *Let  $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}_{<}^2$  be a compatible triple. Let  $k \in \mathbb{N}$  satisfy  $k \leq n$ , and let  $x, y \in \mathbb{R}$ ,  $\bar{u}, \bar{v} \in \mathbb{R}_{\leq}^k$  and  $r > 0$ . Suppose that*

$$nt_{1,2} - 1 \geq \max \left\{ k, 3^{18}c^{-18}, 6^{36}, (t_{1,2} - n^{-1})^{-6} (|u_i + 2^{-1}n^{-2/3} - x| \vee |v_i - 2^{-1}n^{-2/3} - y|)^9 c^{-9}, \right. \\ \left. 2^{18}c^{-18}(t_{1,2} - n^{-1})^{-12} (|u_i + 2^{-1}n^{-2/3} - x| \vee |v_i - 2^{-1}n^{-2/3} - y|)^{18} \right\}$$

where here  $i$  varies over  $\llbracket 1, k \rrbracket$ . Suppose also that  $r \in [4k^2, 9k^2(t_{1,2} - n^{-1})^{1/3}n^{1/3}]$ . Then

$$\mathbb{P} \left( \neg \text{ForBouqReg}_{n,k;(x,t_1)}^{(\bar{u},t_2)} \right) \vee \mathbb{P} \left( \neg \text{BackBouqReg}_{n,k;(\bar{v},t_1)}^{(y,t_2)} \right) \leq 4k^2 C_k \exp \left\{ -2^{-3}k^{-3}c_k r^{3/2} \right\}.$$

The form of the corollary is a little cluttered, not least because the microscopic detail we have introduced forces the bouquets to have lifetime  $t_{1,2} - n^{-1}$  rather than  $t_{1,2}$ . Setting  $n = \infty$  may serve to focus attention on what is essential here, even if it obscures this detail.

**Proof of Corollary 6.4.** We begin by claiming that

$$\text{PropWgt}_{n,k;(x\bar{1},t_1)}^{(\bar{u},t_2)} = \text{Wgt}_{n,k;(x\bar{1},t_1)}^{(\bar{u}+2^{-1}n^{-2/3}\mathbf{1}, t_2-n^{-1})}. \quad (17)$$

To understand why this is true, we consider the multi-polymer that realizes the left-hand side and apply the inverse scaling map  $R_n^{-1}$ , so that it is depicted in unscaled coordinates. The result is a  $k$ -tuple of staircases, each element of which ends in a vertical unit displacement. Omitting these displacements, we obtain a  $k$ -tuple of staircases which is a multi-geodesic: the sum of the energies



of the staircases is maximal given the set of endpoints. Applying the scaling map  $R_n$  to reconsider this multi-geodesic in scaled coordinates, we see that its weight is the above right-hand side. Similar considerations yield the formula

$$\text{PropWgt}_{n,k;(\bar{v},t_1)}^{(y\bar{1},t_2)} = \text{Wgt}_{n,k;(\bar{v}-2^{-1}n^{-2/3}\bar{1},t_1+n^{-1})}^{(y\bar{1},t_2)}. \quad (18)$$

The bound on the first probability in the corollary follows from (17) and an application of the two parts of Proposition 6.3 with parameter settings  $\mathbf{t}_1 = t_1 + n^{-1}$ ,  $\mathbf{t}_2 = t_2$ ,  $\mathbf{x} = x$  and  $\bar{\mathbf{u}} = \bar{u} + 2^{-1}n^{-2/3}\bar{1}$ . The bounds  $C_k \geq C$  and  $c_k \leq c$  are also used. The estimate on the second probability is reduced to this proposition via (18) and by noting the half-circle rotational symmetry of Brownian LPP that arises by reindexing its constituent curves  $B(k, z)$  in the form  $B(-k, -z)$ .  $\square$

**Proof of Proposition 6.3.** The proposition's first assertion is the simpler of the two to derive. To verify it, note that, since  $\rho_{n,k,i;(x\bar{1},t_1)}^{(\bar{u},t_2)}$  has starting and ending points  $(x, t_1)$  and  $(u_i, t_2)$ , the weight of this  $n$ -zigzag is at most  $\text{Wgt}_{n;(x,t_1)}^{(u_i,t_2)}$ . Summing over  $i \in \llbracket 1, k \rrbracket$ , we find that

$$\text{Wgt}_{n,k;(x\bar{1},t_1)}^{(\bar{u},t_2)} \leq \sum_{i=1}^k \text{Wgt}_{n;(x,t_1)}^{(u_i,t_2)}. \quad (19)$$

Set  $w_i = t_{1,2}^{-2/3}(u_i - x)$  and note that

$$\text{Wgt}_{n;(x,t_1)}^{(u_i,t_2)} = \mathcal{L}_{n;(x,t_1)}^{\uparrow,t_2}(1, u_i) = t_{1,2}^{1/3} \cdot t_{1,2}^{-1/3} \mathcal{L}_{n;(x,t_1)}^{\uparrow,t_2}(1, x + t_{1,2}^{2/3}w_i) = t_{1,2}^{1/3} \text{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow,t_2}(1, w_i).$$

Applying *one-point upper tail* Reg(3) with parameter settings  $\mathbf{z} = w_i$  and  $\mathbf{s} = rk^{-1}$  to the  $(nt_{1,2}+1)$ -curve ensemble  $\text{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow,t_2}$ , we see that the event

$$\left\{ t_{1,2}^{-1/3} \cdot \text{Wgt}_{n;(x,t_1)}^{(u_i,t_2)} \geq -2^{-1/2}t_{1,2}^{-4/3}(u_i - x)^2 + r/k \right\} = \left\{ \text{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow,t_2}(1, w_i) + 2^{-1/2}w_i^2 \geq r/k \right\}$$

has  $\mathbb{P}$ -probability at most  $C \exp\{-c(r/k)^{3/2}\}$ , provided that the hypotheses

$$t_{1,2}^{-2/3}|u_i - x| \leq c(nt_{1,2})^{1/9} \text{ and } r \geq k$$

are satisfied. Thus, (19) implies Proposition 6.3(1).

To derive the lower-tail bound Proposition 6.3(2), we will find  $k$  pairwise horizontally separate  $n$ -zigzags of suitable weight, each of which begins at  $(x, t_1)$  and which end successively at  $(u_i, t_2)$ ,  $i \in \llbracket 1, k \rrbracket$ .

Our technique for finding the zigzags is a diagonal argument: see Figure 5. For each  $i \in \llbracket 1, k \rrbracket$ , consider the  $k$ -tuple multi-polymer watermelon

$$\left( \rho_{n,k,i;(x\bar{1},t_1)}^{(u_i\bar{1},t_2)} : i \in \llbracket 1, k \rrbracket \right), \quad (20)$$

where here recall that  $z\bar{1} \in \mathbb{R}^k$  denotes the vector each of whose components equals  $z \in \mathbb{R}$ . Our multi-polymer notation used here is specified in Section 3.4. The object in question is the multi-polymer whose  $k$  components are tethered at both endpoints, to  $(x, t_1)$  and  $(u_i, t_2)$ . As we explained when the notation was introduced, it is Lemma 3.1 that ensures the almost sure existence and uniqueness of the maximizer (20), which for brevity we will denote by  $(\rho_{i,1}, \dots, \rho_{i,k})$ .

We now consider the diagonal vector  $(\rho_{i,i} : i \in \llbracket 1, k \rrbracket)$ . We first verify that this vector's components are pairwise horizontally separate zigzags that consecutively run from  $(x, t_1)$  to  $(u_i, t_2)$ . The

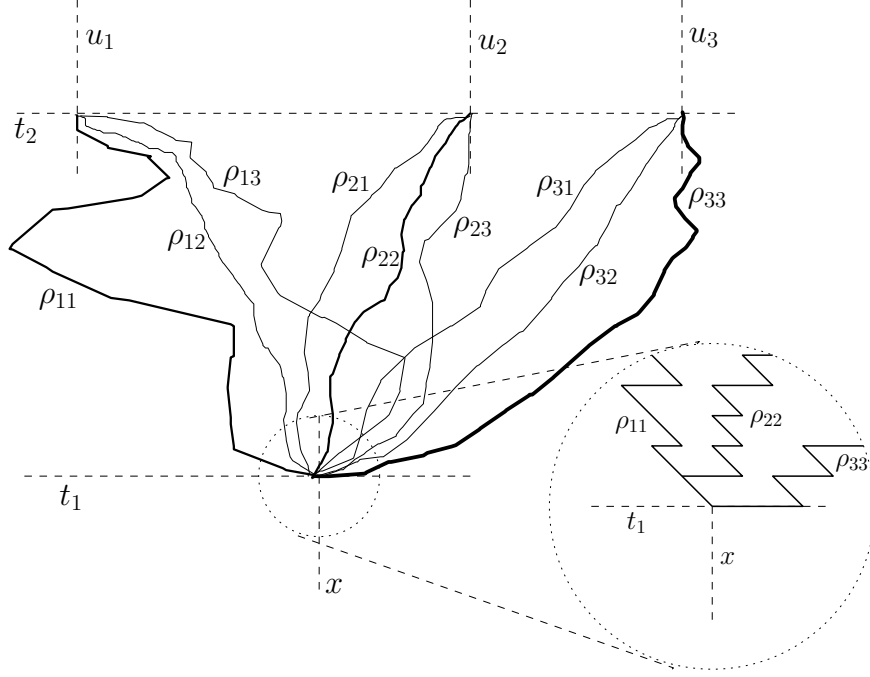


FIGURE 5. The diagonal vector  $(\rho_{1,1}, \dots, \rho_{k,k})$  is illustrated in bold for an example with  $k = 3$ . Under the microscope, we see how it is that these three zigzags may remain horizontally separate despite their sharing the birthplace  $(x, t_1)$ .

endpoint locations are not in doubt, so it is our task to verify the separateness condition between consecutively indexed zigzags. In the language of Section 5.5, we want to check that  $\rho_{i,i} \prec \rho_{i+1,i+1}$  for  $i \in \llbracket 1, k-1 \rrbracket$ . We have that  $\rho_{i,i} \prec \rho_{i+1,i+1}$ , because these players are components of a given multipolymer, and we also have  $\rho_{i,i+1} \preceq \rho_{i+1,i+1}$  almost surely, by Lemma 5.6(1). Thus, Lemma 5.6(2) implies the desired fact.

The confirmed property of the diagonal vector implies that

$$\text{Wgt}_{n,k;(x\bar{1},t_1)}^{(\bar{u},t_2)} \geq \sum_{i=1}^k \text{Wgt}(\rho_{i,i}), \quad (21)$$

where note that  $\text{Wgt}(\rho_{i,i})$  is equal to  $\text{Wgt}_{n,k,i;(x\bar{1},t_1)}^{(u_i\bar{1},t_2)}$ .

We now present a lemma regarding the values of these weights.

**Lemma 6.5.** *Let  $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}_{<}^2$  be a compatible triple. For  $(x, y) \in \mathbb{R}^2$ ,  $y \geq x - 2^{-1}n^{1/3}t_{1,2}$ ,  $k \in \mathbb{N}$  and  $i \in \llbracket 1, k \rrbracket$ ,*

$$\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(1, y) - \sum_{j=2}^k \left( \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(1, y) - \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(j, y) \right) \leq \text{Wgt}_{n,k,i;(x\bar{1},t_1)}^{(y\bar{1},t_2)} \leq \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(1, y). \quad (22)$$

**Proof.** First note that the condition  $y \geq x - 2^{-1}n^{1/3}t_{1,2}$  is assumed simply in order to assure that the ensemble  $\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}$  is well defined at location  $y$ . Note that  $\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(1, y) = \text{Wgt}_{n;(x,t_1)}^{(y,t_2)}$ .

Since  $\rho_{n,k,i;(x\bar{1},t_1)}^{(u_i\bar{1},t_2)}$  is an  $n$ -zigzag from  $(x, t_1)$  to  $(u_i, t_2)$  for  $i \in \llbracket 1, k \rrbracket$ , we have that  $\text{Wgt}_{n,k,i;(x\bar{1},t_1)}^{(y\bar{1},t_2)} \leq \text{Wgt}_{n;(x,t_1)}^{(y,t_2)}$  for such  $i$ . Thus, we obtain the second bound in (22). As for the first, note that, by (7),

$$\sum_{j=1}^k \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(j, y) = \sum_{j=1}^k \text{Wgt}_{n,k,j;(x\bar{1},t_1)}^{(y\bar{1},t_2)},$$

an equality that we may rewrite

$$k \cdot \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(1, y) + \sum_{j=2}^k \left( \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(j, y) - \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(1, y) \right) = \sum_{j=1}^k \text{Wgt}_{n,k,j;(x\bar{1},t_1)}^{(y\bar{1},t_2)}.$$

The right-hand summand  $\text{Wgt}_{n,k,j;(x\bar{1},t_1)}^{(y\bar{1},t_2)}$  is at most  $\text{Wgt}_{n;(x,t_1)}^{(y,t_2)} = \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(1, y)$  for any  $j \in \llbracket 1, k \rrbracket$ . Let  $i \in \llbracket 1, k \rrbracket$ . The lower bound on  $\text{Wgt}_{n,k,i;(x\bar{1},t_1)}^{(y\bar{1},t_2)}$  stated in the lemma is given by the value this term would adopt on the right-hand side of the above equality were each  $j$ -indexed summand on the right set equal to the upper bound value  $\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(1, y)$  for  $j \in \llbracket 1, k \rrbracket$ ,  $j \neq i$ .  $\square$

In light of (21), Proposition 6.3(2) will follow once we show that, under the result's hypotheses,

$$\mathbb{P} \left( t_{1,2}^{-1/3} \sum_{i=1}^k \text{Wgt}_{n,k,i;(x\bar{1},t_1)}^{(u_i\bar{1},t_2)} \leq -2^{-1/2} \sum_{i=1}^k t_{1,2}^{-4/3} (u_i - x)^2 - r \right) \leq 3k^2 C_k \exp \left\{ -2^{-3} k^{-3} c_k r^{3/2} \right\}. \quad (23)$$

To begin showing this, note that the occurrence of the left-hand event entails that the bound

$$t_{1,2}^{-1/3} \text{Wgt}_{n,k,i;(x\bar{1},t_1)}^{(u_i\bar{1},t_2)} \leq -2^{-1/2} t_{1,2}^{-4/3} (u_i - x)^2 - r/k$$

is satisfied for at least one index  $i \in \llbracket 1, k \rrbracket$ . In view of Lemma 6.5's left-hand bound applied with  $\mathbf{y} = u_i$  (for given  $i \in \llbracket 1, k \rrbracket$ ), the last inequality implies that either

$$t_{1,2}^{-1/3} \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(1, u_i) \leq -2^{-1/2} t_{1,2}^{-4/3} (u_i - x)^2 - \frac{r}{2k} \quad (24)$$

or at least one among the inequalities

$$t_{1,2}^{-1/3} \left( \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(1, u_i) - \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(j, u_i) \right) \geq \frac{r}{2k(k-1)}, \quad (25)$$

indexed by  $j \in \llbracket 2, k \rrbracket$ , is satisfied.

Condition (24) asserts that the highest curve of the normalized forward line ensemble, rooted at  $(x, t_1)$  and of duration  $t_{1,2}$ ,

$$\text{Nr} \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(1, w_i) = t_{1,2}^{-1/3} \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(1, x + t_{1,2}^{2/3} w_i),$$

evaluated at  $w_i = t_{1,2}^{-2/3} (u_i - x)$ , is at most  $-2^{-1/2} t_{1,2}^{-4/3} (u_i - x)^2 - 2^{-1} r k^{-1}$ . Applying one-point lower tail  $\text{Reg}(2)$  to the  $(nt_{1,2} + 1)$ -curve ensemble  $\text{Nr} \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}$ , with parameter settings  $\mathbf{z} = w_i$  and  $\mathbf{s} = 2^{-1} r k^{-1}$ , we find that this eventuality has probability at most  $C \exp \left\{ -c 2^{-3/2} k^{-3/2} r^{3/2} \right\}$ . This application of  $\text{Reg}(2)$  may be carried out if the hypotheses

$$t_{1,2}^{-2/3} |u_i - x| \leq c (nt_{1,2})^{1/9} \text{ and } r \in [2k, 2k(nt_{1,2})^{1/3}]$$

are satisfied.

When the bound (25) is rewritten in normalized coordinates, it asserts that

$$\begin{aligned} & \left( \text{Nr} \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(1, w_i) + 2^{-1/2} w_i^2 \right) - \left( \text{Nr} \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(j, w_i) + 2^{-1/2} w_i^2 \right) \\ &= t_{1,2}^{-1/3} \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(1, x + t_{1,2}^{2/3} w_i) - t_{1,2}^{-1/3} \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(j, x + t_{1,2}^{2/3} w_i), \end{aligned}$$

when evaluated at  $w_i = t_{1,2}^{-2/3}(u_i - x)$ , is at least  $\frac{r}{2k(k-1)}$ . The bound forces at least one of two cases: in the above difference of two terms, either the first term is at least  $\frac{r}{4k(k-1)}$  or the second is at most  $-\frac{r}{4k(k-1)}$ .

The first case is handled by applying one-point upper tail Reg(3) to  $\text{Nr} \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}$ , taking  $\mathbf{z} = w_i$  and  $\mathbf{s} = \frac{r}{4k(k-1)}$ , with an upper bound of

$$C \exp \left\{ -c \left( \frac{r}{4k(k-1)} \right)^{3/2} \right\}$$

being found on the probability of the event in question. This use of Reg(3) may be made when

$$t_{1,2}^{-2/3} |u_i - x| \leq c(nt_{1,2})^{1/9} \text{ and } r \geq 4k(k-1).$$

The second case is treated by applying pointwise lower tail Proposition 4.3(1) to the  $(nt_{1,2} + 1)$ -curve ensemble  $\mathcal{L}_n = \text{Nr} \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}$ , with parameter settings  $\mathbf{k} = j$ ,  $\mathbf{z} = w_i$  and  $\mathbf{s} = \frac{r}{4k(k-1)}$ . The event in question is thus found to have  $\mathbb{P}$ -probability at most

$$C_j \exp \left\{ -c_j \left( \frac{r}{4k(k-1)} \right)^{3/2} \right\}.$$

These applications of Proposition 4.3(1) are made for each  $j \in \llbracket 2, k \rrbracket$ , and the  $k-1$  applications may be made provided that

$$nt_{1,2} \geq k \vee (c/3)^{-18} \vee 6^{36}, \quad t_{1,2}^{-2/3} |u_i - x| \leq 2^{-1} c(nt_{1,2})^{1/18} \text{ and } r \leq 8k(k-1)(nt_{1,2})^{1/18}.$$

We find then that the probability that (25) is satisfied for a given pair  $(i, j) \in \llbracket 1, k \rrbracket \times \llbracket 2, k \rrbracket$  is at most

$$C \exp \left\{ -c \left( \frac{r}{4k(k-1)} \right)^{3/2} \right\} + C_j \exp \left\{ -c_j \left( \frac{r}{4k(k-1)} \right)^{3/2} \right\}.$$

Gathering together these inferences by means of a union bound over this set of index pairs, we find that the left-hand side of (23) is at most

$$Ck \exp \left\{ -c2^{-3/2} k^{-3/2} r^{3/2} \right\} + Ck^2 \exp \left\{ -c \left( \frac{r}{4k(k-1)} \right)^{3/2} \right\} + k \sum_{j=2}^k C_j \exp \left\{ -c_j \left( \frac{r}{4k(k-1)} \right)^{3/2} \right\}.$$

Noting that  $C_k$  is increasing,  $c_k$  is decreasing,  $C_k \geq C$  and  $c_k \leq c$ , we verify (23). This completes the proof of Proposition 6.3(2).  $\square$

**6.2. The road map's third challenge: the solution in overview.** We have assembled the elements needed to implement the road map and thus to prove Theorem 6.1, with the exception of addressing the third challenge, which we labelled *final polymer comparison* in Section 2.2.

Recall that the road map advocates the construction of a system of  $k$  near polymers, each running from  $(x, -\epsilon^{3/2})$  to  $(y, 1 + \epsilon^{3/2})$ , and pairwise disjoint otherwise. These will be zigzags, to be called  $\rho_1, \rho_2, \dots, \rho_k$ , that during the time interval  $[0, 1]$  follow the course of the  $k$  polymers, to be called  $\phi_1, \phi_2, \dots, \phi_k$ , whose existence is ensured by the occurrence of the event  $\text{MaxDisjtPoly}_{n;([x-\epsilon, x+\epsilon], 0)}^{([y-\epsilon, y+\epsilon], 1)} \geq k$ . They begin by following elements in a forward bouquet of lifetime  $[-\epsilon^{3/2}, 0]$  with shared endpoint

$(x, -\epsilon^{3/2})$  and end by following elements in a backward bouquet of lifetime  $[1, 1 + \epsilon^{3/2}]$  with shared endpoint  $(y, 1 + \epsilon^{3/2})$ .

Now the third challenge involves arguing that each of the  $\rho_i$  is indeed a near polymer, with a shortfall in weight from the maximum  $\text{Wgt}_{n;(x, -\epsilon^{3/2})}^{(y, 1 + \epsilon^{3/2})}$  of order  $\epsilon^{1/2}$ . As we pointed out in Section 2.2, the third challenge seems merely a restatement of the overall problem. A little more detail is needed to explain even in outline how we propose to solve the third challenge, and we now offer such an outline.

Our task is to establish that, for each  $i \in \llbracket 1, k \rrbracket$ , the quantity  $\text{Wgt}(\rho_{n;(x, -\epsilon^{3/2})}^{(y, 1 + \epsilon^{3/2})}) - \text{Wgt}(\rho_i)$ , which is necessarily non-negative, is in fact at most  $O(\epsilon^{1/2})$ . Now the value  $\text{Wgt}(\rho_i)$  is naturally written as a sum of three terms: the weight of a forward bouquet element, the weight of the polymer  $\phi_i$ , which moves from  $[x - \epsilon, x + \epsilon] \times \{0\}$  to  $[y - \epsilon, y + \epsilon] \times \{1\}$ , and the weight of a backward bouquet element. The first and third weights are known to be of order  $\epsilon^{1/2}$  typically by Corollary 6.4. The weight  $\text{Wgt}(\rho_{n;(x, -\epsilon^{3/2})}^{(y, 1 + \epsilon^{3/2})})$  may correspondingly be split into three terms, by splitting the polymer in question at the pair of times  $(0, 1)$ . To resolve the third challenge, we want to argue that:

- The first and third pieces of the polymer  $\rho_{n;(x, -\epsilon^{3/2})}^{(y, 1 + \epsilon^{3/2})}$ , with lifetimes  $[-\epsilon^{3/2}, 0]$  and  $[1, 1 + \epsilon^{3/2}]$ , have weight of order  $\epsilon^{1/2}$ .
- The second piece, the grand middle section with lifetime  $[0, 1]$ , has a weight that differs from any of the weights  $\text{Wgt}(\phi_i)$  for  $i \in \llbracket 1, k \rrbracket$ , by an order of  $\epsilon^{1/2}$ .

Now to reach these conclusions, we will in fact seek control of the geometry of the polymer  $\rho_{n;(x, -\epsilon^{3/2})}^{(y, 1 + \epsilon^{3/2})}$ . This we will do by invoking one of the main results of the present article, Theorem 1.5. We will learn that, typically, this polymer at time zero is located at distance of order  $\epsilon$  from  $x$ , and at time one, at distance of order  $\epsilon$  from  $y$ . With this understanding, we see that the polymer's first and third pieces do not suffer significant lateral shifts, so that Corollary 5.3 will ensure that these pieces have the desired order  $\epsilon^{1/2}$  weight. The same understanding means that the middle section of  $\rho_{n;(x, -\epsilon^{3/2})}^{(y, 1 + \epsilon^{3/2})}$  and the middle section  $\phi_i$  of any  $\rho_i$  have starting and ending points whose locations differ by an order of  $\epsilon$ : close to  $x$  or to  $y$  respectively. This is crucial information, because it permits us to invoke Theorem 5.4 to conclude that the middle section weights indeed differ by an order of  $\epsilon^{1/2}$ . These ideas are illustrated by Figure 6.

Armed with this elaboration of the third challenge in the road map, we are ready to derive Theorem 6.1.

**6.3. Proof of Theorem 6.1.** By the scaling principle, it suffices to prove the result when  $(t_1, t_2) = (0, 1)$ , and this we now do. Set  $I = [x - \epsilon, x + \epsilon]$  and  $J = [y - \epsilon, y + \epsilon]$ . For a given parameter  $r > 0$ , we also define the extended intervals  $I^+ = [x - (r + 1)\epsilon, x + (r + 1)\epsilon]$  and  $J^+ = [y - (r + 1)\epsilon, y + (r + 1)\epsilon]$  that respectively contain  $I$  and  $J$ .

On the event that  $\text{MaxDisjtPoly}_{n;(I,0)}^{(J,1)} \geq k$ , there exist collections of  $k$  disjoint polymers that make the journey from  $I \times \{0\}$  to  $J \times \{1\}$ . We now seek to choose one of them. Indeed, using the language of Section 5.7, we may choose  $(\bar{U}, \bar{V})$  to be the lexicographically minimal element in the closure of the set  $\text{DisjtIndex}_{n,k;(I,0)}^{(J,1)}$ . This definition makes sense because we are dealing with a closed set. Invoking Lemma 5.8, we see that  $(\bar{U}, \bar{V})$  is an element of  $\text{HorSepIndex}_{n,k;(I,0)}^{(J,1)}$ . That is, we have

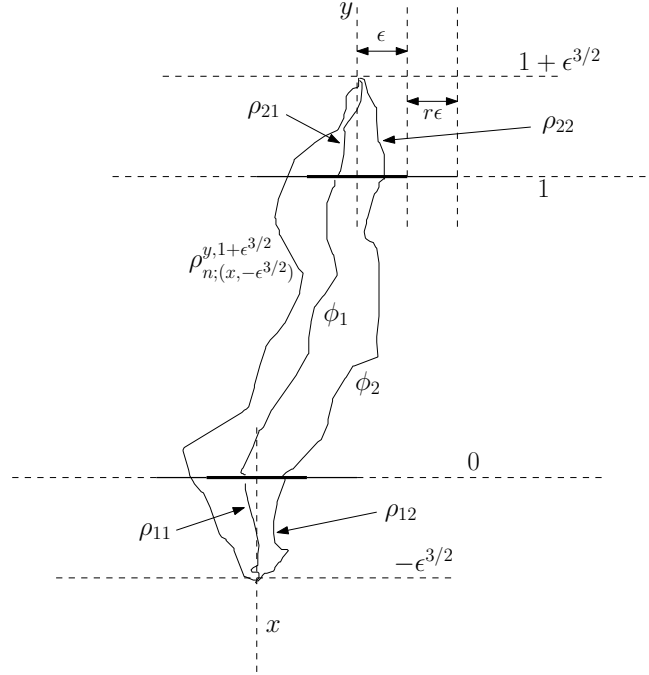


FIGURE 6. Solving the third challenge of the road map. The planar intervals  $I \times \{0\}$  and  $J \times \{1\}$ , with  $I = [x - \epsilon, x + \epsilon]$  and  $J = [y - \epsilon, y + \epsilon]$ , are drawn with thick bold lines. Extended intervals  $I^+ \times \{0\}$  and  $J^+ \times \{1\}$  are also drawn in solid lines. Here,  $I^+ = [x - (r + 1)\epsilon, x + (r + 1)\epsilon]$  and  $J^+ = [y - (r + 1)\epsilon, y + (r + 1)\epsilon]$  with  $r > 0$  given. The notation will be used in the upcoming proof of Theorem 6.1. The polymer  $\rho_{n;(x,-\epsilon^{3/2})}^{(y,1+\epsilon^{3/2})}$  typically visits these extended intervals at times zero and one, and, in this event, the weight of its  $[0, 1]$ -duration subpath may be closely compared to that of the polymers  $\phi_1$  and  $\phi_2$ . The  $\rho_{ij}$  notation will be used in the proof to indicate paths in the forward and backward bouquets.

explicitly selected an  $k$ -tuple of polymers that, while not necessarily pairwise disjoint, is pairwise horizontally separate; this is enough for the upcoming surgery.

It is helpful to recall from the road map that it is our aim to show that the occurrence of the event  $\text{MaxDisjtPoly}_{n;(I,0)}^{(J,1)} \geq k$  typically entails  $\text{NearPoly}_{n,k;(x,-\epsilon^{3/2})}^{(y,1+\epsilon^{3/2})}(\eta)$ , where the parameter  $\eta$  will soon be selected explicitly to have order  $\epsilon^{1/2}$ . What do we mean by typically? Typical behaviour means a collection of provably standard circumstances needed to undertake our rewiring surgery successfully. It has four types:

- **ForBouqReg/BackBouqReg:** The  $\epsilon^{1/2}$ -order weight of bouquets.
- **PolyWgtReg:** The  $\epsilon^{1/2}$ -order weight of any  $[-\epsilon^{3/2}, 0]$ - or  $[1, 1 + \epsilon^{3/2}]$ -lifetime polymer respectively near  $x$  and  $y$ .
- **PolyDevReg:** Control on the geometry of  $\rho_{n;(x,-\epsilon^{3/2})}^{(y,1+\epsilon^{3/2})}$  at times zero and one.
- **LocWgtReg:** The one-half power weight difference for middle section polymers.

We will shortly define a *favourable surgical conditions* event, which specifies exactly what we mean by ‘typical’, as an intersection of events to which the labels just listed correspond.

Before offering the definition, we introduce a variation of the PolyDevReg notation specified in Subsection 1.7.2. When the event  $\text{PolyDevReg}_{n;(x,t_1)}^{(y,t_2)}(a,r)$  is specified in (2), the parameter  $a$  refers to the proportion of the polymer lifetime  $t_{1,2}$  which has elapsed at the intermediate time  $(1-a)t_1 + at_2$ . We now wish to allude to the intermediate time directly. To do so, we will employ a square bracket notation. Taking  $t \in [t_1, t_2] \cap n^{-1}\mathbb{Z}$ , we will write

$$\text{PolyDevReg}_{n;(x,t_1)}^{(y,t_2)}[t,r] = \left\{ \left| \rho_{n;(x,t_1)}^{(y,t_2)}(t) - \ell_{(x,t_1)}^{(y,t_2)}(t) \right| \leq r((t-t_1) \wedge (t_2-t))^{2/3} \right\},$$

so that  $\text{PolyDevReg}_{n;(x,t_1)}^{(y,t_2)}[t,r]$  equals  $\text{PolyDevReg}_{n;(x,t_1)}^{(y,t_2)}((t-t_1)t_{1,2}^{-1}, r)$ . We also extend this notation so that the first square bracket argument  $t$  is replaced by a pair of such times. This notation refers to the intersection of the two single-time PolyDevReg events.

We now may define the favourable surgical conditions event  $\text{FavSurCon}_{n;(I,0)}^{(J,1)}(k; \bar{U}, \bar{V}; \epsilon, r)$ . The quantities  $\epsilon > 0$  and  $r > 0$  will retain their roles as parameters in this event for the remainder of the proof of Theorem 6.1. The event will only be considered when the event  $\text{MaxDisjtPoly}_{n;(I,0)}^{(J,1)} \geq k$  occurs, so that the random vectors  $\bar{U}$  and  $\bar{V}$  that are used as parameters in the event’s definition will always make sense. The new event is defined to equal

$$\begin{aligned} & \text{LocWgtReg}_{n;(I^+,0)}^{(J^+,1)}(2(r+1)\epsilon, r) \cap \text{PolyDevReg}_{n;(x,-\epsilon^{3/2})}^{(y,1+\epsilon^{3/2})}[\{0,1\}, r] \\ & \cap \text{ForBouqReg}_{n,k;(x,-\epsilon^{3/2})}^{(\bar{u},0)}(r) \cap \text{BackBouqReg}_{n,k;(\bar{v},1)}^{(y,1+\epsilon^{3/2})}(r) \\ & \cap \text{PolyWgtReg}_{n;(x,-\epsilon^{3/2})}^{(I^+,0)}(r^2) \cap \text{PolyWgtReg}_{n;(J^+,1)}^{(y,1+\epsilon^{3/2})}(r^2). \end{aligned}$$

The LocWgtReg, ForBouqReg/BackBouqReg and PolyWgtReg events have been defined in Sections 5.4, 6.1 and 5.3. In the use of the new square bracket notation for PolyDevReg, the polymer is being controlled at times zero and one.

Note that the quantity  $\epsilon^{3/2}$  plays the role of the duration of certain zigzags concerned in this definition, these zigzags having lifetime  $[-\epsilon^{3/2}, 0]$  and  $[1, 1 + \epsilon^{3/2}]$ . In order that these zigzags begin and end at vertical coordinates in the  $n^{-1}$ -mesh, we will insist throughout that  $\epsilon > 0$  satisfies  $\epsilon^{3/2} \in n^{-1}\mathbb{Z}$ . In fact, on one occasion, we will have cause to use that  $n\epsilon^{3/2} \in \mathbb{Z}$  is even, and we also insist on this. It would seem then that this condition should enter as a hypothesis of Theorem 6.1, though it does not; we omit it because, in view of the hypotheses (14) and (15), the condition can be forced by a multiplicative adjustment in  $\epsilon$  that differs from one by order say  $10^{-100}$ , and this causes a tiny similar adjustment in the theorem’s conclusion, which the reader may readily confirm is easily absorbed by tightening estimates during the proof. In summary, though  $\epsilon > 0$  is  $n$ -dependent, it may in practice be considered to be fixed at a given small positive value, the upper bound specified by several upcoming demands. We may later omit mention of the mesh membership condition  $\epsilon^{3/2} \in n^{-1}2\mathbb{Z}$ .

The next two results assert that favourable surgical conditions are indeed typical, and that, in the presence of these conditions, the occurrence of  $\text{MaxDisjtPoly}_{n;(I,0)}^{(J,1)} \geq k$  indeed forces the occurrence of  $\text{NearPoly}_{n,k;(x,-\epsilon^{3/2})}^{(y,1+\epsilon^{3/2})}(\eta)$  with  $\eta = O(\epsilon^{1/2})$ .

**Lemma 6.6.** *Suppose that  $n \in 2\mathbb{N}$ ,  $k \in \mathbb{N}$ ,  $x, y \in \mathbb{R}$ ,  $\epsilon > 0$  and  $r > 0$  satisfy*

$$n \geq 10^{37} k c^{-18} \epsilon^{-75/2} \max \{1, |x - y|^{36}\} \text{ and } |x - y| \leq (2(r + 1)\epsilon)^{-1/2}.$$

*Suppose also that*

$$r \geq \max \left\{ 10^9 c_1^{-4/5}, 15C^{1/2}, 4k^2, 70\epsilon^{1/2}|x - y| \right\}, \epsilon \leq 2^{-5}(r + 1)^{-1} \text{ and } r \leq 2^{-4} c \epsilon^{25/6} n^{1/36}.$$

*Then*

$$\begin{aligned} & \mathbb{P} \left( \text{MaxDisjtPoly}_{n;(I,0)}^{(J,1)} \geq k, \neg \text{FavSurCon}_{n;(I,0)}^{(J,1)}(k; \bar{U}, \bar{V}; \epsilon, r) \right) \\ & \leq 14062 k^2 r C_k \exp \left\{ -10^{-11} k^{-3} c_k r^{3/4} \right\}. \end{aligned}$$

**Proposition 6.7.** *Let  $n, k \in \mathbb{N}$ ,  $n \geq k \geq 1$ . Let  $\epsilon > 0$  and  $x, y \in \mathbb{R}$  satisfy  $n\epsilon^{3/2} \geq 10^2$ ,  $|x - y| \leq \epsilon^{-1/2}$ , and let  $r \geq 1$ . Then, whenever  $\left\{ \text{MaxDisjtPoly}_{n;([x-\epsilon, x+\epsilon], 0)}^{(y-\epsilon, y+\epsilon, 1)} \geq k \right\} \cap \text{FavSurCon}_{n;(I,0)}^{(J,1)}(k; \bar{U}, \bar{V}; \epsilon, r)$  occurs,*

$$\text{Wgt}_{n,k;(x\bar{1}, -\epsilon^{3/2})}^{(y\bar{1}, 1+\epsilon^{3/2})} \geq k \cdot \text{Wgt}_{n;(x, -\epsilon^{3/2})}^{(y, 1+\epsilon^{3/2})} - 15kr^2 \epsilon^{1/2}, \quad (26)$$

*which is to say, the event  $\text{NearPoly}_{n,k;(x, -\epsilon^{3/2})}^{(y, 1+\epsilon^{3/2})}(\eta)$  occurs where  $(1 + 2\epsilon^{3/2})^{1/3} \eta = 15kr^2 \epsilon^{1/2}$ .*

We now apply these two results to close out the proof of Theorem 6.1 and then prove them in turn.

Note that, for the value of  $\eta$  specified in Proposition 6.7,  $\text{NearPoly}_{n,k;(x, -\epsilon^{3/2})}^{(y, 1+\epsilon^{3/2})}(\eta)$  is a subset of  $\text{NearPoly}_{n,k;(x, -\epsilon^{3/2})}^{(y, 1+\epsilon^{3/2})}(15kr^2 \epsilon^{1/2})$ . The proposition thus implies that

$$\begin{aligned} & \mathbb{P} \left( \text{MaxDisjtPoly}_{n;(I,0)}^{(J,1)} \geq k, \text{FavSurCon}_{n;(I,0)}^{(J,1)}(k; \bar{U}, \bar{V}; \epsilon, r) \right) \\ & \leq \mathbb{P} \left( \text{NearPoly}_{n,k;(x, -\epsilon^{3/2})}^{(y, 1+\epsilon^{3/2})}(15kr^2 \epsilon^{1/2}) \right), \end{aligned}$$

provided that  $|x - y| \leq \epsilon^{-1/2}$  and  $r \geq 1$ .

We now apply Corollary 5.2 to bound above the right-hand probability. Parameter settings are  $\mathbf{t}_1 = -\epsilon^{3/2}$ ,  $\mathbf{t}_2 = 1 + \epsilon^{3/2}$ ,  $\mathbf{k} = k$ ,  $\mathbf{x} = x$ ,  $\mathbf{y} = y$  and  $\boldsymbol{\eta} = 15kr^2 \epsilon^{1/2}$ . Note that, since  $\boldsymbol{\eta} \geq \epsilon$ , the corollary's hypotheses are satisfied when

$$15kr^2 \epsilon^{1/2} < (\eta_0)^{k^2}, \quad k \geq 2, \quad (1 + 2\epsilon^{3/2})n/2 \geq k \vee (K_0)^{k^2} (\log \epsilon^{-1})^{K_0}$$

and  $(1 + 2\epsilon^{3/2})^{-2/3} |y - x| \leq a_0 n^{1/9}$ . (It is also necessary that  $(n, 1 - \epsilon^{3/2}, 1 + \epsilon^{3/2}) \in \mathbb{N} \times \mathbb{R}_<^2$  be a compatible triple, which amounts to the already imposed condition that  $n\epsilon^{3/2} \in \mathbb{N}$ .) When these conditions are met, the conclusion of the corollary implies that

$$\begin{aligned} & \mathbb{P} \left( \text{NearPoly}_{n,k;(x, -\epsilon^{3/2})}^{(y, 1+\epsilon^{3/2})}(15kr^2 \epsilon^{1/2}) \right) \\ & \leq 15^{k^2} k^{k^2} r^{2k^2} \epsilon^{(k^2-1)/2} \exp \left\{ \beta_k (\log \epsilon^{-1})^{5/6} \right\}, \end{aligned}$$

where we again used  $\eta \geq \epsilon$  to write the final term in the product.

Applying Lemma 6.6, we find that

$$\begin{aligned} \mathbb{P} \left( \text{MaxDisjtPoly}_{n;(I,0)}^{(J,1)} \geq k \right) & \leq 15^{k^2} k^{k^2} r^{2k^2} \epsilon^{(k^2-1)/2} \exp \left\{ \beta_k (\log \epsilon^{-1})^{5/6} \right\} \\ & \quad + 14062 k^2 r C_k \exp \left\{ -10^{-11} k^{-3} c_k r^{3/4} \right\}. \end{aligned} \quad (27)$$



This application requires that  $n \geq 10^{37} k c^{-18} \epsilon^{-75/2} \max\{1, |x-y|^{36}\}$ ,  $|x-y| \leq (2(r+1)\epsilon)^{-1/2}$ ,  
 $r \geq \max\left\{10^9 c_1^{-4/5}, 15C^{1/2}, 4k^2, 70\epsilon^{1/2}|x-y|\right\}$ ,  $\epsilon \leq 2^{-5}(r+1)^{-1}$  and  $r \leq 2^{-4} c \epsilon^{25/6} n^{1/36}$ .

We now set the parameter  $r > 0$  so that  $10^{-11} k^{-3} c_k r^{3/4} = 2^{-1}(k^2 - 1) \log \epsilon^{-1}$ . The choice is made in order that the exponential term in (27) equal  $\epsilon^{(k^2-1)/2}$ . Thus,

$$r = 10^{44/3} k^4 c_k^{-4/3} 2^{-4/3} (k^2 - 1)^{4/3} (\log \epsilon^{-1})^{4/3}. \quad (28)$$

Note that

$$r \leq 10^{15} c_k^{-4/3} k^{20/3} (\log \epsilon^{-1})^{4/3}.$$

We then obtain

$$\begin{aligned} & \mathbb{P}\left(\text{MaxDisjtPoly}_{n;(I,0)}^{(J,1)} \geq k\right) \\ & \leq 15k^2 10^{30k^2} k^{43k^2/3} c_k^{-8k^2/3} (\log \epsilon^{-1})^{4k^2} \epsilon^{(k^2-1)/2} \exp\{\beta_k (\log \epsilon^{-1})^{5/6}\} \\ & \quad + 10^{20} k^{26/3} c_k^{-4/3} C_k (\log \epsilon^{-1})^2 \cdot \epsilon^{(k^2-1)/2} \\ & \leq 10^{32k^2} k^{15k^2} c_k^{-3k^2} C_k (\log \epsilon^{-1})^{4k^2} \epsilon^{(k^2-1)/2} \exp\{\beta_k (\log \epsilon^{-1})^{5/6}\}, \end{aligned}$$

where we used  $c_k \leq c_1$  and  $\epsilon \leq e^{-1}$  in the first inequality and  $k \geq 1$ ,  $C_k \geq 1$ ,  $c_k \leq 1$  and  $\epsilon \leq e^{-1}$  in the second. This completes the proof of Theorem 6.1 in the case that  $t_1 = 0$  and  $t_1 = 1$ ; as we noted at the outset, this special case implies the general one.  $\square$

**Proof of Lemma 6.6.** The FavSurCon event is an intersection of six events. Each of these events is known to be typical, by several results proved or cited in this article. Thus, the present proof need merely gather together the estimates, invoked with suitable parameter settings.

By Corollary 5.5 with  $\epsilon = 2(r+1)\epsilon$ ,  $\mathbf{R} = r$ ,  $\mathbf{x} = x - (r+1)\epsilon$  and  $\mathbf{y} = y - (r+1)\epsilon$ ,

$$\mathbb{P}\left(\neg \text{LocWgtReg}_{n;(I^+,0)}^{(J^+,1)}(2(r+1)\epsilon, r)\right) \leq 10032 C \exp\{-c_1 2^{-21} r^{3/2}\}$$

when  $n \geq 10^{29} c^{-18}$  is even and

$$2(r+1)\epsilon \in (0, 2^{-4}], |x-y| \leq (2(r+1)\epsilon)^{-1/2} \wedge 2^{-5/3} 3^{-1} c n^{1/18} \text{ and } r \in [2 \cdot 10^4, 10^3 n^{1/18}].$$

We apply Proposition 1.6, and the remark following this proposition, in order to find that

$$\mathbb{P}\left(\neg \text{PolyDevReg}_{n;(x,-\epsilon^{3/2})}^{(y,1+\epsilon^{3/2})}[\{0,1\}, r]\right) \leq 22Cr \exp\{-10^{-11} c_1 r^{3/4}\}.$$

When the proposition is applied, it is with parameter settings  $\mathbf{x} = x$ ,  $\mathbf{y} = y$ ,  $\mathbf{t}_1 = -\epsilon^{3/2}$ ,  $\mathbf{t}_2 = 1 + \epsilon^{3/2}$ ,  $\mathbf{r} = r$  and with  $\mathbf{a}$  chosen equal to  $1 - \epsilon^{3/2}(1 + 2\epsilon^{3/2})^{-1}$ . When the remark is applied, we instead take  $\mathbf{a} = \epsilon^{3/2}(1 + 2\epsilon^{3/2})^{-1}$ . Note that  $\mathbf{a} \wedge (1 - \mathbf{a}) \geq \epsilon^{3/2}/2$  since  $\epsilon \leq 2^{-2/3}$ . Using  $\mathbf{a}^{-1} \vee (1 - \mathbf{a})^{-1} \leq 2\epsilon^{-3/2}$  and  $\mathbf{t}_{1,2} \in [1, 2]$ , we see that the hypotheses of this application are met provided that

$$\begin{aligned} n & \geq \max\left\{10^{29} (\epsilon^{3/2}/2)^{-25} c^{-18}, 10^{24} c^{-18} (\epsilon^{3/2}/2)^{-25} |x-y|^{36} (1 + 2\epsilon^{3/2})^{-24}\right\}, \\ r & \geq \max\left\{10^9 c_1^{-4/5}, 15C^{1/2}, 87 (\epsilon^{3/2}/2)^{1/3} (1 + 2\epsilon^{3/2})^{-2/3} |x-y|\right\} \end{aligned}$$

and, since  $\epsilon^{3/2}(1 + 2\epsilon^{3/2})^{-1} \geq 2^{-1} \epsilon^{3/2}$ ,  $r \leq 3(2^{-1} \epsilon^{3/2})^{25/9} n^{1/36} (1 + 2\epsilon^{3/2})^{1/36}$ .

Corollary 6.4 is applied to find that

$$\mathbb{P}\left(\neg \text{ForBouqReg}_{n,k;(x,-\epsilon^{3/2})}^{(\bar{U},0)}(r)\right) + \mathbb{P}\left(\neg \text{BackBouqReg}_{n,k;(\bar{V},1)}^{(y,1+\epsilon^{3/2})}(r)\right) \leq 8k^2 C_k \exp\left\{-2^{-3}k^{-3}c_k r^{3/2}\right\}. \quad (29)$$

Here, parameter settings are  $\mathbf{k} = k$ ,  $\mathbf{t}_1 = -\epsilon^{3/2}$ ,  $\mathbf{t}_2 = 0$ ,  $\mathbf{x} = x$ ,  $\mathbf{y} = y$ ,  $\bar{\mathbf{u}} = \bar{U}$ ,  $\bar{\mathbf{v}} = \bar{V}$  and  $\mathbf{r} = r$ . Since  $|U_i - x| \vee |V_i - y| \leq \epsilon$  for each  $i \in \llbracket 1, k \rrbracket$ , this application may be made provided that

$$n\epsilon^{3/2} - 1 \geq \max\left\{k, 3^{18}c^{-18}, 6^{36}, c^{-9}(1 - n^{-1}\epsilon^{-3/2})^{-6}(1 + 2^{-1}n^{-2/3}\epsilon^{-1})^9, 2^{18}c^{-18}(1 - n^{-1}\epsilon^{-3/2})^{-12}(1 + 2^{-1}n^{-2/3}\epsilon^{-1})^{18}\right\}$$

and  $r \in [4k^2, 9k^2\epsilon^{1/2}(1 - n^{-1}\epsilon^{3/2})^{1/3}n^{1/3}]$ .

There is a subtlety in this application of Corollary 6.4 that deserves mention. In fact, it is in this application that the microscopic detail discussed after Proposition 6.3 is implicated. Our parameter choice  $(\bar{u}, \bar{v}) = (\bar{U}, \bar{V})$  is a random one. Even if, as is the case, our random variable  $(\bar{U}, \bar{V})$  almost surely verifies the necessary bounds for this use, it is not formally admissible in this role. However, recall that  $(\bar{U}, \bar{V})$  has been specified to be measurable with respect to randomness in the region  $\mathbb{R} \times [0, 1]$ . This random variable is thus independent of the randomness in the open region  $\mathbb{R} \times ((-\infty, 0) \cup (1, \infty))$  that specifies the two events in (29) with which the application of the corollary is concerned. It is the use of proper weights in the definition of the events **ForBouqReg** and **BackBouqReg** which ensures that these events are measurable with respect to the randomness in  $\mathbb{R} \times ((-\infty, 0) \cup (1, \infty))$ . It is this independence which renders the application admissible.

Note that

$$\text{PolyWgtReg}_{n;(x,-\epsilon^{3/2})}^{([x-(r+1)\epsilon, x+(r+1)\epsilon], 0)}(r^2) \subseteq \text{PolyWgtReg}_{n;([x, x+\epsilon], -\epsilon^{3/2})}^{([y, y+d\epsilon], 0)}(r^2)$$

where  $y = x - (r+1)\epsilon$  and  $d = \lceil 2(r+1) \rceil$ , and that

$$\text{PolyWgtReg}_{n;([y-(r+1)\epsilon, y+(r+1)\epsilon], 1)}^{(y, 1+\epsilon^{3/2})}(r^2) \subseteq \text{PolyWgtReg}_{n;([z, z+d\epsilon], -\epsilon^{3/2})}^{([y, y+\epsilon], 0)}(r^2)$$

where  $z = y - (r+1)\epsilon$ . We may apply Corollary 5.3 in order to bound the  $\mathbb{P}$ -probability of the two right-hand events. In the first case, we take  $\mathbf{x} = x$ ,  $\mathbf{y} = x - (r+1)\epsilon$ ,  $\mathbf{a} = 1$ ,  $\mathbf{b} = \lceil 2(r+1) \rceil$ ,  $\mathbf{t}_1 = -\epsilon^{3/2}$ ,  $\mathbf{t}_2 = 0$  and  $\mathbf{r} = r^2$  to find that

$$\mathbb{P}\left(\neg \text{PolyWgtReg}_{n;([x, x+\epsilon], -\epsilon^{3/2})}^{([y, y+d\epsilon], 0)}(r^2)\right) \leq (2r+3) \cdot 400C \exp\left\{-c_1 2^{-8}r^3\right\},$$

the application being valid provided that

$$n\epsilon^{3/2} \geq 10^{29} \vee 2(c/3)^{-18}, \quad 3(r+1) \leq 3^{-1}2^{-2/3}cn^{1/18}\epsilon^{1/12} \quad \text{and} \quad r \in [33^{1/2}, 2\epsilon^{1/24}n^{1/36}].$$

In fact, we also need to use that  $n\epsilon^{3/2}$  is an even integer. It was for this reason that we earlier imposed this requirement on  $\epsilon > 0$ .

In the second case, we take  $\mathbf{x} = y - (r+1)\epsilon$ ,  $\mathbf{y} = y$ ,  $\mathbf{a} = \lceil 2(r+1) \rceil$ ,  $\mathbf{b} = 1$ ,  $\mathbf{t}_1 = 1$ ,  $\mathbf{t}_2 = 1 + \epsilon^{3/2}$  and  $\mathbf{r} = r^2$  to find that

$$\mathbb{P}\left(\neg \text{PolyWgtReg}_{n;([z, z+d\epsilon], -\epsilon^{3/2})}^{([y, y+\epsilon], 0)}(r^2)\right) \leq (2r+3) \cdot 400C \exp\left\{-c_1 2^{-8}r^3\right\},$$

the conclusion valid under the same hypotheses as in the first case.

We find then that

$$\mathbb{P}\left(\neg \text{PolyWgtReg}_{n;(x,-\epsilon^{3/2})}^{([x-(r+1)\epsilon, x+(r+1)\epsilon], 0)}(r^2)\right) + \mathbb{P}\left(\neg \text{PolyWgtReg}_{n;([y-(r+1)\epsilon, y+(r+1)\epsilon], 1)}^{(y, 1+\epsilon^{3/2})}(r^2)\right)$$

is at most  $2(2r + 3) \cdot 400C \exp \{ -c_1 2^{-8} r^3 \}$ . Thus,

$$\begin{aligned}
 & \mathbb{P} \left( \text{MaxDisjPoly}_{n;(I,0)}^{(J,1)} \geq k, \neg \text{FavSurCon}_{n;(I,0)}^{(J,1)}(m; \bar{U}, \bar{V}; \epsilon, r) \right) \\
 & \leq 10032 C \exp \{ -c_1 2^{-21} r^{3/2} \} + 22Cr \exp \{ -10^{-11} c_1 r^{3/4} \} \\
 & \quad + 8k^2 C_k \exp \{ -2^{-3} k^{-3} c_k r^{3/2} \} + 2(2r + 3) \cdot 400C \exp \{ -c_1 2^{-8} r^3 \} \\
 & \leq 22Cr \exp \{ -10^{-11} c_k r^{3/4} \} + (10032 + 8 + 4000) k^2 r C_k \exp \{ -2^{-21} k^{-3} c_k r^{3/2} \} \\
 & \leq 14062 k^2 r C_k \exp \{ -10^{-11} k^{-3} c_k r^{3/4} \}.
 \end{aligned}$$

where the second inequality used that the sequence  $c_i$  is decreasing, as well as  $C_k \geq C$ ,  $k \geq 1$  and  $r \geq 1$ . The third used  $r \geq 1$ ,  $k \geq 1$ , and  $C_k \geq C$ . This completes the proof of Lemma 6.6.  $\square$

**Proof of Proposition 6.7.** In this argument, we rigorously implement the resolution of the third challenge of the road map, expressed in outline in the explanation in Section 6.2 that led to the two bullet point comments.

Set  $I = [x - \epsilon, x + \epsilon]$  and  $J = [y - \epsilon, y + \epsilon]$ . When  $\text{MaxDisjPoly}_{n;(I,0)}^{(J,1)} \geq k$ , recall that  $\bar{U} \in I_{\leq}^k$  and  $\bar{V} \in J_{\leq}^k$  are such that the collection  $\{ \rho_{n;(U_i,0)}^{(V_i,1)} : i \in \llbracket 1, k \rrbracket \}$  of  $n$ -polymers is pairwise horizontally separate. Set  $\phi_i = \rho_{n;(U_i,0)}^{(V_i,1)}$  for  $i \in \llbracket 1, k \rrbracket$ . It is the task of surgery to tie together the  $k$  multi-polymer component starting points  $\bar{U} \times \{0\}$  to what we may call the ‘lower knot’,  $(x, -\epsilon^{3/2})$ , and the  $k$  ending points  $\bar{v} \times \{1\}$  to the upper knot  $(y, 1 + \epsilon^{3/2})$ .

Recall the discussion of bouquets and proper weights that followed Proposition 6.3. In surgery, the maximizer  $\rho_{n,k;(x\bar{1}, -\epsilon^{3/2})}^{\text{prop};(\bar{U},0)}$  is selected. (This object is unique, because  $\bar{U}$  is measurable with respect to randomness in the region  $\mathbb{R} \times [0, \infty)$ , so that Lemma 3.1 applies in view of (17) and (18).) The maximizer is the lower bouquet and we denote it by  $(\rho_{11}, \dots, \rho_{1k})$ . Forward bouquet regularity  $\text{ForBouqReg}$  ensures that the bouquet’s weight  $\text{PropWgt}_{n,k;(x\bar{1}, -\epsilon^{3/2})}^{(\bar{U},0)}$  satisfies

$$\left| (\epsilon^{3/2} - n^{-1})^{-1/3} \text{PropWgt}_{n,k;(x\bar{1}, -\epsilon^{3/2})}^{(\bar{U},0)} + 2^{-1/2} (\epsilon^{3/2} - n^{-1})^{-4/3} \sum_{i=1}^k (U_i - x + 2^{-1} n^{-2/3})^2 \right| \leq r.$$

Since  $|U_i - x| \leq \epsilon$  and  $n\epsilon^{3/2} \geq 10^2$ , this leads to the simpler

$$\epsilon^{-1/2} \left| \text{PropWgt}_{n,k;(x\bar{1}, -\epsilon^{3/2})}^{(\bar{U},0)} \right| \leq 2^{1/2} (r + k). \quad (30)$$

The story of the upper bouquet’s construction is no different. This bouquet is the maximizer  $\rho_{n,k;(y\bar{1}, 1 + \epsilon^{3/2})}^{\text{prop};(\bar{V},1)}$ , is denoted by  $(\rho_{21}, \dots, \rho_{2k})$ , and has weight that satisfies

$$\epsilon^{-1/2} \left| \text{PropWgt}_{n,k;(y\bar{1}, 1 + \epsilon^{3/2})}^{(\bar{V},1)} \right| \leq 2^{1/2} (r + k). \quad (31)$$

Surgery is completed by the construction of a  $k$ -tuple  $(\rho_1, \dots, \rho_k)$  of  $n$ -zigzags each running between  $(x, -\epsilon^{3/2})$  and  $(y, 1 + \epsilon^{3/2})$ , where

$$\rho_i = \rho_{1i} \circ \phi_i \circ \rho_{2i} \text{ for } i \in \llbracket 1, k \rrbracket.$$

Given that this  $k$ -tuple has this set of endpoints, the sum of the weights of its elements offers a lower bound on  $\text{Wgt}_{n,k;(x\bar{1},-\epsilon^{3/2})}^{(y\bar{1},1+\epsilon^{3/2})}$ . To show, as Proposition (6.7) asserts, that the latter weight is at least  $k \cdot \text{Wgt}_{n;(x,-\epsilon^{3/2})}^{(y,1+\epsilon^{3/2})} - 15kr^2\epsilon^{1/2}$ , we will now argue that this is true of  $\sum_{i=1}^k \text{Wgt}(\rho_i)$ .

The polymer weight  $\text{Wgt}_{n;(x,-\epsilon^{3/2})}^{(y,1+\epsilon^{3/2})}$  may be written as a sum of three terms,  $\omega_1$ ,  $\omega_2$  and  $\omega_3$ , these being the weights of the three  $n$ -polymers formed by intersecting  $\rho_{n;(x,-\epsilon^{3/2})}^{(y,1+\epsilon^{3/2})}$  with the strips  $\mathbb{R} \times [-\epsilon^{3/2}, 0]$ ,  $\mathbb{R} \times [0, 1]$  and  $\mathbb{R} \times [1, 1 + \epsilon^{3/2}]$ .

Note then that

$$\begin{aligned} & \left| \sum_{i=1}^k \text{Wgt}(\rho_i) - k \cdot \text{Wgt}_{n;(x,-\epsilon^{3/2})}^{(y,1+\epsilon^{3/2})} \right| \\ & \leq \left| \text{Wgt}_{n,k;(x\bar{1},-\epsilon^{3/2})}^{(\bar{u},0)} \right| + \left| \text{Wgt}_{n,k;(x\bar{1},-\epsilon^{3/2})}^{(y\bar{1},1+\epsilon^{3/2})} \right| + \left| \sum_{i=1}^k (\text{Wgt}(\phi_i) - \omega_2) \right| + k(|\omega_1| + |\omega_3|), \end{aligned} \quad (32)$$

since for example  $\sum_{i=1}^k \text{Wgt}(\rho_{1i}) = \text{Wgt}_{n,k;(x\bar{1},-\epsilon^{3/2})}^{(\bar{u},0)}$  and  $\sum_{i=1}^k \text{Wgt}(\rho_{2i}) = \text{Wgt}_{n,k;(x\bar{1},-\epsilon^{3/2})}^{(y\bar{1},1+\epsilon^{3/2})}$ . The occurrence of  $\text{PolyDevReg}_{n;(x,-\epsilon^{3/2})}^{y,1+\epsilon^{3/2}}[\{0, 1\}, r]$  implies that

$$\left| \rho_{n;(x,-\epsilon^{3/2})}^{(y,1+\epsilon^{3/2})}(0) - \left( \left(1 - \frac{\epsilon^{3/2}}{1+2\epsilon^{3/2}}\right)x + \frac{\epsilon^{3/2}}{1+2\epsilon^{3/2}}y \right) \right| \leq r\epsilon(1+2\epsilon^{3/2})^{-2/3}$$

and

$$\left| \rho_{n;(x,-\epsilon^{3/2})}^{(y,1+\epsilon^{3/2})}(1) - \left( \frac{\epsilon^{3/2}}{1+2\epsilon^{3/2}}x + \left(1 - \frac{\epsilon^{3/2}}{1+2\epsilon^{3/2}}\right)y \right) \right| \leq r\epsilon(1+2\epsilon^{3/2})^{-2/3}.$$

Since  $|y - x| \leq \epsilon^{-1/2}$ , these imply that

$$\left| \rho_{n;(x,-\epsilon^{3/2})}^{(y,1+\epsilon^{3/2})}(0) - x \right| \vee \left| \rho_{n;(x,-\epsilon^{3/2})}^{(y,1+\epsilon^{3/2})}(1) - y \right| \leq (r+1)\epsilon. \quad (33)$$

The occurrence of  $\text{PolyWgtReg}_{n;(x,-\epsilon^{3/2})}^{([x-(r+1)\epsilon, x+(r+1)\epsilon], 0)}(r^2) \cap \text{PolyWgtReg}_{n;([y-(r+1)\epsilon, y+(r+1)\epsilon], 1)}^{(y,1+\epsilon^{3/2})}(r^2)$ , which is one of the favourable surgical conditions, then ensures that

$$\max\{|\omega_1|, |\omega_3|\} \leq (r^2 + 2^{-1/2}(r+1)^2)\epsilon^{1/2}. \quad (34)$$

For any given  $i \in \llbracket 1, k \rrbracket$ , the quantity  $\text{Wgt}(\phi_i) - \omega_2$  is the difference in weight of two  $n$ -polymers whose lifetime is  $[0, 1]$ . The first polymer begins at a location in  $[x - \epsilon, x + \epsilon]$  and ends at one in  $[y - \epsilon, y + \epsilon]$  while we see from (33) that the second begins in  $[x - (r+1)\epsilon, x + (r+1)\epsilon]$  and ends in  $[y - (r+1)\epsilon, y + (r+1)\epsilon]$ . Thus,  $\text{LocWgtReg}_{n;(I^+, 0)}^{(J^+, 1)}(2(r+1)\epsilon, r)$  entails that

$$|\text{Wgt}(\phi_i) - \omega_2| \leq r(2(r+1)\epsilon)^{1/2} \quad (35)$$

for each  $i \in \llbracket 1, k \rrbracket$ .

By revisiting the bound (32) equipped with our knowledge of (34) and (35) as well as (30) and (31), we come to learn that

$$\left| \sum_{i=1}^k \text{Wgt}(\rho_i) - k \cdot \text{Wgt}_{n;(x,-\epsilon^{3/2})}^{(y,1+\epsilon^{3/2})} \right| \leq \left( 2^{3/2}(r+k) + kr2^{1/2}(r+1)^{1/2} + 2k(r^2 + 2^{-1/2}(r+1)^2) \right) \epsilon^{1/2}.$$

Since  $r \geq 1$  and  $k \geq 1$ , the right-hand side is at most

$$(2^{3/2}r + 2^{3/2}k + 2kr^{3/2} + 2(1 + 2^{3/2})kr^2)\epsilon^{1/2} \leq 15kr^2\epsilon^{1/2}.$$

This completes the proof of Proposition 6.7.  $\square$

**6.4. Proof of Theorem 6.2.** By the scaling principle, it suffices to consider the case where  $t_1 = 0$  and  $t_2 = 1$  (so that  $t_{1,2} = 1$ ). On the event that  $\text{MaxDisjtPoly}_{n;([x,x+a],0)}^{([y,y+b],1)} \geq m$ , let  $\bar{U} \in [x, x+a]_{\leq}^m$  and  $\bar{V} \in [y, y+b]_{\leq}^m$  be such that the polymer collection  $\{\rho_{n;(U_i,t_1)}^{(V_i,t_2)} : i \in \llbracket 1, m \rrbracket\}$  is pairwise horizontally separate. At the start of Section 6.3 an explicit means for the selection of a measurable choice of  $(\bar{U}, \bar{V})$  is offered.

Recall that we denote  $h = a \vee b$ . Each sequence  $\{U_{i+1} - U_i : i \in \llbracket 1, m-1 \rrbracket\}$  and  $\{V_{i+1} - V_i : i \in \llbracket 1, m-1 \rrbracket\}$  consists of positive terms that sum to at most  $h$ . Let  $k \in \llbracket 1, m-1 \rrbracket$ ; the value of  $k$  will be fixed later. Call an index  $i \in \llbracket 1, m-1 \rrbracket$  *unsuitable* if  $(U_{i+1} - U_i) \vee (V_{i+1} - V_i) > 2h(k+1)(m-1)^{-1}$ . Note that the number of unsuitable indices is less than  $(m-1)(k+1)^{-1}$ , and therefore at most  $\lfloor (m-1)(k+1)^{-1} \rfloor$ . When  $k \in \llbracket 1, m-1 \rrbracket$  is chosen so that  $5k^2 \leq m$ , we claim that there exist  $k$  consecutive indices  $i \in \llbracket 1, m-1 \rrbracket$  such that  $(U_{i+1} - U_i) \vee (V_{i+1} - V_i) \leq 2h(k+1)(m-1)^{-1}$ . Indeed, were there not such an interval, the number of unsuitable indices would be at least  $\lfloor (m-1)k^{-1} \rfloor$ , and this is incompatible with our upper bound, since  $(m-1)k^{-1}$  exceeds  $(m-1)(k+1)^{-1}$  by at least one when  $5k^2 \leq m$ .

Let  $\Theta \in \llbracket 1, m-k \rrbracket$  be chosen so that  $\llbracket \Theta, \Theta+k-1 \rrbracket$  is an interval of such not unsuitable indices. The polymers  $\rho_{n;(U_i,0)}^{(V_i,1)}$  with index  $i \in \llbracket \Theta, \Theta+k-1 \rrbracket$  start and end in the planar intervals

$$[U_{\Theta}, U_{\Theta} + 2hk(k+1)(m-1)^{-1}] \times \{0\} \quad \text{and} \quad [V_{\Theta}, V_{\Theta} + 2hk(k+1)(m-1)^{-1}] \times \{1\}.$$

These planar intervals are contained in

$$[U_{\Theta}^-, U_{\Theta}^- + (2hk+1)(k+1)(m-1)^{-1}] \times \{0\} \quad \text{and} \quad [V_{\Theta}^-, V_{\Theta}^- + (2hk+1)(k+1)(m-1)^{-1}] \times \{1\},$$

where we set

$$U_{\Theta}^- = (k+1)(m-1)^{-1} \lfloor (m-1)(k+1)^{-1} U_{\Theta} \rfloor \quad \text{and} \quad V_{\Theta}^- = (k+1)(m-1)^{-1} \lfloor (m-1)(k+1)^{-1} V_{\Theta} \rfloor$$

to be left-displacements onto a  $(k+1)(m-1)^{-1}$ -mesh. That is,

$$\left\{ \text{MaxDisjtPoly}_{n;([x,x+a],0)}^{([y,y+b],1)} \geq m \right\} \subseteq \left\{ \text{MaxDisjtPoly}_{n;([U_{\Theta}^-, U_{\Theta}^- + (2hk+1)(k+1)(m-1)^{-1}],0)}^{([V_{\Theta}^-, V_{\Theta}^- + (2hk+1)(k+1)(m-1)^{-1}],1)} \geq k \right\}.$$

This is a useful moment to recall that, in a remark in Section 5.7, we permitted that the definition of the  $\text{MaxDisjtPoly}$  random variable be respecified so that the concerned polymers are merely pairwise horizontally separate, rather than pairwise disjoint. We actually need to use this form of the definition to obtain the preceding inequality. Indeed, our explicit selection of  $(\bar{U}, \bar{V})$  means that we are forced to deal with polymers that may be merely horizontally separate.

Let  $X$  denote the set formed by adding to the mesh points in  $(x, x+a]$  the greatest mesh point that is at most  $x$ , and let  $Y$  be the counterpart set where  $(y, y+b]$  replaces  $(x, x+a]$ . The cardinality of  $X$  is at most  $a(m-1)(k+1)^{-1} + 2$  and thus, in view of  $a \geq 1$  and  $m \geq k^2$ , at most  $2amk^{-1}$ .

Similarly,  $|Y| \leq 2bmk^{-1}$ . Since  $U_{\Theta}^- \in U$  and  $V_{\Theta}^- \in V$ , we find that

$$\begin{aligned}
& \mathbb{P}\left(\text{MaxDisjtPoly}_{n;([x,x+a],0)}^{([y,y+b],1)} \geq m\right) \\
& \leq \sum_{u \in X, v \in Y} \mathbb{P}\left(\text{MaxDisjtPoly}_{n;([u,u+(2hk+1)(k+1)(m-1)^{-1}],0)}^{([v,v+(2hk+1)(k+1)(m-1)^{-1}],1)} \geq k\right) \\
& \leq 2amk^{-1} \cdot 2bmk^{-1} \cdot (15hk^2/4 \cdot m^{-1})^{(k^2-1)/2} \\
& \quad \cdot 10^{26k^2} k^{43k^2/3} c_k^{-8k^2/3} C_k(\log mk^{-2})^{4k^2} \exp\{\beta_k(\log mk^{-2})^{5/6}\} \\
& \leq m^{5/2-k^2/2} \cdot h^{(k^2+3)/2} 10^{27k^2} k^{46k^2/3} c_k^{-8k^2/3} C_k(\log m)^{4k^2} \exp\{\beta_k(\log m)^{5/6}\}. \tag{36}
\end{aligned}$$

The last inequality is nothing more than a simplifying of terms that uses  $h = a \vee b$  and  $k \geq 1$ . In the inequality that precedes it, the summand in the second line is bounded above by using Theorem 6.1 with parameter settings  $\mathbf{t}_1 = 0$ ,  $\mathbf{t}_2 = 1$ ,  $\mathbf{x} = u + \frac{1}{2}(2hk + 1)(k + 1)(m - 1)^{-1}$ ,  $\mathbf{y} = v + \frac{1}{2}(2hk + 1)(k + 1)(m - 1)^{-1}$ ,  $\mathbf{m} = k$  and

$$\epsilon = \frac{1}{2}(2hk + 1)(k + 1)(m - 1)^{-1}.$$

In this second inequality, we also make use of the bounds

$$k^2 m^{-1} \leq \epsilon \leq \frac{15}{4} h k^2 m^{-1}, \tag{37}$$

where the latter bound is due to  $k \geq 2$ .

Of course, the hypotheses of Theorem 6.1 must be validated with the above parameter settings for the above bound to hold. In fact, we will now set the value of the parameter  $k \in \mathbb{N}$  in terms of  $m$ , and then justify that, for this choice, these hypotheses are indeed validated.

Recall from Theorem 6.2 that the constant  $\beta$  is set equal to  $e \vee \limsup_{i \in \mathbb{N}} \beta_i^{1/i}$  and that Corollary 5.2 implies that  $\beta < \infty$ . Since  $\beta > 1$ , we may choose  $k \in \mathbb{N}$  to be maximal so that  $\beta^k \leq (\log m)^{1/12}$ . That is,  $k = \lfloor \frac{1}{12 \log \beta} \log \log m \rfloor$ . Recalling that  $\liminf c_i^{1/i}$  is positive, and also that  $c_k \leq c$ , we see that the condition (14) on  $\epsilon$  in Theorem 6.1 is met provided that  $m \geq m_0$ , where  $m_0 \in \mathbb{N}$  is a certain constant; a suitable choice of  $m_0$  also ensures that  $k \geq 2$ .

Also note that  $|\mathbf{x} - \mathbf{y}| = |u - v| \leq |x - y| + 2h$  because  $|u - v| \leq |x - y| + h + (k + 1)(m - 1)^{-1}$  while  $h \geq 1 \geq (k + 1)(m - 1)^{-1}$  due to  $m \geq k + 2$ .

Since  $k \leq (12)^{-1} \log \log m$  (due to  $\beta \geq e$ ),  $K_0 \geq 1$ ,  $\epsilon \geq m^{-1}$  and  $\liminf c_i^{1/i} > 0$ , the bound (15) is verified when  $n$  is at least  $2(K_0)^{(12)^{-2}(\log \log m)^2} (\log m)^{K_0}$  and

$$\max \left\{ 10^{584} ((12)^{-1} \log \log m)^{240} c^{-36} m^{225} \max \{1, (|y - x| + 2h)^{36}\}, a_0^{-9} (|y - x| + 2h) \right\},$$

where we used  $m \geq m_0$  and adjusted the value of  $m_0$  if need be. Finally, the condition

$$|\mathbf{y} - \mathbf{x}|(\mathbf{t}_{1,2})^{-2/3} \leq \epsilon^{-1/2} (\log \epsilon^{-1})^{-2/3} \cdot 10^{-8} c_k^{2/3} k^{-10/3}$$

is verified provided that  $m \geq m_0 \vee (|x - y| + 2h)^3$  in light of  $\epsilon \leq m^{-1+o(1)}$  (with an increase if necessary in the value of  $m_0$ ). This lower bound on  $n$  is implied by Theorem 6.2's hypothesis (16) due to  $\log \log m \leq m$ .

Given the choice of  $k$ , we see that the expression (36) is at most

$$m^{-(145)^{-1}(\log \beta)^{-2}(0 \vee \log \log m)^2} h^{(\log \beta)^{-2}(0 \vee \log \log m)^2/288+3/2} H_m,$$

where  $\{H_i : i \in \mathbb{N}\}$  is a sequence of positive constants such that  $\sup_{i \in \mathbb{N}} H_i \exp\{-2(\log i)^{11/12}\}$  is finite.

This proves Theorem 6.2 when  $t_0 = 0$  and  $t_1 = 1$ ; as we noted at the outset, there is no loss of generality in considering this case.  $\square$

## 7. POLYMER FLUCTUATION: PROVING THEOREM 1.5.

In this section, we will derive the polymer fluctuation Theorem 1.5. The result asserts that any of the polymers that cross between unit-order length intervals separated at unit-order times deviates by a distance  $r$  from the line segment that interpolates its endpoints with probability at most  $\exp\{-O(1)r^{3/4}\}$ , uniformly in high choices of the scaling parameter  $n$ . We may easily reduce to the case where these unit-order intervals are instead singleton sets, however, by a simple application of polymer ordering. In Proposition 1.6, this reduced version of the theorem has been stated. After noting how this proposition implies Theorem 1.5, we will turn to discuss the ideas of the proof of the proposition, and then give the proof itself.

**Proof of Theorem 1.5.** For  $u, v \in \mathbb{R}$ , consider the random variable

$$X_{n;(u,t_1)}^{(v,t_2)}(a) = t_{1,2}^{-2/3} (a \wedge (1-a))^{-2/3} \left( \rho_{n;(u,t_1)}^{(v,t_2)}((1-a)t_1 + at_2) - \ell_{(u,t_1)}^{(v,t_2)}((1-a)t_1 + at_2) \right).$$

Set  $\varphi = t_{1,2}^{2/3} (a \wedge (1-a))^{2/3} r$ , and recall from the theorem's statement that we define  $I = [x, x + \varphi]$  and  $J = [y, y + \varphi]$ . When the event  $\neg \text{PolyDevReg}_{n;(I,t_1)}^{(J,t_2)}(a, 2r)$  occurs, we may choose  $(U, V) \in I \times J$  such that  $|X_{n;(U,t_1)}^{(V,t_2)}(a)| \geq 2\varphi$ . It might seem reassuring to know that this selection may be made measurably, and with a modicum of effort we might show this, but in fact this information is not needed.

Recall the polymer sandwich Lemma 5.7. By applying this result with parameter settings  $\mathbf{x}_1 = x$ ,  $\mathbf{x}_2 = x + \varphi$ ,  $\mathbf{y}_1 = y$ ,  $\mathbf{y}_2 = y + \varphi$ , we readily find that at least one of  $|X_{n;(x,t_1)}^{(y,t_2)}(a)|$  and  $|X_{n;(x+\varphi,t_1)}^{(y+\varphi,t_2)}(a)|$  is at least  $|X_{n;(U,t_1)}^{(V,t_2)}(a)| - \varphi$ . Thus,

$$\begin{aligned} & \mathbb{P}\left(\neg \text{PolyDevReg}_{n;(I,t_1)}^{(J,t_2)}(a, 2r)\right) \\ & \leq \mathbb{P}\left(\neg \text{PolyDevReg}_{n;(x,t_1)}^{(y,t_2)}(a, r)\right) + \mathbb{P}\left(\neg \text{PolyDevReg}_{n;(x+\varphi,t_1)}^{(y+\varphi,t_2)}(a, r)\right). \end{aligned}$$

Two applications of Proposition 1.6, with  $(\mathbf{x}, \mathbf{y})$  equal to  $(x, y)$  and  $(x + \varphi, y + \varphi)$ , complete the proof.  $\square$

We now overview the ideas of the proof of Proposition 1.6. Recall from Section 2.2 the role of the powers of one-half and one-third. The one-half power law is articulated in Theorem 5.4: polymer weight has a Hölder exponent of one-half in response to horizontal displacement of endpoints. The one-third power law is articulated by the scaling principle alongside the regular sequence conditions Reg(2) and Reg(3): a polymer of lifetime  $t$  whose endpoints differ by  $rt^{2/3}$  has weight  $t^{1/3}(U - 2^{-1/2}r^2)$ ; here,  $U$  is a unit-order, random, quantity.

These principles come into conflict unless the two-thirds principle for polymer geometry (which we are trying to prove) also obtains.

To explain this concept in more detail, it is useful to begin by fixing a little notation. We set  $Z = \rho_{n;(x,t_1)}^{(y,t_2)}((1-a)t_1 + at_2)$  and  $z = \ell_{(x,t_1)}^{(y,t_2)}((1-a)t_1 + at_2)$ . Recall that  $a$  is supposed to be close to one, and that we seek to show that  $Z$  is close to  $z$ . Using notation already seen in the proof of Theorem 1.5, we write  $\varphi = t_{1,2}^{2/3}(1-a)^{2/3}r$ . The factor of  $t_{1,2}^{2/3}(1-a)^{2/3}$  is a suitable scale for judging polymer fluctuation between the pairs of times  $(1-a)t_1 + at_2$  and  $t_2$ , and our task is to show that it is a rare event, with a decaying probability expressed in terms of  $r$ , that  $Z \leq z - \varphi$ .

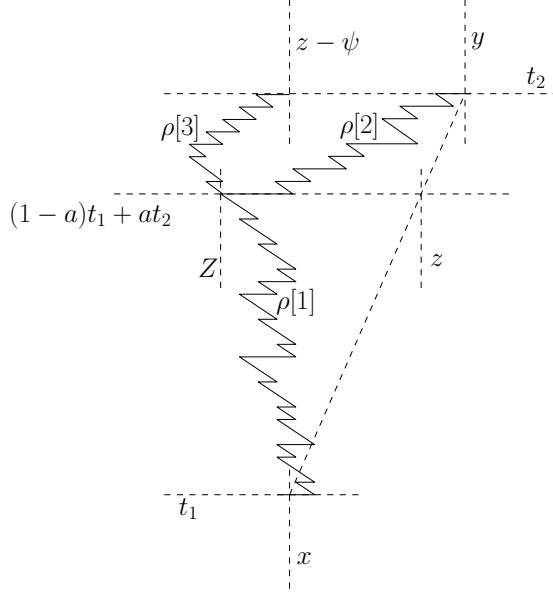


FIGURE 7. The polymer  $\rho = \rho_{n;(x,t_1)}^{(y,t_2)}$  is split at time  $(1-a)t_1 + at_2$  and the resulting subpaths are labelled  $\rho[1]$  and  $\rho[2]$ . The polymer  $\rho[3]$  has a less hectic journey than does  $\rho[2]$  during their shared lifetime. The concatenation  $\rho[1] \circ \rho[3]$  will be labelled  $\hat{\rho}$  in the upcoming proof of Proposition 1.6 and shown in cases such as that depicted to have a significantly greater weight than  $\rho$ .

Abbreviate  $\rho = \rho_{n;(x,t_1)}^{(y,t_2)}$  and consult Figure 7. The polymer  $\rho$  may be split into two pieces,  $\rho[1]$  and  $\rho[2]$ , by cutting it at the point  $(Z, (1-a)t_1 + at_2)$ . If the big fluctuation event  $Z \leq z - \varphi$  occurs, then  $\rho[2]$  makes a rather sudden deviation, and its weight is dictated by parabolic curvature to be of order  $-r^2(1-a)^{1/3}t_{1,2}^{1/3}$ . We may consider the polymer  $\rho[3]$  whose endpoints are the cut location  $(Z, (1-a)t_1 + at_2)$  and  $(z - \varphi, t_2)$ . The latter endpoint being  $(z - \varphi, t_2)$  means that  $\rho[3]$  is not making the sudden deviation that  $\rho[2]$  does, so that its weight is dictated by local randomness to be a random unit-order multiple of  $(1-a)^{1/3}t_{1,2}^{1/3}$ . Consider now the polymer weight profile  $\text{Wgt}_{n;(x,t_1)}^{(v,t_2)}$  as a function of  $v$ . Between the locations  $v = y$  and  $v = z - \varphi$ , which is an order of  $\varphi$  to the left of  $y$ , the weight profile has risen by an order of least  $-r^2(1-a)^{1/3}t_{1,2}^{1/3}$ , because the weight profile at  $v = z - \varphi$  is at least the sum of the weights of  $\rho[1]$  and  $\rho[3]$ , while at  $v = z$ , it equals the sum of the weights of  $\rho[1]$  and  $\rho[2]$ .

Recalling that  $\varphi = t_{1,2}^{2/3}(1-a)^{2/3}r$ , we see that the weight profile is experiencing a change where a factor  $r^{3/2}$  multiplies the square-root  $\Theta(\varphi^{1/2})$  of the horizontal displacement in endpoint location. When  $r$  is large, this is incompatible with the one-half power law principle recalled above.



This idea is important, but it is not adequate to prove Proposition 1.6. To understand why the idea is not the end of the story, examine the form of Corollary 5.5, which articulates the one-half power law for the polymer weight profile. The quantity  $\epsilon$ , whose role is to measure horizontal displacement, must be at most  $2^{-4}$ . Indeed, the one-half power law does not govern the weight profile beyond a unit scale. This means that our idea will work only in a case that we may call *near*: when the quantity  $r$  is large, but not so large that the concerned endpoint displacement goes above a unit scale.

When  $r$  is large enough that the one-half power law estimate cannot be applied, we will note that the polymer weight profile change is unlikely for other reasons. Indeed, since  $r$  is so large, the observed change in the weight profile is very large. In fact, this change is so large that it either forces the profile at value  $v = y$  to be abnormally low or the profile at  $v = z - \varphi$  to be abnormally high. The improbability of these outcomes may then be gauged using the regular ensemble conditions Reg(2) and Reg(3).

In fact, even this is not the end of the story. These two regular ensemble conditions capture the parabolic curvature of the weight profile. But this curvature breaks down on an extremely large scale, as witnessed by the upper bounds on  $|z|$  needed in Reg(2) and Reg(3). This second argument thus works when  $r$  is very large, but not extremely large (where the latter term involves an  $n$  dependence). For this reason, this second argument will be said to apply in the *middle-distant* case.

When  $r$  is extremely high, the linear (but not parabolic) collapse near infinity of the polymer weight profile expressed by Proposition 4.3(4) will replace the use of Reg(3). This third argument applies in what we will call the *far* case.

This then will be the structure of the argument, with three different estimates treating the near, middle-distant and far cases, in increasing order of the value of  $r$ .

**Proof of Proposition 1.6.** Our task is to understand the fluctuation of the polymer  $\rho_{n;(x,t_1)}^{(y,t_2)}$  at time  $(1-a)t_1 + at_2$ . Note first that, since the endpoints are given, we may harmlessly suppose that this polymer is unique, because, as we discussed in Section 1.6, the complementary event has zero probability. Note also that we are working in a case where  $a$  is close to one; specifically,  $a \geq 1/2$ . (The remark following the proposition treats the opposing case.)

For  $K$  a real interval, we write  $\text{Fluc}_{n;(x,t_1)}^{(y,t_2)}(a; K)$  for the event that  $X_{n;(x,t_1)}^{(y,t_2)}(a) \in K$ . We omit the parameters that are fixed from this notation, writing for example  $\text{Fluc}[s, t] = \text{Fluc}_{n;(x,t_1)}^{(y,t_2)}(a; [s, t])$  for  $(s, t) \in \mathbb{R}_{<}^2$ .

We will argue that, under the proposition's hypotheses,

$$\mathbb{P}\left(\text{Fluc}(-\infty, -r]\right) \leq 22Cr \exp\{-10^{-11}c_1r^{3/4}\}, \quad (38)$$

and that the same bound holds on  $\mathbb{P}(\text{Fluc}[r, \infty))$ . Indeed, since the event  $\neg \text{PolyDevReg}_{n;(x,t_1)}^{(y,t_2)}(a, r)$  equals the union of  $\text{Fluc}(-\infty, -r)$  and  $\text{Fluc}(r, \infty)$ , these bounds suffice to prove Proposition 1.6.

To derive the two bounds, it suffices to prove (38). The two parts of the next proposition and the following lemma are the three principal tools: they treat the near, middle-distant and far cases. The reader should bear in mind that, in these assertions and later ones, it is supposed, as it was in Proposition 1.6, that  $(n, t_1, t_2)$  is a compatible triple, and that  $at_{1,2} \in n^{-1}\mathbb{Z}$ .

**Proposition 7.1.** *Set  $\hat{n} = n(1-a)t_{1,2}$ . Suppose that*

$$\hat{n} \geq 10^{29}c^{-18}, \quad r \in \left[8 \vee 87(1-a)^{1/3}t_{1,2}^{-2/3}|x-y|, 3\hat{n}^{1/36}\right] \quad (39)$$

and

$$|x-y|t_{1,2}^{-2/3} \leq 2^{-8/3}3^{-1}c(nt_{1,2})^{1/18}. \quad (40)$$

(1) *Suppose in addition that*

$$r \in \left[2800, \frac{1}{18}(1-a)^{-2/3}\right]$$

and  $a \geq 1 - 10^{-3}$ . Then

$$\mathbb{P}\left(\text{Fluc}[-(r+1), -r]\right) \leq (10178 + 3r)C \exp\left\{-c_1 2^{-25}3^{-3}5^{3/2}r^{9/4}\right\}$$

(2) *Now suppose in addition to (39) and (40) that  $r \geq 6(1-a)^{-1/2}$  and  $a \geq 1 - 10^{-2}$ . Then*

$$\mathbb{P}\left(\text{Fluc}[-(r+1), -r]\right) \leq (148C + 9r/4) \exp\left\{-c_1(1-a)^{1/2}3^{-3/2}5^{3/2}2^{-33/4}r^3\right\}.$$

**Lemma 7.2.** *Suppose that  $a \in (1/2, 1)$ , and that  $nt_{1,2}$  is at least the maximum of*

$$(1-a)^{-25}t_{1,2}^{-24}|x-y|^{36}, \quad c^{-9}t_{1,2}^{-6}|x-y|^9 \quad \text{and} \quad 10^5(1-a)^{-17}c^{-12}.$$

Then

$$\mathbb{P}\left(\text{Fluc}(-\infty, -2\hat{n}^{1/36})\right) \leq 13C \exp\left\{-2^{-15/4}c(1-a)^{25/12}t_{1,2}^{1/12}n^{1/12}\right\}.$$

We now derive (38) using this proposition and lemma. The interval  $(-\infty, -r]$  may be partitioned into a *far* interval  $(-\infty, -2\hat{n}^{1/36}]$  and a collection of disjoint unit intervals of the form  $(-t-1, -t]$  where  $t-r$  varies over a finite initial sequence of integers; i.e.,  $t = -r$ , then  $t = -r-1$ , and so on. (In fact, one of the unit intervals may overlap with the far interval.) The unit intervals to the left of  $-\frac{1}{18}(1-a)^{-2/3}$  will be called *middle-distant*, and the remaining ones, closer to the origin, will be called *near*. To each of these intervals  $I$  corresponds an event  $\text{Fluc}I$ . In the case of the far interval, the event's probability is bounded above by Lemma 7.2. Proposition 7.1(2) provides the bound for the middle-distant intervals, and Proposition 7.1(1) for the near intervals. Note that Proposition 7.1(2) is applicable to the middle-distant intervals because the condition  $\frac{1}{18}(1-a)^{-2/3} \geq 5(1-a)^{-1/2}$ , i.e.,  $a \geq 1 - 90^{-6}$ , is implied by Proposition 1.6's hypothesis that  $a \geq 1 - 10^{-11}c_1^2$  alongside  $c_1 \leq 1/8$ .

We are summing the right-hand sides of both parts of Proposition 7.1. The next two lemmas provide an upper bound on these sums. The calculational proofs appear in the appendix.

**Lemma 7.3.** *Suppose that  $r \geq 10^7c_1^{-4/5}$ . We have that*

$$\sum (10178 + 3r)C \exp\left\{-c_1 2^{-25}3^{-3}5^{3/2}s^{9/4}\right\} \leq 8Cr \exp\left\{-c_1 10^{-8}r^{9/4}\right\},$$

where the sum is taken over values of  $s \in \mathbb{R}$  that satisfy  $s \in r + \mathbb{N}$ .

**Lemma 7.4.** *Suppose now that  $r \geq \frac{1}{18}(1-a)^{-2/3} \vee 15C^{1/2}$  as well as  $a \geq 1 - 10^{-10}c_1^2$ . Setting  $\alpha = (5/3)^{3/2}2^{-33/4}$ , we have that*

$$\begin{aligned} & \sum (148C + 9s/4) \exp\left\{-c_1(1-a)^{3/2}(5/3)^{3/2} \cdot 2^{-33/4}s^3\right\} \\ & \leq \exp\left\{-10^{-2}c_1\alpha r^{9/4}\right\}, \end{aligned}$$

where the sum is again over  $s \in r + \mathbb{N}$ .

The bound (38) may now be obtained using Lemma 7.2 alongside the two preceding lemmas. Indeed, we obtain an upper bound on  $\mathbb{P}(\text{Fluc}(-\infty, -r])$  of the form

$$9Cr \exp \left\{ -10^{-8} c_1 r^{3/4} \right\} + 13C \exp \left\{ -2^{-15/4} c_1 (1-a)^{25/12} t_{1,2}^{1/12} n^{1/12} \right\}$$

because this expression is, in view of  $C \geq 1$ ,  $r \geq 1$  and  $c_1 \leq c$ , an upper bound on the sum of the right-hand sides in Lemmas 7.2, 7.3 and 7.4. That this expression is under the hypotheses of Proposition 1.6 at most the right-hand side of (38) is a calculational matter. This assertion is stated as Lemma C.3 in the appendix and derived there.

This completes the proof of (38), and Proposition 1.6, subject to confirming Proposition 7.1 and Lemma 7.2.

We now prepare to prove this proposition and lemma. Recall throughout that we are supposing that  $a \geq 1/2$ .

Recalling notation that was used in the overview that preceded the present proof, we write  $\varphi = t_{1,2}^{2/3} (1-a)^{2/3} r$ . We also set  $Z = \rho_{n;(x,t_1)}^{(y,t_2)}((1-a)t_1 + at_2)$  and  $z = \ell_{(x,t_1)}^{(y,t_2)}((1-a)t_1 + at_2)$ .

Abbreviate  $\rho = \rho_{n;(x,t_1)}^{(y,t_2)}$ . We now split  $\rho$  at the point  $(Z, (1-a)t_1 + at_2)$  according to polymer splitting as described in Section 5.6. Indeed, we may specify  $\rho[1]$  and  $\rho[2]$  to be the polymers  $\rho_{n;(x,t_1)}^{(Z, (1-a)t_1 + at_2)}$  and  $\rho_{n;(Z, (1-a)t_1 + at_2)}^{(y,t_2)}$ . Thus,  $\rho = \rho[1] \circ \rho[2]$ . Let  $\rho[3]$  denote the polymer  $\rho_{n;(Z, (1-a)t_1 + at_2)}^{(z-\varphi, t_2)}$ . Let  $\hat{\rho} = \rho[1] \circ \rho[3]$ . Thus,  $\rho$  and  $\hat{\rho}$  are two polymers that begin from  $(x, t_1)$  and follow a shared course until time  $(1-a)t_1 + at_2$ . They continue on possibly separate trajectories until a shared ending time  $t_2$ , with  $\rho$  then reaching  $y$  and  $\hat{\rho}$  reaching  $z - \varphi = y + (1-a)(x - y) - r(1-a)^{2/3} t_{1,2}^{2/3}$ .

The next lemma is a tool to prove the near and middle-distant Proposition 7.1. It asserts that, when  $\text{Fluc}[-(r+1), -r]$  occurs, the polymer  $\rho[3]$  is unlikely to have a significantly negative weight, while  $\rho[2]$  is unlikely to have a weight much exceeding a certain constant multiple of  $-r^2$ ; thus, when  $r$  is chosen so that either the near and middle-distant case obtains,  $\rho[3]$  is typically much heavier than  $\rho[2]$  on the event  $\text{Fluc}[-(r+1), -r]$ .

**Lemma 7.5.** *Suppose that the hypotheses (39) and (40) hold.*

(1) *We have that*

$$\mathbb{P}\left(\text{Fluc}[-(r+1), -r], t_{1,2}^{-1/3} (1-a)^{-1/3} \text{Wgt}(\rho[3]) \leq -2^{-1/2} - s\right) \leq 140C \exp\{-c_1 s^{3/2}\}.$$

(2) *We also have*

$$\begin{aligned} & \mathbb{P}\left(\text{Fluc}[-(r+1), -r], t_{1,2}^{-1/3} (1-a)^{-1/3} \text{Wgt}(\rho[2]) \geq -2^{-1/2} (7r/8)^2 + s\right) \\ & \leq (3r/8 + 1) \cdot 6C \exp\{-2^{9/2} c s^{3/2}\}. \end{aligned}$$

**Proof.** Note that

$$\text{Wgt}(\rho[3]) = \text{Wgt}_{n;(Z, (1-a)t_1 + at_2)}^{(z-\varphi, t_2)} = \mathcal{L}_{n;(1-a)t_1 + at_2}^{\downarrow; (z-\varphi, t_2)}(1, Z) = (1-a)^{1/3} t_{1,2}^{1/3} \text{Nr} \mathcal{L}_{n;(1-a)t_1 + at_2}^{\downarrow; (z-\varphi, t_2)}(1, V)$$

where  $V = (1-a)^{-2/3}t_{1,2}^{-2/3}(Z - (z - \varphi))$ . Note that the event  $\text{Fluc}[-(r+1), -r]$  is characterized by  $-(r+1) \leq (1-a)^{-2/3}t_{1,2}^{-2/3}(Z - z) \leq -r$  and thus also by the condition that  $V \in [-1, 0]$ . Thus,

$$\begin{aligned} & \mathbb{P}\left(\text{Fluc}[-(r+1), -r], t_{1,2}^{-1/3}(1-a)^{-1/3}\text{Wgt}(\rho[3]) \leq -2^{-1/2} - s\right) \\ &= \mathbb{P}\left(\text{Fluc}[-(r+1), -r], \text{Nr}\mathcal{L}_{n;(1-a)t_1+at_2}^{\downarrow;(z-\varphi,t_2)}(1, V) \leq -2^{-1/2} - s\right) \\ &\leq \mathbb{P}\left(\text{Fluc}[-(r+1), -r], \text{Nr}\mathcal{L}_{n;(1-a)t_1+at_2}^{\downarrow;(z-\varphi,t_2)}(1, V) + 2^{-1/2}V^2 \leq -s\right) \\ &\leq \mathbb{P}\left(\inf_{-1 \leq v \leq 0} \left(\text{Nr}\mathcal{L}_{n;(1-a)t_1+at_2}^{\downarrow;(z-\varphi,t_2)}(1, v) + 2^{-1/2}v^2\right) \leq -s\right) \\ &\leq \left(1/2 \vee 5 \vee 5^{1/2}(3 - 2^{3/2})^{-1}\right) 10C \exp\{-c_1 s^{3/2}\} \leq 140C \exp\{-c_1 s^{3/2}\}, \end{aligned}$$

so that we obtain Lemma 7.5(1). The final inequality was obtained by applying Proposition 4.3(2) with the parameter choices  $\mathbf{k} = 1$ ,  $\mathbf{y} = -1/2$ ,  $\mathbf{t} = 1/2$  and  $\mathbf{r} = s$  to the  $(c, C)$ -regular ensemble  $\text{Nr}\mathcal{L}_{n;(1-a)t_1+at_2}^{\downarrow;(z-r,t_2)}$ . The ensemble has  $\hat{n} + 1$  curves, where recall that we denote  $\hat{n} = n(1-a)t_{1,2}$ . As such, this application of Proposition 4.3(2) may be made provided that

$$1/2 \leq c/2 \cdot \hat{n}^{1/18}, 1/2 \leq \hat{n}^{1/18}, s \in [2^{3/2}, 2\hat{n}^{1/18}] \text{ and } \hat{n} \geq 1 \vee (c/3)^{-18} \vee 6^{36}.$$

We have the equality

$$\text{Wgt}(\rho[2]) = (1-a)^{1/3}t_{1,2}^{1/3} \text{Nr}\mathcal{L}_{n;(1-a)t_1+at_2}^{\downarrow;(y,t_2)}(1, V') \quad (41)$$

where we set  $V'$  equal  $(1-a)^{-2/3}t_{1,2}^{-2/3}(Z - y)$ . Indeed, the polymer weight  $\text{Wgt}(\rho[2])$  equals  $\text{Wgt}_{n;(Z,(1-a)t_1+at_2)}^{\downarrow;(y,t_2)} = \mathcal{L}_{n;(1-a)t_1+at_2}^{\downarrow;(y,t_2)}(1, Z)$ , whence we obtain (41).

We claim that, when  $r \geq 2 \vee 8(1-a)^{1/3}t_{1,2}^{-2/3}|x - y|$ ,

$$\text{Fluc}[-(r+1), -r] \subseteq \left\{ V' \in r \cdot [-13/8, -7/8] \right\}. \quad (42)$$

To see this, note that, when  $\text{Fluc}[-(r+1), -r]$  occurs,

$$Z \in z - (1-a)^{2/3}t_{1,2}^{2/3} \cdot [r, r+1] = y + (1-a)(x - y) - (1-a)^{2/3}t_{1,2}^{2/3} \cdot [r, r+1]$$

and, in light of our lower bound on  $r$ ,

$$Z \in y - (1-a)^{2/3}t_{1,2}^{2/3} \cdot [r - r/8, r + r/8 + 1] \subseteq y - \varphi \cdot [7/8, 13/8]$$

where we used  $r \geq 2$  in the form  $r + 1 \leq 3r/2$ . Thus, we confirm (42).

We thus obtain Lemma 7.5(2):

$$\begin{aligned} & \mathbb{P}\left(\text{Fluc}[-(r+1), -r], t_{1,2}^{-1/3}(1-a)^{-1/3}\text{Wgt}(\rho[2]) \geq -2^{-1/2}(7r/8)^2 + s\right) \\ &= \mathbb{P}\left(\text{Fluc}[-(r+1), -r], \text{Nr}\mathcal{L}_{n;(1-a)t_1+at_2}^{\downarrow;(y,t_2)}(1, V') \geq -2^{-1/2}(7r/8)^2 + s\right) \\ &\leq \mathbb{P}\left(\text{Fluc}[-(r+1), -r], \text{Nr}\mathcal{L}_{n;(1-a)t_1+at_2}^{\downarrow;(y,t_2)}(1, V') + 2^{-1/2}(V')^2 \geq s\right) \\ &\leq \mathbb{P}\left(\sup_{-13r/8 \leq v \leq -7r/8} \left(\text{Nr}\mathcal{L}_{n;(1-a)t_1+at_2}^{\downarrow;(y,t_2)}(1, v) + 2^{-1/2}v^2\right) \geq s\right) \\ &\leq (3r/8 + 1) \cdot 6C \exp\{-2^{9/2}cs^{3/2}\}. \end{aligned}$$

Here, the final inequality is due to an application of Proposition 4.3(3) to the  $(c, C)$ -regular ensemble  $\text{Nr}\mathcal{L}_{n;(1-a)t_1+at_2}^{\downarrow;(y,t_2)}$  which shares a curve cardinality of  $\hat{n} + 1$  with the ensemble that we considered a few moments ago. Our parameter choice for the application is  $\mathbf{y} = -5r/4$ ,  $\mathbf{r} = 3r/8$  and  $\mathbf{t} = s$ . This application may be made provided that

$$5r/4 \leq c/2 \cdot \hat{n}^{1/9}, \quad 3r/8 \leq c/4 \cdot \hat{n}^{1/9}, \quad s \in [2^{7/2}, 2\hat{n}^{1/3}] \quad \text{and} \quad \hat{n} \geq c^{-18}.$$

This completes the proof of Lemma 7.5.  $\square$

**Proof of Proposition 7.1(1).** Here we implement the *near* case argument outlined before the proof of Proposition 1.6. Consider the event

$$\begin{aligned} & \text{Fluc}[-(r+1), -r] \cap \left\{ t_{1,2}^{-1/3}(1-a)^{-1/3} \text{Wgt}(\rho[3]) > -2^{-1/2} - s \right\} \\ & \cap \left\{ t_{1,2}^{-1/3}(1-a)^{-1/3} \text{Wgt}(\rho[2]) < -2^{-1/2}(7r/8)^2 + s \right\}. \end{aligned} \quad (43)$$

Note that  $\text{Wgt}(\rho[3]) - \text{Wgt}(\rho[2]) = \text{Wgt}(\hat{\rho}) - \text{Wgt}(\rho)$ .

Note that  $-2^{-1/2} - s = (-2^{-1/2}(7r/8)^2 + s) + s$  is solved by  $s = 3^{-1} \cdot 2^{-1/2}((7r/8)^2 - 1)$ . Setting the value of  $s$  in this way, note that  $s \geq 5/3 \cdot 2^{-7/2}r^2$  since  $r \geq 4$ . We find then that the  $\mathbb{P}$ -probability of the above event is bounded above by

$$\mathbb{P}\left(t_{1,2}^{-1/3}(1-a)^{-1/3}(\text{Wgt}(\hat{\rho}) - \text{Wgt}(\rho)) \geq dr^2\right), \quad (44)$$

where here we write  $d = 5/3 \cdot 2^{-7/2}$ .

As a temporary notation, we denote the *maximum weight difference*  $\text{Max}\Delta\text{Wgt}_{n;(I,t_1)}^{(J,t_2)}$  to be the supremum over  $x_1, x_2 \in I$  and  $y_1, y_2 \in J$  of  $|\text{Wgt}_{n;(x_1,t_1)}^{(y_1,t_2)} - \text{Wgt}_{n;(x_2,t_1)}^{(y_2,t_2)}|$ .

Set  $I = \{x\}$  and  $J = [y - 9\varphi/8, y]$ . We claim that, when  $r \geq 8(1-a)^{1/3}t_{1,2}^{-2/3}|x-y|$ ,

$$\left\{ t_{1,2}^{-1/3}(1-a)^{-1/3}(\text{Wgt}(\hat{\rho}) - \text{Wgt}(\rho)) \geq dr^2 \right\} \subseteq \left\{ \text{Max}\Delta\text{Wgt}_{n;(I,t_1)}^{(J,t_2)} \geq dt_{1,2}^{1/3}(1-a)^{1/3}r^2 \right\}. \quad (45)$$

Indeed, note that  $\text{Wgt}_{n;(x,t_1)}^{(z-\varphi,t_2)} \geq \text{Wgt}(\hat{\rho})$  while  $\text{Wgt}_{n;(x,t_1)}^{(y,t_2)} = \text{Wgt}(\rho)$ . Moreover, the lower bound on  $r$  ensures that  $z - \varphi \geq y - 9\varphi/8$ , so that  $\text{Max}\Delta\text{Wgt}_{n;(I,t_1)}^{(J,t_2)} \geq \text{Wgt}(\hat{\rho}) - \text{Wgt}(\rho)$ . This implies (45).

Since  $\varphi = t_{1,2}^{2/3}(1-a)^{2/3}r$ , the right-hand event in (45) equals

$$t_{1,2}^{-1/3} \text{Max}\Delta\text{Wgt}_{n;(I,t_1)}^{(J,t_2)} \geq (8/9)^{1/2} dr^{3/2} \cdot (9t_{1,2}^{-2/3}\varphi/8)^{1/2}. \quad (46)$$

We seek to bound above the probability of this event by using Theorem 5.4. Two obstacles are that this result is stated for the special case where the start time is zero and the end time is one, and the presence of parabolic terms  $Q$  in its statement. Regarding the first difficulty, we use the scaling principle from Section 5.1 to obtain a general form for Theorem 5.4 which will be applicable to the time pair  $(t_1, t_2)$  with which we work. The conclusion of this general form of the result is that

$$\mathbb{P}\left(\sup_{\substack{u_1, u_2 \in [x, x + \epsilon t_{1,2}^{2/3}] \\ v_1, v_2 \in [y, y + \epsilon t_{1,2}^{2/3}]}} \left| t_{1,2}^{-1/3} \text{Wgt}_{n;(u_2,t_1)}^{(v_2,t_2)} + Q\left(\frac{v_2 - u_2}{t_{1,2}^{2/3}}\right) - t_{1,2}^{-1/3} \text{Wgt}_{n;(u_1,t_1)}^{(v_1,t_2)} - Q\left(\frac{v_1 - u_1}{t_{1,2}^{2/3}}\right) \right| \geq \epsilon^{1/2} R \right)$$

is at most  $10032 C \exp \{ -c_1 2^{-19} R^{3/2} \}$ ; while the new result's hypotheses are those of Theorem 5.4 after making the replacements  $n \rightarrow nt_{1,2}$  and  $(x, y) \rightarrow t_{1,2}^{-2/3} \cdot (x, y)$ .

We will find an upper bound on the event (46) by applying this general version of Theorem 5.4, with parameter choices with  $\epsilon = 9t_{1,2}^{-2/3} \varphi/8$ , which also equals  $9(1-a)^{2/3}r/8$ ; as well as  $\mathbf{x} = x$ ,  $\mathbf{y} = y - 9\varphi/8$ , and  $\mathbf{R} = 2^{-1}(8/9)^{1/2}dr^{3/2}$ . In order that this indeed provide such an upper bound, we must first overcome the second obstacle just mentioned, confirming that parabolic curvature is suitably controlled. The relevant bound is

$$\sup_{\substack{u_1, u_2 \in [\mathbf{x}, \mathbf{x} + \epsilon t_{1,2}^{2/3}] \\ v_1, v_2 \in [\mathbf{y}, \mathbf{y} + \epsilon t_{1,2}^{2/3}]}} |Q(t_{1,2}^{-2/3}(v_2 - u_2)) - Q(t_{1,2}^{-2/3}(v_1 - u_1))| \leq 2^{-1}(8/9)^{1/2}dr^{3/2} \cdot (9t_{1,2}^{-2/3}\varphi/8)^{1/2}.$$

This bound holds because this supremum is at most

$$2\epsilon t_{1,2}^{2/3} \cdot 2^{1/2}(|\mathbf{y} - \mathbf{x}| + 2\epsilon t_{1,2}^{2/3})t_{1,2}^{-4/3} = 2^{-3/2} \cdot 9(1-a)^{2/3}r \left( |x - y|t_{1,2}^{-2/3} + 9(1-a)^{2/3}r/4 \right).$$

Expanding the right-hand bracket, the resulting two terms are both at most  $4^{-1}(8/9)^{1/2}dr^{3/2} \cdot (9t_{1,2}^{-2/3}\varphi/8)^{1/2}$  (which suffices for our purpose) respectively provided that

$$r \geq 3^3 2^4 5^{-1} (1-a)^{1/3} |x - y| t_{1,2}^{-2/3} \quad \text{and} \quad 1-a \leq 5 \cdot 3^{-5} 2^{-2}$$

since  $t_{1,2}^{-2/3} \varphi = (1-a)^{2/3}r$  and  $d = 5/3 \cdot 2^{-7/2}$ .

We further mention that, for the application in question to be made, it suffices that

$$9(1-a)^{2/3}r/8 \leq 2^{-4}, \quad nt_{1,2} \geq 10^{29}c^{-18},$$

$$|x - y|t_{1,2}^{-2/3} + 9(1-a)^{2/3}r/8 \leq 2^{-5/3}3^{-1}c(nt_{1,2})^{1/18} \quad \text{and} \quad 2^{1/2}/3 \cdot dr^{3/2} \in [10^4, 10^3(nt_{1,2})^{1/18}],$$

as well as  $nt_{1,2}$  being even.

With these pieces in place, we apply the general version of Theorem 5.4, learning that the event in (46) has probability at most

$$10032 C \exp \{ -c_1 2^{-25} 3^{-3} 5^{3/2} r^{9/4} \}.$$

Recall that this quantity is thus an upper bound on the probability of the event (43). Combining this information with the two parts of Lemma 7.5, we find that

$$\begin{aligned} \mathbb{P} \left( \text{Fluc} [-(r+1), -r] \right) &\leq 140C \exp \{ -c_1 s^{3/2} \} + (3r/8 + 1) \cdot 6C \exp \{ -2^{9/2} c s^{3/2} \} \\ &\quad + 10032 C \exp \{ -c_1 2^{-25} 3^{-3} 5^{3/2} r^{9/4} \}. \end{aligned}$$

Recalling that  $s \geq 5/3 \cdot 2^{-7/2}r^2$ , we obtain

$$\begin{aligned} \mathbb{P} \left( \text{Fluc} [-(r+1), -r] \right) &\leq (146 + 9r/4)C \exp \{ -c_1 2^{-21/4} 3^{-3/2} 5^{3/2} r^3 \} \\ &\quad + 10032 C \exp \{ -c_1 2^{-25} 3^{-3} 5^{3/2} r^{9/4} \} \\ &\leq (10178 + 3r)C \exp \{ -c_1 2^{-25} 3^{-3} 5^{3/2} r^{9/4} \}, \end{aligned}$$

where we used  $c_1 \leq c$  in the first inequality and  $r \geq 1$  in the second. This completes the proof of Proposition 7.1(1).  $\square$

**Proof of Proposition 7.1(2).** We now provide an alternative upper bound on the probability of the event in the display beginning at (43), one that treats the middle-distant case. Recalling that

this probability is at most (44), we will find a new upper bound on the probability (44). Recalling that we set  $d = 5/3 \cdot 2^{-7/2}$ , the occurrence of the event whose probability is in question entails that

$$\begin{aligned} \text{either} \quad & t_{1,2}^{-1/3} \text{Wgt}(\hat{\rho}) \geq 2^{-1} d (1-a)^{1/3} r^2 - 2^{-1/2} (x-y)^2 t_{1,2}^{-4/3} \\ \text{or} \quad & t_{1,2}^{-1/3} \text{Wgt}(\rho) \leq -2^{-1} d (1-a)^{1/3} r^2 - 2^{-1/2} (x-y)^2 t_{1,2}^{-4/3}. \end{aligned}$$

We now state and prove a claim regarding the probabilities of these two outcomes.

**Claim: (1).** The former outcome has  $\mathbb{P}$ -probability at most  $C \exp \{ -c(1-a)^{1/2} 5^{3/2} 3^{-3/2} 2^{-33/4} r^3 \}$ .  
**(2).** The latter outcome has  $\mathbb{P}$ -probability at most  $C \exp \{ -c(1-a)^{1/2} 5^{3/2} 3^{-3/2} 2^{-27/4} r^3 \}$ .

First we prove Claim (2). Since  $\text{Wgt}_{n;(x,t_1)}^{(y,t_2)} = \text{Wgt}(\rho)$ , the latter outcome entails

$$\text{Nr}\mathcal{L}_{n;t_1}^{\downarrow;(y,t_2)}(1, \omega') + 2^{-1/2} \omega'^2 \leq -2^{-1} d (1-a)^{1/3} r^2$$

where we set  $\omega' = t_{1,2}^{-2/3} (x-y)$ . The probability of the outcome is thus bounded above by

$$\begin{aligned} & \mathbb{P} \left( \text{Nr}\mathcal{L}_{n;t_1}^{\downarrow;(y,t_2)}(1, \omega') + 2^{-1/2} \omega'^2 \leq -5/3 \cdot 2^{-9/2} (1-a)^{1/3} r^2 \right) \\ & \leq C \exp \{ -c(1-a)^{1/2} (5/3)^{3/2} \cdot 2^{-27/4} r^3 \}, \end{aligned}$$

where the displayed inequality is a consequence of an application of one-point lower tail Reg(2) with the ensemble equalling the  $(nt_{1,2} + 1)$ -curve  $\text{Nr}\mathcal{L}_{n;t_1}^{\downarrow;(y,t_2)}$ , and where the parameters are chosen to be  $\mathbf{z} = \omega'$  and  $\mathbf{s} = 5/3 \cdot 2^{-9/2} (1-a)^{1/3} r^2$ . This application may be made when  $|\omega'| \leq c(nt_{1,2})^{1/9}$  and  $5/3 \cdot 2^{-9/2} (1-a)^{1/3} r^2 \in [1, (nt_{1,2})^{1/3}]$ .

We now derive Claim (1). This is an assertion about  $\hat{\rho}$ , an  $n$ -zigzag with starting and ending points  $(x, t_1)$  and  $(z - \varphi, t_2)$ . Note that the quadratic correction term in the assertion is  $2^{-1/2} (y-x)^2 t_{1,2}^{-4/3}$ , whereas we would like to instead work with  $2^{-1/2} (z - \varphi - x)^2 t_{1,2}^{-4/3}$ , this being the natural quadratic correction associated to the starting and ending points of  $\hat{\rho}$ . We begin then by noting that the difference is suitably small:

$$\left| 2^{-1/2} (y-x)^2 t_{1,2}^{-4/3} - 2^{-1/2} (z - \varphi - x)^2 t_{1,2}^{-4/3} \right| \leq 4^{-1} d (1-a)^{1/3} r^2. \quad (47)$$

An explicit condition that suffices to ensure this bound is that  $r \geq 11(1-a)^{1/3} |y-x| t_{1,2}^{-2/3}$  and  $a \geq 1 - 10^{-2}$ . See Lemma C.4.

We see from (47) and  $\text{Wgt}_{n;(x,t_1)}^{(z-\varphi,t_2)} \geq \text{Wgt}(\hat{\rho})$  that the outcome that Claim (1) concerns entails that

$$\text{Nr}\mathcal{L}_{n;t_1}^{\downarrow;(z-\varphi,t_2)}(1, \omega) + 2^{-1/2} \omega^2 \geq 4^{-1} d (1-a)^{1/3} r^2$$

where now we set  $\omega = t_{1,2}^{-2/3} (x-z+\varphi)$ . Next we apply one-point upper tail Reg(3) to the  $(nt_{1,2} + 1)$ -curve ensemble  $\text{Nr}\mathcal{L}_{n;t_1}^{\downarrow;(z-\varphi,t_2)}$  with parameter settings  $\mathbf{z} = \omega$  and  $\mathbf{s} = 4^{-1} d (1-a)^{1/3} r^2$ . We learn that

$$\mathbb{P} \left( \text{Nr}\mathcal{L}_{n;t_1}^{\downarrow;(z-\varphi,t_2)}(1, \omega) + 2^{-1/2} \omega^2 \geq 4^{-1} d (1-a)^{1/3} r^2 \right) \leq C \exp \left\{ -c(1-a)^{1/2} (5/3)^{3/2} \cdot 2^{-33/4} r^3 \right\}.$$

This application of Reg(3) may be made when  $|\omega| \leq c(nt_{1,2})^{1/9}$  and  $(1-a)^{1/3} \cdot 5/3 \cdot 2^{-11/2} r^2 \geq 1$ . We have proved the Claim.

The sum of the two expressions in Claim (1) and (2) the sought alternative upper bound on the probability of the event in the display beginning at (43). Combining again with the two parts of Lemma 7.5, we find that

$$\begin{aligned} \mathbb{P}\left(\text{Fluc}[-(r+1), -r]\right) &\leq 140C \exp\{-c_1 s^{3/2}\} + (3r/8 + 1) \cdot 6C \exp\{-2^{9/2} c s^{3/2}\} \\ &\quad + 2C \exp\{-c(1-a)^{1/2} 5^{3/2} 3^{-3/2} 2^{-33/4} r^3\}. \end{aligned}$$

Recalling that  $s \geq 5/3 \cdot 2^{-7/2} r^2$ , we obtain

$$\begin{aligned} \mathbb{P}\left(\text{Fluc}[-(r+1), -r]\right) &\leq (146 + 9r/4)C \exp\{-c_1 2^{-21/4} 3^{-3/2} 5^{3/2} r^3\} \\ &\quad + 2C \exp\{-c(1-a)^{1/2} 3^{-3/2} 5^{3/2} 2^{-33/4} r^3\} \\ &\leq (148C + 9r/4) \exp\{-c_1 (1-a)^{1/2} 3^{-3/2} 5^{3/2} 2^{-33/4} r^3\}, \end{aligned}$$

where we used  $c_1 \leq c$  in the both inequalities. This completes the proof of Proposition 7.1(2).  $\square$

**Proof of Lemma 7.2.** Now we treat the far case. In outline, the one-point lower tail  $\text{Reg}(2)$  shows that neither the weight of  $\rho[1]$  nor of  $\rho[2]$  can be extremely low, except with a tiny probability. In the opposing case, the polymer weight  $\text{Wgt}(\rho) = \text{Wgt}(\rho[1]) + \text{Wgt}(\rho[2])$  is not highly negative. However, this eventuality is unlikely in the far case, where  $Z$  is very much less than  $z$ , due to collapse near infinity Proposition 4.3(4).

Naturally, here we are continuing to denote  $Z = \rho((1-a)t_1 + at_2)$  and  $z = (1-a)x + ay$ . To begin the formal argument, note that the far case event  $\text{Fluc}(-\infty, -2\hat{n}^{1/36}]$  occurs if and only if  $Z - z \leq -t_{1,2}^{2/3} (1-a)^{2/3} \cdot 2\hat{n}^{1/36}$ . Note that  $\hat{n}^{1/36} \geq (1-a)^{1/3} t_{1,2}^{-2/3} |x-y|$  implies that  $|z-y| \leq \hat{n}^{1/36}$ .  $t_{1,2}^{2/3} (1-a)^{2/3}$ , so that the occurrence of  $\text{Fluc}(-\infty, -2\hat{n}^{1/36}]$  entails that  $Z - y \leq -t_{1,2}^{2/3} (1-a)^{2/3} \hat{n}^{1/36}$ . Using again the notation  $V' = (1-a)^{-2/3} t_{1,2}^{-2/3} (Z - y)$ , the latter condition is given by  $V' \leq -\hat{n}^{1/36}$ .

Recall that in (41), the polymer weight  $\text{Wgt}(\rho[2])$  is expressed in terms of  $\text{Nr}\mathcal{L}_{n;(1-a)t_1+at_2}^{\downarrow;(y,t_2)}(1, V')$ . We thus find that

$$\begin{aligned} &\mathbb{P}\left(\text{Fluc}(-\infty, -2\hat{n}^{1/36}], t_{1,2}^{-1/3} (1-a)^{-1/3} \text{Wgt}(\rho[2]) \geq (-2^{-1/2} + 2^{-5/2})(1-a)^{1/18} t_{1,2}^{1/18} n^{1/18}\right) \\ &\leq \mathbb{P}\left(\sup_{x \leq -\hat{n}^{1/36}} \text{Nr}\mathcal{L}_{n;(1-a)t_1+at_2}^{\downarrow;(y,t_2)}(1, x) \geq -(2^{-1/2} - 2^{-5/2})(1-a)^{1/18} t_{1,2}^{1/18} n^{1/18}\right). \end{aligned} \quad (48)$$

Recall that  $\hat{n} = n(1-a)t_{1,2}$ . To find an upper bound on the probability (48), we now apply collapse near infinity Proposition 4.3(4) to the  $(c, C)$ -regular  $(\hat{n} + 1)$ -curve ensemble  $\text{Nr}\mathcal{L}_{n;(1-a)t_1+at_2}^{\downarrow;(y,t_2)}$ . We set the parameter  $\boldsymbol{\eta}$  so that  $\boldsymbol{\eta}(\hat{n} + 1)^{1/9} = \hat{n}^{1/36}$ . For the application to be made, it is enough that

$$(n(1-a)t_{1,2})^{-1/12} \leq c \quad \text{and} \quad n(1-a)t_{1,2} \geq (2^{5/4} c^{-1})^9.$$

Note then that

$$\ell(-\hat{n}^{-1/12}(\hat{n} + 1)^{1/9}) = (-2^{-1/2} + 2^{-5/2})(\hat{n}^{-1/12})^2 (\hat{n} + 1)^{2/9} = (-2^{-1/2} + 2^{-5/2})(\hat{n}^{-1/12})^2 (\hat{n} + 1)^{2/9}.$$

Since  $\ell(x)$  is increasing for  $x \leq 0$ , we find that  $\ell(-\hat{n}^{-1/36}) \leq -(2^{-1/2} - 2^{-5/2})\hat{n}^{1/18}$ . Thus, the expression (48) is found to be at most

$$6C \exp\left\{-c\boldsymbol{\eta}^3 2^{-15/4} (\hat{n} + 1)^{1/3}\right\} = 6C \exp\left\{-2^{-15/4} c \hat{n}^{1/12}\right\}. \quad (49)$$



Set  $\tilde{V} = a^{-2/3}t_{1,2}^{-2/3}(Z - x)$ . Note that

$$\text{Wgt}(\rho[1]) = a^{1/3}t_{1,2}^{1/3}\text{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow;(1-a)t_1+at_2}(1, \tilde{V}), \quad (50)$$

since both of these quantities equal  $\text{Wgt}_{n;(x,t_1)}^{(Z,(1-a)t_1+at_2)}$ .

We claim that

$$\text{Fluc}(-\infty, -2\hat{n}^{1/36}] \subseteq \left\{ \tilde{V} \leq -(1-a)^{2/3}\hat{n}^{1/36} \right\}. \quad (51)$$

Indeed, note that the left-hand event entails that

$$Z - x \leq a|x - y| - t_{1,2}^{2/3}(1-a)^{2/3} \cdot 2\hat{n}^{1/36} \leq -t_{1,2}^{2/3}(1-a)^{2/3}\hat{n}^{1/36}$$

since  $|x - y| \leq t_{1,2}^{2/3}(1-a)^{2/3}\hat{n}^{1/36}$  and  $a \leq 1$ . From another use of  $a \leq 1$ , we obtain (51).

From (50) and (51), we find that

$$\begin{aligned} & \mathbb{P}\left(\text{Fluc}(-\infty, -2\hat{n}^{1/36}], t_{1,2}^{-1/3}a^{-1/3}\text{Wgt}(\rho[1]) \geq -(2^{-1/2} - 2^{-5/2})(1-a)^{25/18}t_{1,2}^{1/18}n^{1/18}\right) \\ & \leq \mathbb{P}\left(\sup_{x \leq -(1-a)^{2/3}\hat{n}^{1/36}} \text{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow;(1-a)t_1+at_2}(1, x) \geq -(2^{-1/2} - 2^{-5/2})(1-a)^{25/18}t_{1,2}^{1/18}n^{1/18}\right). \end{aligned} \quad (52)$$

To find an upper bound on the latter probability, we apply Proposition 4.3(4) to the  $(c, C)$ -regular ensemble  $\text{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow;(1-a)t_1+at_2}$ . This ensemble has  $\tilde{n} + 1$  curves, where here we introduce the shorthand  $\tilde{n} = nat_{1,2}$ . We set the Proposition 4.3(4) parameter  $\eta$  so that  $\eta(\tilde{n} + 1)^{1/9} = (1-a)^{2/3}\hat{n}^{1/36}$ . Recalling that  $\hat{n} = n(1-a)t_{1,2}$ , we see that  $\eta \leq n^{-1/12}a^{-1/9}(1-a)^{25/36}t_{1,2}^{-1/12}$ . For Proposition 4.3(4) to be applied in this way, it suffices then that

$$n^{-1/12}a^{-1/9}(1-a)^{25/36}t_{1,2}^{-1/12} \leq c \text{ as well as } nat_{1,2} \geq (2^{5/4}c^{-1})^9.$$

Note then that

$$\begin{aligned} \ell(-(1-a)^{2/3}\hat{n}^{1/36}) &= -(2^{-1/2} - 2^{-5/2})\left((1-a)^{2/3}\hat{n}^{1/36}(\tilde{n} + 1)^{-1/9}\right)^2(\tilde{n} + 1)^{2/9} \\ &= -(2^{-1/2} - 2^{-5/2})(1-a)^{25/18}t_{1,2}^{1/18}n^{1/18}. \end{aligned}$$

Also using that  $\ell(x)$  is increasing for  $x \leq -(1-a)^{2/3}\hat{n}^{1/36}$ , we see that our application of Proposition 4.3(4) implies that the probability (52) is at most

$$6C \exp\left\{-c\eta^3 2^{-15/4}(\tilde{n} + 1)^{1/3}\right\} = 6C \exp\left\{-2^{-15/4}c(1-a)^{25/12}t_{1,2}^{1/12}n^{1/12}\right\}. \quad (53)$$

Note that  $t_{1,2}^{1/3}\text{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(1, t_{1,2}^{-2/3}(y-x)) = \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(1, y) = \text{Wgt}(\rho) = \text{Wgt}(\rho[1]) + \text{Wgt}(\rho[2])$ . Thus,

$$\begin{aligned} & \mathbb{P}\left(t_{1,2}^{-1/3}a^{-1/3}\text{Wgt}(\rho[1]) \leq -(2^{-1/2} - 2^{-5/2})(1-a)^{25/18}t_{1,2}^{1/18}n^{1/18},\right. \\ & \quad \left.t_{1,2}^{-1/3}(1-a)^{-1/3}\text{Wgt}(\rho[2]) \leq -(2^{-1/2} - 2^{-5/2})(1-a)^{1/18}t_{1,2}^{1/18}n^{1/18}\right) \\ & \leq \mathbb{P}\left(\text{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow}(1, t_{1,2}^{-2/3}(y-x)) \leq -(2^{-1/2} - 2^{-5/2})\left((1-a)^{25/18}a^{1/3} + (1-a)^{1/18+1/3}\right)t_{1,2}^{1/18}n^{1/18}\right) \\ & \leq C \exp\left\{-c(2^{-1/2} - 2^{-5/2})^{3/2}\left((1-a)^{25/18}a^{1/3} + (1-a)^{17/18}\right)^{3/2}t_{1,2}^{1/12}n^{1/12}\right\}, \end{aligned} \quad (54)$$

where in the latter inequality, we applied one-point lower tail  $\text{Reg}(2)$  to the  $(nt_{1,2}+1)$ -curve ensemble  $\text{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}$ . In this application, we take

$$\mathbf{z} = t_{1,2}^{-2/3}(y-x) \quad \text{and} \quad \mathbf{s} = (2^{-1/2} - 2^{-5/2})((1-a)^{25/18}a^{1/3} + (1-a)^{17/18})t_{1,2}^{1/18}n^{1/18}.$$

The application may be made provided that

$$|y-x|t_{1,2}^{-2/3} \leq c(nt_{1,2})^{1/9} \quad \text{and} \quad (2^{-1/2} - 2^{-5/2})((1-a)^{25/18}a^{1/3} + (1-a)^{17/18})t_{1,2}^{1/18}n^{1/18} \in [1, (nt_{1,2})^{1/3}].$$

Recalling the upper bounds on the probabilities in the first lines of the displays (48) and (52) offered by (49) and (53), and combining these with the bound (54), we find that  $\mathbb{P}(\text{Fluc}(-\infty, -2\hat{n}^{1/36}))$  is at most

$$\begin{aligned} & 6C \exp \left\{ -2^{-15/4}c(1-a)^{1/12}t_{1,2}^{1/12}n^{1/12} \right\} + 6C \exp \left\{ -2^{-15/4}c(1-a)^{25/12}t_{1,2}^{1/12}n^{1/12} \right\} \\ & + C \exp \left\{ -c(2^{-1/2} - 2^{-5/2})^{3/2} \left( (1-a)^{25/18}a^{1/3} + (1-a)^{17/18} \right)^{3/2} t_{1,2}^{1/12}n^{1/12} \right\} \\ & \leq 13C \exp \left\{ -2^{-15/4}c(1-a)^{25/12}t_{1,2}^{1/12}n^{1/12} \right\}. \end{aligned}$$

This completes the proof of Lemma 7.2.  $\square$

## APPENDIX A. MULTI-GEODESIC ORDERING

The multi-geodesic with given endpoints was defined in Section 3.1 and further discussed in Section 3.2, where its almost surely uniqueness was stated in Lemma 3.1. The principal aim of this appendix is to establish in Lemma A.1 that this multi-geodesic satisfies a natural monotonicity property, moving to the right in response to rightward displacement of endpoints. The multi-polymer counterpart, Lemma 5.6(1), will follow immediately.

We introduce two ordering relations on sets of staircases that share their starting and ending heights.

Let  $(i, j) \in \mathbb{Z}_{<}^2$  and  $(x_1, y_1), (x_2, y_2) \in \mathbb{R}_{<}^2$  satisfy  $x_1 \leq x_2$  and  $y_1 \leq y_2$ . Consider two staircases.  $\phi_1 \in SC_{i,j}(x_1, y_1)$  and  $\phi_2 \in SC_{i,j}(x_2, y_2)$ . We define two relations,  $\prec$  and  $\preceq$ , that  $\phi_1$  and  $\phi_2$  may satisfy.

- We say that  $\phi_1 \prec \phi_2$  if, whenever  $(z, t) \in \mathbb{R} \times [i, j]$  is an element of  $\phi_2$ , the open planar line segment that emanates rightwards from  $(z, t)$ , namely  $(z, \infty) \times \{t\}$ , is disjoint from  $\phi_1$ .
- We say that  $\phi_1 \preceq \phi_2$  if, whenever  $(z, t) \in \mathbb{R} \times [i, j]$  is an element of  $\phi_1$ , the closed planar line segment running rightwards from  $(z, t)$ ,  $[z, \infty) \times \{t\}$ , intersects  $\phi_2$ .

In fact, each of these two conditions is equivalent to the conditions seen in Section 5.5 when the statements there are made for zigzags, rather than staircases.

Two basic properties are readily verified:

- $\phi_1 \prec \phi_2$  implies  $\phi_1 \preceq \phi_2$ ;
- and  $\phi_1 \preceq \phi_2$  and  $\phi_2 \prec \phi_3$ , or for that matter  $\phi_1 \prec \phi_2$  and  $\phi_2 \preceq \phi_3$ , imply that  $\phi_1 \prec \phi_3$ .

When two staircases  $\phi_1$  and  $\phi_2$  share their pair  $(i, j) \in \mathbb{Z}_{<}^2$  of starting and ending heights, we may define their maximum  $\phi_1 \vee \phi_2$ . Indeed, suppose that  $\phi_1 \in SC_{i,j}(x_1, y_1)$  and  $\phi_2 \in SC_{i,j}(x_2, y_2)$  where  $(x_1, y_1), (x_2, y_2) \in \mathbb{R}_{<}^2$ ; contrary to a moment ago, we permit the cases that  $x_2 < x_1$  or  $y_2 < y_1$ . The maximum  $\phi_1 \vee \phi_2$  is a staircase from  $(x_1 \vee x_2, i)$  to  $(y_1 \vee y_2, j)$ . It is equal to the unique staircase

with these starting and ending points which is contained in the union of the horizontal and vertical segments of  $\phi_1$  and  $\phi_2$  and which dominates these two staircases in the  $\preceq$  ordering. To these two staircases is also associated a minimum  $\phi_1 \wedge \phi_2$ , a staircase that makes its way from  $(x_1 \wedge x_2, i)$  to  $(y_1 \wedge y_2, j)$ . It is analogously defined and is instead dominated by  $\phi_1$  and  $\phi_2$  according to  $\preceq$ .

**Lemma A.1.** *For  $k \in \mathbb{N}$ , let  $\bar{u}, \bar{v}, \bar{x}, \bar{y} \in \mathbb{R}_{\leq}^k$  be four non-decreasing lists such that  $\bar{v} - \bar{u} \in [0, \infty)^k$  and  $\bar{y} - \bar{x} \in [0, \infty)^k$ . Let  $(j_1, j_2) \in \mathbb{Z}_{<}^2$ . Suppose that these parameters are such that the two maximizers*

$$\left( P_{(\bar{u}, j_1 \mathbf{1}) \rightarrow (\bar{x}, j_2 \mathbf{1}); i}^k : i \in \llbracket 1, k \rrbracket \right) \quad \text{and} \quad \left( P_{(\bar{v}, j_1 \mathbf{1}) \rightarrow (\bar{y}, j_2 \mathbf{1}); i}^k : i \in \llbracket 1, k \rrbracket \right)$$

*exist uniquely. Then*

$$P_{(\bar{u}, j_1 \mathbf{1}) \rightarrow (\bar{x}, j_2 \mathbf{1}); i}^k \preceq P_{(\bar{v}, j_1 \mathbf{1}) \rightarrow (\bar{y}, j_2 \mathbf{1}); i}^k \text{ for } i \in \llbracket 1, k \rrbracket.$$

(In this appendix, we write  $\mathbf{1}$ , rather than  $\bar{\mathbf{1}}$ , for a  $k$ -tuple whose components equal one.)

This result permits us to derive Lemma 5.6.

**Proof of Lemma 5.6: (1).** This assertion is the multi-polymer counterpart to Lemma A.1 and follows immediately from this lemma.

**(2).** This result is a direct consequence of the definitions of the relations  $\prec$  and  $\preceq$ .  $\square$

**Proof of Lemma A.1.** For convenience, denote the two maximizers by  $(P_{1,i} : i \in \llbracket 1, k \rrbracket)$  and  $(P_{2,i} : i \in \llbracket 1, k \rrbracket)$  in such a way that we seek to show that  $P_{1,i} \preceq P_{2,i}$  for  $i \in \llbracket 1, k \rrbracket$ .

Suppose that this ordering property fails. Our plan is to argue that this assumption leads to a violation of the almost sure uniqueness of multi-geodesics with given endpoints asserted by Lemma 3.1.

Let  $i \in \llbracket 1, k \rrbracket$  be maximal such that  $P_{1,i} \not\preceq P_{2,i}$ . Let  $j \in \llbracket 1, i \rrbracket$  be minimal such that  $P_{1,j} \not\preceq P_{2,i}$ . (The stroke through the symbol,  $\preceq$ , and later  $\prec$ , is used to indicate that the relation does not hold.)

We now set the value of a parameter  $L$  in the interval  $\llbracket 0, j - 1 \rrbracket$ . If  $P_{1,j-1} \prec P_{2,i}$ , then  $L$  is set equal to zero. In the other case, we set  $L \in \llbracket 1, j - 1 \rrbracket$  so that

$$P_{1,j-m-1} \not\preceq P_{2,i-m} \text{ for each } m \in \llbracket 0, L - 1 \rrbracket,$$

and, if  $L \neq j - 1$ , then  $P_{1,j-L-1} \prec P_{2,i-L}$ .

Whatever the value of  $L$ , we now specify two  $k$ -vectors of staircases,  $(P'_{1,\ell} : \ell \in \llbracket 1, k \rrbracket)$  and  $(P'_{2,\ell} : \ell \in \llbracket 1, k \rrbracket)$ , according to the formulas

$$P'_{1,\ell} = \begin{cases} P_{1,\ell} & \ell \in \llbracket 1, k \rrbracket \setminus \llbracket j - L, j \rrbracket, \\ P_{1,j-m} \wedge P_{2,i-m} & \ell = j - m \text{ with } m \in \llbracket 0, L \rrbracket, \end{cases}$$

and

$$P'_{2,\ell} = \begin{cases} P_{2,\ell} & \ell \in \llbracket 1, k \rrbracket \setminus \llbracket i - L, i \rrbracket, \\ P_{1,j-m} \vee P_{2,i-m} & \ell = i - m \text{ with } m \in \llbracket 0, L \rrbracket. \end{cases}$$

Using such notation as  $P'_{1,\cdot} = (P'_{1,i} : i \in \llbracket 1, k \rrbracket)$ , we now make a claim with four parts.

*Claim.*

- (1)  $P'_{1,\cdot}$  does not equal  $P_{1,\cdot}$ , and nor does  $P'_{2,\cdot}$  equal  $P_{2,\cdot}$ .
- (2) The vector  $P'_{1,\cdot}$  is an element of  $D_{(x_1 \mathbf{1}, j_1 \mathbf{1}) \rightarrow (\bar{u}, j_2 \mathbf{1})}^k$ .

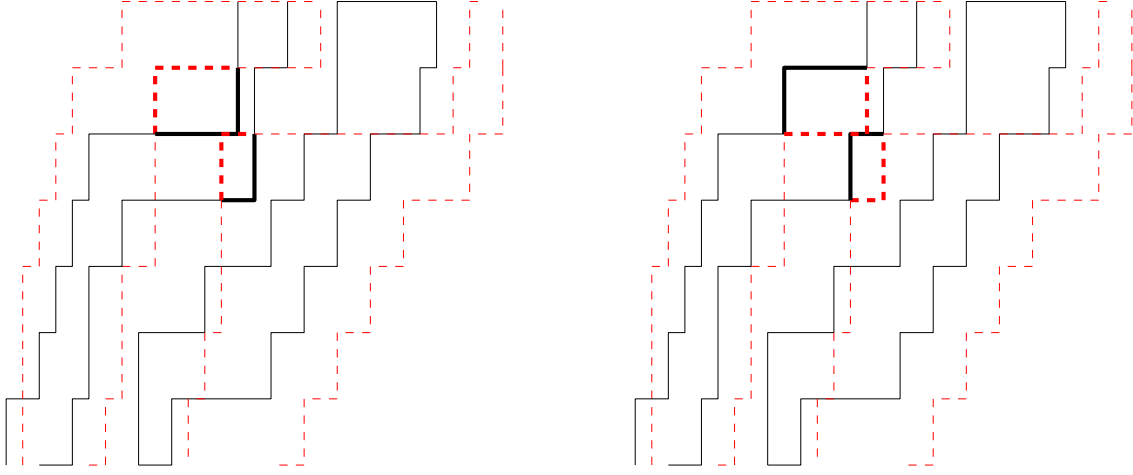


FIGURE 8. Illustrating the proof of Lemma A.1. In this example,  $\ell = 4$ . On the left, the consecutive staircases  $\{P_{1,i} : i \in \llbracket 1, 4 \rrbracket\}$  are indicated in unbroken black. The four components of  $P_{2,\cdot}$  are shown in dashed red. On the right, the  $P'$  counterparts are similarly indicated. We have  $i = 3$ ,  $j = 2$  and  $L = 1$ . Substaircases that change hands are highlighted. There are two such rearrangements. One leads to  $P'_{1,1} = P_{1,1} \wedge P_{2,2}$  and  $P'_{2,2} = P_{1,1} \vee P_{2,2}$ , and the other to  $P'_{1,2} = P_{1,2} \wedge P_{2,3}$  and  $P'_{2,3} = P_{1,2} \vee P_{2,3}$ . Otherwise,  $P'$  components coincide with their  $P$  counterparts.

(3) The vector  $P'_{2,\cdot}$  belongs to  $D^k_{(x_{1,j_1}\mathbf{1}) \rightarrow (\bar{v}, j_2\mathbf{1})}$ .

(4) These two vectors are maximizers of their respective sets.

We verify the four assertions in turn.

*Verifying Claim (1):* Since  $P'_{1,j} = P_{1,j} \wedge P_{2,i}$  and  $P_{1,j} \not\leq P_{2,i}$ , we see that, although  $P'_{1,j} \preceq P_{1,j}$ , we also have  $P'_{1,j} \neq P_{1,j}$ . Similarly, since  $P'_{2,i} = P_{1,j} \vee P_{2,i}$  while  $P_{1,j} \not\leq P_{2,i}$ , we see that, although  $P'_{2,i} \preceq P_{2,i}$ , we also have  $P'_{2,i} \neq P_{2,i}$ .

*Verifying Claim (2):* For each  $i \in \llbracket 1, k \rrbracket$ , the staircase  $P_{2,i}$  staircase begins and ends at or to the right of  $P_{1,i}$ . For this reason, each staircase  $P'_{1,i}$  shares its starting and ending points with  $P_{1,i}$ . In light of this, it is enough in order to show that  $P'_{1,\cdot} \in D^k_{(x_{1,j_1}\mathbf{1}) \rightarrow (\bar{u}, j_2\mathbf{1})}$  that it be demonstrated that  $P'_{1,\ell} \prec P'_{1,\ell+1}$  for each  $\ell \in \llbracket 0, k-1 \rrbracket$ . This statement follows from the ordering under  $\prec$  of the components of  $P_{1,\cdot}$  in the case that  $\ell \notin \llbracket j-L-1, j \rrbracket$ .

We have that  $P'_{1,j} \prec P'_{1,j+1}$  since  $P_{1,j} \prec P_{1,j+1}$ ; and, for  $m \in \llbracket 1, L \rrbracket$ , that  $P'_{1,j-m} \prec P'_{1,j-m+1}$  since  $P_{1,j-m} \prec P_{1,j-m+1}$  and  $P_{2,i-m} \prec P_{2,i-m+1}$ . Finally, when  $L < j-1$ , we have that  $P'_{1,j-L-1} \prec P'_{1,j-L}$  since  $P_{1,j-L-1} \prec P_{1,j-L}$  and  $P_{1,i-L-1} \prec P_{2,i-L}$  (an inequality which is indeed valid when  $L \neq j-1$ ). Note that when  $L = j-1$ ,  $j-L-1$  equals zero, so that the verification in the preceding sentence is not made in this case.

Thus, we verify that  $P'_{1,\cdot} \in D^k_{(x_{1,j_1}\mathbf{1}) \rightarrow (\bar{u}, j_2\mathbf{1})}$ .

*Verifying Claim (3):* For each  $i \in \llbracket 1, k \rrbracket$ ,  $P'_{2,i}$  shares its starting and ending points with  $P_{2,i}$ , for the same reason that  $P'_{1,i}$  and  $P_{1,i}$  do.

It is thus enough to verify that  $P'_{2,\cdot} \in D^k_{(x_1, j_1 \mathbf{1}) \rightarrow (\bar{v}, j_2 \mathbf{1})}$ . This is confirmed in view of:

- $P'_{2,\ell} \prec P'_{2,\ell+1}$  for  $\ell \in \llbracket 1, k \rrbracket \setminus \llbracket i-L, i \rrbracket$  since  $P_{2,\ell} \prec P_{2,\ell+1}$ ;
- $P'_{2,i} \prec P'_{2,i+1}$  since  $P_{2,i} \prec P_{2,i+1}$  and, by  $j < i+1$  and then  $i$ 's maximality,  $P_{1,j} \prec P_{1,i+1} \preceq P_{2,i+1}$ ;
- $P'_{2,i-m-1} \prec P'_{2,i-m}$  for  $m \in \llbracket 1, L-1 \rrbracket$  since  $P_{1,j-m-1} \prec P_{1,j-m}$  and  $P_{2,i-m-1} \prec P_{2,i-m}$ ;
- and  $P'_{2,i-L-1} \prec P'_{2,i-L}$  since  $P_{2,i-L-1} \prec P_{2,i-L}$ .

*Verifying Claim (4):* To argue that the two  $k$ -vectors  $P'_{1,\cdot}$  and  $P'_{2,\cdot}$  are maximizers of their respective sets, we first note that

$$W(P'_1) + W(P'_2) = W(P_1) + W(P_2).$$

Since  $P'_1 \in D^k_{(x_1, j_1 \mathbf{1}) \rightarrow (\bar{u}, j_2 \mathbf{1})}$  and  $P_1$  is a maximizer of this set,  $W(P'_1) \leq W(P_1)$ . Likewise,  $W(P'_2) \leq W(P_2)$ . This circumstance forces  $W(P'_1)$  to equal  $W(P_1)$  and  $W(P'_2)$  to equal  $W(P_2)$ . Thus,  $P'_1$  and  $P'_2$  is each a maximizer of the set to which it belongs.

The proof of the claim complete, we need now merely invoke the maximizer uniqueness Lemma 3.1 to learn that  $P'_1 = P_1$ , and also that  $P'_2 = P_2$ . However, we have seen that neither of these equalities holds. From this contradiction, we learn that  $P_{1,i} \preceq P_{2,i}$  for  $i \in \llbracket 1, k \rrbracket$ , as we sought to do.  $\square$

## APPENDIX B. NORMALIZED ENSEMBLES ARE REGULAR: DERIVING PROPOSITION 4.2

In order to prove Proposition 4.2, it is convenient to expand a little on the review in Section 4.1 of the definition of the scaled forward ensemble  $\mathcal{L}^{\uparrow; t_2}_{n; (x, t_1)}$  and its normalized counterpart. It is natural to view the first object as the scaled counterpart to an unscaled forward ensemble which we now specify.

To do so, let  $(m_1, m_2) \in \mathbb{N}_{\leq}^2$  and  $u \in \mathbb{R}$ . Define the unscaled forward ensemble

$$L^{\uparrow; m_2}_{(u, m_1)} : \llbracket 1, m_2 - m_1 + 1 \rrbracket \times [u, \infty) \rightarrow \mathbb{R}$$

with base-point  $(u, m_1)$  and end height  $m_2$  by insisting that, for each  $k \in \llbracket 1, m_2 - m_1 + 1 \rrbracket$  and  $v \geq u$ ,

$$\sum_{i=1}^k L^{\uparrow; m_2}_{(u, m_1)}(i, v) = M^k_{(u, m_1) \rightarrow (v, m_2)}.$$

The relation of this object to its scaled counterpart may be described by letting  $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}_{\leq}^2$  be a compatible triple, and  $x \in \mathbb{R}$ . The scaled forward ensemble

$$\mathcal{L}^{\uparrow; t_2}_{n; (x, t_1)} : \llbracket 1, nt_{1,2} + 1 \rrbracket \times [x - 2^{-1}n^{1/3}t_{1,2}, \infty) \rightarrow \mathbb{R}$$

is seen to satisfy, for each  $k \in \llbracket 1, nt_{1,2} + 1 \rrbracket$  and  $y \geq x - 2^{-1}n^{1/3}t_{1,2}$ , the identity

$$\mathcal{L}^{\uparrow; t_2}_{n; (x, t_1)}(k, y) = 2^{-1/2}n^{-1/3} \left( L^{\uparrow; nt_2}_{(nt_1 + 2n^{2/3}x, nt_1)}(k, nt_2 + 2n^{2/3}y) - 2nt_{1,2} - 2n^{2/3}(y - x) \right). \quad (55)$$

A further, parabolic, change of coordinates (9) then specifies the normalized forward ensemble  $\text{Nr}\mathcal{L}^{\uparrow; t_2}_{n; (x, t_1)}$  in terms of the scaled one  $\mathcal{L}^{\uparrow; t_2}_{n; (x, t_1)}$ .

A similar story holds for the backward ensemble. Indeed, considering now  $(m_1, m_2) \in \mathbb{N}_{\leq}^2$  and  $v \in \mathbb{R}$ , we may define the unscaled backward ensemble

$$L_{m_1}^{\downarrow;(v, m_2)} : \llbracket 1, m_2 - m_1 + 1 \rrbracket \times (-\infty, v] \rightarrow \mathbb{R}$$

with base-point  $(v, m_2)$  and end height  $m_1$  by insisting that, for each  $k \in \llbracket 1, m_2 - m_1 + 1 \rrbracket$  and  $u \leq v$ ,

$$\sum_{i=1}^k L_{m_1}^{\downarrow;(v, m_2)}(i, u) = M_{(u, m_1) \rightarrow (v, m_2)}^k.$$

Again fixing a compatible triple  $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}_{\leq}^2$ , and now letting  $y \in \mathbb{R}$ , we note that the scaled backward ensemble

$$\mathcal{L}_{n; t_1}^{\downarrow;(y, t_2)} : \llbracket 1, nt_{1,2} + 1 \rrbracket \times (-\infty, y + 2^{-1}n^{1/3}t_{1,2}] \rightarrow \mathbb{R}$$

satisfies, for each  $k \in \llbracket 1, nt_{1,2} + 1 \rrbracket$  and  $x \leq y + 2^{-1}n^{1/3}t_{1,2}$ ,

$$\mathcal{L}_{n; t_1}^{\downarrow;(y, t_2)}(k, x) = 2^{-1/2}n^{-1/3} \left( L_{nt_1}^{\downarrow;(nt_2 + 2n^{2/3}y, nt_2)}(k, nt_1 + 2n^{2/3}x) - 2nt_{1,2} - 2n^{2/3}(y - x) \right).$$

Now consider again a compatible triple  $(n, t_1, t_2)$  as well as  $x \in \mathbb{R}$ , and write  $N = nt_{1,2} + 1 \in \mathbb{N}$ . Consider the ensemble  $L_N : \llbracket 1, N \rrbracket \times [0, \infty) \rightarrow \mathbb{R}$ , given by

$$L_N(i, u) = L_{(nt_1 + 2n^{2/3}x, nt_1)}^{\uparrow; nt_2}(i, nt_1 + 2n^{2/3}x + u) \quad \text{for } (i, u) \in \llbracket 1, N \rrbracket \times [0, \infty).$$

By [OY02, Theorem 7], this ensemble has the law of an  $N$ -curve Dyson's Brownian motion each of whose curves begins at the origin at time zero. The ensemble has a scaled counterpart  $\mathcal{L}_N^{\text{sc}} : \llbracket 1, N \rrbracket \times [-2^{-1}N^{1/3}, \infty) \rightarrow \mathbb{R}$ , given by

$$\mathcal{L}_N^{\text{sc}}(i, u) = 2^{-1/2}N^{-1/3} \left( L_N(i, N + 2N^{2/3}u) - 2N - 2N^{2/3}u \right).$$

By [Ham17a, Proposition 2.5], we have the

*Key fact:* there exist constants  $C, c \in (0, \infty)$  such that, for all choices of the parameters  $(n, t_1, t_2, x)$ , the ensemble  $\mathcal{L}_N^{\text{sc}}$  is  $(c, C)$ -regular.

There is only one, very mundane, difference between the ensemble  $\mathcal{L}_N^{\text{sc}}$  and the normalized forward ensemble  $\text{Nr}\mathcal{L}_{n; (x, t_1)}^{\uparrow; t_2}$  that is the subject of Proposition 4.2. With our choice of  $N = nt_{1,2} + 1$ , the latter ensemble is simply equal to  $\mathcal{L}_{N-1}^{\text{sc}}$ . The key fact thus bears the essence of the proof of Proposition 4.2. The formal proof, which will now be given, is merely a matter of verifying that the adjustment of the parameter value  $N \rightarrow N - 1$  is inconsequential for the purpose of checking the conditions enjoyed by a  $(c, C)$ -regular ensemble.

**Proof of Proposition 4.2.** Note from (55) and (9) that  $\text{Nr}\mathcal{L}_{n; (x, t_1)}^{\uparrow; t_2}(k, z)$  equals

$$2^{-1/2}(nt_{1,2})^{-1/3} \left( L_{(nt_1 + 2n^{2/3}x, nt_1)}^{\uparrow; nt_2}(k, nt_2 + 2n^{2/3}x + 2(nt_{1,2})^{2/3}z) - 2nt_{1,2} - 2(nt_{1,2})^{2/3}z \right).$$

Note that

$$\begin{aligned}
& L_{(nt_1+2n^{2/3}x, nt_1)}^{\uparrow; nt_2}(k, nt_2 + 2n^{2/3}x + 2(nt_{1,2})^{2/3}z) - 2nt_{1,2} - 2(nt_{1,2})^{2/3}z \\
&= L_N(k, nt_{1,2} + 2(nt_{1,2})^{2/3}z) - 2nt_{1,2} - 2(nt_{1,2})^{2/3}z \\
&= L_N(k, N + 2N^{2/3}z(nt_{1,2}N^{-1})^{2/3} + nt_{1,2} - N) \\
&\quad - 2N - 2N^{2/3}z - 2(nt_{1,2} - N) - 2((nt_{1,2})^{2/3} - N^{2/3})z \\
&= L_N(k, N + 2N^{2/3}\tilde{z}) - 2N - 2N^{2/3}\tilde{z} + \phi_1,
\end{aligned}$$

where  $\tilde{z} = z(nt_{1,2}N^{-1})^{2/3} - 2^{-1}N^{-2/3}$  satisfies  $\tilde{z} = z(1 + O(N^{-1})) + O(N^{-2/3})$ . Recalling that  $N = nt_{1,2} + 1$ , the error  $\phi_1$  is seen to equal  $2 - 2((N - 1)^{2/3} - N^{2/3})z - 2N^{2/3}(z - \tilde{z})$  and thus to satisfy

$$\phi_1 = O(1) + |z|O(N^{-1/3}) + |z|O(N^{-1/3}) + O(1) = |z|O(N^{-1/3}) + O(1).$$

We see then that  $\text{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(k, z)$  equals

$$2^{-1/2}(nt_{1,2})^{-1/3} \left( L_N(k, N + 2N^{2/3}\tilde{z}) - 2N - 2N^{2/3}\tilde{z} + \phi_1 \right) = \mathcal{L}_N^{\text{sc}}(k, \tilde{z})\phi_2 - 2^{-1/2}(nt_{1,2})^{-1/3}\phi_1,$$

where  $\phi_2 = (nt_{1,2})^{-1/3}N^{1/3} = 1 + O(N^{-1})$ . We find then that

$$\text{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(k, z) = \mathcal{L}_N^{\text{sc}}(k, \tilde{z})(1 + O(N^{-1})) + |z|O(N^{-2/3}) + O(N^{-1/3}),$$

where  $\tilde{z}$  differs from  $z$  by  $|z|O(N^{-1}) + O(N^{-2/3})$ . We may thus note that

$$\begin{aligned}
& \text{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(k, z) + 2^{-1/2}z^2 \\
&= \mathcal{L}_N^{\text{sc}}(k, \tilde{z})(1 + O(N^{-1})) + 2^{-1/2}\tilde{z}^2 + |z|^2O(N^{-1}) + |z|O(N^{-2/3}) + |z|O(N^{-2/3}) + O(N^{-1/3}) \\
&= \mathcal{L}_N^{\text{sc}}(k, \tilde{z})(1 + O(N^{-1})) + 2^{-1/2}\tilde{z}^2 + |z|^2O(N^{-1}) + |z|O(N^{-2/3}) + O(N^{-1/3}).
\end{aligned}$$

When  $z = O(N^{1/2})$ , the triple sum of error terms in the last expression is  $O(1)$ . The three Reg properties hold for  $\mathcal{L}_N^{\text{sc}}$  by the key fact stated before the proof. We want to learn that these hold also for  $\text{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}$ . The bound on the triple sum error permits the passage of Reg(2) and Reg(3) from  $\mathcal{L}_N^{\text{sc}}$  to  $\text{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}$ , provided that  $C < \infty$  is increased, and  $c > 0$  decreased, in order that the case of small  $N$  is included. The property Reg(1) also makes the passage, because the left endpoint modulus  $z_{\mathcal{L}}$  differs by a factor of  $(nt_{1,2}N^{-1})^{1/3} = 1 + O(N^{-1})$  between the two ensembles; again,  $c > 0$  is being decreased in order to capture the case of small  $N$ . Thus, the ensemble  $\text{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}$  is  $(c, C)$ -regular, as we sought to show.  $\square$

## APPENDIX C. CALCULATIONAL DERIVATIONS

In Subsection 5.8.2, we explained that one aspect of the proofs of most of our results has been separated from the rest. These are the ‘calculational derivations’ of these results. Here, we present these derivations. In the case of each concerned result, it is now our task to verify that the result’s hypotheses are sufficient for the purpose of verifying every condition that is invoked during the course of the result’s proof.

There are six results for which a calculational derivation is owed. They are treated in the six ensuing sections. The first two sections treat the short derivations needed for the non-technical versions Theorems 1.1 and 1.4 of the polymer disjointness results. The third and fourth sections

treat Proposition 6.3 and Lemma 6.6, which are tools in the proof of Theorem 6.1. The calculational derivation of Theorem 6.1 appears in the fifth section. In the final section, we present the derivation of Proposition 1.6, including such a treatment for several results that were used in the proof of this proposition.

There is some variation in the manner in which the derivations are presented. A guideline of the format is that first all of the conditions invoked in the proof of the result in question are recorded. These conditions may then be split into lists, according to which parameters are being bounded. Each list is then separately simplified to produce a shorter list that implies the original. Sometimes in carrying out this simplification, new conditions, called *further conditions*, will be introduced. The simplified lists may then be compared, with some further simplifications being noted. The derivation ends when a final list is drawn up whose conditions coincide with the hypotheses of the result under review.

Some notational conventions govern the presentation of the derivations. Conditions are given square bracketed names such as [1] or [n4], shown on the left. These names may be recycled from one calculational derivation to the next. The notation [1, n4] means ‘conditions [1] and [n4]’. Implication is denoted by a right arrow, so that [1, n4]  $\rightarrow$  [r3] means ‘conditions [1] and [n4] imply condition [r3]’.

The notation [n2, 4, 5] is used as a shorthand for [n2, n4, n5]. This meaning would be ambiguous, were there named conditions [4] or [5], but we employ this condition only when the meaning is unambiguous.

**C.1. Theorem 1.1: derivation.** This result emerges from simplifying Theorem 6.1. The conditions (14) and (15) are replaced by  $\epsilon \leq G^{-4k^2}$  and  $nt_{1,2} \geq G^{k^2}(1 + |x - y|^{36}t_{1,2}^{-24})\epsilon^{-G}$  by using  $\liminf c_i^{1/i} > 0$ . The condition  $|y - x|t_{1,2}^{-2/3} \leq \epsilon^{-1/2}(\log \epsilon^{-1})^{-2/3} \cdot 10^{-8}c_k^{2/3}k^{-10/3}$  is replaced by  $|y - x|t_{1,2}^{-2/3} \leq \epsilon^{-1/2}(\log \epsilon^{-1})^{-2/3}G^{-k}$  in view of  $\liminf c_i^{1/i} > 0$ . The derivation also uses that

$$10^{32k^2}k^{15k^2}c_k^{-3k^2}C_k(\log \epsilon^{-1})^{4k^2} \exp\{\beta_k(\log \epsilon^{-1})^{5/6}\}$$

is bounded above by  $G^{k^3} \exp\{G^k(\log \epsilon^{-1})^{5/6}\}$ . This holds due to  $\limsup C_i^{1/i^2} < \infty$ ,  $\liminf c_i^{1/i} > 0$  and  $\limsup \beta_i^{1/i} < \infty$ .

**C.2. Theorem 1.4: derivation.** This theorem is obtained by simplifying Theorem 6.2. By such simplifications, which use  $c \leq 1$ , the next, intermediate, result is obtained.

**Theorem C.1.** *There exist constants  $G \geq g > 0$  such that the following holds. Let  $(t_1, t_2) \in \mathbb{R}_{<}^2$ ,  $x, y \in \mathbb{R}$ ,  $h \in \mathbb{N}$ , and  $m, n \in \mathbb{N}$ . Suppose that*

$$m \geq G \vee (|x - y|t_{1,2}^{-2/3} + 2h)^3,$$

and  $nt_{1,2} \geq G \max\{1, |x - y|t_{1,2}^{-2/3} + 2h\}^{36}m^G$ . Then

$$\mathbb{P}\left(\text{MaxDisjtPoly}_{n;(I,t_1)}^{(J,t_2)} \geq m\right) \leq m^{-g(\log \log m)^2}Q,$$

where  $I$  denotes the interval  $[x, x + ht_{1,2}^{2/3}]$  and  $J$  the interval  $[y, y + ht_{1,2}^{2/3}]$ . The quantity  $Q$  is a correction term that is bounded above by  $(Gh)^{G(\log \log m)^2}$ . Thus, if we further suppose that  $m \geq (Gh)^{2g^{-1}G}$ , then

$$\mathbb{P}\left(\text{MaxDisjtPoly}_{n;(I,t_1)}^{(J,t_2)} \geq m\right) \leq m^{-2^{-1}g(\log \log m)^2}.$$



An adjustment is now needed to arrive at Theorem 1.4. Note that the hypothesis of the latter claim in Theorem C.1, namely  $m \geq (Gh)^{2g^{-1}G}$ , is also a hypothesis in Theorem 1.4. Thus, the latter theorem holds with  $g/2$  replacing  $g$  in its conclusion. Theorem 1.4 follows by relabelling  $g$ .

**C.3. Proposition 6.3: derivation.** Proposition 6.3(1) hypotheses are that  $nt_{1,2}$  is at least  $t_{1,2}^{-6}|u_i - x|^9 c^{-9}$  for  $i \in \llbracket 1, k \rrbracket$ , and that  $r \geq k$ . These are invoked in order to permit the application of Reg(3) to the  $(nt_{1,2} + 1)$ -curve ensemble  $\text{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}$ .

We turn now to Proposition 6.3(2).

We begin by recording the various conditions that were invoked during the proof of this result.

Note first that the conditions

$$u_i \geq x - 2^{-1}n^{1/3}t_{1,2}, i \in \llbracket 1, k \rrbracket,$$

are assumed, in order that the various polymer or zigzag weights or such ensemble values as  $\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(1, u_i)$  are well defined.

In the application of Reg(2) to  $\text{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}$ , we suppose that

$$t_{1,2}^{-2/3}|u_i - x| \leq c(nt_{1,2})^{1/9} \text{ and } r \in [2k, 2k(nt_{1,2})^{1/3}].$$

In the application of Reg(3) to  $\text{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}$ , we suppose that

$$t_{1,2}^{-2/3}|u_i - x| \leq c(nt_{1,2})^{1/9} \text{ and } r \geq 4k(k-1).$$

In the  $k-1$  applications of Proposition 4.3(1) to  $\mathcal{L}_n = \text{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}$ , we suppose that

$$nt_{1,2} \geq k \vee (c/3)^{-18} \vee 6^{36}, t_{1,2}^{-2/3}|u_i - x| \leq 2^{-1}c(nt_{1,2})^{1/18} \text{ and } r \leq 8k(k-1)(nt_{1,2})^{1/18}.$$

We now record this set of conditions in a list:

- [0]  $u_i \geq x - 2^{-1}n^{1/3}t_{1,2}, i \in \llbracket 1, k \rrbracket$
- [1]  $t_{1,2}^{-2/3}|u_i - x| \leq c(nt_{1,2})^{1/9}, i \in \llbracket 1, k \rrbracket$
- [2]  $r \geq 2k$
- [3]  $r \leq 2k(nt_{1,2})^{1/3}$
- [4]  $t_{1,2}^{-2/3}|u_i - x| \leq c(nt_{1,2})^{1/9}, i \in \llbracket 1, k \rrbracket$
- [5]  $r \geq 4k(k-1)$
- [6]  $nt_{1,2} \geq k \vee (c/3)^{-18} \vee 6^{36}$
- [7]  $t_{1,2}^{-2/3}|u_i - x| \leq 2^{-1}c(nt_{1,2})^{1/18}$
- [8]  $r \leq 8k(k-1)(nt_{1,2})^{1/18}.$

*Further conditions:*

- [9]  $r \geq 4k^2$
- [10]  $r \leq 9k^2(nt_{1,2})^{1/3}$

Inferences:  $[1] = [4]$ .  $[9] \rightarrow [2, 5]$ .  $[10] \rightarrow 3$ .  $[6, 10] \rightarrow [8]$ .  $[6, 7] \rightarrow [0, 4]$ .

Thus,  $[6, 7, 9, 10] \rightarrow [0, \dots, 8]$ , so that  $[6, 7, 9, 10]$  may form the hypothesis list of Proposition 6.3(2).

Since [7] is equivalent to

$$nt_{1,2} \geq 2^{18}c^{-18}t_{1,2}^{-12}|u_i - x|^{18},$$

we see that the hypotheses of Proposition 6.3(2) may be chosen to be that  $nt_{1,2}$  is at least the maximum of  $k$ ,  $(c/3)^{-18}$ ,  $6^{36}$  and  $2^{18}c^{-18}t_{1,2}^{-12}|u_i - x|^{18}$  (where  $i$  varies over  $\llbracket 1, k \rrbracket$ ), and that  $r \in [4k^2, 9k^2(nt_{1,2})^{1/3}]$ . Since these are indeed this result's hypotheses, the calculational derivation of Proposition 6.3(2) is complete.

**C.4. Lemma 6.6: derivation.** We begin by recalling the various conditions that were invoked in applying other results in order to obtain this one.

In the application of Corollary 5.5 used to bound above  $\mathbb{P}\left(\neg \text{LocWgtReg}_{n;(I^+,0)}^{(J^+,1)}(2(r+1)\epsilon, r)\right)$ , these conditions are:

$$2(r+1)\epsilon \in (0, 2^{-4}], \quad n \geq 10^{29}c^{-18}, \quad |x - y| \leq 2^{-5/3}3^{-1}cn^{1/18} \quad \text{and} \quad r \in [2 \cdot 10^4, 10^3n^{1/18}].$$

In fact, the hypotheses that  $n$  is even and that  $|x - y| \leq (2(r+1)\epsilon)^{-1/2}$  are also used, and the bound  $\epsilon \leq 2^{-2/3}$  is invoked during the proof of Lemma 6.6. Since these two hypotheses are explicitly stated in Lemma 6.6, and the latter bound is implied by  $2(r+1)\epsilon \in (0, 2^{-4}]$ , we will not further record any of these three conditions.

In the application of Proposition 1.6, and the remark following this proposition, in order to bound above  $\mathbb{P}\left(\neg \text{PolyDevReg}_{n;(x,-\epsilon^{3/2})}^{(y,1+\epsilon^{3/2})}[\{0, 1\}, r]\right)$ , the conditions that enable the application are:

$$n \geq \max \left\{ 10^{29}(\epsilon^{3/2}/2)^{-25}c^{-18}, \quad 10^{24}c^{-18}(\epsilon^{3/2}/2)^{-25}|x - y|^{36}(1 + 2\epsilon^{3/2})^{-24} \right\},$$

$$r \geq \max \left\{ 10^9c_1^{-4/5}, \quad 15C^{1/2}, \quad 87(\epsilon^{3/2}/2)^{1/3}(1 + 2\epsilon^{3/2})^{-2/3}|x - y| \right\}$$

and

$$r \leq 3(2^{-1}\epsilon^{3/2})^{25/9}n^{1/36}(1 + 2\epsilon^{3/2})^{1/36}.$$

When Corollary 6.4 is applied to bound above

$$\mathbb{P}\left(\neg \text{ForBouqReg}_{n,k;(x,-\epsilon^{3/2})}^{(\bar{u},0)}(r)\right) + \mathbb{P}\left(\neg \text{BackBouqReg}_{n,k;(\bar{v},1)}^{(y,1+\epsilon^{3/2})}(r)\right),$$

the application makes use of

$$n\epsilon^{3/2} - 1 \geq \max \left\{ k, \quad 3^{18}c^{-18}, \quad 6^{36}, \quad c^{-9}(1 - n^{-1}\epsilon^{-3/2})^{-6}(1 + 2^{-1}n^{-2/3}\epsilon^{-1})^9, \right. \\ \left. 2^{18}c^{-18}(1 - n^{-1}\epsilon^{-3/2})^{-12}(1 + 2^{-1}n^{-2/3}\epsilon^{-1})^{18} \right\}$$

and

$$r \in [4k^2, 9k^2\epsilon^{1/2}(1 - n^{-1}\epsilon^{3/2})^{1/3}n^{1/3}].$$

Two applications of Corollary 5.3 are made. The conditions used are:

$$n\epsilon^{3/2} \geq 10^{29} \vee 2(c/3)^{-18}, \quad 3(r+1) \leq 3^{-1}2^{-2/3}cn^{1/18}\epsilon^{1/12} \quad \text{and} \quad r \in [(33)^{1/2}, 2\epsilon^{1/24}n^{1/36}].$$

(The condition that  $n\epsilon^{3/2}$  is an even integer is also used. The need for this condition has been noted in the proof of Theorem 6.1, of which Lemma 6.6 forms part, and it need not be mentioned further in this calculational derivation.)

Before we proceed, we mention that the conditions involved in the use of Corollary 6.4 are complicated by the use of a  $\epsilon^{3/2} - n^{-1}$  parameter in this application. We prefer to now find simpler sufficient conditions for this application.

The lower bounds on  $n\epsilon^{3/2} - 1$  are ensured by

$$n\epsilon^{3/2} \geq 2 \max \left\{ k, 3^{18}c^{-18}, 6^{36}, c^{-9}, 2^{18}c^{-18} \right\},$$

provided that  $n\epsilon^{3/2} \geq (1 - 2^{-1/2})^{-1}$  and  $(1 + 2^{-1}n^{-2/3}\epsilon^{-1})^{3/2}(1 - n^{-1}\epsilon^{-3/2}) \leq 2^{1/24}$ . These last conditions are implied by  $n\epsilon^{3/2} \geq 2 \cdot 6^{36}$  which appears in the display.

The bounds on  $r$  are implied by

$$r \in [4k^2, 8k^2\epsilon^{1/2}n^{1/3}]$$

when  $n\epsilon^{3/2} \geq 2 \cdot 6^{36}$  holds.

Thus, the two preceding displays may replace the listed conditions for the use of Corollary 6.4.

We now partition this collection of conditions into several lists:

- the  $[r]$  list contains bounds on  $r$  that do not involve  $n$ ;
- the  $[rn]$  list contains the remaining bounds on  $r$ , each of which is an upper bound in terms of  $n$ ;
- the  $[n]$  list contains the remaining bounds on  $n$ , each of which is a lower bound.

*The  $[r]$  list.* Lower bounds are listed first.

$$\begin{aligned} [r1] \quad r &\geq 2 \cdot 10^4 \\ [r2] \quad r &\geq 10^9 c_1^{-4/5} \\ [r3] \quad r &\geq 15C^{1/2} \\ [r4] \quad r &\geq 87(\epsilon^{3/2}/2)^{1/3}(1 + 2\epsilon^{3/2})^{-2/3}|x - y| \\ [r5] \quad r &\geq 4k^2 \\ [r6] \quad r &\geq (33)^{1/2} \\ [r7] \quad 2(r + 1)\epsilon &\leq 2^{-4} \end{aligned}$$

*The  $[rn]$  list.*

$$\begin{aligned} [rn1] \quad r &\leq 10^3 n^{1/18} \\ [rn2] \quad r &\leq 3(2^{-1}\epsilon^{3/2})^{25/9} n^{1/36}(1 + 2\epsilon^{3/2})^{1/36} \\ [rn3] \quad r &\leq 8k^2\epsilon^{1/2}n^{1/3} \\ [rn4] \quad 3(r + 1) &\leq 3^{-1}2^{-2/3}cn^{1/18}\epsilon^{1/12} \\ [rn5] \quad r &\leq 2\epsilon^{1/24}n^{1/36} \end{aligned}$$

*The  $[n]$  list.*

$$\begin{aligned}
[n1] \quad n &\geq 10^{29} c^{-18} \\
[n2] \quad |x - y| &\leq 2^{-5/3} 3^{-1} c n^{1/18} \\
[n3] \quad n &\geq 10^{29} (\epsilon^{3/2}/2)^{-25} c^{-18} \\
[n4] \quad n &\geq 10^{24} c^{-18} (\epsilon^{3/2}/2)^{-25} |x - y|^{36} (1 + 2\epsilon^{3/2})^{-24} \\
[n5] \quad n\epsilon^{3/2} &\geq 2k \\
[n6] \quad n\epsilon^{3/2} &\geq 2 \cdot 3^{18} c^{-18} \\
[n7] \quad n\epsilon^{3/2} &\geq 2 \cdot 6^{36} \\
[n8] \quad n\epsilon^{3/2} &\geq 2c^{-9} \\
[n9] \quad n\epsilon^{3/2} &\geq 2^{19} c^{-18} \\
[n10] \quad n\epsilon^{3/2} &\geq 10^{29} \\
[n11] \quad n\epsilon^{3/2} &\geq 2(c/3)^{-18}
\end{aligned}$$

### Simplifying the lists one-by-one.

We now provide calculations that lead to shorter presentations of the three lists. In each case, a shorter list will be provided whose conditions collectively imply all those on the original list.

*The [r] list: calculations.*

We consider two further conditions.

$$\begin{aligned}
[r8] \quad r &\geq 70\epsilon^{1/2}|x - y| \\
[r9] \quad \epsilon &\leq 2^{-5}(r + 1)^{-1}
\end{aligned}$$

Since  $11 \cdot 2^{-1/3} = 8.73 \dots$ ,  $[r8] \rightarrow [r4]$  since  $87 \cdot 2^{-1/3} = 69.05 \dots$ .

Since  $c_1 \leq 1$ ,  $[r2] \rightarrow [r1]$ .  $[r1] \rightarrow [r6]$ .  $[r7] = [r9]$ .

$$\begin{aligned}
[r1] \quad r &\geq 2 \cdot 10^4 \\
[r2] \quad r &\geq 10^9 c_1^{-4/5} \\
[r3] \quad r &\geq 15C^{1/2} \\
[r4] \quad r &\geq 11(\epsilon^{3/2}/2)^{1/3} (1 + 2\epsilon^{3/2})^{-2/3} |x - y| \\
[r5] \quad r &\geq 4k^2 \\
[r6] \quad r &\geq (33)^{1/2} \\
[r7] \quad 2(r + 1)\epsilon &\leq 2^{-4}
\end{aligned}$$

That is, the shorter [r] list, which implies the original, is  $[r2, 3, 5, 7, 8, 9]$ . It is implied by the conditions that

$$r \geq \max \left\{ 10^9 c_1^{-4/5}, 15C^{1/2}, 4k^2, 70\epsilon^{1/2}|x - y| \right\} \quad \text{and} \quad \epsilon \leq 2^{-5}(r + 1)^{-1}. \quad (56)$$

*The [rn] list: calculations.*

Each  $[rn]$  condition is an upper bound. We express each upper bound as a triple product of a numerical constant, a factor in  $\epsilon$  and a factor in  $n$ :

$$\begin{aligned} [rn1] \quad r &\leq 10^3 \cdot 1 \cdot n^{1/18} \\ [rn2] \quad r &\leq (3 \cdot 2^{-25/9}) \cdot \epsilon^{25/6} (1 + 2\epsilon^{3/2})^{1/36} \cdot n^{1/36} \\ [rn3] \quad r &\leq 8k^2 \cdot \epsilon^{1/2} \cdot n^{1/3} \\ [rn4] \quad r + 1 &\leq 3^{-2} 2^{-2/3} c \cdot \epsilon^{1/12} \cdot n^{1/18} \\ [rn5] \quad r &\leq 2 \cdot \epsilon^{1/24} \cdot n^{1/36} \end{aligned}$$

Since  $\epsilon \leq 1$ ,  $n \geq 1$ ,  $k \geq 1$  and  $c \leq 1$ , these conditions are all implied by

$$r + 1 \leq 3^{-2} 2^{-2/3} c \epsilon^{25/6} n^{1/36}.$$

Since  $r \geq 1$  follows from the  $[r]$  list, a sufficient condition may be chosen to be  $r \leq 3^{-2} 2^{-5/3} c \epsilon^{25/6} n^{1/36}$  or, since,  $3^2 2^{5/3} = 14.28 \dots$ , more simply,

$$r \leq 2^{-4} c \epsilon^{25/6} n^{1/36}. \quad (57)$$

*The  $[n]$  list: calculations.*

We express each condition as a lower bound on  $n$  given by a product of a numerical constant, an  $\epsilon$ -dependent term and a power of  $|x - y|$ :

$$\begin{aligned} [n1] \quad n &\geq 10^{29} c^{-18} \\ [n2] \quad n &\geq 2^{30} 3^{18} c^{-18} \cdot |x - y|^{18} \\ [n3] \quad n &\geq 10^{29} 2^{25} c^{-18} \cdot \epsilon^{-75/2} \\ [n4] \quad n &\geq 10^{24} 2^{25} c^{-18} \cdot \epsilon^{-75/2} (1 + 2\epsilon^{3/2})^{-24} \cdot |x - y|^{36} \\ [n5] \quad n &\geq 2k \cdot \epsilon^{-3/2} \\ [n6] \quad n &\geq 2 \cdot 3^{18} c^{-18} \cdot \epsilon^{-3/2} \\ [n7] \quad n &\geq 2 \cdot 6^{36} \cdot \epsilon^{-3/2} \\ [n8] \quad n &\geq 2c^{-9} \cdot \epsilon^{-3/2} \\ [n9] \quad n &\geq 2^{19} c^{-18} \cdot \epsilon^{-3/2} \\ [n10] \quad n &\geq 10^{29} \cdot \epsilon^{-3/2} \\ [n11] \quad n &\geq 2 \cdot 3^{18} c^{-18} \cdot \epsilon^{-3/2} \end{aligned}$$

Since  $c \leq 1$ ,  $\epsilon \in (0, 1)$ , all these conditions are implied by

$$n \geq 10^{29} 2^{25} k c^{-18} \epsilon^{-75/2} \max \{1, |x - y|^{36}\}$$

Note that the inclusion of 1 among the quantities over which we take a maximum has permitted to assume that  $|x - y|$  is at least one. Since  $2^{25} = 3.3 \dots \times 10^7$ , we see that the  $[n]$  list has been reduced to

$$n \geq 10^{37} k c^{-18} \epsilon^{-75/2} \max \{1, |x - y|^{36}\} \quad (58)$$

*Summary.* We earlier noted that the hypotheses that  $n$  is even and that  $|x - y| \leq (2(r+1)\epsilon)^{-1/2}$  are invoked in the proof of Lemma 6.6. these conditions, alongside the short lists (56), (57) and (58), comprise the hypotheses of Lemma 6.6. This completes the calculational derivation of the lemma.

**C.5. Theorem 6.1: derivation.** First recall that by the scaling principle this result must be validated only in the case that  $t_1 = 0$  and  $t_2 = 1$ . For this reason, we are seeking to show that the set of the hypotheses of the result, after we set  $t_{1,2}$  equal to one, is adequate for validating all of the conditions during the proof of the theorem.

We will begin by recalling all of these conditions. However, we first mention the mesh membership condition  $n\epsilon^{3/2} \in 2\mathbb{N}$ , which is invoked during the proof of the theorem, but which is not stated as a hypothesis. This matter is discussed early in the proof of the theorem, after the FavSurCon event is defined, and we do not discuss it further here.

Now the needed conditions. For the application of Corollary 5.2, we use:

$$\begin{aligned} 15kr^2\epsilon^{1/2} &< (\eta_0)^{k^2} \\ k &\geq 2 \\ (1 + 2\epsilon^{3/2})n/2 &\geq k \vee (K_0)^{k^2} (\log \epsilon^{-1})^{K_0} \\ (1 + 2\epsilon^{3/2})^{-2/3}|y - x| &\leq a_0 n^{1/9}. \end{aligned}$$

For the application of Proposition 6.7, we use:

$$\begin{aligned} n\epsilon^{3/2} &\geq 10^2 \\ |x - y| &\leq \epsilon^{-1/2} \\ r &\geq 1 \end{aligned}$$

For the application of Lemma 6.6, we use:

$$\begin{aligned} n &\geq 10^{37}kc^{-18}\epsilon^{-75/2} \max\{1, |x - y|^{36}\} \\ |x - y| &\leq (2(r + 1)\epsilon)^{-1/2} \\ r &\geq \max\left\{10^9c_1^{-4/5}, 15C^{1/2}, 4k^2, 70\epsilon^{1/2}|x - y|\right\} \\ \epsilon &\leq 2^{-5}(r + 1)^{-1} \\ r &\leq 2^{-4}c\epsilon^{25/6}n^{1/36}. \end{aligned}$$

The value of  $r$  is set equal to:

$$r = 10^{44/3}k^4c_k^{-4/3}2^{-4/3}(k^2 - 1)^{4/3}(\log \epsilon^{-1})^{4/3}$$

It satisfies

$$[r] \quad r \leq 10^{15}c_k^{-4/3}k^{20/3}(\log \epsilon^{-1})^{4/3}.$$

We begin our analysis by labelling the conditions used in the application of Corollary 5.2 and analysing them. The other conditions will then be drawn in.

$$\begin{aligned} [1] \quad &15kr^2\epsilon^{1/2} < (\eta_0)^{k^2} \\ [2] \quad &k \geq 2 \\ [3] \quad &(1 + 2\epsilon^{3/2})n/2 \geq k \vee (K_0)^{k^2} (\log \epsilon^{-1})^{K_0} \\ [4] \quad &(1 + 2\epsilon^{3/2})^{-2/3}|y - x| \leq a_0 n^{1/9} \end{aligned}$$

*Further conditions:*

$$\begin{aligned} [5] \quad &15 \cdot 10^{30}c_k^{-8/3}k^{43/3}(\log \epsilon^{-1})^4\epsilon^{1/2} < (\eta_0)^{k^2} \\ [6] \quad &\epsilon^{1/4} < (\eta_0)^{k^2} \end{aligned}$$

$$\begin{aligned}
[7] \quad & 15 \cdot 10^{30} c_k^{-8/3} k^{43/3} (\log \epsilon^{-1})^4 \leq \epsilon^{-1/4} \\
[8] \quad & (\log \epsilon^{-1})^4 \leq \epsilon^{-1/8} \\
[9] \quad & \epsilon < (34!)^{-16} \\
[10] \quad & 15 \cdot 10^{30} c_k^{-8/3} k^{43/3} \leq \epsilon^{-1/8} \\
[11] \quad & \epsilon \leq 15^{-8} \cdot 10^{-240} c_k^{64/3} k^{-344/3} \\
[12] \quad & \epsilon \leq (\eta_0)^{4k^2} \\
[13] \quad & \epsilon \leq 10^{-616} c_k^{22} k^{-115}
\end{aligned}$$

Inferences:

[5]  $\rightarrow$  [1] due to [r].

[6, 7]  $\rightarrow$  [5].

[9]  $\rightarrow$  [8] as follows:  $(\log \epsilon^{-1})^4 \leq \epsilon^{-1/8}$  is equivalent to  $\epsilon^{-1} \leq \exp \{ \epsilon^{-1/32} \}$ . Since  $\exp \{ \epsilon^{-1/32} \} \geq (34!)^{-1} \epsilon^{-17/16}$ , the last condition is implied by  $\epsilon \leq (34!)^{-16}$ .

[8, 10]  $\rightarrow$  [7]

[11]  $\rightarrow$  [10]

[12] = [6]

[13]  $\rightarrow$  [9] since  $c_k \leq 1$  and  $(34!)^{16} = 3.33 \dots \times 10^{615}$  (which is implied by  $34! = 2.952 \dots \times 10^{38}$ ).

[13]  $\rightarrow$  [11] since  $c_k \leq 1$

Thus, [12, 13]  $\rightarrow$  [1].

Also consider:

$$\begin{aligned}
[3'] \quad & n \geq 2k \vee 2(K_0)^{k^2} (\log \epsilon^{-1})^{K_0} \\
[4'] \quad & |y - x| \leq a_0 n^{1/9}
\end{aligned}$$

[3']  $\rightarrow$  [3] and [4']  $\rightarrow$  [4].

Thus, [2, 3', 4', 12, 13]  $\rightarrow$  [1, 2, 3, 4].

*Analysing bounds involving r.* Next we focus on bounds on  $r$  among the applied conditions, beyond [1]. These bounds are:

$$\begin{aligned}
[r1] \quad & r \geq 1 \\
[r2] \quad & r \geq \max \left\{ 10^9 c_1^{-4/5}, 15C^{1/2}, 4k^2, 70\epsilon^{1/2}|x - y| \right\} \\
[r3] \quad & r \leq 2^{-4} c \epsilon^{25/6} n^{1/36} . \\
[r\epsilon] \quad & \epsilon \leq 2^{-5} (r + 1)^{-1} \\
[rxy\epsilon] \quad & |x - y| \leq (2(r + 1)\epsilon)^{-1/2}
\end{aligned}$$

*Further conditions:*

$$\begin{aligned}
[r4] \quad & n \geq 2^{144} r^{36} c^{-36} \epsilon^{-150} \\
[r5] \quad & n \geq 2^{144} 10^{540} c_k^{-48} k^{240} (\log \epsilon^{-1})^{72} c^{-36} \epsilon^{-150} \\
[r6] \quad & n \geq 10^{584} c_k^{-48} k^{240} c^{-36} \epsilon^{-222}
\end{aligned}$$

$$\begin{aligned}
[r7] \quad \epsilon &\leq 2^{-6} r^{-1} \\
[r8] \quad \epsilon &\leq 2^{-6} \cdot 10^{-15} c_k^{4/3} k^{-20/3} (\log \epsilon^{-1})^{-2} \\
[r9] \quad \epsilon^{15/16} &\leq 2^{-6} \cdot 10^{-12} c_k^{4/3} k^{-20/3} \\
[r10] \quad \epsilon &\leq 10^{-15} c_k^2 k^{-8} \\
[xy\epsilon] \quad |x - y| &\leq \epsilon^{-1/2} (\log \epsilon^{-1})^{-2/3} 10^{-8} c_k^{2/3} k^{-10/3}
\end{aligned}$$

Inferences:  $[r4] = [r3]$ .  $[r5] \rightarrow [r4]$  by recalling that

$$[r] \quad r \leq 10^{15} c_k^{-4/3} k^{20/3} (\log \epsilon^{-1})^{4/3}.$$

$[r6] \rightarrow [r5]$  due to  $2^{144} \in [10^{43}, 10^{44}]$  and  $\log \epsilon^{-1} \leq \epsilon^{-1}$ .

$[r2] \rightarrow [r1]$  since  $c_1 \leq 1$ .  $[r1, r7] \rightarrow [r\epsilon]$ .

$[r8] \rightarrow [r7]$  by upper bound  $[r]$ .

$[r9] \rightarrow [r8]$  due to  $\log \epsilon^{-1} \leq \epsilon^{-1/32}$  by [8].

$[r10] \rightarrow [r9]$  due to  $(2^6 10^{12})^{16/15} = 5.32 \dots \times 10^{14}$  and  $c_k \leq 1$ .

$[13] \rightarrow [r10]$ .

Since  $k \geq 1$ ,  $\epsilon \leq e^{-1}$  and  $c_k \leq 1$ ,  $[r]$  implies that  $[rxy\epsilon]$  follows from

$$|x - y| \leq \left( 10^{16} c_k^{-4/3} k^{20/3} (\log \epsilon^{-1})^{4/3} \epsilon \right)^{-1/2}$$

which is equivalent to  $[xy\epsilon]$ .

Thus, given that we have chosen  $r$  so that  $[r]$  holds, we have:  $[r2, r6, 13, xy\epsilon] \rightarrow [r1, r3, r\epsilon, rxy\epsilon]$ .

To discuss  $[r2]$ , we label one of the remaining applied conditions:

$$[r11] \quad |x - y| \leq \epsilon^{-1/2}$$

and introduce a further condition

$$[r12] \quad \epsilon \leq \exp \{ -C^{3/8} \}$$

$[xy\epsilon] \rightarrow [r11]$  since  $\epsilon \leq e^{-1}$ ,  $k \geq 1$  and  $c_k \leq 1$ .

The condition  $[r2]$  consists of four lower bounds on  $r$  which we label  $[r2.1]$ ,  $[r2.2]$ ,  $[r2.3]$  and  $[r2.4]$ . Recall that the value of  $r$  is given by

$$r = 10^{44/3} k^4 c_k^{-4/3} 2^{-4/3} (k^2 - 1)^{4/3} (\log \epsilon^{-1})^{4/3}.$$

Since  $k \geq 2$  is known by [2], we see that  $[r2.1]$  and  $[r2.3]$  are implied by  $\epsilon \leq e^{-1}$  and  $c_k \leq 1$ . Note that  $[r12]$  is equivalent to  $(\log \epsilon^{-1})^{4/3} \geq C^{1/2}$  and thus  $[r2.2]$  is implied by the value of  $r$  alongside  $k \geq 2$  and  $c_k \leq 1$ . Note also that  $[r11] \rightarrow [r2.4]$  since the value of  $r$  satisfies  $r \geq 70$ .

Thus, given the value of  $r$ ,  $[r11, r12, 2] \rightarrow [r2]$ .

Thus,  $[r6, r12, 2, 13, xy\epsilon] \rightarrow [r1, r2, r3, r\epsilon, rxy\epsilon]$ .

Recalling that  $[2, 3', 4', 12, 13] \rightarrow [1, 2, 3, 4]$ , our summary of deductions to date is

$$[2, 3', 4', 12, 13, r6, r12, xy\epsilon] \rightarrow [1, 2, 3, 4, r1, r2, r3, r\epsilon, r11, rxy\epsilon].$$



The right-hand side lists all of the conditions invoked during the proof of Theorem 6.1, with the exception of

$$[n] \quad n \geq 10^{37} k c^{-18} \epsilon^{-75/2} \max \{1, |x - y|^{36}\}$$

and  $n \epsilon^{3/2} \geq 10^2$ , which is implied by  $[n]$ .

If we introduce

$$[r13] \quad n \geq 10^{584} c_k^{-48} k^{240} c^{-36} \epsilon^{-222} \max \{1, |x - y|^{36}\},$$

we see that  $[r13] \rightarrow [r6, n]$  since  $c_k, c, \epsilon \leq 1$ .

Thus,

$$[2, 3', 4', 12, 13, r12, r13, xy\epsilon] \rightarrow [1, 2, 3, 4, r1, r2, r3, r\epsilon, r11, rxy\epsilon, n].$$

Thus, the left-hand list is a sufficient set of conditions for all applications in the proof of Theorem 6.1. These conditions are now recorded:

$$\begin{aligned} k &\geq 2 \\ n &\geq \max \left\{ 2k, 2(K_0)^{k^2} (\log \epsilon^{-1})^{K_0}, 10^{584} c_k^{-48} k^{240} c^{-36} \epsilon^{-222} \max \{1, |x - y|^{36}\} \right\} \\ |y - x| &\leq a_0 n^{1/9} \\ \epsilon &\leq \min \left\{ (\eta_0)^{4k^2}, 10^{-616} c_k^{22} k^{-115}, \exp \{-C^{3/8}\} \right\} \\ |x - y| &\leq \epsilon^{-1/2} (\log \epsilon^{-1})^{-2/3} 10^{-8} c_k^{2/3} k^{-10/3} \end{aligned}$$

The condition  $n \geq 2k$  is unnecessary, because it is implied by the third lower bound on  $n$ . Dispensing with this condition, we obtain the set of hypotheses of Theorem 6.1 with  $t_{1,2}$  set equal to one. As we noted at the outset, this is sufficient for the purpose of proving the result. The calculational derivation of Theorem 6.1 is complete.

**C.6. Proposition 1.6: derivation.** This section is devoted to the calculational derivation of Proposition 1.6, including the derivations of the various results needed during its proof. The three results whose calculational derivations are given here are Proposition 7.1, Lemma 7.2 and Proposition 1.6 itself; these appear next. There are also four results that are invoked in Proposition 1.6's proof whose proofs, while not calculational derivations in the sense in which we use this term, are a purely calculational matter. These are Lemmas 7.3, 7.4, C.3 and C.4. Their proofs appear at the end of the section.

**Proposition 7.1: derivation**

To prove this result we first state a variant, Proposition C.2. The variant differs from Proposition 7.1 only in the form of its hypotheses. We will carry out the calculations needed to derive the variant result and then show that the variant implies Proposition 7.1 by some further calculations.

**Proposition C.2.** *Set  $\hat{n} = n(1 - a)t_{1,2}$ . Suppose that*

$$\hat{n} \geq (c/3)^{-18} \vee 6^{36} \quad \text{and} \quad r \in \left[ 8 \vee 8(1 - a)^{1/3} t_{1,2}^{-2/3} |x - y|, 3\hat{n}^{1/36} \right]. \quad (59)$$

(1) Set  $n' = nt_{1,2}$ . Suppose further that

$$\begin{aligned} n' &\geq 10^{29}c^{-18}, \\ r &\in [2800, \frac{1}{18}(1-a)^{-2/3} \wedge 591n'^{1/27}], \\ |x-y| &\leq t_{1,2}^{2/3} \cdot 2^{-8/3}3^{-1}cn'^{1/18}, \\ r &\geq 3^32^45^{-1}(1-a)^{1/3}|x-y|t_{1,2}^{-2/3}, \end{aligned}$$

and

$$1-a \leq 5 \cdot 3^{-5}2^{-2}.$$

Then

$$\mathbb{P}\left(\text{Fluc}[-(r+1), -r]\right) \leq (10178 + 3r)C \exp\{-c_12^{-25}3^{-3}5^{3/2}r^{9/4}\}.$$

(2) Now suppose in addition to (59) that

$$\begin{aligned} r &\in [6(1-a)^{-1/2} \vee 11(1-a)^{1/3}|y-x|t_{1,2}^{-2/3}, 3n'^{1/6}(1-a)^{-1/2} \wedge (1-a)^{-2/3} \cdot 2^{-1} \cdot cn'^{1/9}], \\ |x-y| &\leq 2^{-1}t_{1,2}^{2/3} \cdot cn'^{1/9}, \end{aligned}$$

and

$$a \geq 1 - 10^{-2}.$$

Then

$$\mathbb{P}\left(\text{Fluc}[-(r+1), -r]\right) \leq (148C + 9r/4) \exp\{-c_1(1-a)^{1/2}3^{-3/2}5^{3/2}2^{-33/4}r^3\}.$$

**Calculational derivation.** The hypotheses listed in the proposition may be split into three parts: the *general* hypotheses that appear before either part of the lemma, namely (59), and then the hypotheses that are specific to the *first* and *second* part. We use the italicized terms to refer to the three types of hypotheses. We will argue that each collection of hypotheses is adequate for the needed uses in the proof of Proposition 7.1.

**General hypotheses.** Now we collect together all the assumptions needed for the applications in the preparation for the proof of Proposition 7.1. These applications are made in order to derive Lemma 7.5.

The application of Proposition 4.3(2) makes use of:  $1/2 \leq c/2 \cdot \hat{n}^{1/18}$ ;  $1/2 \in [0, \hat{n}^{1/18}]$ ;

$$3^{-1} \cdot 2^{-1/2}((7r/8)^2 - 1) \in [2^{3/2}, 2\hat{n}^{1/18}];$$

and  $\hat{n} \geq (c/3)^{-18} \vee 6^{36}$ .

To obtain (42), we use

$$r \geq 8(1-a)^{1/3}t_{1,2}^{-2/3}|x-y|.$$

We also use  $r \geq 2$  to prove (42); in fact, we later use  $r \geq 4$ .

The application of Proposition 4.3(3) makes use of:  $5r/4 \leq c/2 \cdot \hat{n}^{1/9}$ ;  $3r/8 \in [0, c/4 \cdot \hat{n}^{1/9}]$ ;

$$3^{-1} \cdot 2^{-1/2}((7r/8)^2 - 1) \in [2^{7/2}, 2\hat{n}^{1/3}];$$

and  $\hat{n} \geq c^{-18}$ .

We now label the above list  $[1, \dots, 10]$ . We will then argue that  $[1, 5, 14, 15]$ , which are the actual general hypotheses of Proposition C.2, imply them.

Set  $\hat{n} = n(1-a)t_{1,2}$ .

$$\begin{aligned}
[1] \quad & r \geq 8(1-a)^{1/3}t_{1,2}^{-2/3}|x-y|. \\
[2] \quad & 1/2 \leq c/2 \cdot \hat{n}^{1/18}, \\
[3] \quad & 1/2 \in [0, \hat{n}^{1/18}], \\
[4] \quad & 3^{-1} \cdot 2^{-1/2}((7r/8)^2 - 1) \in [2^{3/2}, 2\hat{n}^{1/18}] \\
[5] \quad & \hat{n} \geq (c/3)^{-18} \vee 6^{36}. \\
[6] \quad & r \geq 4. \\
[7] \quad & 5r/4 \leq c/2 \cdot \hat{n}^{1/9}, \\
[8] \quad & 3r/8 \in [0, c/4 \cdot \hat{n}^{1/9}], \\
[9] \quad & 3^{-1} \cdot 2^{-1/2}((7r/8)^2 - 1) \in [2^{7/2}, 2\hat{n}^{1/3}] \\
[10] \quad & \hat{n} \geq c^{-18}.
\end{aligned}$$

And some further conditions:

$$\begin{aligned}
[11] \quad & 3^{-1} \cdot 2^{-1/2}((7r/8)^2 - 1) \in [2^{7/2}, 2\hat{n}^{1/18}] \\
[12] \quad & (7r/8)^2 \geq 3 \cdot 2^4 + 1 \\
[13] \quad & 3^{-1} \cdot 2^{-1/2}(7r/8)^2 \leq 2\hat{n}^{1/18} \\
[14] \quad & r \leq 3\hat{n}^{1/36} \\
[15] \quad & r \geq 8
\end{aligned}$$

Note that [10]  $\rightarrow$  [2]. [10]  $\rightarrow$  [3] since  $c \leq 2$ . [11]  $\rightarrow$  [4, 9]. [12, 13]  $\rightarrow$  [11]. [14]  $\rightarrow$  [13] since  $2^{15/4}3^{1/2}7^{-1} \geq 3.32908$ . [15]  $\leftrightarrow$  [12]. [15]  $\rightarrow$  [6]. [5]  $\rightarrow$  [10]. [10, 14]  $\rightarrow$  [7] since assuming [14], it is sufficient for [7] that  $\hat{n} \geq (2c/15)^{12}$ , a bound which follows from [10] and  $c \leq 1$ . [7]  $\rightarrow$  [8]. Thus, we conclude that

$$[1, 5, 14, 15] \rightarrow [1, \dots, 15].$$

**Hypotheses specific to the first part of Proposition C.2.** The conditions needed in the proof of the proposition's first part are invoked in the application of the general version of Theorem 5.4. Setting  $n' = nt_{1,2}$ , these hypotheses are as follows:

$$(1-a)^{2/3}r \leq 8/9 \cdot 2^{-4}, \quad n' \geq 10^{29}c^{-18}$$

$$|x-y|t_{1,2}^{-2/3} + 9/8 \cdot (1-a)^{2/3}r \leq 2^{-5/3}3^{-1}cn'^{1/18}$$

and  $2^{1/2}/3 \cdot 5/3 \cdot 2^{-7/2} \cdot r^{3/2} \in [10^4, 10^3n'^{1/18}]$ .

Moreover, the next bounds were used to control parabolic curvature in preparation for the application in question:

$$[par \ r] \quad r \geq 3^32^45^{-1}(1-a)^{1/3}|x-y|t_{1,2}^{-2/3} \quad \text{and} \quad 1-a \leq 5 \cdot 3^{-5}2^{-2}$$

Condition [par r] will not be analysed further for now, because it is included in the hypotheses of Proposition C.2's first part.

The remaining conditions may be written:

$$\begin{aligned}
[A] \quad & (1-a)^{2/3}r \leq 1/18, \\
[B] \quad & n' \geq 10^{29}c^{-18},
\end{aligned}$$

$$\begin{aligned}
[C] \quad & |x - y| t_{1,2}^{-2/3} + 9/8 \cdot (1 - a)^{2/3} r \leq 2^{-5/3} 3^{-1} c n^{1/18} \\
[D] \quad & 2^{1/2}/3 \cdot 5/3 \cdot 2^{-7/2} \cdot r^{3/2} \geq 10^4 \\
[E] \quad & 2^{1/2}/3 \cdot 5/3 \cdot 2^{-7/2} \cdot r^{3/2} \leq 10^3 n^{1/18}.
\end{aligned}$$

Further conditions:

$$\begin{aligned}
[F] \quad & r \geq 2800 \\
[G] \quad & r \leq 591 n^{1/27} \\
[H] \quad & |x - y| t_{1,2}^{-2/3} + 1/16 \leq 2^{-5/3} 3^{-1} c n^{1/18} \\
[I] \quad & |x - y| t_{1,2}^{-2/3} \leq 2^{-1} \cdot 2^{-5/3} 3^{-1} c n^{1/18} \\
[J] \quad & 1/16 \leq 2^{-1} \cdot 2^{-5/3} 3^{-1} c n^{1/18}
\end{aligned}$$

[F]  $\rightarrow$  [D] since  $(72/5 \cdot 10^4)^{2/3} \leq 2.75 \dots \times 10^3$ . [G]  $\rightarrow$  [E] since  $\lfloor (72/5 \cdot 10^3)^{2/3} \rfloor \geq 591$ . [A, H]  $\rightarrow$  [C]. [I, J]  $\rightarrow$  [H]. [B]  $\rightarrow$  [J]. Thus, [A, B, F, G, I]  $\rightarrow$  [A,  $\dots$ , E].

We thus see that the conditions [A, B, F, G, I, *par r*] are sufficient for the applications specific to the first part of the lemma. These conditions may be expressed as:

$$\begin{aligned}
n' &\geq 10^{29} c^{-18}, \\
r &\in [2800, \frac{1}{18} (1 - a)^{-2/3} \wedge 591 n^{1/27}], \\
|x - y| t_{1,2}^{-2/3} &\leq 2^{-8/3} 3^{-1} c n^{1/18}
\end{aligned}$$

and [*par r*].

The last three displays and [*par r*] are the hypotheses stated in Proposition C.2's first part.

**Hypotheses specific to the second part of Proposition C.2.** Recall that  $n' = n t_{1,2}$ . In the proof of Claim (2), Reg(2) is applied. The conditions enabling this application are

$$|x - y| \leq t_{1,2}^{2/3} \cdot c n^{1/9}$$

and  $3^{-15} \cdot 2^{-9/2} (1 - a)^{1/3} r^2 \in [1, n^{1/3}]$ .

To ensure that (47) holds, we make use of

$$r \geq 11 (1 - a)^{1/3} |y - x| t_{1,2}^{-2/3}$$

and

$$a \geq 1 - 10^{-2}.$$

In the proof of Claim (1), Reg(3) is applied. Here, we make use of

$$\left| x - z + t_{1,2}^{2/3} (1 - a)^{2/3} r \right| \leq t_{1,2}^{2/3} \cdot c n^{1/9},$$

and  $(1 - a)^{1/3} \cdot 5/3 \cdot 2^{-11/2} r^2 \geq 1$ . Recalling that  $z = y + (1 - a)(x - y)$ , the last display is given by

$$\left| a(x - y) + t_{1,2}^{2/3} (1 - a)^{2/3} r \right| \leq t_{1,2}^{2/3} \cdot c n^{1/9}.$$

The list of conditions that are used may thus be recorded as:

$$\begin{aligned}
[a] \quad & \left| a(x - y) + t_{1,2}^{2/3} (1 - a)^{2/3} r \right| \leq t_{1,2}^{2/3} \cdot c n^{1/9}, \\
[b] \quad & 3^{-15} \cdot 2^{-11/2} (1 - a)^{1/3} r^2 \geq 1
\end{aligned}$$

$$\begin{aligned}
[c] \quad & |x - y| \leq t_{1,2}^{2/3} \cdot cn^{1/9} \\
[d] \quad & 3^{-1}5 \cdot 2^{-9/2}(1-a)^{1/3}r^2 \in [1, n^{1/3}]. \\
[e] \quad & r \geq 11(1-a)^{1/3}|y-x|t_{1,2}^{-2/3} \\
[f] \quad & a \geq 1 - 10^{-2}.
\end{aligned}$$

Further conditions:

$$\begin{aligned}
[g] \quad & 3^{-1}5 \cdot 2^{-9/2}(1-a)^{1/3}r^2 \geq 1 \\
[h] \quad & 3^{-1}5 \cdot 2^{-9/2}(1-a)^{1/3}r^2 \leq n^{1/3} \\
[i] \quad & |x - y| \leq 2^{-1}t_{1,2}^{2/3} \cdot cn^{1/9} \\
[j] \quad & (1-a)^{2/3}r \leq 2^{-1} \cdot cn^{1/9}
\end{aligned}$$

$[g, h] \rightarrow [d]$ .  $[b] \rightarrow [g]$ .  $[i, j] \rightarrow [a]$  since  $a \in [0, 1]$ .  $[i] \rightarrow [c]$ .

Thus,  $[b, h, i, j] \rightarrow [a, b, c, d]$ , and  $[b, h, i, j, e, f]$  is a sufficient set of conditions for the applications in question.

Since

$$(3/5 \cdot 2^{9/2})^{1/2} = 3.684 \dots \in [3, 4], \quad (3/5 \cdot 2^{11/2})^{1/2} = 5.210 \dots \in [5, 6],$$

$[b, fH]$  is implied by

$$r \in (1-a)^{-1/2} \cdot [6, 3n^{1/6}].$$

Thus,  $[b, h, i, j, e, f]$  is implied by

$$\begin{aligned}
r \in & [6(1-a)^{-1/2} \vee 11(1-a)^{1/3}|y-x|t_{1,2}^{-2/3}, 3n^{1/6}(1-a)^{-1/2} \wedge (1-a)^{-2/3} \cdot 2^{-1} \cdot cn^{1/9}], \\
& |x - y| \leq 2^{-1}t_{1,2}^{2/3} \cdot cn^{1/9},
\end{aligned}$$

and

$$a \geq 1 - 10^{-2}.$$

The last three displays are the hypotheses stated in Proposition C.2's second part.

This completes the calculational derivation of Proposition C.2.

**Calculational derivation of Proposition 7.1 from Proposition C.2.** The hypotheses of these two propositions may once again be divided into three types: the general hypotheses, and those specific to the first and second part. We now label the conditions, using the symbols  $A$ ,  $B$  and  $C$  for the respective types. Recalling that  $\hat{n} = n(1-a)t_{1,2}$  and  $n' = nt_{1,2}$ , the hypothesis list of Proposition C.2 is when thus represented:

$$\begin{aligned}
[A1] \quad & \hat{n} \geq (c/3)^{-18} \vee 6^{36} \\
[A2] \quad & r \geq 8 \\
[A3] \quad & r \geq 8(1-a)^{1/3}t_{1,2}^{-2/3}|x-y| \\
[A4] \quad & r \leq 3\hat{n}^{1/36} \\
[B1] \quad & n' \geq 10^{29}c^{-18}, \\
[B2] \quad & r \geq 2800, \\
[B3] \quad & r \leq \frac{1}{18}(1-a)^{-2/3} \\
[B4] \quad & r \leq 591n^{1/27} \\
[B5] \quad & |x - y| \leq t_{1,2}^{2/3} \cdot 2^{-8/3}3^{-1}cn^{1/18}.
\end{aligned}$$

$$\begin{aligned}
[B6] \quad r &\geq 3^3 2^4 5^{-1} (1-a)^{1/3} |x-y| t_{1,2}^{-2/3}, \\
[B7] \quad 1-a &\leq 5 \cdot 3^{-5} 2^{-2}. \\
[C1] \quad r &\geq 6(1-a)^{-1/2} \\
[C2] \quad r &\geq 11(1-a)^{1/3} |y-x| t_{1,2}^{-2/3} \\
[C3] \quad r &\leq 3n^{1/6} (1-a)^{-1/2} \\
[C4] \quad r &\leq (1-a)^{-2/3} \cdot 2^{-1} c n^{1/9}, \\
[C5] \quad |x-y| &\leq 2^{-1} t_{1,2}^{2/3} \cdot c n^{1/9}, \\
[C6] \quad a &\geq 1 - 10^{-2}.
\end{aligned}$$

Our task now is to simplify these hypotheses within each of the three types in order to verify that they are respectively implied by the hypotheses in each grouping in Proposition 7.1.

*Further conditions.* We consider

$$\begin{aligned}
[D1] \quad \hat{n} &\geq 10^{29} c^{-18}. \\
[D2] \quad r &\geq 87(1-a)^{1/3} |y-x| t_{1,2}^{-2/3}. \\
[D3] \quad a &\geq 1 - 10^{-3}.
\end{aligned}$$

*Calculations.* [A4]  $\rightarrow$  [B4] since  $\hat{n} \leq n'$ . [A1, A4]  $\rightarrow$  [C4] since  $a \in [0, 1]$ ,  $\hat{n} \leq n'$  and  $2^{-1} c \hat{n}^{1/9} \geq 3 \hat{n}^{1/36}$  if and only if  $\hat{n}^{1/9-1/36} \geq 6c^{-1}$  which is implied by [A1].

[A1, A4]  $\rightarrow$  [C3]. [B1]  $\rightarrow$  [A1]. [D1]  $\rightarrow$  [A1, B1]. [B1, B5]  $\rightarrow$  [C5]. [C2]  $\rightarrow$  [A3].

[D2]  $\rightarrow$  [B6, C2] since  $3^3 2^4 5^{-1} = 86 \cdot 4$ . [D3]  $\rightarrow$  [B7] since  $5 \cdot 3^{-5} 2^{-2} = 0.0051 \dots$ .

We may thus record a sufficient set of hypothesis conditions in the three categories as follows:

General hypotheses: [D1, A2, D2, A4, B5].

Hypotheses specific to the proposition's first part: [B2, 3, D3].

Hypotheses specific to the proposition's second part: [C1, 6].

Since these are indeed the three set of hypotheses of Proposition 7.1. the calculational derivation of this proposition is complete.

**Calculational derivation of Lemma 7.2.** We begin by collecting together all the conditions used in this proof.

To derive  $V' \leq -\hat{n}^{1/36}$ , we use

$$(n(1-a)t_{1,2})^{1/36} \geq (1-a)^{1/3} t_{1,2}^{-2/3} |x-y|.$$

In the first application of Proposition 4.3(4), we use

$$(n(1-a)t_{1,2})^{-1/12} \leq c \quad \text{and} \quad n(1-a)t_{1,2} \geq (2^{5/4} c^{-1})^9.$$

In deriving (51), we use  $|x-y| \leq t_{1,2}^{2/3} (1-a)^{2/3} \hat{n}^{1/36}$ .

In the second application of Proposition 4.3(4), we use

$$n^{-1/12} a^{-1/9} (1-a)^{25/36} t_{1,2}^{-1/12} \leq c \quad \text{as well as} \quad n a t_{1,2} \geq (2^{5/4} c^{-1})^9.$$

In the application of Reg(2) used to obtain (54), we use

$$|y-x|t_{1,2}^{-2/3} \leq c(nt_{1,2})^{1/9} \quad \text{and} \quad (2^{-1/2}-2^{-5/2})((1-a)^{25/18}a^{1/3}+(1-a)^{17/18})t_{1,2}^{1/18}n^{1/18} \in [1, (nt_{1,2})^{1/3}].$$

We now record all these conditions in a list:

$$\begin{aligned} [1] \quad n^{1/36} &\geq (1-a)^{1/3-1/36}t_{1,2}^{-2/3-1/36}|x-y| \\ [2] \quad (n(1-a)t_{1,2})^{-1/12} &\leq c \\ [3] \quad n(1-a)t_{1,2} &\geq (2^{5/4}c^{-1})^9 \\ [4] \quad |x-y| &\leq t_{1,2}^{2/3+1/36}(1-a)^{2/3+1/36}n^{1/36} \\ [5] \quad n^{-1/12}a^{-1/9}(1-a)^{25/36}t_{1,2}^{-1/12} &\leq c \\ [6] \quad nat_{1,2} &\geq (2^{5/4}c^{-1})^9 \\ [7] \quad |y-x|t_{1,2}^{-2/3-1/9} &\leq cn^{1/9} \\ [8] \quad (2^{-1/2}-2^{-5/2})((1-a)^{25/18}a^{1/3}+(1-a)^{17/18})t_{1,2}^{1/18}n^{1/18} &\geq 1 \\ [9] \quad (2^{-1/2}-2^{-5/2})((1-a)^{25/18}a^{1/3}+(1-a)^{17/18}) &\leq n^{1/3-1/18}t_{1,2}^{1/3-1/18} \end{aligned}$$

Of these conditions, [1, 4, 7] are lower bounds on  $n$  expressed in terms of  $|x-y|$ . Note that we may reexpress [4] and [7] as follows:

$$\begin{aligned} [4] \quad nt_{1,2} &\geq (1-a)^{-25}t_{1,2}^{-24}|x-y|^{36} \\ [7] \quad nt_{1,2} &\geq c^{-9}t_{1,2}^{-6}|x-y|^9 \end{aligned}$$

Note that [1], which may be written  $nt_{1,2} \geq (1-a)^{-11}t_{1,2}^{-24}|x-y|^{36}$ , is implied by [4] due to  $1-a \leq 1$ .

Thus, [4, 7]  $\rightarrow$  [1, 4, 7].

The remaining conditions, [2, 3, 5, 6, 8, 9], are other lower bounds on  $n$ . We introduce two further conditions,

$$[10] \quad nt_{1,2} \geq (1-a)^{-1}2^{45/4}c^{-12}$$

and

$$[11] \quad nt_{1,2} \geq 10^5(1-a)^{-17}.$$

We now argue that [10]  $\rightarrow$  [2, 3, 5, 6, 9] and that [11]  $\rightarrow$  [8]. We treat each of [2, 3, 5, 6, 8, 9] in turn to reach these conclusions.

[2] is given by  $nt_{1,2} \geq (1-a)^{-1}c^{-12}$  and thus is implied by [10].

[3] is given by  $nt_{1,2} \geq (1-a)^{-1}2^{45/4}c^{-9}$  and is implied by [10] since  $c \leq 1$ .

[5] is given by  $nt_{1,2} \geq (1-a)^{25/3}a^{-4/3}c^{-12}$ . Since  $a \geq 1/2$ , this is implied by  $nt_{1,2} \geq 2^{4/3}c^{-12}$  and thus by [10].

[6] is given by  $nt_{1,2} \geq a^{-1}2^{45/4}c^{-9}$ . Since  $(1-a)^{-1} \geq a^{-1}$  (due to  $a \geq 1/2$ ), and  $c \leq 1$ , [10]  $\rightarrow$  [6].

[8] is implied by  $nt_{1,2} \geq (2^{-1/2}-2^{-5/2})^{-18}(1-a)^{-17}$ . Since  $\lfloor (2^{-1/2}-2^{-5/2})^{-18} \rfloor = 90817$ , [11]  $\rightarrow$  [8].

[9] is implied by  $nt_{1,2} \geq (2^{1/2}-2^{-3/2})^{18/5}$ . Since  $(2^{1/2}-2^{-3/2})^{18/5} = 1.236\dots$ , [10]  $\rightarrow$  [9].

If we further introduce

$$[12] \quad nt_{1,2} \geq 10^5(1-a)^{-17}c^{-12}$$

then  $[12] \rightarrow [10]$ , and  $[12] \rightarrow [11]$  since  $c \leq 1$ .

In summary,  $[4, 7, 12]$  are a sufficient set of conditions for the proof of Lemma 7.2. Since these alongside  $a \geq 1/2$  are indeed the hypotheses of the lemma, the calculational derivation of Lemma 7.2 is complete.

### Calculational derivation of Proposition 1.6

This proposition is derived by using Proposition 7.1 and Lemma 7.2 alongside the calculational Lemmas 7.3, 7.4, C.3 and C.4. Moreover, the bound  $a \geq 1 - 10^{-11}c_1^2$  is used in the derivation. As such, it is our task to verify that the hypotheses of these results are verified under the assumptions of Proposition 1.6.

*Lower bounds on  $nt_{1,2}$ :* in Lemma 7.2, it is demanded that  $nt_{1,2}$  is at least the maximum of

$$[a] \quad nt_{1,2} \geq (1-a)^{-25}t_{1,2}^{-24}|x-y|^{36},$$

$$[b] \quad nt_{1,2} \geq c^{-9}t_{1,2}^{-6}|x-y|^9$$

and

$$[c] \quad nt_{1,2} \geq 10^5(1-a)^{-17}c^{-12},$$

while in Proposition 7.1, the condition

$$[d] \quad nt_{1,2} \geq (1-a)^{-1}10^{29}c^{-18}$$

is used. In addition, Proposition 7.1 demands  $|x-y|t_{1,2}^{-2/3} \leq 2^{-8/3}3^{-1}c(nt_{1,2})^{1/18}$  or equivalently

$$nt_{1,2} \geq 2^{48}3^{18}c^{-18}|x-y|^{18}t_{1,2}^{-12},$$

which since  $2^{48}3^{18} = 1.09 \dots \times 10^{23}$  is implied by

$$[e] \quad nt_{1,2} \geq 10^{24}c^{-18}|x-y|^{18}t_{1,2}^{-12}.$$

We now write

$$[f] \quad nt_{1,2} \geq 10^{29}(1-a)^{-25}c^{-18},$$

Note that  $[f] \rightarrow [c, d]$  and also that  $[f]$  implies  $[a, b, e]$  in the special case where  $|x-y|^{3/2}t_{1,2}^{-1} \leq 1$ . In the case where  $|x-y|^{3/2}t_{1,2}^{-1} > 1$ ,  $[a, b, e]$  are implied by

$$[g] \quad nt_{1,2} \geq 10^{24}c^{-18}(1-a)^{-25}|x-y|^{36}t_{1,2}^{-24}.$$

Thus,  $[f, g] \rightarrow [a, b, c, d, e]$ . For this reason,  $[f, g]$  is the lower bound on  $nt_{1,2}$  hypothesised in Proposition 1.6.

*Lower bounds on  $r$ :*

In Proposition 7.1, we suppose

$$[A] \quad r \geq 8 \vee 87(1-a)^{1/3}t_{1,2}^{-2/3}|x-y|$$

as well as  $r \geq 2800$ . The latter is superceded by Lemma 7.3's hypothesis that

$$[B] \quad r \geq 10^9c_1^{-4/5}$$

In Lemma 7.4, we use

$$[C] \quad r \geq 15C^{1/2}$$

Lemma C.4 uses

$$[D] \quad r \geq 11(1-a)^{1/3}|y-x|t_{1,2}^{-2/3}$$



Note that Lemma 7.4 also uses  $r \geq \frac{1}{18}(1-a)^{-2/3}$ . However, this condition need not be a consequence of the lower bound on  $r$  in Proposition 1.6 because the lemma is only applied when  $r$  is in addition supposed to satisfy this bound.

That is, the lower bound on  $r$  hypothesised in Proposition 1.6 should imply  $[A, B, C, D]$ . We may and do take this lower bound to be

$$\max \left\{ 10^9 c_1^{-4/5}, 15C^{1/2}, 87(1-a)^{1/3} t_{1,2}^{-2/3} |x-y| \right\}.$$

*Upper bounds on  $r$ :*

Proposition 7.1 supposes that

$$r \leq 3n^{1/36}(1-a)^{1/36} t_{1,2}^{1/36}$$

Lemma C.3 supposes that

$$r \leq 10^{10}(1-a)^{25/9} t_{1,2}^{1/9} n^{1/9}$$

In addition, Proposition 7.1(1)  $r \leq \frac{1}{18}(1-a)^{-2/3}$ . This hypothesis need not be included as a necessary consequence of Proposition 1.6's assumptions because the value of  $r$  considered in the use of Proposition 7.1(1) always satisfies this bound. (When  $r$  becomes higher, Proposition 7.1(2) is instead applied. In that case,  $r \geq 6(1-a)^{-1/2}$  must be satisfied. It has been argued in the proof of Proposition 1.6 that there is no gap here, i.e., that  $6(1-a)^{-1/2} \leq \frac{1}{18}(1-a)^{-2/3}$  on the basis of a smallness assumption on  $1-a$  hypothesised in Proposition 1.6.)

In summary, then, the required upper bounds on  $r$  are the two displayed inequalities. Since  $nt_{1,2} \geq 1$ , these are implied by

$$r \leq 3(1-a)^{25/9} n^{1/36} t_{1,2}^{1/36}$$

which is the upper bound on  $r$  hypothesised in Proposition 1.6.

*Upper bounds on  $1-a$ :*

Proposition 7.1 uses  $a \geq 1 - 10^{-2}$  and  $a \geq 1 - 10^{-3}$ . Lemma 7.4 uses  $a \geq 1 - 10^{-10} c_1^2$ . Also, the bound  $a \geq 1 - 10^{-11} c_1^2$  is used in the proof of Proposition 1.6. The last condition implies the other two and is the hypothesised upper bound on  $1-a$  in the proposition.

This completes the calculational derivation of Proposition 1.6.

**Proof of Lemma 7.3.** Note that

$$(10178 + 3r) C \exp \left\{ -c_1 2^{-25} 3^{-3} 5^{3/2} r^{9/4} \right\} \leq 4Cr \exp \left\{ -c_1 2^{-25} 3^{-3} 5^{3/2} r^{9/4} \right\} \quad (60)$$

when  $r \geq 10^7$  (which is implied by  $r \geq 10^9 c_1^{-4/5}$  and  $c_1 \leq 1$ ), since  $10178 \cdot 10^{-9} \leq 1$ . The ratio of this right-hand side evaluated at  $r+1$  and  $r$  equals

$$(1+r^{-1}) \exp \left\{ -c_1 2^{-25} 3^{-3} 5^{3/2} ((r+1)^{9/4} - r^{9/4}) \right\}$$

Note that  $(r+1)^{9/4} - r^{9/4} = r^{9/4}((1+r^{-1})^{9/4} - 1) \geq 9/4 \cdot r^{5/4}$  since  $(1+x)^a \geq 1+ax$  for  $x \geq 0$  and  $a \geq 1$  (taking  $x = r^{-1}$  and  $a = 9/4$ ). Thus, this ratio is at most

$$(1+r^{-1}) \exp \left\{ -c_1 2^{-25} 3^{-3} 5^{3/2} \cdot 9/4 \cdot r^{5/4} \right\} \leq (1+r^{-1}) \exp \left\{ -c_1 10^{-8} \cdot r^{5/4} \right\}$$

since  $2^{25}3^35^{-3/2} \cdot 4/9 = 3.600 \cdots \times 10^7$ . Thus the ratio is at most  $1/2$  provided that  $r^{5/4} \geq 10^8 c_1^{-1}$  and thus if  $r \geq 10^7 c_1^{-4/5}$ . Recalling (60), we find that, when  $r \geq 10^7 c_1^{-4/5}$ ,

$$\sum_{s \in r + \mathbb{N}} (10178 + 3s) C \exp \left\{ -c_1 2^{-25} 3^{-3} 5^{3/2} s^{9/4} \right\} \leq 2 \cdot 4Cr \exp \left\{ -c_1 10^{-8} r^{9/4} \right\}$$

since  $2^{-25} 3^{-3} 5^{3/2} \geq 10^{-8}$ . This completes the proof of Lemma 7.3.  $\square$

**Proof of Lemma 7.4.** Recall that we set  $\alpha = (5/3)^{3/2} 2^{-33/4}$ . Our task is to find an upper bound on

$$\sum_{s \in r + \mathbb{N}} (148C + 9s/4) \exp \left\{ -c_1 (1-a)^{1/2} \alpha s^3 \right\} \quad (61)$$

under the assumptions that  $r \geq \frac{1}{18} (1-a)^{-2/3} \vee 15C^{1/2}$  and  $a \geq 1 - 10^{-10} c_1^2$ .

We begin by arguing that, for the values of  $s$  in question,

$$148C + 9s/4 \leq (s-1)^2. \quad (62)$$

Since the derivative of the right-hand side exceeds the left's whenever  $2(s-1) \geq 9/4$ , i.e., when  $s \geq 17/8$ , it is enough to argue that (62) holds when  $s = s_0$  with  $s_0 = 15C^{1/2}$ . The required bound  $148C + (9/4 + 2)s_0 \leq s_0^2 + 1$  is implied by  $17/4 \cdot 15 \leq (225 - 148)C^{1/2}$ , which follows from  $C \geq 1$ . This proves (62).

Using this bound, we see that (61) is at most

$$\begin{aligned} & \sum_{s \in r + \mathbb{N}} (s-1)^2 \exp \left\{ -c_1 (1-a)^{1/2} \alpha s^3 \right\} \\ & \leq \int_{r-1}^{\infty} x^2 \exp \left\{ -c_1 (1-a)^{1/2} \alpha x^3 \right\} dx \\ & \leq 3^{-1} c_1^{-1} (1-a)^{-1/2} \alpha^{-1} \exp \left\{ -c_1 (1-a)^{1/2} \alpha (r-1)^3 \right\} \end{aligned} \quad (63)$$

which is at most

$$3^{-1} c_1^{-1} (1-a)^{-1/2} \alpha^{-1} \exp \left\{ -2^{-1} c_1 (1-a)^{1/2} \alpha r^3 \right\} \quad (64)$$

since  $(r-1)^3 \geq 2^{-1} r^3$  is equivalent to  $r \geq 1 - 2^{-1/3} = 4.8 \cdots$  and thus is implied by  $r \geq 15C^{1/2}$  and  $C \geq 1$ .

Since  $r \geq (18)^{-1} (1-a)^{-2/3}$ , we may note that

$$2^{-1} c_1 (1-a)^{1/2} \alpha r^3 \geq 2^{-1} c_1 (1-a)^{1/2} \alpha \left( (18)^{-1} (1-a)^{-2/3} \right)^{3/4} r^{9/4} = 2^{-1} c_1 (18)^{-3/4} \alpha r^{9/4}$$

Thus, (64) is bounded above by

$$\exp \left\{ -4^{-1} (18)^{-3/4} c_1 \alpha r^{9/4} \right\} \quad (65)$$

provided that

$$3^{-1} c_1^{-1} (1-a)^{-1/2} \alpha^{-1} \leq \exp \left\{ 4^{-1} (18)^{-3/4} c_1 \alpha r^{9/4} \right\},$$

a condition which is implied by

$$3^{-1} c_1^{-1} (1-a)^{-1/2} \alpha^{-1} \leq 4^{-1} (18)^{-3/4} c_1 \alpha r^{9/4}$$

which since  $r \geq (18)^{-1} (1-a)^{-2/3}$  is implied by

$$1-a \leq 3/4 \cdot (18)^{-3} c_1^2 \alpha^2 = 6^{-5} \alpha^2 c_1^2$$

Now  $\alpha = (5/3)^{3/2}2^{-33/4} = 7.06 \dots \times 10^{-3}$  and  $6^5 = 7776$ , so that  $6^{-5}\alpha^2 = 6.4 \dots \times 10^{-9}$ . Thus the last condition is implied by our assumption that

$$a \geq 1 - 10^{-10}c_1^2$$

Since  $4^{-1}(18)^{-3/4} = 0.0286 \dots$ , we see that (65) is bounded above by

$$\exp \left\{ -10^{-2}c_1\alpha r^{9/4} \right\}$$

This completes the proof of Lemma 7.4. □

**Lemma C.3.** *When  $r \in [1, 10^{10}(1-a)^{25/9}t_{1,2}^{1/9}n^{1/9}]$ , we have that*

$$9Cr \exp \left\{ -10^{-8}c_1r^{3/4} \right\} + 13C \exp \left\{ -2^{-15/4}c(1-a)^{25/12}t_{1,2}^{1/12}n^{1/12} \right\}$$

is at most

$$22Cr \exp \left\{ -10^{-11}c_1r^{3/4} \right\}.$$

**Proof.** Since  $r \geq 1$ , it is enough to argue that

$$10^{-8}c_1r^{3/4} \leq 2^{-15/4}c(1-a)^{25/12}t_{1,2}^{1/12}n^{1/12}. \quad (66)$$

Recall that  $c_1 = 2^{-5/2}c \wedge 1/8$ , so that  $c_1 \leq c$ . Thus, (66) is implied by

$$r \leq 10^{32/3} \cdot 2^{-5}(1-a)^{25/9}t_{1,2}^{1/9}n^{1/9},$$

which bound is a consequence of our assumed upper bound on  $r$ , namely that  $r$  is at most

$$10^{10}(1-a)^{25/9}t_{1,2}^{1/9}n^{1/9}.$$

□

**Lemma C.4.** *Suppose that  $r \geq 11(1-a)^{1/3}|y-x|t_{1,2}^{-2/3}$  and  $a \geq 1 - 10^{-2}$ . Then*

$$\left| 2^{-1/2}(y-x)^2t_{1,2}^{-4/3} - 2^{-1/2}(z-x-\varphi)^2t_{1,2}^{-4/3} \right| \leq 4^{-1}d(1-a)^{1/3}r^2.$$

**Proof.** Recall that  $z-x = a(y-x)$ . The left-hand side is thus bounded above by  $A_1 + A_2 + A_3$ , where

$$A_1 = 2^{-1/2}(1-a^2)(y-x)^2t_{1,2}^{-4/3}, \quad A_2 = 2^{1/2}a|y-x|\varphi t_{1,2}^{-4/3} \quad \text{and} \quad A_3 = 2^{-1/2}\varphi^2t_{1,2}^{-4/3}.$$

We will argue that each of these terms is at most one-third of  $4^{-1}d(1-a)^{1/3}r^2$ . In the first case, this claim is

$$2^{-1/2}(1-a^2)(y-x)^2t_{1,2}^{-4/3} \leq (12)^{-1}5 \cdot 3^{-1}2^{-7/2}(1-a)^{1/3}r^2;$$

in other words, that

$$(1+a)(y-x)^2t_{1,2}^{-4/3} \leq 5 \cdot 3^{-2}2^{-5}(1-a)^{-2/3}r^2;$$

since  $a \leq 1$ , it is thus implied by

$$(y-x)^2t_{1,2}^{-4/3} \leq 5 \cdot 3^{-2}2^{-6}(1-a)^{-2/3}r^2,$$

or equivalently

$$r \geq 3 \cdot 2^3 5^{-1/2}(1-a)^{1/3}|x-y|t_{1,2}^{-2/3}.$$

Recall that  $\varphi = t_{1,2}^{2/3}(1-a)^{2/3}r$ . As such, the bound we seek on  $A_2$  may be expressed in the form

$$2^{1/2}a(1-a)^{2/3}|y-x|t_{1,2}^{-2/3} \leq 5 \cdot 3^{-1}2^{-7/2}(1-a)^{1/3}r$$

or equivalently

$$a(1-a)^{1/3}|y-x|t_{1,2}^{-2/3} \leq 5 \cdot 3^{-1}2^{-4}r.$$

Since  $a \leq 1$ , this is implied by

$$r \geq 3 \cdot 2^4 5^{-1}(1-a)^{1/3}|y-x|t_{1,2}^{-2/3}.$$

The sought bound on  $A_3$  may be written

$$2^{-1/2}(t_{1,2}^{2/3}(1-a)^{2/3}r)^2 t_{1,2}^{-4/3} \leq (12)^{-1}5 \cdot 3^{-1}2^{-7/2}(1-a)^{1/3}r^2$$

or equivalently

$$a \geq 1 - 5 \cdot 3^{-2}2^{-5}.$$

Since  $3 \cdot 2^3 5^{-1/2} = 10.733 \dots$ ,  $3 \cdot 2^4 5^{-1} = 9.6$  and  $5 \cdot 3^{-2}2^{-5} = 0.0173 \dots$ , we see that the conclusion of Lemma C.4 holds provided that

$$r \geq 11(1-a)^{1/3}|y-x|t_{1,2}^{-2/3}$$

and  $a \geq 1 - 10^{-2}$ . □

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