MODULUS OF CONTINUITY OF POLYMER WEIGHT PROFILES IN BROWNIAN LAST PASSAGE PERCOLATION

ALAN HAMMOND

ABSTRACT. In last passage percolation models lying in the KPZ universality class, the energy of long energy-maximizing paths may be studied as a function of the paths' pair of endpoint locations. Scaled coordinates may be introduced, so that these maximizing paths, or polymers, now cross unit distances with unit-order fluctuations, and have scaled energy, or weight, of unit order. In this article, we consider Brownian last passage percolation in these scaled coordinates. In the narrow wedge case, one endpoint of such polymers is fixed, say at $(0,0) \in \mathbb{R}^2$, and the other is varied horizontally, over $(z,1)$, $z \in \mathbb{R}$, so that the polymer weight profile is a function of $z \in \mathbb{R}$. This profile is known to manifest a one-half power law, having $1/2-$Hölder continuity. The polymer weight profile may be defined beginning from a much more general initial condition. In this article, we present a more general assertion of this one-half power law, as well as a bound on the poly-logarithmic correction. The polymer weight profile admits a modulus of continuity of order $x^{1/2}\big(\log x^{-1}\big)^{2/3}$, with a high degree of uniformity in the scaling parameter and over a very broad class of initial data.

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1. Introduction

The $1 + 1$ dimensional Kardar-Parisi-Zhang (KPZ) universality class includes a wide range of interface models suspended over a one-dimensional domain, in which growth in a direction normal to the surface competes with a smoothening surface tension in the presence of a local randomizing force that roughens the surface. Such surfaces typically grow linearly, with fluctuations after that linear growth is subtracted being described by scaling exponents: if linear growth has order n, then interface height above a given point has typical deviation from the mean of order $n^{1/3}$, while non-trivial correlations in this height as the spatial coordinate is varied are encountered on scale $n^{2/3}$. Moreover, an exponent of one-half dictates the interface's regularity, with the interface height being expected to vary between a pair of locations at distance of order at most $n^{2/3}$ on the order of the square root of the distance between these locations.

Such growth models may be initiated at time zero with a given interface profile. In the narrow wedge case, when growth is initiated from a unique point, a limiting description of the late time interface, suitably scaled in light of the one-third and two-thirds powers and up to the subtraction of a parabola, is offered by the Airy_2 process, which is a random function $\mathcal{A} : \mathbb{R} \to \mathbb{R}$, whose finite dimensional distributions are specified by Fredholm determinants, that was introduced by [PS02]. Another well-known initial condition is the flat case, when growth begins from a zero initial condition. Here, the Airy_1 process describes the interface at late time. The one-half power law for interface regularity is expressed by the Hölder-$1/2-$-continuity of the processes Airy_1 and Airy_2, which was proved in [QR13].

Growth may be initiated from a much more general initial condition than in these narrow wedge or flat cases. For initial conditions that grow at most linearly, it has been anticipated that a limiting description of the suitably scaled late-time interface should exist in these cases also. Indeed, in a recent preprint [MQR17], Matetski, Quastel and Remenik have utilized a biorthogonal ensemble representation found by [Sas05, BFPS07] associated to the totally asymmetric exclusion process in order to find Fredholm determinant formulas for the multi-point distribution of the height function of this growth process begun from an arbitrary initial condition. Using these formulas to take the KPZ scaling limit, the authors construct a scale invariant Markov process that lies at the heart of the KPZ universality class. The time-one evolution of this Markov process may be applied to very general initial data, and the result is the scaled profile begun from such data, which generalizes the Airy_1 and Airy_2 processes seen in the flat and narrow wedge cases. These more general limiting processes also enjoy Hölder-$1/2-$-continuity: see [MQR17, Theorem 4.4].

The broad range of interface models that are rigorously known or expected to lie in the KPZ universality class includes many last passage percolation models. Such an LPP model comes equipped with a planar random environment, which is independent in disjoint regions. Directed paths, that are permitted say to move only in a direction in the first quadrant, are then assigned energy via this randomness, by say integrating the environment's value along the path. For a given pair of planar points, the path attaining the maximum energy over directed paths with such endpoints is

called a geodesic. The random interface model that we alluded to at the outset is then specified as the maximum geodesic energy when one geodesic endpoint is varied and the other held fixed, in the narrow wedge case, or when the other is free to vary and is rewarded according to the initial condition, in the more general case. The one-third and two-thirds power laws for typical deviation of maximum energy and for lateral correlation have been rigorously demonstrated for only a few LPP models, each of which enjoys an integrable structure: the seminal work of Baik, Deift and Johansson [BDJ99] rigorously established the one-third exponent, and moreover obtained the GUE Tracy-Widom distributional limit, for the case of Poissonian last passage percolation, while the two-thirds power law for transversal fluctuation was derived for this model by Johansson [Joh00]. For models in which these two exponents have been rigorously identified, the exponent pair dictates a system of scaled coordinates in which the concerned maximizing paths and their weights are unit-order, random, quantities: the scaled geodesics may be called polymers, and their scaled energies, weights.

Brownian last passage percolation is an LPP model with attractive integrable and probabilistic features. In this article, we study the scaled interface profile (that is, the polymer weight profile) in Brownian LPP begun from a very general initial condition. We present results proving a more precise version of the one-half power law for interface regularity than has been established hitherto. Here are two of the main conclusions:

- In Theorem 1.1, we prove that the maximum difference in the weight of two point-to-point polymers whose endpoints differ by at most a small scaled quantity ϵ exceeds $\epsilon^{1/2}R$ with probability at most $\exp\left\{-O(1)R^{3/2}\right\}$ for a very broad range of values of R, uniformly in the scaling parameter for Brownian LPP.

- In Theorem 1.4, we prove that any weak limit point of the scaled interface profiles, as the scaling parameter tends to infinity, has sample paths that admit a modulus of continuity of the order of $x^{1/2}\left(\log x^{-1}\right)^{2/3}$. This assertion, alongside a finite-n counterpart, Theorem 1.3, is proved uniformly over a large class of the data that initiates the random growth.

For a given choice of initial condition, the weak limit point in Theorem 1.4 may be expected to be unique and to coincide with the interface profile obtained from this initial data by evolving for a given duration the Markov operator constructed in [MQR17]. However, this Markov operator has been constructed as a limit of totally asymmetric exclusion, so at present this assertion is not proved. Were the techniques of [MQR17] to be adapted to hold for Brownian last passage percolation, it would then presumably be possible to assert the upper bound of order $x^{1/2}\left(\log x^{-1}\right)^{2/3}$ on modulus of continuity for general initial condition interface profiles under the KPZ fixed point.

The strongly on-scale assertion of the one-half power law for profile regularity in Theorem 1.1 plays a significant role in two companion papers. In [Ham17b], it is harnessed to prove that, in Brownian last passage percolation, it is a superpolynomial rarity that a large number of disjoint polymers coexist in a unit-order scaled region. In [Ham17c], this assertion is exploited to make a strong unit-order Brownian comparison for polymer weight profiles (about which more momentarily).

Beyond the conclusions just discussed, the present article also presents a useful tool, Proposition 1.5. Although the weight of a polymer is random, this weight is dictated in the large by parabolic curvature, with the randomness playing a unit-order role once this curvature is accounted for. The proposition shows that the discrepancy between polymer weight and parabola is controlled uniformly as the polymer's endpoints are varied over compact intervals lying in a very broad region.

This tool is needed in the present article and in [Ham17b]. For exponential or Poissonian last passage percolation, a similar tool has been developed, in [BSS16, Propositions 10.1 and 10.5].

We also mention that an alternative expression of the one-half power law for interface regularity is the assertion that Airy processes such as Airy_1 and Airy_2, or scaled interface models in the last passage percolation setting, locally resemble Brownian motion. Such statements may be understood in a local limit, when Gaussianity of a process $\mathcal{A}$ is proved for the low ϵ limit for the random variable $\epsilon^{-1/2}\big(\mathcal{A}(x+\epsilon) - \mathcal{A}(x)\big)$ associated to any given $x \in \mathbb{R}$. Finite dimensional distributional convergence to Brownian motion (of diffusion rate two) in this limit has been proved for the Airy_2 process in [Häg08], for the Airy_1 process in [QR13], and for the more general versions of these Airy processes constructed in [MQR17] in Theorem 4.4 of that paper; in [Pim17], similar local limit results for general initial condition profiles have been obtained for geometric last passage percolation models. Comparison to Brownian motion may also be made without taking such a local limit. In [CH14], the Airy_2 process was understood to be absolutely continuous with respect to Brownian motion on a unit-order interval, by a technique in which this process is embedded as the uppermost curve in a random ensemble of, in effect, mutually avoiding Brownian motions. (This *Brownian Gibbs* technique will play a fundamental role in the present article, and we will return to it.) This inference was improved in [Ham17a], where the implied Radon-Nikodym derivative is shown to lie in all L^p-spaces for $p \in (1, \infty)$, albeit after an affine shift is applied to the Airy_2 process, so that comparison is made not to Brownian motion but to Brownian bridge. In a companion paper to the present article [Ham17c], the problem of unit-order scale Brownian comparison is made for the class of Brownian LPP polymer weight profiles, begun from general initial data, that are the subject of the present article. It is in essence shown there that a given unit-order interval may be split into a random but controlled number of intervals in such a way that the profile when restricted to the smaller intervals has, after affine adjustment, a Radon-Nikodym derivative with respect to Brownian bridge that lies in L^p for $p \in (1, 3)$.

1.1. Brownian last passage percolation [LPP].

We now define this model. On a probability space carrying a law labelled $\mathbb{P}$, let $B : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ denote an ensemble of independent two-sided standard Brownian motions $B(k, \cdot) : \mathbb{R} \to \mathbb{R}$, $k \in \mathbb{Z}$.

Let $i, j \in \mathbb{Z}$ with $i \leq j$. We denote the integer interval $\{i, \cdots, j\}$ by $[\![i, j]\!]$. Further let $x, y \in \mathbb{R}$ with $x \leq y$. Consider the collection of non-decreasing lists $\{z_k : k \in [\![i+1, j]\!]\}$ of values $z_k \in [x, y]$. With the convention that $z_i = x$ and $z_{j+1} = y$, we associate an energy $\sum_{k=i}^{j} \big(B(k, z_{k+1}) - B(k, z_k)\big)$ to any such list. We then define the maximum energy

$$M^1_{(x,i)\to(y,j)} = \sup\left\{ \sum_{k=i}^{j} \Big(B(k, z_{k+1}) - B(k, z_k)\Big) \right\},$$

where the supremum is taken over all such lists. The random process $M^1_{(0,1)\to(\cdot,n)} : [0, \infty) \to \mathbb{R}$ was introduced by [GW91] and further studied in [OY02].

The one-third and two-thirds KPZ scaling considerations that we outlined earlier in the introduction are manifest in Brownian LPP. When the ending height j exceeds the starting height i by a large quantity $n \in \mathbb{N}$, and the location y exceeds x also by n, then the maximum energy grows linearly, at rate $2n$, and has a fluctuation about this mean of order $n^{1/3}$. Moreover, if y is permitted to vary from this location, then it is changes of $n^{2/3}$ in its value that result in a non-trivial correlation of the maximum energy with its original value.

These facts prompt us to introduce scaled coordinates to describe the two endpoint locations, and a notion of scaled maximum energy, which we will refer to as weight. Let $n \in \mathbb{N}$, and suppose that $x, y \in \mathbb{R}$ satisfy $y \geq x - 2^{-1}n^{1/3}$. Define

$$\mathsf{Wgt}_{n;(x,0)}^{(y,1)} = 2^{-1/2}n^{-1/3}\Big(M^1_{(2n^{2/3}x,0)\to(n+2n^{2/3}y,n)} - 2n - 2n^{2/3}(y-x) \Big). \tag{1}$$

(Clearly, n must be positive. In fact, $\mathbb{N}$ will denote $\{1, 2, \cdots\}$ throughout.)

Consistently with the facts just mentioned, the quantity $\mathsf{Wgt}_{n;(x,0)}^{(y,1)}$ may be expected to be, for given real choices of x and y, a unit-order random quantity, whose law is tight in the scaling parameter $n \in \mathbb{N}$. The quantity describes, in units chosen to achieve this tightness, the maximum possible energy associated to journeys which in the original coordinates occur between $(2n^{2/3}x, 0)$ and $(n + 2n^{2/3}y, n)$. In scaled coordinates, this is a journey between $(x, 0)$ and $(y, 1)$. We view the first coordinate as space and the second as time, so this journey is between x and y over the unit time interval $[0, 1]$.

Underlying this definition is a geometric picture of scaled maximizing paths, or polymers, that achieve these weight values. We will defer explicitly defining these polymers, but it is useful to bear in mind that $\mathsf{Wgt}_{n;(x,0)}^{(y,1)}$ equals the weight of a polymer that travels between locations that in scaled coordinates equal $(x, 0)$ and $(y, 1)$.

1.2. **Main results.** In four subsections, we present the principal conclusions: on polymer weight difference under horizontal perturbation of endpoints; our finite-n assertion concerning the modulus of continuity of polymer weight profiles from general initial condition; the inference made about weak limit points of such profiles in the high n limit; and a general tool, on the rarity of deviation from parabolic curvature by polymer weights.

1.2.1. *Polymer weight change under horizontal perturbation of endpoints.* Set $Q : \mathbb{R} \to \mathbb{R}$, $Q(z) = 2^{-1/2}z^2$. The polymer weight $\mathsf{Wgt}_{n;(x,0)}^{(y,1)}$ has a globally parabolic profile, hewing to the shape $-Q(y - x)$. When this parabolic term is added to the polymer weight, the result is a random process in (x, y) which typically suffers changes of order $\epsilon^{1/2}$ when x or y are varied on a small scale $\epsilon > 0$. Our first main result gives rigorous expression to this statement, uniformly in $(n, x, y) \in \mathbb{N} \times \mathbb{R} \times \mathbb{R}$ for which the difference $|y - x|$ is permitted to inhabit an expanding region about the origin, of scale $n^{1/18}$.

Theorem 1.1. *Let $\epsilon \in (0, 2^{-4}]$. Let $n \in \mathbb{N}$ satisfy $n \geq 10^{32}c^{-18}$ and let $x, y \in \mathbb{R}$ satisfy $|x - y| \leq 2^{-2}3^{-1}cn^{1/18}$. Let $R \in [10^4, 10^3 n^{1/18}]$. Then*

$$\mathbb{P}\left(\sup_{\substack{u_1, u_2 \in [x, x+\epsilon] \\ v_1, v_2 \in [y, y+\epsilon]}} \left| \mathsf{Wgt}_{n;(u_2,0)}^{(v_2,1)} + Q(v_2 - u_2) - \mathsf{Wgt}_{n;(u_1,0)}^{(v_1,1)} - Q(v_1 - u_1) \right| \geq \epsilon^{1/2}R \right) \tag{2}$$

is at most $10032\, C \exp\left\{ -c_1 2^{-21} R^{3/2} \right\}$.

Here, we set $c_1 = 2^{-5/2}c \wedge 1/8$, where $\wedge$ denotes minimum. Bounds in Theorem 1.1, and many later results, have been expressed explicitly up to two positive constants c and C. See Subsection 3.3.1 for a discussion of the role of this pair of constants.

The imposition in Theorem 1.1 that $R \in \left[10^4, 10^3 n^{1/18}\right]$ is rather weak, with the case where R is fixed being of interest; and indeed, the decay rate asserted by the theorem is already very fast when R is of order $n^{1/18}$.

1.2.2. *Maximum local variation of polymer weight profiles from general initial data.* What do we mean by such polymer weight profiles? The random function $y \to \mathsf{Wgt}_{n;(0,0)}^{(y,1)}$ may be viewed as the weight profile obtained by scaled maximizing paths that travel from the origin at time zero to the variable location y at time one. This insistence that the paths must begin at the origin, called the narrow wedge by physicists, is of course rather special. We now make a more general definition, of the f-rewarded line-to-point polymer weight $\mathsf{Wgt}_{n;(*:f,0)}^{(y,1)}$. Here, f is an initial condition, defined on the real line. Paths may begin anywhere on the real line at time zero; they travel to $y \in \mathbb{R}$ at time one. (Because they are free at the beginning and fixed at the end, we refer to these paths as 'line-to-point'.) They begin with a reward given by evaluating f at the starting location, and then gain the weight associated to the journey they make. The value $\mathsf{Wgt}_{n;(*:f,0)}^{(y,1)}$, which we will define momentarily, denotes the maximum f-rewarded weight of all such paths. In the notation $\mathsf{Wgt}_{n;(*:f,0)}^{(y,1)}$, we again use subscript and superscript expressions to refer to space-time pairs of starting and ending locations. The starting spatial location is being denoted $* : f$. The star is intended to refer to the free time-zero endpoint, which may be varied, and the $: f$ to the reward offered according to where this endpoint is placed.

The next definition specifies essentially the broadest class of f suitable for a study of the weight profiles $y \to \mathsf{Wgt}_{n;(*:f,0)}^{(y,1)}$ for all sufficiently high $n \in \mathbb{N}$.

Definition 1.2. Writing $\overline{\Psi} = \left(\Psi_1, \Psi_2, \Psi_3\right) \in (0,\infty)^3$ for a triple of positive reals, we let $\mathcal{I}_{\overline{\Psi}}$ denote the set of measurable functions $f : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ such that $f(x) \leq \Psi_1\left(1 + |x|\right)$ and $\sup_{x \in [-\Psi_2, \Psi_2]} f(x) > -\Psi_3$.

For f lying in one of the function spaces $\mathcal{I}_{\overline{\Psi}}$, we now formally define the f-rewarded line-to-point polymer weight $\mathsf{Wgt}_{n;(*:f,0)}^{(y,1)}$ to be

$$\sup_{x \in (-\infty, 2^{-1} n^{1/3} + y]} \left(\mathsf{Wgt}_{n;(x,0)}^{(y,1)} + f(x)\right).$$

Our second main result asserts that the maximal variation in f-rewarded n-polymer weight over length $\epsilon > 0$ intervals in $[-1,1]$ is a controlled random multiple of $\epsilon^{1/2}\left(\log \epsilon^{-1}\right)^{2/3}$. The bound on probability, above scale $e^{-O(1)n^{1/12}}$, is asserted uniformly in initial data, and in (n, ϵ), except for very small $\epsilon \leq e^{-O(1)n^{1/12}}$.

Theorem 1.3. *For $\overline{\Psi} \in (0,\infty)^3$, some $c' = c'(\overline{\Psi})$, $r_0 = r_0(\overline{\Psi}) > 0$ and all $f \in \mathcal{I}_{\overline{\Psi}}$, $n \in \mathbb{N}$ and $r \geq r_0$,*

$$\mathbb{P}\left(\sup_{\substack{y,z \in [-1,1], \\ 2\exp\{-c'n^{1/12}\} < z-y < e^{-1}}} \frac{\left|\mathsf{Wgt}_{n;(*:f,0)}^{(z,1)} - \mathsf{Wgt}_{n;(*:f,0)}^{(y,1)}\right|}{(z-y)^{1/2}\left(\log(z-y)^{-1}\right)^{2/3}} \geq r\right) \leq 2^{47} c^{-4/3} r^{-2}\left(\log r\right)^{4/3} \vee 4 e^{-c'n^{1/12}}.$$

$$(3)$$

1.2.3. *Modulus of continuity of weak limits of weight profiles from general initial data.* Let $n \in \mathbb{N}$, $\overline{\Psi} \in (0, \infty)^3$ and $f \in \mathcal{I}_{\overline{\Psi}}$. Let $\nu_{n;(*:f,0)}^{([-1,1],1)}$ denote the law of the random function

$$[-1, 1] \to \mathbb{R} : y \to \mathsf{Wgt}_{n;(*:f,0)}^{(y,1)} .$$

The control offered by Theorem 1.3 is certainly sufficient to show that the curves of any weak limit point of $\nu_{n;(*:f,0)}^{([-1,1],1)}$ as $n \to \infty$ admit modulus of continuity $z^{1/2} \big(\log z^{-1} \big)^{2/3}$, up to a random factor that is controlled uniformly in the choice of limit point.

To formulate a theorem in this regard, let $\mathcal{A}$ be an arbitrary index set, and let $\big\{ \nu_{n,\alpha} : n \in \mathbb{N} \big\}$, $\alpha \in \mathcal{A}$, be an $\mathcal{A}$-indexed collection of sequences of probability measures on the Borel σ-algebra of a given Hausdorff topological space. The collection is here called $\mathcal{A}$-uniformly tight if, for each $\epsilon > 0$, there exist $n_0 \in \mathbb{N}$ and a compact set K such that $\nu_{n,\alpha}(K) \geq 1 - \epsilon$ whenever $n \geq n_0$ and $\alpha \in \mathcal{A}$.

Theorem 1.4. *Let $\overline{\Psi} \in (0, \infty)^3$ denote a triple of positive reals.*

(1) *Suppose that $n \in \mathbb{N}$ satisfies $n > 2^{-3/2} \Psi_1^3 \vee 8(\Psi_2 + 1)^3$. Let $f \in \mathcal{I}_{\overline{\Psi}}$. Then the measure $\nu_{n;(*:f,0)}^{([-1,1],1)}$ is supported on the space $\mathcal{C}$ of continuous real-valued functions on $[-1, 1]$.*

(2) *The collection of sequences of probability measures $\big\{ \nu_{n;(*:f,0)}^{([-1,1],1)} : n \in \mathbb{N} \big\}$ indexed by $f \in \mathcal{I}_{\overline{\Psi}}$ is $\mathcal{I}_{\overline{\Psi}}$-uniformly tight. Here, the space $\mathcal{C}$ is endowed with the topology of uniform convergence.*

(3) *A law on $\mathcal{C}$ is said to belong to the weak limit point set $\mathrm{WLP}_{\overline{\Psi}}$ if, for some sequence $f_n \in \mathcal{I}_{\overline{\Psi}}$, $n \in \mathbb{N}$, it equals the weak limit of the laws $\nu_{n;(*:f_n,0)}^{([-1,1],1)}$ along some subsequence of $n \in \mathbb{N}$. By (2), $\mathrm{WLP}_{\overline{\Psi}} \neq \emptyset$. For any $\nu \in \mathrm{WLP}_{\overline{\Psi}}$, let X be ν-distributed. Then, for $r \geq r_0$,*

$$\nu \left(\sup_{\substack{x,y \in [-1,1], \\ x < y < x + e^{-1}}} \frac{\big| X(y) - X(x) \big|}{(y - x)^{1/2} \big(\log(y-x)^{-1} \big)^{2/3}} \geq r \right) \leq 2^{47} c^{-4/3} r^{-2} \big(\log r \big)^{4/3}, \tag{4}$$

where Theorem 1.3 provides the constant $r_0 = r_0(\overline{\Psi})$.

Brownian motion on a unit interval has modulus of continuity of order $x^{1/2} \big(\log x^{-1} \big)^{1/2}$, and it may be expected that some version of Theorem 1.4(3) is valid with the logarithmic power of two-thirds replaced by one-half. Indeed, in the special case of narrow wedge initial data, such a result has been proved: see [Ham17a, Theorem 2.13], or [Ham17a, Theorem 1.11(1)] for a result concerning the Airy_2 process.

1.2.4. *Tail behaviour of polymer weight suprema and infima.* Theorem 1.1 quantifies polymer weight changes in response to horizontal endpoint perturbation after the weight has been adjusted by the addition of the parabola $Q(z) = 2^{-1/2} z^2$. In our fourth result, we verify that the point-to-point polymer weight profile indeed strongly hews to this given parabola. The regime where this is verified is that in which the polymer endpoints differ by at most an order of $n^{1/18}$. Within this zone, the inference is made uniformly as the endpoints vary over any given unit-order region.

Proposition 1.5. *Let $n \in \mathbb{N}$ satisfy $n \geq 10^{29} \vee 2(c/3)^{-18}$. Let $x, y \in \mathbb{R}$ satisfy $|x - y| \leq 6^{-1} cn^{1/18}$. Let $t \in \big[34 , 4n^{1/18} \big]$. Then*

$$\mathbb{P}\left(\sup_{u,v \in [0,1]} \Big(\mathsf{Wgt}_{n;(x+u,0)}^{(y+v,1)} + Q(y + v - x - u) \Big) \geq t \right) \leq 139 C \exp \big\{ - c_1 2^{-10} t^{3/2} \big\} \tag{5}$$

and

$$\mathbb{P}\left(\inf_{u,v\in[0,1]}\left(\mathsf{Wgt}_{n;(x+u,0)}^{(y+v,1)}+Q(y+v-x-u)\right)\le -t\right)\le 261C\exp\left\{-c_1 2^{-3}t^{3/2}\right\}. \tag{6}$$

In [BSS16, Propositions 10.1 and 10.5], comparable bounds are proved for exponential and Poissonian LPP, with bounds of the form $\exp\left\{-O(1)t\right\}$. These propositions have the flexibility of treating extremal weights of polymers whose endpoints are permitted to vary over compact regions in space as well as time, rather than merely time, as it is the case for Proposition 1.5.

1.3. **The road map.** Theorem 1.1, which is a key result underlying Theorems 1.3 and 1.4, is proved using ideas similar to the proof of the Kolmogorov continuity criterion. The authors of [QR13] note in Section 1.1 that the task of checking the Kolmogorov criterion on the basis of suitable two-point information for such processes as Airy_1 has turned out to be surprisingly difficult. Similar subtleties arise in our context: two-point information has to be presented in a way that is valid on arbitrarily small scales, without the index n needing to rise. The crucial tool that will enable the derivation of Theorem 1.1 is a powerful two-point estimate with the necessary attributes, Proposition 3.5.

Section 2 introduces notation for the use of scaled coordinates and presents some basic results about polymer weight.

In Section 3, the engine for our main results, Proposition 3.5, namely the two-point estimate for narrow wedge weight profiles, is stated and proved. In this section, we will explain how the narrow wedge profile may be embedded as the uppermost curve in a certain system of ordered random continuous curves called a line ensemble. A suitably normalized version of any such line ensemble has the *Brownian Gibbs* property, which in essence means it is a system of mutually avoiding Brownian bridges. Its curves moreover have a globally parabolic shape, and a definition of *regular* ensemble is made to capture these attributes. The short proof of Proposition 3.5 harnesses the Brownian Gibbs property in an essential way. Certain further properties of regular ensembles are needed, and these also appear in Section 3 as Proposition 3.3, quoted from [Ham17a].

In three further sections are then respectively proved Proposition 1.5; Theorem 1.1; and Theorems 1.3 and 1.4.

1.3.1. *Comment on the companion papers* [Ham17a], [Ham17b] *and* [Ham17c]. Via the upcoming Proposition 3.3 and Lemma 3.4, this article draws on Brownian Gibbs results developed in [Ham17a]. That article is long and it is worth pointing out that the concerned results in [Ham17a] are simple and have short proofs. The present article's main conclusions about scaled Brownian LPP are applied in the later two companion papers. The article may be read on its own, or viewed as part of this four-paper study, an overview of which appears in [Ham17a, Section 1.2].

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2. The basics: notation, scaling, polymers and their weight

In consecutive subsections, we introduce notation; describe Brownian LPP in scaled coordinates; offer a principle that aids in working with these coordinates, and an application; discuss basics about polymers; and provide a simple result about them.

2.1. **General notation and structure.**

2.1.1. *Notation.* Let $i, j \in \mathbb{Z}$ with $i \leq j$. Recall that $[\![i, j]\!]$ denotes the integer interval $\{i, \cdots, j\}$.

For $k \geq 1$, we write $\mathbb{R}^k_{\leq}$ for the subset of $\mathbb{R}^k$ whose elements $(z_1, \cdots, z_k)$ are non-decreasing sequences. When the sequences are increasing, we instead write $\mathbb{R}^k_{<}$. We also use the notation $A^k_{\leq}$ and $A^k_{<}$. Here, $A \subset \mathbb{R}$ and the sequence elements are supposed to belong to A. We will typically use this notation when $k = 2$.

A bar over a symbol indicates a vector, as in the usage $\overline{\Psi} = (\Psi_1, \Psi_2, \Psi_3)$ in Theorems 1.3 and 1.4.

2.1.2. *The role of hypotheses invoked during proofs.* Our proofs invoke several inputs, notably three Reg conditions that specify the notion of a regular ensemble, and Propositions 3.3 and 3.5. Whenever such results are invoked, certain conditions on the concerned hypotheses will be needed. We will always note explicitly what these conditions are, whenever such an application is made. Clearly, it is necessary that the hypotheses of the result that is being proved collectively imply all the conditions that are invoked during its proof. The work needed to do this for a given result may be called the *calculational derivation* of that result. These derivations have almost no conceptual content, reach conclusions that in their overall form are plausible, consist of largely trivial steps, and will be of interest to only the most committed of readers (perhaps only those who are actually applying the results). In some cases, however, the derivations occupy a fair amount of space. We have chosen to separate the principal calculational derivations from the body of the proofs in this article. The concerned results are Theorem 1.1, Proposition 1.5, Proposition 5.1, and Lemmas 6.2 and 6.4. Their calculational derivations are presented in a supplement to this paper, Appendix A. This supplement also contains the details of a rather mundane variation related a question of index parity needed in the proof of Proposition 1.5.

2.2. **Scaling: staircases to zigzags, energy to weight, and geodesics to polymers.**

2.2.1. *Staircases.* Taking $i, j \in \mathbb{N}$ with $i \leq j$, and $x, y \in \mathbb{R}^2_{\leq}$, we have ascribed in Section 1.1 an energy to any non-decreasing list $\{z_k : k \in [\![i+1, j]\!]\}$ of values $z_k \in [x, y]$. In order to emphasise the geometric aspects of this definition, and in the hope that it may aid the visualization of the concerned concepts, we associate to each list a subset of $[x, y] \times [i, j] \subset \mathbb{R}^2$, which will be the range of a piecewise affine path, that we call a staircase.

To define the staircase associated to $\{z_k : k \in [\![i+1, j]\!]\}$, we again adopt the convention that $z_i = x$ and $z_{j+1} = y$. The staircase is specified as the union of certain horizontal planar line segments, and certain vertical ones. The horizontal segments take the form $[z_k, z_{k+1}] \times \{k\}$ for $k \in [\![i, j]\!]$. The right and left endpoints of each consecutive pair of horizontal segments are interpolated by a vertical planar line segment of unit length. It is this collection of vertical line segments that form the vertical segments of the staircase.

The resulting staircase may be depicted as the range of an alternately rightward and upward moving path from starting point (x, i) to ending point (y, j). The set of staircases with these starting and ending points will be denoted by $SC_{(x,i) \to (y,j)}$. Such staircases are in bijection with the collection of non-decreasing lists already considered. Thus, any staircase $\phi \in SC_{(x,i) \to (y,j)}$ is assigned an energy $E(\phi) = \sum_{k=i}^{j} (B(k, z_{k+1}) - B(k, z_k))$ via the associated z-list.

2.2.2. *Energy maximizing staircases are called geodesics.* A staircase $\phi \in SC_{(x,i) \to (y,j)}$ whose energy attains the maximum value $M^1_{(x,i) \to (y,j)}$ is called a geodesic from (x,i) to (y,j). It is a simple consequence of the continuity of the constituent Brownian paths $B(k, \cdot)$ that this geodesic exists for all choices of $(x,y) \in \mathbb{R}^2_{\leq}$. It is also true, and is proved in [Ham17c, Lemma A.1], that, for any given such choice of the pair (x,y), there is an almost surely unique geodesic from (x,i) to (y,j). However, this uniqueness will not be needed in the present article.

2.2.3. *The scaling map.* For $n \in \mathbb{N}$, consider the n-indexed *scaling* map $R_n : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$R_n(v_1, v_2) = \left(2^{-1} n^{-2/3}(v_1 - v_2), \, v_2/n \right).$$

The scaling map acts on subsets C of $\mathbb{R}^2$ by setting $R_n(C) = \{ R_n(x) : x \in C \}$.

2.2.4. *Scaling transforms staircases to zigzags.* The image of any staircase under R_n will be called an n-zigzag. The starting and ending points of an n-zigzag Z are defined to be the image under R_n of such points for the staircase S for which $Z = R_n(S)$.

Note that the set of horizontal lines is invariant under R_n, while vertical lines are mapped to lines of gradient $-2n^{-1/3}$. As such, an n-zigzag is the range of a piecewise affine path from the starting point to the ending point which alternately moves rightwards along horizontal line segments and northwesterly along sloping line segments, where each sloping line segment has gradient $-2n^{-1/3}$.

2.2.5. *Compatible triples.* Let $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}^2_{<}$, which is to say that $n \in \mathbb{N}$ and $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$. Taking $x, y \in \mathbb{R}$, does there exist an n-zigzag from (x, t_1) and (y, t_2)? As far as the data (n, t_1, t_2) is concerned, such an n-zigzag may exist only if

$$t_1 \text{ and } t_2 \text{ are integer multiplies of } n^{-1}. \tag{7}$$

We say that data $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}^2_{<}$ is a *compatible triple* if it verifies the last condition. We will consistently impose this condition, whenever we seek to study n-zigzags whose lifetime is $[t_1, t_2]$. The use of compatible triples should be considered to be a fairly minor, microscopic, detail. As the index n increases, the n^{-1}-mesh becomes finer, so that the space of n-zigzags better approximates a field of functions, defined on arbitrary finite intervals of the vertical coordinate, and taking values in the horizontal coordinate.

An important piece of notation associated to a compatible triple is $t_{1,2}$, which will denote the difference $t_2 - t_1$. The law of the underlying Brownian ensemble $B : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is invariant under integer shifts in the first, curve indexing, coordinate. This translates to an invariance in law of scaled objects under vertical shifts by multiples of n^{-1}, something that makes the parameter $t_{1,2}$ of far greater relevance than t_1 or t_2.

2.2.6. *Staircase energy scales to zigzag weight.* Let $n \in \mathbb{N}$ and $(i,j) \in \mathbb{N}^2_{<}$. Any n-zigzag Z from $(x, i/n)$ to $(y, j/n)$ is ascribed a scaled energy, which we will refer to as its weight, $\mathsf{Wgt}(Z) = \mathsf{Wgt}_n(Z)$, given by

$$\mathsf{Wgt}(Z) = 2^{-1/2} n^{-1/3} \Big(E(S) - 2(j - i) - 2n^{2/3}(y - x) \Big) \tag{8}$$
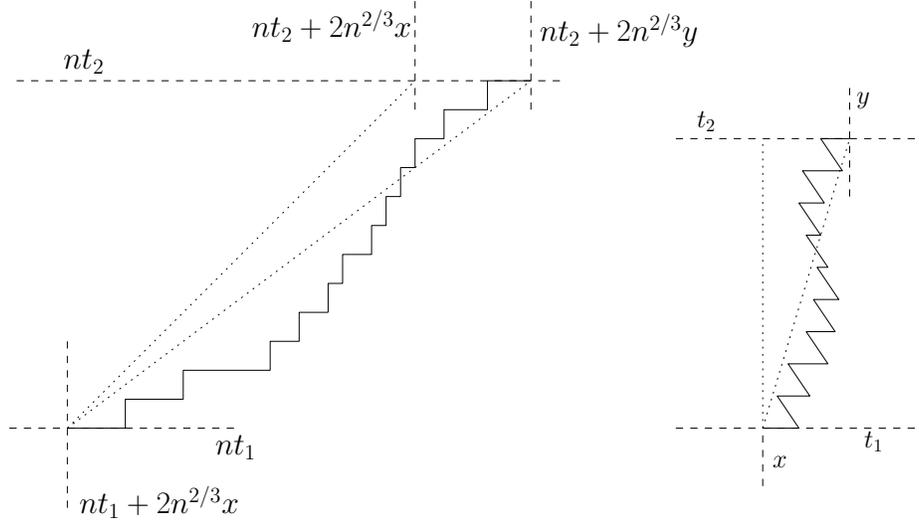
where Z is the image under R_n of the staircase S.

FIGURE 1. Let (n, t_1, t_2) be a compatible triple and let $x, y \in \mathbb{R}$. The endpoints of the geodesic in the left sketch are such that, when the scaling map R_n is applied to produce the right sketch, the result is an n-polymer from (x, t_1) to (y, t_2).

2.2.7. *Maximum weight.* Let $n \in \mathbb{N}$. The quantity $\mathsf{Wgt}^{(y,1)}_{n;(x,0)}$ specified in (1) is nothing other than the maximum weight ascribed to any n-zigzag from $(x, 0)$ to $(y, 1)$.

Let $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}^2_<$ be a compatible triple. Suppose that $x, y \in \mathbb{R}$ satisfy $y \geq x - 2^{-1} n^{1/3} t_{1,2}$. We now offer a definition of $\mathsf{Wgt}^{(y,t_2)}_{n;(x,t_1)}$ such that this quantity equals maximum weight of any n-zigzag from (x, t_1) to (y, t_2). We must set

$$\mathsf{Wgt}^{(y,t_2)}_{n;(x,t_1)} = 2^{-1/2} n^{-1/3} \left(M^1_{(nt_1 + 2n^{2/3}x, nt_1) \to (nt_2 + 2n^{2/3}y, nt_2)} - 2nt_{1,2} - 2n^{2/3}(y - x) \right). \qquad (9)$$

2.2.8. *Highest weight zigzags are called polymers.* An n-zigzag that attains this maximum will be called an n-polymer, or usually, simply a polymer. Thus, geodesics map to polymers under the scaling map. We will write $\rho^{(y,t_2)}_{n;(x,t_1)}$ for *any* n-polymer (x, t_1) to (y, t_2) (see Figure 1), since we do not invoke polymer uniqueness results in this article.

This usage of the term 'polymer' for 'scaled geodesic' is apt for our study, due to the central role played by these objects. The usage is not, however, standard: the term 'polymer' is often used to refer to typical realizations of the path measure in LPP models at positive temperature.

2.2.9. *Parameter settings in applications of results will be indicated in boldface.* Often we apply results involving several parameters. Typically these include the index n, times t_1 and t_2, and spatial locations such as x and y. Whenever we apply results, we will always state what these parameter settings are. When we do so, we will use boldface to indicate the variables in the result being applied, expressing these in terms of non-boldface variables, which assume their values from the current context. This device permits notational conflicts to be deescalated (so that notational choices need not proliferate, as they would were these conflicts to be eliminated by other means).

2.3. **The scaling principle.** Let $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}^2_<$ be a compatible triple. The quantity $nt_{1,2}$ is a positive integer, in view of the defining property (7). The scaling map $R_k : \mathbb{R}^2 \to \mathbb{R}^2$ has been defined whenever $k \in \mathbb{N}$, and thus we may speak of R_n and $R_{nt_{1,2}}$. The map R_n is the composition of $R_{nt_{1,2}}$ and the transform $S_{t_{1,2}^{-1}}$ given by $\mathbb{R}^2 \to \mathbb{R}^2 : (a, b) \to \left(at_{1,2}^{-2/3}, bt_{1,2}^{-1} \right)$. That is, the system of $nt_{1,2}$-zigzags is transformed into the system of n-zigzags by an application of $S_{t_{1,2}^{-1}}$. Note from (9) that $\mathsf{Wgt}_{n;(x,t_1)}^{(y,t_2)} = t_{1,2}^{1/3} \mathsf{Wgt}_{nt_{1,2};(xt_{1,2}^{-2/3},\kappa)}^{(yt_{1,2}^{-2/3},\kappa+1)}$, where $\kappa = t_1 t_{1,2}^{-1}$; indeed this weight transformation law is valid for all zigzags, rather than just polymers, in view of (8).

We may summarise these inferences by saying that the system of $nt_{1,2}$-zigzags, including their weight data, is transformed into the n-zigzag system, and its accompanying weight data, by the transformation $(a, b, c) \to \left(at_{1,2}^{-1/3}, bt_{1,2}^{-2/3}, ct_{1,2}^{-1} \right)$, where the coordinates refer to the changes suffered in weight, horizontal and vertical coordinates. This fact leads us to what we call the *scaling principle.*

The scaling principle. Let $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}^2_<$ be a compatible triple. Any statement concerning the system of n-zigzags, including weight information, is equivalent to the corresponding statement concerning the system of $nt_{1,2}$-zigzags, provided that the following changes are made:

- the index n is replaced by $nt_{1,2}$;

- any time is multiplied by $t_{1,2}^{-1}$;

- any weight is multiplied by $t_{1,2}^{1/3}$;

- and any horizontal distance is multiplied by $t_{1,2}^{-2/3}$.

2.3.1. *The scaling principle applied: uniform control on polymer weight for a general time-pair.* Proposition 1.5 provides a uniform control on polymer weights whose starting and ending points lie in $[x, x+1] \times \{0\}$ and $[y, y+1] \times \{1\}$. We now illustrate the scaling principle by using it to extend the proposition to treat the case where these intervals are replaced by $[x, x+t_{1,2}^{2/3}] \times \{t_1\}$ and $[y, y+t_{1,2}^{2/3}] \times \{t_2\}$ for a general time pair (t_1, t_2).

We phrase this more general result as an upper bound on the probability of a *polymer weight regularity* event. To define the new event, we again consider a compatible triple $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}^2_<$. For $x, y \in \mathbb{R}$, $w_1, w_2 \geq 0$ and $r > 0$, let $\mathsf{PolyWgtReg}_{n;([x,x+w_1],t_1)}^{([y,y+w_2],t_2)}(r)$ denote the event that, for all $(u, v) \in [0, w_1] \times [0, w_2]$,

$$\left| t_{1,2}^{-1/3} \mathsf{Wgt}_{n;(x+u,t_1)}^{(y+v,t_2)} + 2^{-1/2} t_{1,2}^{-4/3} \left(y + v - x - u \right)^2 \right| \leq r.$$

We write $\neg A$ for the complement of the event A.

Corollary 2.1. *Let $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}^2_<$ be a compatible triple for which $nt_{1,2} \in \mathbb{N}$ is at least $10^{29} \vee 2(c/3)^{-18}$. Let $x, y \in \mathbb{R}$ and $a, b \in \mathbb{N}$ be such that $\left| x - y \right| t_{1,2}^{-2/3} + \max\{a, b\} - 1 \leq 6^{-1} c(nt_{1,2})^{1/18}$. Let $r \in \left[34, 4(nt_{1,2})^{1/18} \right]$. Then*

$$\mathbb{P}\left(\neg \mathsf{PolyWgtReg}_{n;([x,x+at_{1,2}^{2/3}],t_1)}^{([y,y+bt_{1,2}^{2/3}],t_2)}(r) \right) \leq ab \cdot 400C \exp\left\{ -c_1 2^{-10} r^{3/2} \right\}.$$

Proof. It is immediate from Proposition 1.5 that

$$\mathbb{P}\Big(\neg\,\mathsf{PolyWgtReg}^{([y,y+1],1)}_{n;([x,x+1],0)}(r)\Big) \le 400C\exp\big\{-c_1 2^{-10}r^{3/2}\big\}. \tag{10}$$

when $n \ge 10^{29} \vee 2(c/3)^{-18}$, $|x-y| \le 6^{-1}cn^{1/18}$ and $r \in \big[34\,,\,4n^{1/18}\big]$.

By the scaling principle and invariance under vertical shift, we know that

$$\mathbb{P}\Big(\mathsf{PolyWgtReg}^{([y,y+t^{2/3}_{1,2}],t_2)}_{n;([x,x+t^{2/3}_{1,2}],t_1)}(r)\Big) = \mathbb{P}\Big(\mathsf{PolyWgtReg}^{\big([(y-x)t^{-2/3}_{1,2},(y-x)t^{-2/3}_{1,2}+1],1\big)}_{nt_{1,2};([0,1],0)}(r)\Big).$$

Consider (10) with parameter settings $\mathbf{n} = nt_{1,2}$, $\mathbf{x} = 0$, $\mathbf{y} = (y-x)t^{-2/3}_{1,2}$ and $\mathbf{r} = r$. (This is the first use of boldface notation as specified in Subsection 2.2.9. Its use here deprives $\mathbf{n} = nt_{1,2}$ of ambiguity of meaning, for example.) We find then that

$$\mathbb{P}\Big(\neg\,\mathsf{PolyWgtReg}^{([y,y+t^{2/3}_{1,2}],t_2)}_{n;([x,x+t^{2/3}_{1,2}],t_1)}(r)\Big) \le 400C\exp\big\{-c_1 2^{-10}r^{3/2}\big\} \tag{11}$$

provided that $nt_{1,2} \ge 10^{29} \vee 2(c/3)^{-18}$, $|x-y|t^{-2/3}_{1,2} \le 6^{-1}c(nt_{1,2})^{1/18}$ and $r \in \big[34\,,\,4(nt_{1,2})^{1/18}\big]$.

This last bound is then summed over the ab choices of pairs

$$(\mathbf{x},\mathbf{y}) \in \big\{x, x+t^{2/3}_{1,2}, \cdots, x+(a-1)t^{2/3}_{1,2}\big\} \times \big\{y, y+t^{2/3}_{1,2}, \cdots, y+(b-1)t^{2/3}_{1,2}\big\}$$

in order to obtain the corollary. Note that we hypothesise that $|x-y|t^{-2/3}_{1,2} + \max\{a,b\} - 1$ be at most $6^{-1}c(nt_{1,2})^{1/18}$ in order that the bound (11) be valid for these parameter choices. $\square$

Actually, when we apply this corollary in the present article, it will be in the case that $t_1 = 0$ and $t_2 = 1$, so in fact the use of the scaling principle is unnecessary in this regard. It is useful, however, to have a general form for the corollary: for example, it is used in [Ham17b].

2.4. **Polymers.**

2.4.1. *Polymer concatenation.* Let $n \in \mathbb{N}$ and $(t_1, t_2, t_3) \in \mathbb{R}^3_\le$ be such that (n, t_1, t_2) and (n, t_2, t_3) are compatible triples. Let $x, y, z \in \mathbb{R}$. In accordance with the convention stated in Subsection 2.2.8, we may consider n-polymers $\rho^{(y,t_2)}_{n;(x,t_1)}$ and $\rho^{(z,t_3)}_{n;(y,t_2)}$. The union of these two subsets of $\mathbb{R}^2$ is clearly an n-zigzag from (x, t_1) and (z, t_3). In the union, the journey over the latter polymer follows that over the former. For this reason, we regard the union polymer as the concatenation of the two given polymers, and denote it by $\rho^{(y,t_2)}_{n;(x,t_1)} \circ \rho^{(z,t_3)}_{n;(y,t_2)}$. The new polymer's weight equals $\mathsf{Wgt}^{(y,t_2)}_{n;(x,t_1)} + \mathsf{Wgt}^{(z,t_3)}_{n;(y,t_2)}$.

2.4.2. *Polymer splitting.* Opposite to the operation of polymer concatenation is the splitting of a given polymer into two pieces. Let $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}^2_\le$ be a compatible triple, and let $(x, y) \in \mathbb{R}^2$ satisfy $y \ge x - 2^{-1}n^{1/3}t_{1,2}$. Let $t \in (t_1, t_2)$ be such that (n, t_1, t) and (n, t, t_2) are also compatible triples. For any polymer $\rho^{(y,t_2)}_{n;(x,t_1)}$, we may select an element $(z, t) \in \rho^{(y,t_2)}_{n;(x,t_1)}$. The removal of (z, t) from $\rho^{(y,t_2)}_{n;(x,t_1)}$ creates two connected components. Taking the closure of each of these amounts to adding the point (z, t) to each of them. The resulting sets are n-zigzags from (x, t_1) to (z, t), and from (z, t) to (y, t_2); in fact, it is straightforward to see that they are n-polymers.

2.4.3. *Polymer crossing and rewiring.* We now make some comments about the implications of the event that two polymers cross. We do so for line-to-point polymers. For $\bar{\Psi} \in (0, \infty)^3$, let $f \in \mathcal{I}_{\bar{\Psi}}$. Let $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}^2_{\leq}$ be a compatible triple, and let $y \in \mathbb{R}$. An n-zigzag ϕ from (x, t_1), where $x \in \mathbb{R}$, to (y, t_2), whose f-rewarded weight $\mathsf{Wgt}(\phi) + f(x)$ attains the maximum value $\mathsf{Wgt}^{(y,t_2)}_{n;(*:f,t_1)}$, is called an f-rewarded line-to-point polymer. Such polymers are born free, but not equal: an endowment of f is bestowed according to the place of birth. Pursuing a similar convention to that used in the point-to-point case, *any* such polymer will be denoted by $\rho^{(y,t_2)}_{n;(*:f,t_1)}$.

Suppose that two f-rewarded line-to-point polymers cross. That is, suppose that $(x_1, x_2) \in \mathbb{R}^2_{<}$ and $(y_1, y_2) \in \mathbb{R}^2_{<}$ are such that there exist such polymers, labelled $\rho^{(y_1,t_2)}_{n;(*:f,t_1)}$ and $\rho^{(y_2,t_2)}_{n;(*:f,t_1)}$ by our convention, whose journeys are $(x_2, t_1) \to (y_1, t_2)$ and $(x_1, t_1) \to (y_2, t_2)$. The two polymers necessarily meet, and indeed the union of the horizontal segments of the two polymers also meet. If (z, t) is such a point of intersection, the operation of polymer splitting at (z, t) may be applied to the two polymers, resulting in decompositions that may be respectively labelled $\rho_1 \circ \rho_2$ and $\rho_3 \circ \rho_4$. The zigzags ρ_1 and ρ_3 share their f-rewarded weights, $\mathsf{Wgt}(\rho_1) + f(x_2)$ and $\mathsf{Wgt}(\rho_3) + f(x_1)$, because the weight maximality of $\rho_1 \circ \rho_2$ and $\rho_3 \circ \rho_4$ each enforce one of the two inequalities between these quantities. Thus, $\rho_1 \circ \rho_2$ and $\rho_3 \circ \rho_2$ share their f-rewarded weight, and so do $\rho_3 \circ \rho_4$ and $\rho_1 \circ \rho_4$. The new, rewired, zigzags $\rho_3 \circ \rho_2$ and $\rho_1 \circ \rho_4$ are thus seen to be f-rewarded line-to-point polymers.

In summary, when two f-rewarded line-to-point polymers cross, the rewiring just undertaken results in an alternative pair of such polymers so that the old pair and the new share their set of starting and ending points.

2.5. **A simple lemma concerning polymer weight.**

Lemma 2.2. *Let $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}^2_{<}$ be a compatible triple.*

(1) *The random function $(x, y) \to \mathsf{Wgt}^{(y,t_2)}_{n;(x,t_1)}$, which is defined on the set of $(x, y) \in \mathbb{R}^2$ satisfying $y \geq x - 2^{-1} n^{1/3} t_{1,2}$, is continuous almost surely.*

(2) *Further consider an intermediate time $t \in (t_1, t_2)$ such that (n, t_1, t) and (n, t, t_2) are compatible triples. Let $x, y, z \in \mathbb{R}$. Then*
$$\mathsf{Wgt}^{(y,t_2)}_{n;(x,t_1)} \geq \mathsf{Wgt}^{(z,t)}_{n;(x,t_1)} + \mathsf{Wgt}^{(y,t_2)}_{n;(z,t)},$$
provided that these three weights are well defined. (The explicit conditions that ensure that the definitions make sense are $y \geq x - 2^{-1} n^{1/3} t_{1,2}$, $z \geq x - 2^{-1} n^{1/3} (t - t_1)$ and $y \geq z - 2^{-1} n^{1/3} (t_2 - t)$.)

(3) *Let $\bar{\Psi} \in (0, \infty)^3$ and $f \in \mathcal{I}_{\bar{\Psi}}$. Suppose that $n \in \mathbb{N}$ satisfies $n > 2^{-3/2} \Psi_1^3 \vee 8(\Psi_2 + 1)^3$. Then $[-1, 1] \to \mathbb{R} : y \to \mathsf{Wgt}^{(y,1)}_{n;(*:f,0)}$ is almost surely finite and continuous.*

Proof. (1): By (9), it is enough to prove, for each $i, j \in \mathbb{N}$, $i \leq j$, that $M^1_{(x,i) \to (y,j)}$ is a continuous function of $(x, y) \in \mathbb{R}^2_{\leq}$. Let x_1 and x_2 satisfy $x_1 \leq x_2 \leq y$. It is a simple matter to verify that
$$B(i, x_2) - B(i, x_1) \leq M^1_{(x_1,i) \to (y,j)} - M^1_{(x_2,i) \to (y,j)} \leq \sup_{k \in [\![i,j]\!]} M^1_{(x_1,i) \to (x_2,k)} \, ;$$

$$\text{and} \quad \sup_{k \in [\![i,j]\!]} M^1_{(x_1,i) \to (x_2,k)} \leq \sum_{k=i}^{j} \Big(\sup_{z \in [x_1,x_2]} B(k, z) - \inf_{z \in [x_1,x_2]} B(k, z) \Big).$$

Continuity of $M^1_{(x,i)\to(y,j)}$ in the x-variable thus follows from continuity of the two-sided Brownian motions $B(k, \cdot)$. This continuity in the y-variable follows similarly.

(2): The weight $\mathsf{Wgt}^{(z,t)}_{n;(x,t_1)} + \mathsf{Wgt}^{(y,t_3)}_{n;(z,t)}$ of the concatenation of two polymers $\rho^{(z,t)}_{n;(x,t_1)}$ and $\rho^{(y,t_3)}_{n;(z,t)}$ offers a lower bound on $\mathsf{Wgt}^{(y,t_2)}_{n;(x,t_1)}$.

(3): In the proof of [Ham17c, Lemma 4.6(2)], which appears in [Ham17c, Appendix A], it is noted that $\mathsf{W}^{(y,1)}_{n;(*:f,0)}$ equals

$$2^{-1/2}n^{-1/3} \sup_{u\in(-\infty,n+2n^{2/3}y]} \left(M^1_{(u,0)\to(n+2n^{2/3}y,n)} - n - 2n^{2/3}y + u + h(u)\right).$$

Here, $h : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is given by $h(x) = 2^{1/2}n^{1/3}f\left(n^{-2/3}x/2\right)$. Using a notation for unscaled line-to-point energy, namely

$$M^1_{(*:g,0)\to(y,n)} := \sup\left\{E(\phi) + g(x) : \phi \in D^1_{(x,0)\to(y,n)}, \ x \le y\right\},$$

the quantity $\mathsf{W}^{(y,1)}_{n;(*:f,0)}$ is seen to equal $2^{-1/2}n^{-1/3}M^1_{(*:g,0)\to(n+2n^{2/3}y,n)}$ where the function g is given by $g(u) = -n - 2n^{2/3}y + u + h(u)$. It is further noted in the same proof that $\limsup_{u\to-\infty} g(u)/|u| < 0$ is satisfied when $n > 2^{-3/2}\Psi_1^3$; and that, since $f \in \mathcal{I}_{\overline{\Psi}}$, the condition that $g(u) > -\infty$ for some $u \le n + 2n^{2/3}y$ (where $y \in [-1,1]$ is given) is verified when $n \ge 8(\Psi_2 - y)^3$. We may thus apply [Ham17c, Lemma A.2] to learn that $\mathsf{W}^{(y,1)}_{n;(*:f,0)}$ almost surely assumes finite real values whenever $y \in [-1,1]$ under our present hypotheses. Regarding the continuity of this function of $y \in [-1,1]$, note first that, for n given satisfying these hypotheses, the location of the maximizer '$* : f$' is tight as f varies over $\mathcal{I}_{\overline{\Psi}}$: this is a consequence of the square-root growth of $M^1_{(x,0)\to(y,n)}$ in the variable $y - x$, which is explained in and after equation (28) of [Ham17c], which growth cannot compete with the linear decrease in the function g. The question of the continuity of $y \to \mathsf{W}^{(y,1)}_{n;(*:f,0)}$ has in essence been reduced to the first part of the present lemma; we omit the details of this reduction. $\square$

3. Line ensembles, their Brownian Gibbs property, and a key two-point estimate

In four subsections: we discuss the embedding of the narrow wedge polymer weight profile in a certain line ensemble; explain the Brownian Gibbs property enjoyed by a normalized version of that line ensemble, and an associated notion of *regular* ensemble; we gather the needed inputs from elsewhere, which assert that the narrow wedge profile is embedded in a regular ensemble (Proposition 3.2), and certain useful properties of regular ensembles (Proposition 3.3 and Lemma 3.4); and we give the short, Brownian Gibbs, proof of the vital two-point estimate, Proposition 3.5.

3.1. Polymer weight profiles as the uppermost curves in line ensembles. Let $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}^2_{\le}$ be a compatible triple and let $x \in \mathbb{R}$. Define the forward polymer weight profile

$$\mathcal{L}^{\uparrow;t_2}_{n;(x,t_1)}(1, \cdot) : \left[x - 2^{-1}n^{1/3}t_{1,2}, \infty\right) \to \mathbb{R}$$

with base-point (x, t_1) and end height t_2 by setting $\mathcal{L}^{\uparrow;t_2}_{n;(x,t_1)}(1, y) = \mathsf{Wgt}^{(y,t_2)}_{n;(x,t_1)}$ for $y \ge x - 2^{-1}n^{1/3}t_{1,2}$. We call this weight profile 'forward', and adorn the notation with the symbol $\uparrow$, to reflect that it is the spatial location y associated to the more advanced time t_2 that is treated as the variable: we stand at (x, t_1) and look forward in time to witness the weight profile as a function of $(\cdot, t_2)$.

Retaining the triple $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}^2_{\leq}$ but now fixing $y \in \mathbb{R}$ (and treating $x \in \mathbb{R}$ as a variable), we also introduce the *backward* polymer weight profile

$$\mathcal{L}_{n;t_1}^{\downarrow;(y,t_2)}(1, \cdot) : \left(-\infty, y + 2^{-1} n^{1/3} t_{1,2} \right] \to \mathbb{R}$$

with base-point (y, t_2) and end height t_1 by setting $\mathcal{L}_{n;t_1}^{\downarrow;(x,t_2)}(1, x) = \mathsf{Wgt}_{n;(x,t_1)}^{(y,t_2)}$ for each $x \leq y + 2^{-1} n^{1/3} t_{1,2}$. In other words, we now stand at (y, t_2) and look backwards in time at those polymers, ending at our location, which begin at time (and height) t_1; it is the weight profile of these polymers that is being recorded.

This new and elaborate looking notation may seem merely to describe the already denoted weight profile $\mathsf{Wgt}_{n;(x,t_1)}^{(y,t_2)}$, viewed as a function either of x or y. Its conceptual significance is suggested by our calling the argument of either profile '$(1, \cdot)$' rather than simply '$(\cdot)$'. Indeed, we will view $\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}$ as an ensemble of $nt_{1,2} + 1$ curves of which the lowest indexed curve, just defined, is the uppermost; and the backward object is just the same. Either ensemble collectively has the Brownian Gibbs property, a fundamental tool for the analysis of the weight profile that is its uppermost curve.

It is useful to retain a vivid picture of both of the processes $\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(1, \cdot)$ and $\mathcal{L}_{n;t_1}^{\downarrow;(x,t_2)}(1, \cdot)$ as random curves that locally resemble Brownian motion but that globally follow the shape of a parabola. The parabola in question is $-2^{-1/2}(y - x)^2 t_{1,2}^{-4/3}$. The forward process adopts values of order $t_{1,2}^{1/3}$ for argument values y that differ from x by order $t_{1,2}^{2/3}$, and is forced downwards rapidly by parabolic curvature outside this region. When $t_{1,2}$ is small, for example, the weight profile is sharply peaked, and it broadens out as $t_{1,2}$ rises. (This description neglects the role of the index n, but roughly it develops accuracy as n rises.)

It is valuable to bring the forward and backward weight profiles for differing values of $t_{1,2}$ on to the same footing, by using a parabolic change of coordinates that, for example, flattens out the sharp peak witnessed when $t_{1,2}$ is small. The coordinate change will also bring the peak centre to the origin (from x or y, according to the forward or backward case). The above weight profiles are already scaled objects, and so we introduce the term *normalized* to refer to the profiles viewed after this new, $t_{1,2}$-determined, change of coordinates.

Indeed, we define the normalized forward polymer weight profile

$$\mathsf{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(1, \cdot) : \left[-2^{-1}(nt_{1,2})^{1/3}, \infty \right) \to \mathbb{R} \, ,$$

by setting

$$\mathsf{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(1, z) = t_{1,2}^{-1/3} \mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}\left(x + t_{1,2}^{2/3} z \right) .$$

Its backwards counterpart $\mathsf{Nr}\mathcal{L}_{n;t_1}^{\downarrow;(y,t_2)}(1, \cdot) : \left(-\infty, 2^{-1}(nt_{1,2})^{1/3} \right] \to \mathbb{R}$ is obtained by setting $\mathsf{Nr}\mathcal{L}_{n;t_1}^{\downarrow;(y,t_2)}(1, z)$ equal to $t_{1,2}^{-1/3} \mathcal{L}_{n;t_1}^{\downarrow;(y,t_2)}(1, y + t_{1,2}^{2/3} z)$.

Brownian motion is invariant under the parabolic change of coordinates, while the parabola $x \to -2^{-1/2}(y - x)^2 t_{1,2}^{-4/3}$ maps to $x \to -2^{1/2} x^2$. Thus, our normalized processes should be pictured as locally Brownian as before, but with curvature dictated by the curve $-2^{-1/2} x^2$. This picture in fact expands its domain of validity as the index increases, encompassing an expanding region about the origin, where the relevant indexing variable is now $nt_{1,2}$, rather than n. These heuristic comments find rigorously expressed counterparts in the next section.

3.2. **Brownian Gibbs ensembles and a regularity property.** Our weight profiles $\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(1,\cdot)$, $\mathsf{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(1,\cdot)$, and their backward counterparts, may be embedded as uppermost curves in systems (or 'ensembles') of random curves. (In [Ham17b, Figure 4], such a scaled and a normalized forward ensemble are illustrated.) The normalized forward and backward ensembles satisfy the Brownian Gibbs property; moreover, these objects adhere well enough to the informal description of being locally Brownian and globally parabolic that we describe them as *regular* ensembles.

In the next paragraphs:

- we offer in outline the definition of the curves, indexed by higher values $k > 1$, in ensembles such as $\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(k,\cdot)$ and $\mathsf{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(k,\cdot)$ (our treatment is informal because only the uppermost ensemble curves, indexed by $k = 1$, concern us in this article);

- we explain the implication for the uppermost curve of an ensemble being Brownian Gibbs;

- and we specify in Definition 3.1 what it means for a Brownian Gibbs ensemble to be regular.

3.2.1. *Embedding weight profiles into ensembles as the uppermost curve.* For $(i,j) \in \mathbb{N}_{\leq}^2$ and $(x,y) \in \mathbb{R}_{\leq}^2$, recall that $M_{(x,i)\to(y,j)}^1$ is the energy maximum over staircases from (x,i) to (y,j). It has a counterpart $M_{(x,i)\to(y,j)}^k$, the maximum collective energy of a k-tuple of staircases with these endpoints satisfying a natural disjointness condition. The scaling relation (9) is extended to define the weight associated to the maximizing k-tuple after scaling. This weight is then defined to equal the sum (over i) of the k lowest indexed ensemble curves $\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}(i,\cdot)$ evaluated at $\cdot = y$.

We attempt a heuristic explanation of the upcoming appearance of the Brownian Gibbs property. The ensemble $[\![1, n+1]\!] \times [0, \infty) \to \mathbb{R} : (k,y) \to M_{(0,0)\to(y,n)}^k - M_{(0,0)\to(y,n)}^{k-1}$ (where $M^0 = 0$ is taken) is a microscopic counterpart to the scaled system $[\![1, n+1]\!] \times [0, \infty) \to \mathbb{R} : (k,y) \to \mathcal{L}_{n;(0,0)}^{\uparrow;1}(k,y)$, because it is the inverse image under the scaling map R_n of the latter ensemble. In [OY02], this microscopic counterpart was identified in law as Dyson's Brownian motion with $n + 1$ particles: a system of $n + 1$ one-dimensional Brownian motions, all begun at the origin at time zero, and conditioned on mutual avoidance at all positive times. The linear scaling map R_n preserves the diffusion rate of locally Brownian processes, so that the scaled ensemble $\mathcal{L}_{n;(0,0)}^{\uparrow;1}$ maintains the basic character of Dyson's Brownian motion: it is an ordered system of standard Brownian motions, conditioned on mutual avoidance, with boundary conditions that ensure that the first few uppermost curves are at unit distance, with the system's curves globally following the parabola $-2^{-1/2}x^2$.

3.2.2. *Line ensembles and their Brownian Gibbs property.* Let $n \in \mathbb{N}$ and let $I \subseteq \mathbb{R}$ be closed. A $[\![1, n]\!]$-indexed line ensemble defined on I is a random collection of continuous curves $\mathcal{L} : [\![1, n]\!] \times I \to \mathbb{R}$ specified under a probability measure $\mathbb{P}$. The ith curve is thus $\mathcal{L}(i, \cdot) : I \to \mathbb{R}$. (The adjective 'line' has been applied to these systems perhaps because of their origin in such models as Poissonian LPP, where the counterpart object has piecewise constant curves. We will omit it henceforth.) An ensemble is called *ordered* if $\mathcal{L}(i, x) > \mathcal{L}(i + 1, x)$ whenever $i \in [\![1, n - 1]\!]$ and x lies in the interior of I. The curves may thus assume a common value at any finite endpoint of I. We will consider ordered ensembles that satisfy a key condition called the Brownian Gibbs property. We specify this property only in the special case that we need, in regard to the uppermost curve $\mathcal{L}(1, \cdot)$.

For $a, b \in \mathbb{R}$, $a < b$, and $y, z \in \mathbb{R}$, let $\mathcal{B}_{y,z}^{[a,b]}$ denote the law of Brownian bridge $B : [a, b] \to \mathbb{R}$ with $B(a) = y$ and $B(b) = z$, given by conditioning standard Brownian motion to have these endpoints.

For any $[a, b] \subseteq I$, and $y, z \in \mathbb{R}$, consider the conditional distribution of the marginal process $\mathcal{L}(1, \cdot) : [a, b] \to \mathbb{R}$ given that $\mathcal{L}(1, a) = x$ and $\mathcal{L}_N(1, b) = y$ and further given the form $g : [a, b] \to \mathbb{R}$ of $\mathcal{L}_N(2, \cdot)$ on $[a, b]$. The Brownian Gibbs property of $\mathcal{L}$ asserts that this law equals Brownian bridge B under $\mathcal{B}_{y,z}^{[a,b]}$ conditioned by $B(u) > g(u)$ for all $u \in [a, b]$.

3.2.3. *Defining (c, C)-regular ensembles.* The next definition specifies a $(\bar{\phi}, c, C)$-regular ensemble from [Ham17a, Definition 2.4], in the special case where the vector $\bar{\phi}$ equals $(1/3, 1/9, 1/3)$.

Definition 3.1. Consider a Brownian Gibbs ensemble of the form

$$\mathcal{L} : [\![1, N]\!] \times \big[-z_{\mathcal{L}}, \infty\big) \to \mathbb{R},$$

defined on a probability space under the law $\mathbb{P}$. The number $N = N(\mathcal{L})$ of ensemble curves and the absolute value $z_{\mathcal{L}}$ of the finite endpoint may take any values in $\mathbb{N}$ and $[0, \infty)$.

Let C and c be two positive constants. The ensemble $\mathcal{L}$ is said to be (c, C)-regular if the following conditions are satisfied.

(1) **Endpoint escape.** $z_{\mathcal{L}} \geq cN^{1/3}$.

(2) **One-point lower tail.** If $z \in [-z_{\mathcal{L}}, \infty)$ satisfies $|z| \leq cN^{1/9}$, then

$$\mathbb{P}\Big(\mathcal{L}(1, z) + 2^{-1/2}z^2 \leq -s\Big) \leq C \exp\big\{-cs^{3/2}\big\}$$

for all $s \in \big[1, N^{1/3}\big]$.

(3) **One-point upper tail.** If $z \in [-z_{\mathcal{L}}, \infty)$ satisfies $|z| \leq cN^{1/9}$, then

$$\mathbb{P}\Big(\mathcal{L}(1, z) + 2^{-1/2}z^2 \geq s\Big) \leq C \exp\big\{-cs^{3/2}\big\}$$

for all $s \in [1, \infty)$.

A Brownian Gibbs ensemble of the form

$$\mathcal{L} : [\![1, N]\!] \times \big(-\infty, z_{\mathcal{L}}\big] \to \mathbb{R}$$

is also said to be (c, C)-regular if the reflected ensemble $\mathcal{L}(\cdot, -\cdot)$ is. This is equivalent to the above conditions when instances of $[-z_{\mathcal{L}}, \infty)$ are replaced by $(-\infty, z_{\mathcal{L}}]$.

We will refer to these three regular ensemble conditions as Reg(1), Reg(2) and Reg(3).

3.3. **Inputs concerning regular ensembles.**

3.3.1. *The normalized forward and backward ensembles are (c, C)-regular.* In Subsection 3.2.1, we informally described how $\mathcal{L}_{n;(0,0)}^{\uparrow;1}$ is a globally parabolic object whose curves are mutually avoiding Brownian motions with a boundary condition suitable to ensuring that for example $\mathcal{L}_n(1, 0)$ and $\mathcal{L}_n(1, 0) - \mathcal{L}_n(2, 0)$ are random but unit-order quantities. We capture this notion by asserting that this ensemble is *regular*. When the time-pair $(t_1, t_2) \in \mathbb{R}_{\leq}^2$ (as well as $x \in \mathbb{R}$) is general, it is the normalized ensemble $\mathsf{Nr}\mathcal{L}_{n;(x,t_1)}^{\uparrow;t_2}$ which is regular. Our assertion to this effect is [Ham17b, Proposition 4.2].

Proposition 3.2. *Let $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}^2_<$ be a compatible triple, and let $x \in \mathbb{R}$. The normalized forward weight profile $\mathsf{NrL}^{\uparrow;t_2}_{n;(x,t_1)}(1, \cdot)$, defined on $\left[-2^{-1}(nt_{1,2})^{1/3}, \infty \right)$, may be represented as the lowest indexed curve in an ensemble*

$$\mathsf{NrL}^{\uparrow;t_2}_{n;(x,t_1)} : [\![1, nt_{1,2} + 1]\!] \times \left[-2^{-1}(nt_{1,2})^{1/3}, \infty \right) \to \mathbb{R}$$

that enjoys the Brownian Gibbs property. Denoting this ensemble by $\mathcal{L}$, we naturally have $N(\mathcal{L}) = nt_{1,2} + 1$ and $z_{\mathcal{L}} = 2^{-1}(nt_{1,2})^{1/3}$.

There exist positive constants C and c, which may be chosen independently of all such choices of the parameters t_1, t_2, x and n, such that the ensemble $\mathcal{L}$ is (c, C)-regular.

Similarly, the backward weight profile $\mathsf{NrL}^{\downarrow;(y,t_2)}_{n;t_1}(1, \cdot) : \left(-\infty, 2^{-1}(nt_{1,2})^{1/3} \right] \to \mathbb{R}$ may be embedded in an ensemble

$$\mathsf{NrL}^{\downarrow;(y,t_2)}_{n;t_1}(1, \cdot) : [\![1, nt_{1,2} + 1]\!] \times \left(-\infty, 2^{-1}(nt_{1,2})^{1/3} \right] \to \mathbb{R}.$$

This new ensemble also enjoys the properties just described for its forward counterpart, uniformly in the concerned parameters.

Hypothesis bounds in most of our results have been stated explicitly up to the appearance of two positive constants c and C. The value is this pair is fixed by Proposition 3.2. Since bounding the constants would render hypotheses to be explicit, we mention that they are determined in [Ham17a, Appendix A.1] via Ledoux [Led07, (5.16)] and Aubrun's [Aub05, Proposition 1] bounds on the lower and upper tail of the maximum eigenvalue of a matrix in the Gaussian unitary ensemble. In applications, we will harmlessly suppose that $C \geq 1$ and $c \leq 1/2$.

3.3.2. *Basic properties of (c, C)-regular ensembles.* Recall from Subsection 1.2.1 that $c_1 = 2^{-5/2}c \wedge \frac{1}{8}$.

Proposition 3.3. *Suppose that $\mathcal{L} = \mathcal{L}_N$, mapping either $[\![1, N]\!] \times [-z_{\mathcal{L}}, \infty)$ or $[\![1, N]\!] \times (-\infty, z_{\mathcal{L}}]$, to $\mathbb{R}$, is a (c, C)-regular ensemble, where $N \in \mathbb{N}$ and $z_{\mathcal{L}} \geq 0$.*

(1) *(Uniform curve lower bound) Whenever $(t, r, y) \in \mathbb{R}$ satisfy $N \geq (c/3)^{-18} \vee 6^{36}$, $t \in \left[0, N^{1/18} \right]$, $r \in \left[2^{3/2}, 2N^{1/18} \right]$ and $|y| \leq 2^{-1}cN^{1/18}$,*

$$\mathbb{P}\left(\inf_{x \in [y-t, y+t]} \left(\mathcal{L}_N(1, x) + 2^{-1/2}x^2 \right) \leq -r \right) \leq \left(t \vee (3 - 2^{3/2})^{-1}5^{1/2} \right) \cdot 10C \exp \left\{ -c_1 r^{3/2} \right\}.$$

(2) *(No Big Max) For $|y| \leq 2^{-1}cN^{1/9}$, $r \in \left[0, 4^{-1}cN^{1/9} \right]$, $t \in \left[2^{7/2}, 2N^{1/3} \right]$ and $N \geq c^{-18}$,*

$$\mathbb{P}\left(\sup_{x \in [y-r, y+r]} \left(\mathcal{L}_N(1, x) + 2^{-1/2}x^2 \right) \geq t \right) \leq (r + 1) \cdot 6C \exp \left\{ -2^{-11/2}ct^{3/2} \right\}.$$

(3) *(Collapse near infinity) For $\eta \in (0, c]$, let $\ell = \ell_\eta : \mathbb{R} \to \mathbb{R}$ denote the even function which is affine on $[0, \infty)$ and has gradient $-5 \cdot 2^{-3/2}\eta N^{1/9}$ on this interval, and which satisfies $\ell(\eta N^{1/9}) = \left(-2^{-1/2} + 2^{-5/2} \right)\eta^2 N^{2/9}$. If $N \geq 2^{45/4}c^{-9}$, then*

$$\mathbb{P}\left(\mathcal{L}_N(1, z) > \ell(z) \text{ for some } z \in D \setminus \left[-\eta N^{1/9}, \eta N^{1/9} \right] \right)$$
$$\leq 6C \exp \left\{ -c\eta^3 2^{-15/4} N^{1/3} \right\}.$$

The set D is the spatial domain of $\mathcal{L}$, either $[-z_{\mathcal{L}}, \infty)$ or $(-\infty, z_{\mathcal{L}}]$.

These three assertions are proved in [Ham17a]. Respectively, they appear as, or are special cases of, the following results in that article: Proposition $A.2$, Proposition 2.28, and Proposition 2.30.

A few words about the assertions' meaning. The first is a bound on the lower tail of the minimum value of the lowest indexed curve on a compact interval. The result is a strengthening of the defining property Reg(2), which treats the one-point case. The second is a similar strengthening of the one-point upper tail Reg(3). In regard to the third, note that Reg(2) and Reg(3) do not assert that the lowest indexed curve hews to the parabola $-2^{-1/2}z^2$ globally, but only in an expanding region about the origin, of width $2cN^{1/9}$ centred at the origin, where N is the ensemble curve cardinality. Proposition 3.3(3) offers a substitute control on curves far from the origin, showing them to decay at a rapid but nonetheless linear rate in the region beyond scale $N^{1/9}$.

Two further basic properties of regular ensembles are needed. One concerns a *parabolic symmetry* for whose explanation a little notation is helpful. Write $Q : \mathbb{R} \to \mathbb{R}$ for the parabola $Q(u) = 2^{-1/2}u^2$, and let $l : \mathbb{R}^2 \to \mathbb{R}$ be given by $l(u, v) = -2^{-1/2}v^2 - 2^{1/2}v(u - v)$. Note that $u \to l(u, v)$ is the tangent line of the parabola $u \to -Q(u)$ at the point $\big(v, -Q(v)\big)$. Note also that, for any $u, v \in \mathbb{R}$,

$$Q(u) = -l(u, v) + Q(u - v). \tag{12}$$

For $z_{\mathcal{L}} \geq 0$, consider a (c, C)-regular ensemble $\mathcal{L}_N : [\![1, N]\!] \times [-z_{\mathcal{L}}, \infty) \to \mathbb{R}$. For any $y_N > -z_{\mathcal{L}}$, define $\mathcal{L}_{N,y_N}^{\mathrm{shift}} : [\![1, N]\!] \times [-z_{\mathcal{L}} - y_N, \infty) \to \mathbb{R}$ to be the shifted ensemble given by

$$\mathcal{L}_{N,y_N}^{\mathrm{shift}}(i, u) = \mathcal{L}_N(i, u + y_N) - l(u + y_N, y_N).$$

Lemma 3.4. *Let $c, C > 0$ and $N \in \mathbb{N}$. Suppose that $\mathcal{L}_N : [\![1, N]\!] \times [-z_{\mathcal{L}}, \infty) \to \mathbb{R}$ is a $\big(c, C\big)$-regular ensemble.*

(1) *Whenever $y_N \in \mathbb{R}$ satisfies $|y_N| \leq c/2 \cdot N^{1/9}$, the ensemble $\mathcal{L}_{N,y_N}^{\mathrm{shift}}$ is $\big(c/2, C\big)$-regular.*

(2) *For any $[a, b] \subseteq [-z_{\mathcal{L}}, \infty)$, and $y, z \in \mathbb{R}$, the conditional distribution of the marginal process $\mathcal{L}_N(1, \cdot) : [a, b] \to \mathbb{R}$ given that $\mathcal{L}_N(1, a) = y$ and $\mathcal{L}_N(1, b) = z$ stochastically dominates the Brownian bridge law $\mathcal{B}_{y,z}^{[a,b]}$.*

Proof (1). This is [Ham17a, Lemma 2.26].

Proof (2). By the Brownian Gibbs property of $\mathcal{L}_N$, the conditional distribution of the marginal process $\mathcal{L}_N(1, \cdot) : [a, b] \to \mathbb{R}$ given that $\mathcal{L}_N(1, a) = x$ and $\mathcal{L}_N(1, b) = y$ and further given the form $g : [a, b] \to \mathbb{R}$ of $\mathcal{L}_N(2, \cdot)$ on $[a, b]$ equals Brownian bridge B under $\mathcal{B}_{y,z}^{[a,b]}$ conditioned by $B(u) > g(u)$ for all $u \in [a, b]$. A stochastic monotonicity result [CH14, Lemma 2.6] implies that this conditional distribution stochastically dominates the law specified by $g \equiv -\infty$. The latter law is $\mathcal{B}_{y,z}^{[a,b]}$. $\square$

3.4. **Two-point estimate for the uppermost curve in a regular ensemble.** Here is the critical input for the proof of Theorem 1.1.

Proposition 3.5. *Suppose that $\mathcal{L} = \mathcal{L}_N$, mapping either $[\![1, N]\!] \times [-z_{\mathcal{L}}, \infty)$ or $[\![1, N]\!] \times (-\infty, z_{\mathcal{L}}]$, to $\mathbb{R}$, is a (c, C)-regular ensemble, where $N \in \mathbb{N}$ and $z_{\mathcal{L}} \geq 0$. For $x \geq -z_{\mathcal{L}} + 2$ and $t > 0$, define*

$$\mathsf{G}_t(x) = \bigcap_{x-2 \leq y \leq x+2} \left\{ -t \leq \mathcal{L}_N(1, y) + 2^{-1/2}y^2 \leq t \right\}.$$

If $|x| \leq 2^{-1}cN^{1/9}$, $\epsilon \in (0, 1]$, $K \geq 9$ and $N \geq 6^3 c^{-3}$, then

$$\mathbb{P}\left(\left| \mathcal{L}_N(1, x + \epsilon) + 2^{-1/2}(x + \epsilon)^2 - \mathcal{L}_N(1, x) + 2^{-1/2}x^2 \right| \geq K\epsilon^{1/2}, \, \mathsf{G}_{K/4}(x) \right)$$

is at most $2^{3/2}\pi^{-1/2}K^{-1}\exp\left\{-2^{-3}K^2\right\}$.

Theorem 1.1 (and Theorem 1.4), and Proposition 3.5 applied via Proposition 3.2, all give expression to the one-half power law that governs polymer weight: when the endpoints of polymers are varied by short horizontal displacements of order ϵ, the change in polymer weight has an order of $\epsilon^{1/2}$. Proposition 3.5 is notably flexible, in that the parameters for horizontal scale, ϵ, and scaled fluctuation, K, may be selected without imposing any dependence on the lower bound demanded on the ensemble curve cardinality N. This favourable feature comes at the price that the result gauges the small probability of high two-point difference only when we impose a global boundedness event $\mathsf{G}_{K/4}(x)$ on the ensemble $\mathcal{L}_N$. We will have more to say about the role of Proposition 3.5 and the implications of its strengths and its drawback early in Section 5, when Theorem 1.1 is proved.

Proof of Proposition 3.5. We first argue that an application of Lemma 3.4(1) reduces to the case that $x = 0$. To see this, note that (12) implies that $\mathcal{L}^{\text{shift}}_{N,y_N}(i,u)+Q(u)$ equals $\mathcal{L}_N(i,u+y_N)+Q(u+y_N)$, whenever $y_N > -z_{\mathcal{L}}$. Selecting y_N in Lemma 3.4(1) to be x in Proposition 3.5, the proposition's conclusion is seen to be unchanged when the adjustments $x \to 0$ and $\mathcal{L}_N \to \mathcal{L}^{\text{shift}}_{N,x}$ are made. For this reason, we may, and will, consider only $x = 0$. We must also work now under the assumption that $\mathcal{L}_N$ is $(c/2, C)$-regular. This information is used alongside the hypothesis $N \geq 6^3 c^{-3}$ to ensure that $c/2 \cdot N^{1/3} \geq 3$, so that the interval $[-3, 3]$ lies in the spatial domain $[-z_{\mathcal{L}}, \infty)$ or $(-\infty, z_{\mathcal{L}}]$ of $\mathcal{L}_N$.

For a stochastic process X whose domain of definition includes $[0, \epsilon]$, define the events

$$\mathsf{Fall}_{\epsilon,K}[X] = \left\{ X(\epsilon) + 2^{-1/2}\epsilon^2 \leq X(0) - K\epsilon^{1/2} \right\}$$

and

$$\mathsf{Rise}_{\epsilon,K}[X] = \left\{ X(\epsilon) + 2^{-1/2}\epsilon^2 \geq X(0) + K\epsilon^{1/2} \right\}.$$

For brevity, we write $W : [-z_{\mathcal{L}}, \infty) \to \mathbb{R}$, $W(u) = \mathcal{L}_N(1, u)$, so that W maps $[-z_{\mathcal{L}}, \infty)$ or $(-\infty, z_{\mathcal{L}}]$ to $\mathbb{R}$; and for the same reason, we also denote $t = K/4$. To obtain Proposition 3.5 for $x = 0$, it is enough to verify two bounds:

$$\mathbb{P}\left(\mathsf{Fall}_{\epsilon,K}[W], \, \mathsf{G}_t(0) \right) \leq 2^{1/2}\pi^{-1/2}K^{-1}\exp\left\{-8^{-1}K^2\right\} \tag{13}$$

and

$$\mathbb{P}\left(\mathsf{Rise}_{\epsilon,K}[W], \, \mathsf{G}_t(0) \right) \leq 2^{1/2}\pi^{-1/2}K^{-1}\exp\left\{-8^{-1}K^2\right\}. \tag{14}$$

Deriving (13). Note that

$$\mathbb{P}\left(\mathsf{Fall}_{\epsilon,K}[W], \, \mathsf{G}_t(0) \right) \leq \mathbb{P}\left(\mathsf{Fall}_{\epsilon,K}[W], \, W(0) \leq t, \, W(2) + 2^{3/2} \geq -t \right)$$

$$\leq \sup \left\{ \mathbb{P}\left(\mathsf{Fall}_{\epsilon,K}[W] \,\big|\, W(0) = y, \, W(2) = z \right) : y \leq t, \, z \geq -t - 2^{3/2} \right\}$$

$$\leq \sup \left\{ \mathcal{B}^{[0,2]}_{y,z}\left(\mathsf{Fall}_{\epsilon,K}[B] \right) : y \leq t, \, z \geq -t - 2^{3/2} \right\} = \mathcal{B}^{[0,2]}_{t,-t-2^{3/2}}\left(\mathsf{Fall}_{\epsilon,K}[B] \right).$$

The third bound (which is highlighted $\leq$) follows from Lemma 3.4(2), while the final equality is a consequence of the coupling of Brownian bridge laws via affine shift. Brownian bridge $B : [0, 2] \to \mathbb{R}$ subject to $B(0) = t$ and $B(2) = -t - 2^{3/2}$ may be mapped via a further affine shift to Brownian

bridge $\mathcal{B}_{0,0}^{0,2}$ with vanishing endpoint values. We thus see that

$$\mathcal{B}_{t,-t-2^{3/2}}^{[0,2]}\Big(\mathsf{Fall}_{\epsilon,K}[B]\Big) \leq \mathcal{B}_{t,-t-2^{3/2}}^{[0,2]}\Big(B(\epsilon) \leq t - K\epsilon^{1/2}\Big)$$

$$= \mathcal{B}_{0,0}^{[0,2]}\Big(B(\epsilon) \leq -K\epsilon^{1/2} + \epsilon/2 \cdot \big(2t + 2^{3/2}\big)\Big) \leq \mathcal{B}_{0,0}^{[0,2]}\Big(B(\epsilon) \leq -2^{-1}K\epsilon^{1/2}\Big), \qquad (15)$$

where the bound $\epsilon \leq (2t + 2^{3/2})^{-2}K^2$ that permits the final inequality is due to $K = 4t$, $t \geq 2^{1/2}$ (in the guise that $2t + 2^{3/2} \leq 4t$) and $\epsilon \leq 1$. Since $B(\epsilon)$ under $\mathcal{B}_{0,0}^{[0,2]}$ is normally distributed with mean zero and variance $2^{-1}\epsilon(2-\epsilon) \leq \epsilon$, the right-hand side of (15) is at most $\nu(x,\infty) \leq (2\pi)^{-1/2}x^{-1}e^{-x^2/2}$ with $x = K/2$, where ν is the standard normal distribution, and the inequality [Wil91, Section 14.8] is classical. We have obtained (13).

Deriving (14). In summary of the last argument, W may fall no more suddenly between zero and ϵ than does a suitable Brownian bridge that begins at time zero. But neither may it rise suddenly between this same pair of times, because such a rise is a fall when viewed from right to left, and this fall entails a fall that is at least as great on the part of a suitable Brownian bridge whose rightmost time is ϵ. That is, the argument for (14) is almost symmetrical to that for (13). It takes the form

$$\mathbb{P}\Big(\mathsf{Rise}_{\epsilon,K}[W], \, \mathsf{G}_t(0)\Big) \leq \mathbb{P}\Big(\mathsf{Rise}_{\epsilon,K}[W], \, W(-2) + 2^{3/2} \geq -t, \, W(\epsilon) + 2^{-1/2}\epsilon^2 \leq t\Big)$$

$$\leq \sup\Big\{\mathbb{P}\Big(\mathsf{Rise}_{\epsilon,K}[W]\,\Big|\,W(-2) = y, \, W(\epsilon) = z\Big) : y \geq -t - 2^{3/2}, \, z \leq t - 2^{-1/2}\epsilon^2\Big\}$$

$$\leq \sup\Big\{\mathcal{B}_{y,z}^{[-2,\epsilon]}\Big(\mathsf{Rise}_{\epsilon,K}[B]\Big) : y \geq -t - 2^{3/2}, \, z \leq t - 2^{-1/2}\epsilon^2\Big\} = \mathcal{B}_{-t-2^{3/2},t-2^{-1/2}\epsilon^2}^{[-2,\epsilon]}\Big(\mathsf{Rise}_{\epsilon,K}[B]\Big)$$

$$= \mathcal{B}_{0,0}^{[0,2+\epsilon]}\Big(B(\epsilon) \leq -K\epsilon^{1/2} + 2^{-1/2}\epsilon^2 + \tfrac{\epsilon}{2+\epsilon}(2t + 2^{3/2} - 2^{-1/2}\epsilon^2)\Big) \leq \mathcal{B}_{0,0}^{[0,2]}\Big(B(\epsilon) \leq -2^{-1}K\epsilon^{1/2}\Big),$$

where Lemma 3.4(2) again implies the third bound, which is again highlighted $\leq$. To obtain the latter equality (which is highlighted $=$), consider the map that sends $B : [-2,\epsilon] \to \mathbb{R}$ to $W : [0, 2 + \epsilon] \to \mathbb{R}$, where W is the affine shift of $[0, 2 + \epsilon] \to \mathbb{R} : y \to B(\epsilon - y)$ for which $W(0) = W(2 + \epsilon) = 0$, and note that this map sends the law $\mathcal{B}_{-t-2^{3/2},t-2^{-1/2}\epsilon^2}^{[-2,\epsilon]}$ to $\mathcal{B}_{0,0}^{[0,2+\epsilon]}$ and the event $\mathsf{Rise}_{\epsilon,K}[B]$ to the event that $W(\epsilon) \leq -K\epsilon^{1/2} + 2^{-1/2}\epsilon^2 + \tfrac{\epsilon}{2+\epsilon}(2t + 2^{3/2} - 2^{-1/2}\epsilon^2)$. It is now $\epsilon \leq 1 \wedge (t + 2^{1/2} + 2^{-1/2})^{-2}2^{-2}K^2$ that permits the final displayed inequality, with the assumptions that $t \geq 2^{1/2} + 2^{-1/2}$ (in the guise that $t + 2^{1/2} + 2^{-1/2}$ is at most $2t$), $K = 4t$ and $\epsilon \leq 1$ implying this condition. Since the last displayed quantity is the right-hand side of (15), the rest of the derivation of (13) applies, and (14) is obtained.

In summary, (13) and (14) imply Proposition 3.5 with $x = 0$, to whose derivation the proof of this proposition has been reduced. $\qquad\square$

4. Collective control on polymer weights: the proof of Proposition 1.5

Proposition 1.5 will be proved first in the case that the parameter $n \in \mathbb{N}$ is even; the case of odd n will then be obtained by reducing to the preceding one. In this section, we prove the proposition for even n, and then explain in outline the argument for n odd. The detailed argument for odd n appears in Appendix A.

In order to permit the reduction argument, we will in fact prove the result when $n \in 2\mathbb{N}$ under slightly weaker hypotheses and with a slightly stronger conclusion.

That is, we will now prove the proposition in the case that $n \in \mathbb{N}$ is even, when n is supposed to satisfy $n \geq 10^{29} \vee 2(c/3)^{-18}$; with $x, y \in \mathbb{R}$ satisfying $|x - y| \leq 3^{-1}2^{-2/3}cn^{1/18}$; and with $t \in [33, 4n^{1/18}]$. We will derive the conclusions (5) and (6) where these bounds are strengthened by the respective replacements $2^{-10} \to 2^{-9}$ and $2^{-3} \to 2^{-5/2}$ on their right-hand sides.

The planar line segment with endpoints $(x, 0)$ and $(y, 1)$ crosses the horizontal line whose height is one-half at the location $(x+y)/2$. We record this location in the form $z = (x+y)/2$. We will prove the lower-tail bound (6) by bounding the polymer weight $\mathsf{Wgt}_{n;(x+u,0)}^{(y+v,1)}$ for any given $(u, v) \in [0, 1]^2$ below by considering routes from $(x + u, 0)$ to $(y + v, 1)$ that pass via $(z, 1/2)$. The two polymer weights in the lower bound concern journeys from $(z, 1/2)$ to $(x+u, 0)$ (after time reversal) and from $(z, 1/2)$ to $(y + v, 1)$. Rooting in this way at $(z, 1/2)$, we may gauge the probability of low weight values for polymers emanating from $(z, 1/2)$ and ending in a compact interval at time zero or one by applying Proposition 3.3(1) to the duration one-half normalized ensembles rooted at $(z, 1/2)$, of forward or backward type according to whether a time one or time zero endpoint is being considered.

Thus, we let $u, v \in [0, 1]$. Note that

$$2^{-1/2}\big(2^{2/3}(z - x - u)\big)^2 + 2^{-1/2}\big(2^{2/3}(y + v - z)\big)^2$$
$$= 2^{-1/6}(y + v - x - u)^2 + 2^{-1/6}(u + v)^2 \,.$$

Note further that

$$\mathsf{Wgt}_{n;(x+u,0)}^{(y+v,1)} \geq \mathsf{Wgt}_{n;(x+u,0)}^{(z,1/2)} + \mathsf{Wgt}_{n;(z,1/2)}^{(y+v,1)} = \mathcal{L}_{n;0}^{\downarrow;(z,1/2)}(1, x + u) + \mathcal{L}_{n;(z,1/2)}^{\uparrow;1}(1, y + v)$$
$$= 2^{-1/3}\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(z,1/2)}\big(1, 2^{2/3}(x + u - z)\big) + 2^{-1/3}\mathsf{Nr}\mathcal{L}_{n;(z,1/2)}^{\uparrow;1}\big(1, 2^{2/3}(y + v - z)\big) \,.$$

The inequality here invokes Lemma 2.2(2) with $\mathbf{t_1} = 0$, $\mathbf{t_2} = 1$, $\mathbf{t} = 2^{-1}$, $\mathbf{x} = x + u$, $\mathbf{y} = y + v$ and $\mathbf{z} = z$. The two equalities invoke the definitions of the four ensembles whose top curves are being evaluated. Regarding the inequality, we may note that the use of Lemma 2.2(2) entails that certain bounds on $\mathbf{z}$ be satisfied. We omit reference to these bounds now because they are anyway implicated later in the argument, when we come to analyse the above two right-hand terms. Finally, note that, in order to enable our use of Lemma 2.2(2) with the choice $\mathbf{t} = 2^{-1}$, we have imposed in the present case of the proof of Proposition 1.5 that $n \in \mathbb{N}$ be even, in order that the triples $(n, 0, 2^{-1})$ and $(n, 2^{-1}, 1)$ be compatible.

Adding to the above inequality the $2^{-1/3}$-rd multiple of the display that preceded it, we find that

$$\mathsf{Wgt}_{n;(x+u,0)}^{(y+v,1)} + 2^{-1/2}(y + v - x - u)^2 + 2^{-1/2}(u + v)^2$$
$$\geq 2^{-1/3}\Big(\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(z,1/2)}\big(1, 2^{2/3}(x + u - z)\big) + 2^{-1/2}\big(2^{2/3}(z - x - u)\big)^2\Big) \tag{16}$$
$$+ 2^{-1/3}\Big(\mathsf{Nr}\mathcal{L}_{n;(z,1/2)}^{\uparrow;1}\big(1, 2^{2/3}(y + v - z)\big) + 2^{-1/2}\big(2^{2/3}(y + v - z)\big)^2\Big) \,.$$

Note that

$$\mathbb{P}\bigg(\inf_{u \in [0,1]} \Big(\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(z,1/2)}\big(1, 2^{2/3}(x + u - z)\big) + 2^{-1/2}\big(2^{2/3}(z - x - u)\big)^2\Big) \leq -s\bigg) \tag{17}$$
$$= \mathbb{P}\bigg(\inf_{u \in [0,2^{2/3}]} \Big(\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(z,1/2)}\big(1, 2^{2/3}(x - z) + u\big) + 2^{-1/2}\big(2^{2/3}(x - z) + u\big)^2\Big) \leq -s\bigg) \,.$$

The latter term equals

$$\mathbb{P}\left(\inf_{x' \in [y-t',y+t']} \left(\mathcal{L}_N(1,x') + 2^{-1/2}(x')^2 \right) \leq -r \right)$$

when $t' = 2^{-1/3}$; when $t' \geq 2^{-1/3}$, the new expression is an upper bound. Here, we take $\mathcal{L}_N = \mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(z,1/2)}$, $y = 2^{2/3}(x-z) + 2^{-1/3}$ and $r = s$.

We seek then to apply Proposition 3.3(1) to the (c,C)-regular ensemble $\mathcal{L}_N = \mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(z,1/2)}$, doing so with the choice of $(\mathbf{r},\mathbf{y},\mathbf{t}) = (r,y,t')$. It is Proposition 3.2 that permits this choice of ensemble. If we set $t_1 = 0$ and $t_2 = 1/2$, so that $t_{1,2} = 1/2$, the number of curves in the ensemble $\mathcal{L}_N = \mathsf{Nr}\mathcal{L}_{n;t_1}^{\downarrow;(z,t_2)}$ equals $nt_{1,2} + 1$ and thus is at least $nt_{1,2}$. For the application to be valid, our parameters must thus satisfy

$$t_{1,2}n \geq 1 \vee (c/3)^{-18} \vee 6^{36}, \ 2^{-1/3} \leq (nt_{1,2})^{1/18}, \ s \in \left[2^{3/2}, 2(nt_{1,2})^{1/18} \right], \tag{18}$$

and $\left| 2^{2/3}(x-z) + 2^{-1/3} \right| = \left| 2^{2/3}(x-y)/2 + 2^{-1/3} \right| \leq c/2 \cdot (nt_{1,2})^{1/18}$. From this application of Proposition 3.3(1), we find that the probability in (17) is at most

$$\left(2^{-1/3} \vee (3 - 2^{3/2})^{-1} 5^{1/2} \right) \cdot 10C \exp\left\{ -c_1 s^{3/2} \right\}. \tag{19}$$

By applying Proposition 3.3(1) to the ensemble $\mathcal{L}_N = \mathsf{Nr}\mathcal{L}_{n;(z,1/2)}^{\uparrow;1}$, with $\mathbf{y}$ now chosen equal to $2^{2/3}(y-z) + 2^{-1/3}$, and with $(\mathbf{r},\mathbf{t})$ again set to be $\left(s, 2^{-1/3} \right)$, we find that the quantity

$$\mathbb{P}\left(\inf_{v \in [0,1]} \left(\mathsf{Nr}\mathcal{L}_{n;(z,1/2)}^{\uparrow;1}\left(1, 2^{2/3}(y+v-z)\right) + 2^{-1/2}\left(2^{2/3}(y+v-z)\right)^2 \right) \leq -s \right)$$

is also bounded above by (19). This application of the proposition requires in addition to the bounds in (18) that $\left| 2^{2/3}(y-z) + 2^{-1/3} \right| = \left| 2^{2/3}(y-x)/2 + 2^{-1/3} \right| \leq c/2 \cdot (nt_{1,2})^{1/18}$.

Since $2^{-1/2}(u+v)^2 \leq 2^{3/2}$ whenever $u,v \in [0,1]$, we find from the inequality (16) and the upper bound by (19) on the two probabilities that

$$\mathbb{P}\left(\inf_{u,v \in [0,1]} \left(\mathsf{Wgt}_{n;(x+u,0)}^{(y+v,1)} + 2^{-1/2}(y+v-x-u)^2 \right) \leq -2 \cdot 2^{-1/3} s - 2^{3/2} \right)$$

$$\leq \ 2\left(2^{-1/3} \vee (3 - 2^{3/2})^{-1} 5^{1/2} \right) \cdot 10C \exp\left\{ -c_1 s^{3/2} \right\}.$$

Setting $t = 2 \cdot 2^{2/3} s$ and using $s \geq 2^{5/6}$ (so that $t \geq 2^{2/3}s + 2^{3/2}$), and noting $20\left(2^{-1/3} \vee 5^{1/2}(3 - 2^{3/2})^{-1} \right) \leq 261$, we obtain (6).

Let $\mathsf{High}_{n;([x,x+1],0)}^{([y,y+1],1)}(t)$ denote the event that $\sup_{u,v \in [0,1]} \left(\mathsf{Wgt}_{n;(x+u,0)}^{(y+v,1)} + 2^{-1/2}(y+v-x-u)^2 \right) \geq t$. Clearly it is this event whose probability we must bound above as we turn to derive (5). The event entails the presence of a high weight polymer that crosses a square, but both of its endpoints may have exceptional locations. The derivation of (5) will proceed by noting that, typically, one of the endpoints can be made typical. Indeed, when the High event occurs, so that a high weight polymer runs between say $(x+U,0)$ and $(y+V,1)$, where the pair $(U,V) \in [0,1]^2$ is random, a fairly high weight polymer will also typically exist between the deterministic location $(2x-y,-1)$ and $(y+V,1)$. This is because a rather high lower bound on the weight of such a polymer is obtained by considering the pair of polymers, from $(2x-y,-1)$ to $(x+U,0)$, and from $(x+U,0)$ to $(y+V,1)$, whose weights are typically not too low, and high. The probability of the presence of this fairly high weight polymer may then be gauged by the No Big Max Proposition 3.3(2), because this event

entails that the duration-two normalized forward ensemble rooted at $(2x - y, -1)$ assumes a high value within a compact interval.

To begin implementing this approach, we consider the event $\mathrm{High}_{n;([x,x+1],0)}^{([y,y+1],1)}(t)$, and let $(U, V) \in [0, 1]^2$ be the lexicographically minimal pair of $(u, v) \in [0, 1]^2$ that realize this event (a definition which makes sense because, by Lemma 2.2(1), the set of such pairs is closed).

Now *reset* the value of z to be $2x - y$, so that the planar line segment interpolating $(z, -1)$ and $(y, 1)$ passes through $(x, 0)$. Note that

$$2^{-1/2}(z - x - U)^2 + 2^{-1/2}(x + U - y - V)^2 \tag{20}$$
$$= 2^{-3/2}(z - y - V)^2 + 2^{-1/2}(2U^2 + V^2/2 - 2UV).$$

Let $\mathrm{NotLow}_{n;(z,-1)}^{([x,x+1],0)}(t/2)$ denote the event that $\inf_{u \in [0,1]} \left(\mathsf{Wgt}_{n;(z,-1)}^{(x+u,0)} + 2^{-1/2}(x + u - z)^2 \right) \geq -t/2$. Note that

$$\neg \, \mathrm{NotLow}_{n;(z,-1)}^{([x,x+1],0)}(t/2) \; = \; \left\{ \inf_{u \in [0,1]} \left(\mathsf{Nr}\mathcal{L}_{n;(z,-1)}^{\uparrow;0}(1, x + u - z) + 2^{-1/2}(x + u - z)^2 \right) < -t/2 \right\}.$$

We apply Proposition 3.3(1) to the ensemble $\mathsf{Nr}\mathcal{L}_{n;(z,-1)}^{\uparrow;0}$, the choice admissible by Proposition 3.2, in order to find an upper bound on the probability of the displayed event. The application is made with $\mathbf{y} = x - z + 1/2$, $\mathbf{t} = 1/2$ and $\mathbf{r} = t/2$. Since the ensemble $\mathsf{Nr}\mathcal{L}_{n;(z,-1)}^{\uparrow;0}$ has $n + 1$, and therefore at least n, curves, we see that the next bounds suffice for the application to be made: $n \geq (c/3)^{-18} \vee 6^{36}$, $1/2 \leq n^{1/18}$, $t/2 \in \left[2^{3/2}, 2n^{1/18} \right]$ and $|x - z + 1/2| = |y - x + 1/2| \leq 2^{-1}cn^{1/18}$. (In regard to the latter, recall that z is now $2x - y$.) We learn from the application that

$$\mathbb{P}\left(\neg \, \mathrm{NotLow}_{n;(z,-1)}^{([x,x+1],0)}(t/2) \right) \; \leq \; (3 - 2^{3/2})^{-1} 5^{1/2} \cdot 10C \exp \left\{ -c_1 2^{-3/2} t^{3/2} \right\}.$$

When $\mathrm{NotLow}_{n;(z,-1)}^{([x,x+1],0)}(t/2) \cap \mathrm{High}_{n;([x,x+1],0)}^{([y,y+1],1)}(t)$ occurs, consider the concatenation $\rho_{n;(z,-1)}^{(x+U,0)} \circ \rho_{n;(x+U,0)}^{(y+V,1)}$ of any pair of polymers with the endpoints implied by our convention governing this notation. The concatenation has weight at least

$$\left(-t/2 - 2^{-1/2}(x + U - z)^2 \right) + \left(t - 2^{-1/2}(y + V - x - U)^2 \right)$$
$$\geq \; t/2 - 2^{-3/2}(z - y - V)^2 - 2^{-1/2}(2U^2 + V^2/2 - 2UV)$$
$$\geq \; t/2 - 2^{-3/2}(z - y - V)^2 - 5 \cdot 2^{-3/2},$$

the displayed inequalities due to (20) and $U, V \in [0, 1]$. Thus,

$$\mathrm{High}_{n;([x,x+1],0)}^{([y,y+1],1)}(t) \cap \mathrm{NotLow}_{n;(z,-1)}^{([x,x+1],0)}(t/2) \subseteq \left\{ \sup_{v \in [0,1]} \left(\mathsf{Wgt}_{n;(z,-1)}^{(y+v,1)} + 2^{-3/2}(z - y - v)^2 \right) \geq t/2 - 5 \cdot 2^{-3/2} \right\}.$$

The right-hand event equals

$$\left\{ \sup_{v \in [0,1]} \left(\mathcal{L}_{n;(z,-1)}^{\uparrow;1}(1, y + v) + 2^{-3/2}(z - y - v)^2 \right) \geq t/2 - 5 \cdot 2^{-3/2} \right\} \tag{21}$$
$$= \; \left\{ \sup_{v \in [0,1]} \left(\mathsf{Nr}\mathcal{L}_{n;(z,-1)}^{\uparrow;1}(1, 2^{-2/3}(y + v - z)) + 2^{-1/2}(2^{-2/3}(z - y - v))^2 \right) \geq 2^{-4/3}t - 5 \cdot 2^{-11/6} \right\}$$
$$= \; \left\{ \sup_{v \in [0,2^{-2/3}]} \left(\mathsf{Nr}\mathcal{L}_{n;(z,-1)}^{\uparrow;1}(1, 2^{-2/3}(y - z) + v) + 2^{-1/2}(2^{-2/3}(z - y) - v)^2 \right) \geq 2^{-4/3}t - 5 \cdot 2^{-11/6} \right\}.$$

We now apply the No Big Max Proposition 3.3(2) to the ensemble $\mathcal{L}_N = \mathsf{Nr}\mathcal{L}_{n;(z,-1)}^{\uparrow;1}$, the application permitted by Proposition 3.2. In the application, we set $\mathbf{y} = 2^{-2/3}(y-z) + 2^{-5/3}$, $\mathbf{r} = 2^{-5/3}$ and $\mathbf{t} = 2^{-4/3}t - 5 \cdot 2^{-11/6}$. The curve cardinality of the ensemble in queston is $2n+1 \geq 2n$. As such, it is sufficient for the application to be valid that

$$\left|2^{-2/3}(y-z) + 2^{-5/3}\right| = \left|2^{1/3}(y-x) + 2^{-5/3}\right| \leq c/2 \cdot (nt_{1,2})^{1/9}, \ 2^{-5/3} \leq c/4 \cdot (nt_{1,2})^{1/9}$$

and $2^{-4/3}t - 5 \cdot 2^{-11/6} \in \left[2^{7/2}, 2(nt_{1,2})^{1/3}\right]$ as well as $t_{1,2}n \geq c^{-18}$ where here $t_{1,2}$ equals 2 (in accordance with the time-pair $t_1 = -1$ and $t_2 = 1$ being considered). This application tells us that the $\mathbb{P}$-probability of the event (21) is at most

$$\left(2^{-5/3} + 1\right) \cdot 6C \exp\left\{-2^{-11/2}c\left(2^{-4/3}t - 5 \cdot 2^{-11/6}\right)^{3/2}\right\} \leq 8C \exp\left\{-2^{-9}ct^{3/2}\right\}, \tag{22}$$

where we used $2^{-4/3}t - 5 \cdot 2^{-11/6} \geq 2^{-7/3}t$ when $t \geq 5 \cdot 2^{1/2}$.

We find then that

$$\mathbb{P}\left(\mathsf{High}_{n;([x,x+1],0)}^{([y,y+1],1)}(t)\right) \leq \mathbb{P}\left(\mathsf{High}_{n;([x,x+1],0)}^{([y,y+1],1)}(t) \cap \mathsf{NotLow}_{n;(z,-1)}^{([x,x+1],0)}(t/2)\right) + \mathbb{P}\left(\neg \ \mathsf{NotLow}_{n;(z,-1)}^{([x,x+1],0)}(t/2)\right)$$

is bounded above by

$$8C \exp\left\{-2^{-9}ct^{3/2}\right\} + 10(3 - 2^{3/2})^{-1}5^{1/2}C \exp\left\{-c_1 2^{-3/2}t^{3/2}\right\}.$$

We now use $c_1 \leq c$ and $8 + 10(3 - 2^{3/2})^{-1}5^{1/2} \leq 139$ to obtain (5). This completes the proof of the slightly strengthened form of Proposition 1.5 specified at the start of this section in the case that $n \in \mathbb{N}$ is even.

It remains to derive the proposition in the case that $n \in \mathbb{N}$ is odd. The argument is simple enough and here we outline it; but its details occupy a few pages and are deferred to Appendix A.

The new case is handled by reducing to that where n is even; this involves retreating to unscaled coordinates. The argument harnesses a trivial bound on unscaled energy, namely that $M_{(z_1,0)\to(z_2,m)}^1 \leq M_{(z_1,0)\to(z_2,m+1)}^1$ whenever $m \in \mathbb{N}$ and $z_1, z_2 \in \mathbb{R}$ satisfy $z_1 \leq z_2$. Presently given are $n \in \mathbb{N}$ odd; $x, y \in \mathbb{R}$; and $u, v \in [0, 1]$. First we seek to obtain (5). Applying the unscaled inequality with $m = n$, $z_1 = 2n^{2/3}(x+u)$ and $z_2 = n + 2n^{2/3}(y+v)$, and writing the resulting bound in scaled coordinates via (1), we find that

$$\mathsf{Wgt}_{n;(x+u,0)}^{(y+v,1)} \leq \left(1 + n^{-1}\right)^{1/3} \mathsf{Wgt}_{n+1;(x'+u',0)}^{(y'+v',1)} + 2^{-1/2}n^{-1/3}, \tag{23}$$

where the new variables $x', y', u', v' \in \mathbb{R}$ satisfy $x' + u' = \left(1 + n^{-1}\right)^{-2/3}(x+u)$ and $y' + v' = \left(1+n^{-1}\right)^{-2/3}(y+v) - 2^{-1}(n+1)^{-2/3}$. Thus the weight term in (5) may be replaced by $\mathsf{Wgt}_{n+1;(x'+u',0)}^{(y'+v',1)}$ at the expense of adjustments that are negligible when n is high, with the new term exceeding the old. Similarly may the parabolic term $Q(y + v - x - u)$ be replaced by $Q(y' + v' - x' - u')$. Thus we are able to derive (5) in the present case of odd n, by using (23) and the already derived (5) with $\mathbf{n} = n + 1$, $\mathbf{x} = \left(1 + n^{-1}\right)^{-2/3}x$ and $\mathbf{y} = \left(1 + n^{-1}\right)^{-2/3}y - 2^{-1}(n+1)^{-2/3}$; and with $\mathbf{t}$ set to be t multiplied by a factor that is slightly less than one. Proposition 1.5 has slightly stronger hypotheses on parameters, and slightly weaker conclusions, in its general case $n \in \mathbb{N}$ than in the form obtained when n is even. It is these changes that permit us to apply the even version of (5) in order to obtain (5) when n is odd. The deriviation of (6) with n odd similarly proceeds from the unscaled inequality noted above and reduces to (6) with $\mathbf{n} = n - 1$. The details are similar enough that we defer further discussion to Appendix A. $\qquad\square$

The reader may have noticed that every application of a result concerning a (c, C)-regular ensemble in the preceding proof invoked Proposition 3.2 in order to justify that the ensemble in question indeed enjoys this property. Every subsequent such application is no different, and henceforth we omit mention of Proposition 3.2's role.

5. POLYMER WEIGHT REGULARITY: PROVING THEOREM 1.1

We begin this section by introducing a notation $\Delta^\cup \mathsf{Wgt}$ to denote the difference in *parabolically adjusted* weight of two polymers crossing between the opposite endpoints of two intervals. (The superscript $\cup$ is intended to suggest a parabola.) When (x_1, x_2) and (y_1, y_2) belong to $\mathbb{R}^2_{\leq}$, we set

$$\Delta^\cup \mathsf{Wgt}^{([y_1,y_2],1)}_{n;([x_1,x_2],0)} := \left(\mathsf{Wgt}^{(y_2,1)}_{n;(x_2,0)} + 2^{-1/2}(y_2 - x_2)^2 \right) - \left(\mathsf{Wgt}^{(y_1,1)}_{n;(x_1,0)} + 2^{-1/2}(y_1 - x_1)^2 \right). \quad (24)$$

We will abuse this notation when one of the concerned intervals collapses a point, writing for example
$\Delta^\cup \mathsf{Wgt}^{([y_1,y_2],1)}_{n;(x_1,0)} = \mathsf{Wgt}^{(y_2,1)}_{n;(x_1,0)} - \mathsf{Wgt}^{(y_1,1)}_{n;(x_1,0)} + 2^{-1/2}(y_2 - x_1)^2 - 2^{-1/2}(y_1 - x_1)^2.$

We also write $y + U = \{y + u : u \in U\}$ when $y \in \mathbb{R}$ and $U \subset \mathbb{R}$.

Any integer is called a dyadic rational of scale zero. A dyadic rational of scale $i \in \mathbb{N}$, $i \geq 1$, has for the form $p2^{-i}$ where $p \in \mathbb{Z}$ is odd. A dyadic interval of scale $i \in \mathbb{N}$ is a closed interval of length 2^{-i} that has an endpoint which is a dyadic rational of scale i.

Recall from Subsection 2.3.1 the polymer weight regularity events $\mathsf{PolyWgtReg}$.

Proposition 5.1. *Let $n \in \mathbb{N}$, $n \geq (4/c)^9$, let $x, y \in \mathbb{R}$ satisfy $|x - y| \leq c/4 \cdot n^{1/9}$, and let $K_0 \geq 9$. Let $i, k_0 \in \mathbb{N}$ with $i \geq k_0$ (which is at least one). Consider the quantities*

$$\mathbb{P}\left(\sup \left| \Delta^\cup \mathsf{Wgt}^{(y+U,1)}_{n;(x+z,0)} \right| \geq K_0 2^{-i/2} , \ \mathsf{PolyWgtReg}^{([y-2,y+3],1)}_{n;([x,x+1],0)}(K_0/4) \right) \quad (25)$$

and

$$\mathbb{P}\left(\sup \left| \Delta^\cup \mathsf{Wgt}^{(y+z,1)}_{n;(x+U,0)} \right| \geq K_0 2^{-i/2} , \ \mathsf{PolyWgtReg}^{([y,y+1],1)}_{n;([x-2,x+3],0)}(K_0/4) \right), \quad (26)$$

where in each case the supremum is taken over all dyadic rationals $z \in [0, 2^{-k_0}]$ of scale i and dyadic intervals $U \subseteq [0, 2^{-k_0}]$ of scale at least i. Each of these quantities is at most $2^{2(i-k_0)} \exp\left\{ -2^{-3} K_0^2 \right\}$.

This result is naturally a key technical ingredient in the proof of Theorem 1.1. The theorem concerns polymer weight differences when small changes are made in endpoint locations. The proposition is similar, but restricts to endpoint locations that are dyadic rationals. The theorem will follow from the proposition by noting that one may skip between the two nearby endpoint locations x and $x + \epsilon$ (and similarly for y and $y + \epsilon$) by jumping through a possibly infinite sequence of intermediate dyadic rational locations where no dyadic scale need be visited more than twice and the minimal dyadic scale is of the order of the difference ϵ. This property of this 'stepping stone' sequence makes the union bounds over the estimate in Proposition 5.1 manageable. This inference of the theorem from the proposition is similar to the derivation of the Kolmogorov continuity criterion, in which moment bounds on the difference of a stochastic process between a generic pair of times imply Hölder continuity of the process: see [Dur10, Thoerem 8.13].

One further aspect of the plan for proving Theorem 1.1 deserves mention before we proceed. Note that in the theorem the parameter R, which measures the degree of polymer weight fluctuation, must verify an n-dependent upper bound. Although this bound in a sense is insignificant for the purpose of analysing high n behaviour, Proposition 5.1 has been stated so that there is no comparable hypothesis: the quantities K_0 and 2^{-i} are counterparts to R and ϵ, and the proposition holds for

all high n, where the lower bound on n deteriorates neither as the dyadic scale 2^{-i} decreases, nor as the parameter K_0 increases. Now this is a valuable property, but it comes at a certain price, about which more in a moment. The reason that the property is valuable is that, for a given high value of n, Proposition 5.1 may be applied as i increases to infinity, while in the meantime, K_0 also increases; indeed, this is how we will derive Theorem 1.1, with a union bound over the infinite number of applications of the proposition indexed by i being controllable due to the ongoing increase in K_0. As for the price to be paid, we mention that, in order that the property obtains, it has been necessary in the events whose probabilities are gauged in (25) and (26) to include the global polymer weight regularity events $\mathsf{PolyWgtReg}$; this in turn is because the proposition's proof will invoke the two-point Proposition 3.5, in which the spatial scale $\epsilon > 0$ is permitted to be arbitrarily small without forcing the ensemble curve cardinality N to be higher in an ϵ-dependent way, a favourable circumstance which is only possible at the expense of introducing the global regularity event $\mathsf{G}_{K/4}$, counterpart to the above $\mathsf{PolyWgtReg}$ events, into the probability upper bound in that result. In any case, it would seem that we can derive Theorem 1.1 from Proposition 5.1 only by invoking the $\mathsf{PolyWgtReg}$ event. In fact, we do impose this event, but, at the very end of the derivation, we gauge the probability of the complementary event $\neg\,\mathsf{PolyWgtReg}$ via Corollary 2.1; thus, this probability cost is paid only once, rather than with each of the infinitely many applications of Proposition 5.1.

Proof of Proposition 5.1. Consider a given dyadic interval $U \subseteq [0, 2^{-k_0}]$ of a scale j such that $j \geq i$ and write $U = [u_1, u_2]$. For a given dyadic rational $z \in [0, 2^{-k_0}]$, note that

$$\Delta^{\cup}\mathsf{Wgt}_{n;(x+z,0)}^{(y+U,1)} = \mathcal{L}_{n;(x+z,0)}^{\uparrow;1}\big(1, y+u_2\big) + 2^{-1/2}(y+u_2-x+z)^2 - \mathcal{L}_{n;(x+z,0)}^{\uparrow;1}\big(1, y+u_1\big) - 2^{-1/2}(y+u_1-x+z)^2$$

and that

$$\Delta^{\cup}\mathsf{Wgt}_{n;(x+U,0)}^{(y+z,1)} = \mathcal{L}_{n;0}^{\downarrow;(y+z,1)}\big(1, x+u_2\big) + 2^{-1/2}(x+u_2-y+z)^2 - \mathcal{L}_{n;0}^{\downarrow;(y+z,1)}\big(1, x+u_1\big) - 2^{-1/2}(x+u_1-y+z)^2 \,.$$
$$(27)$$

Let $K > 0$ be a parameter that we will later specify. We now seek to apply Proposition 3.5 in order to bound the $\mathbb{P}$-probability that the last two displayed quantities exceed $K|U|^{1/2} = K2^{-j/2}$. In order to do so for the first quantity, the proposition will be applied to the ensemble $\mathcal{L}_N = \mathsf{Nr}\mathcal{L}_{n;(x+z,0)}^{\uparrow;1}$. Proposition 3.5's parameters in this case are set $\mathbf{x} = y + u_1 - x - z$, $\boldsymbol{\epsilon} = 2^{-j}$ and $\mathbf{K} = K$. Note that the event whose probability is bounded above by the proposition is a subset of $\mathsf{G}_{K/4}(\mathbf{x})$. With the present choice of ensemble $\mathcal{L}_n$, the event $\mathsf{G}_{K/4}(\mathbf{x})$ equals $\mathsf{PolyWgtReg}_{n;(\{x+z\},0)}^{([y+u_1-2,y+u_1+2],1)}(K/4)$. (Note that here $\{x+z\}$ is a singleton set, so that a space of polymers emanating from the point $(x+z, 0)$ is at stake.) Thus, $\mathsf{PolyWgtReg}_{n;([x,x+1],0)}^{([y-2,y+3],1)}(K/4) \subseteq \mathsf{G}_{K/4}(\mathbf{x})$. As such, Proposition 3.5 implies that

$$\mathbb{P}\bigg(\Big|\Delta^{\cup}\mathsf{Wgt}_{n;(x+z,0)}^{(y+U,1)}\Big| \geq K2^{-j/2}, \, \mathsf{PolyWgtReg}_{n;([x,x+1],0)}^{([y-2,y+3],1)}(K/4)\bigg)$$

$$\leq \quad \mathbb{P}\bigg(\Big|\Delta^{\cup}\mathsf{Wgt}_{n;(x+z,0)}^{(y+U,1)}\Big| \geq K2^{-j/2}, \, \mathsf{G}_{K/4}(\mathbf{x})\bigg) \leq 2^{3/2}\pi^{-1/2}K^{-1}\exp\big\{-2^{-3}K^2\big\}\,. \quad (28)$$

The hypotheses of Proposition 3.5 that are invoked to obtain this bound are

$$n + 1 \geq 6^3 c^{-3}\,, \ |y + u_1 - x - z| \leq 2^{-1}cn^{1/9}\,, \ 2^{-j} \in (0, 1] \ \text{and} \ K \geq 9\,;$$

note that $j \geq 0$ is used to validate the third of these.

Another application of Proposition 3.5 is made in regard to the quantity (27). On this occasion, the ensemble $\mathcal{L}_n$ is set equal to $\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(y+z,1)}$, and the parameters are set: $\mathbf{x} = x + u_1 - y - z$, $\boldsymbol{\epsilon} = 2^{-j}$ and $\mathbf{K} = K$. In this instance, the event $\mathsf{G}_{K/4}(\mathbf{x})$ equals $\mathsf{PolyWgtReg}_{n;([x+u_1-2,x+u_1+2],0)}^{(y+z,1)}(K/4)$, so

that $\mathsf{PolyWgtReg}_{n;([x-2,x+3],0)}^{([y,y+1],1)}(K/4) \subseteq \mathsf{G}_{K/4}(\mathbf{x})$. The outcome of the application in this case is the conclusion that

$$\mathbb{P}\left(\left|\Delta^{\cup}\mathsf{Wgt}_{n;(x+U,0)}^{(y+z,1)}\right| \geq K2^{-j/2}\,,\, \mathsf{PolyWgtReg}_{n;([x-2,x+3],0)}^{([y,y+1],1)}(K/4)\right)$$

$$\leq\ 2^{3/2}\pi^{-1/2}K^{-1}\exp\left\{-2^{-3}K^2\right\}, \tag{29}$$

while the hypothesis $|x + u_1 - y - z| \leq 2^{-1}cn^{1/9}$ is used to make the application.

The dyadic scale j is at least the scale i by assumption, and we now denote $\ell = j - i \geq 0$. We recall from the statement of Proposition 5.1 that we consider a parameter $K_0 \geq 9$. We now set the value of our parameter K, choosing it to be $K_0 2^{(j-i)/2}$. Our conclusions (28) and (29) tell us that

$$\mathbb{P}\left(\left|\Delta^{\cup}\mathsf{Wgt}_{n;(x+z,0)}^{(y+U,1)}\right| \geq K_0 2^{-i/2}\,,\, \mathsf{PolyWgtReg}_{n;([x,x+1],0)}^{([y-2,y+3],1)}(K_0/4)\right)$$

and

$$\mathbb{P}\left(\left|\Delta^{\cup}\mathsf{Wgt}_{n;(x+U,0)}^{(y+z,1)}\right| \geq K_0 2^{-i/2}\,,\, \mathsf{PolyWgtReg}_{n;([x-2,x+3],0)}^{([y,y+1],1)}(K_0/4)\right) \tag{30}$$

are both at most $2^{3/2}\pi^{-1}K_0^{-1}\exp\left\{-2^{-3}\cdot 2^{\ell}K_0^2\right\}$, where we used $K/4 \geq K_0/4$.

We will now sum the stated bound on quantities of the form (30) in order to find an upper bound on the expression (26) where recall that, in this expression, the supremum is taken over choices of scale i dyadic rational $z \in [0, 2^{-k_0}]$ and dyadic interval $U = [u_1, u_2] \subset [0, 2^{-k_0}]$ of scale at least i. Indeed, since the number of dyadic rationals of scale i in $[0, 2^{-k_0}]$ is at most $2^{i-k_0} + 1$, while the number of scale j dyadic intervals contained in $[0, 2^{-k_0}]$ equals 2^{j-k_0}, we may sum over $j \geq i$, also using that $K_0 \geq 2^{3/2}$, to find that (26) is at most

$$(2^{i-k_0} + 1) \cdot 2^{i-k_0} \cdot \tfrac{1}{e-2} \cdot 2^{3/2}\pi^{-1}K_0^{-1}\exp\left\{-2^{-3}K_0^2\right\}.$$

Using $i \geq k_0$ and $K_0 \geq 9$, this quantity is found to be bounded above by

$$2^{2(i-k_0)+5/2}3^{-2}(e-2)^{-1}\pi^{-1}\exp\left\{-2^{-3}K_0^2\right\} \leq 2^{2(i-k_0)}\exp\left\{-2^{-3}K_0^2\right\}.$$

We have obtained the upper bound claimed in Proposition 5.1 on the probability (26) and, since this upper bound on (25) is similarly obtained, we have completed the proof of this proposition. $\qquad\square$

We now state an estimate also needed for the proof of Theorem 1.1. Suppose that $n \in \mathbb{N}$ satisfies

$$n \geq 10^{29} \vee 2(c/3)^{-18}\,,\, |x-y| + 4 \leq 6^{-1}cn^{1/18} \text{ and } r_0 \in \left[34\,,\, 4n^{1/18}\right]. \tag{31}$$

Corollary 2.1 with $\mathbf{n} = n$, $\mathbf{t}_1 = 0$, $\mathbf{t}_2 = 1$, $\mathbf{x} = x - 2$, $\mathbf{y} = y - 2$, $\mathbf{a} = \mathbf{b} = 5$ and $\mathbf{r} = r_0$ implies that

$$\mathbb{P}\left(\neg\,\mathsf{PolyWgtReg}_{n;([x-2,x+3],0)}^{([y-2,y+3],1)}(r_0)\right) \leq 5^2 \cdot 400\,C\exp\left\{-c_1 2^{-10}r_0^{3/2}\right\}. \tag{32}$$

Note that $|x-y| + 4 \leq 6^{-1}cn^{1/18}$ is implied by $|x-y| \leq 2^{-1}6^{-1}cn^{1/18}$ and $4 \leq 2^{-1}6^{-1}cn^{1/18}$ and the latter is implied by $n \geq \left(48c^{-1}\right)^{18}$.

Proof of Theorem 1.1. Write $I = [x, x+\epsilon]$ and $J = [y, y+\epsilon]$, and let $u \in I$ and $v \in J$ be arbitrary. Recalling that $\epsilon \leq 2^{-4}$ is less than one, we consider the binary expansion $u - x = \sum_{j=1}^{\infty} s_j 2^{-j}$. If the expansion is not unique, we choose its terminating version for definiteness. Let the increasing sequence $u_0, u_1, \cdots$ enumerate the set

$$\left\{x + \sum_{j=1}^{k} s_j 2^{-j} : k \in \mathbb{N}\right\}.$$

This sequence may be finite or infinite. It begins $u_0 = x$, and, when it is infinite, each term u_n equals the sum of x and the quantity given by the truncation of the binary expansion of $u - x$ that contains n instances of the digit one. Let $n_1 \in \mathbb{N} \cup \{\infty\}$ denote the maximal index of a term in the u-sequence. If $n_1 < \infty$, then $u_{n_1} = u$, and if $n_1 = \infty$, then $u_n \nearrow u$, and we set $u_\infty = u$.

Similarly, we specify an increasing sequence $v_0, v_1, \cdots$ by replacing (x, u) by (y, v) above, and let n_2 denote the maximal index of a term in the v-sequence. If $n_2 = \infty$, set $v_\infty = v$.

Call the planar points $\{(u_i, 0) : 0 \leq i \leq n_1\}$ lower pegs, and the points $\{(v_i, 1) : 0 \leq i \leq n_2\}$ upper pegs. Think of a planar line segment *cord* that runs in the first instance between $(u_0, 0) = (x, 0)$ and $(v_0, 1) = (y, 1)$. Permitting the cord to be pegged at its lower and upper end to any of the pegs, we see that the cord begins in its leftmost possible location. The rightmost available location is given by lower peg $(u_\infty, 0) = (u, 0)$ and upper peg $(v_\infty, 1) = (v, 1)$. We now specify a possibly infinite sequence of cord moves by which the cord will achieve, or at least converge towards, this rightmost location. Let $(L_i, U_i) \in \{0, \cdots, n_1\} \times \{0, \cdots, n_2\}$ denote the indices of lower and upper peg locations at step $i \in \mathbb{N}$, where the original location is indexed by $i = 0$, so that $(L_0, U_0) = (0, 0)$. Let $k \in \mathbb{N}$ and consider the value of (L_k, U_k). If this value is (n_1, n_2), then the cord movement is complete and the value of (L_{k+1}, U_{k+1}) is not recorded. In the other case, there are two possible moves for the cord at the next step: a lower peg advance, in which $L_{k+1} = L_k + 1$ and $U_{k+1} = U_k$, or an upper peg advance, in which $U_{k+1} = U_k + 1$ and $L_{k+1} = L_k$. It may be that one of these moves is inadmissible, because $L_k = n_1$, which renders the lower peg advance unavailable, or $U_k = n_2$, which does likewise for the upper peg advance. If this is so, then (L_{k+1}, U_{k+1}) is set equal to the value given by the only available advance. In the case where both advances are possible, note that each move entails displacing a peg to the right by a distance of the form 2^{-i} for some $i \in \mathbb{N}$. The decision of which advance to make is taken so that this distance is the larger for the available two advances, with say the upper advance being made if the distances are equal. In this way, we specify the value of (L_{k+1}, U_{k+1}); we may also record the dyadic scale, $D_{k+1} \in \mathbb{N}$, of the advance associated to this index increase $k \to k+1$: this scale is the value of $i \in \mathbb{N}$ such that the peg displacement made in the peg advance resulting in the new peg locations (L_{k+1}, U_{k+1}) equals 2^{-i}.

The sequence of location pairs (u_{L_k}, v_{U_k}) either reaches its terminal state (u, v) after finitely many moves, or it converges to this state as k increases. Note also that the dyadic scale sequence $D_1, D_2, \cdots$ is a non-decreasing $\mathbb{N}$-valued sequence that assumes any given value at most twice. This sequence depends on the pair (u, v), and we may indicate this dependence by writing $D_k = D_k(u, v)$.

Moreover, we define a dyadic scale $i_0 \in \mathbb{N}$ by setting $i_0 \geq 0$ to be minimal such that $2^{i_0}\epsilon \geq 1$. Then, since the first peg is displaced by a distance 2^{-D_1} which is at most $\epsilon < 2^{1-i_0}$, we see that $D_1 \geq i_0$. We see then that the dyadic scales $\{D_j : j \geq 1\}$ of the distances of peg moves for the advancing cord grow linearly from around i_0:

$$D_j \geq i_0 - 1 + j/2 \quad \text{for} \quad j \geq 1. \tag{33}$$

To any cord location we may associate the parabolically adjusted weight $\mathsf{Wgt}_{n;(z,0)}^{(z',1)} + 2^{-1/2}(z' - z)^2$ of the polymer whose endpoints $(z, 0)$ and $(z', 1)$ are the pegs to which the cord is pinned. Using the notation (24), we may further set, for $k \geq 0$,

$$W_{k+1} = W_{k+1}(u, v) = \Delta^\cup \mathsf{Wgt}_{n;([u_{L_k}, u_{L_{k+1}}], 0)}^{([v_{U_k}, v_{U_{k+1}}], 1)}, \tag{34}$$

this being the difference in parabolically adjusted weight of this polymer as a result of the cord move from its index k to $k+1$ location.

Lemma 2.2(1) implies that, for given n, $\lim_k \mathsf{Wgt}_{n;(L_k,0)}^{(U_k,1)} = \mathsf{Wgt}_{n;(u,0)}^{(v,1)}$. Thus,

$$\mathsf{Wgt}_{n;(u,0)}^{(v,1)} - \mathsf{Wgt}_{n;(x,0)}^{(y,1)} + 2^{-1/2}(v-u)^2 - 2^{-1/2}(y-x)^2 = \sum_{k=1}^{\infty} W_k \,, \tag{35}$$

where it is understood that the right-hand sum may have only finitely non-zero terms.

For each $k \in \mathbb{N}$ (at least one), we set $W^*[k]$ to be the supremum of the values of $|W_k|$ over all choices of (u,v) in our construction. We now argue that there exists a pair $(u^*[k], v^*[k]) \in I \times J$ such that $W^*[k] = \big|W_k(u^*[k], v^*[k])\big|$; in other words, we argue that the supremum specifying $W^*[k]$ is attained. First note that, if $W^*[k] = 0$, any choice of $(u^*[k], v^*[k]) \in I \times J$ works. Suppose instead then that $W^*[k] > 0$. For any given $(u,v) \in I \times J$, the value $W_k(u,v)$ takes one of the forms $\Delta^{\cup}\mathsf{Wgt}_{n;(x+\Theta,0)}^{(y+z_1,1)}$ or $\Delta^{\cup}\mathsf{Wgt}_{n;(x+z_2,0)}^{(y+\Lambda,1)}$, where in the former case, Θ is a dyadic rational interval and $z_1 \in [0, \epsilon]$ is a dyadic rational whose scale is by our construction at most that of Θ; and, in the latter, these statements are equally true of Λ and z_2. Now the number of such expressions for which the length of the interval Θ or Λ in the indexing pair (Θ, z_1) or (z_2, Λ) exceeds an arbitrary given positive value δ is finite. On the other hand, since the map $I \times J \to \mathbb{R} : (u,v) \to \mathsf{Wgt}_{n;(u,0)}^{(v,1)}$ is uniformly continuous, there exists a random value $\delta > 0$ such that none of the parabolically adjusted weight differences whose indexing pair contains an interval Θ or Λ of length less than δ has value exceeding the positive quantity $W^*[k]/2$. The value $W^*[k]$ is thus seen to be the maximum of a certain finite set of such weight differences, and thus, it is indeed achieved.

For given $(u,v) \in I \times J$, there are many choices of $(u^*[k], v^*[k]) \in I \times J$, and we pick the lexicographically minimal pair for definiteness. We further specify the dyadic scale $D^*[k] = D_k(u^*[k], v^*[k])$.

By (33), $D_\ell(u,v) \geq i_0 - 1 + k$ provided that $\ell \geq 2k$, whatever the value of $(u,v) \in [x, x+\epsilon] \times [y, y+\epsilon]$ may be. Thus, when $\ell \in \{2k, 2k+1\}$, the quantity $W^*[\ell]$ takes the form $\big|\Delta^{\cup}\mathsf{Wgt}_{n;(x+z,0)}^{(y+U,1)}\big|$ or $\big|\Delta^{\cup}\mathsf{Wgt}_{n;(x+U,0)}^{(y+z,1)}\big|$ where $z \in [0, 2^{1-i_0})$ is a dyadic rational of scale $D^*[\ell]$ at least $i_0 - 1 + k$ and $U \subset [0, 2^{1-i_0})$ is a dyadic interval whose scale is at least that of z. For this reason, it is tempting – though, as we shortly explain, mistaken – to use Proposition 5.1 with $\mathbf{k_0} = i_0 - 1$ and $\mathbf{i} = D^*[\ell]$ to find an upper bound on the probability

$$\mathbb{P}\Big(W^*[\ell] \geq K_0 \, 2^{-D^*[\ell]/2} \,, \, \mathsf{PolyWgtReg}_{n;([x-2,x+3],0)}^{([y-2,y+3],1)}\big(2^{-2}K_0\big)\Big),$$

where recall from the proposition that $K_0 \geq 9$ is a parameter. Indeed, the proposition formally implies that, whenever $k \in \mathbb{N}$ and $\ell \in \{2k, 2k+1\}$, this probability is at most

$$2 \cdot 2^{2(D^*[\ell]-i_0+1)} \cdot \exp\big\{ - 2^{-3}K_0^2\big\}.$$

This application is erroneous, however, because $D^*[\ell]$ is random and thus the choice $\mathbf{i} = D^*[\ell]$ is inadmissible. We adjust to cope with this problem by introducing a parameter $j \in \mathbb{N}$ that is supposed to be at least $i_0 - 1 + k$. For any given such j, we may now apply Proposition 5.1 with $\mathbf{k_0} = i_0 - 1$ and $\mathbf{i} = j$. We thus find that, for $k \in \mathbb{N}$, $\ell \in \{2k, 2k+1\}$ and $j \geq i_0 - 1 + k$,

$$\mathbb{P}\Big(W^*[\ell] \geq K_0 \, 2^{-j/2} \,, \, D^*[\ell] = j \,, \, \mathsf{PolyWgtReg}_{n;([x-2,x+3],0)}^{([y-2,y+3],1)}\big(2^{-2}K_0\big)\Big)$$
$$\leq \quad 2^{1+2(j-i_0+1)} \exp\big\{ - 2^{-3}K_0^2\big\}.$$

With such j remaining fixed, we now choose the parameter K_0 as a function of $\ell \in \mathbb{N}$ and j, setting $K_0 = S \cdot 2^{-i_0/2 - k/4} \cdot 2^{j/2}$. (Since $k = \lfloor \ell/2 \rfloor$, this choice is indeed determined by ℓ.) For reasons soon to be explained, we set the new quantity S equal to $2^{9/2}$. Using $j \geq i_0 - 1 + k$, we see that

$K_0 \geq 2^{k/4-1/2}S$ (and thus $K_0 \geq 9$, as is needed). Since the event $\mathsf{PolyWgtReg}_{n;([x-2,x+3],0)}^{([y-2,y+3],1)}(K_0/4)$ is increasing in $K_0 > 0$, we find that

$$\mathbb{P}\Big(W^*[\ell] \geq S \cdot 2^{-i_0/2-k/4} \,,\, D^*[\ell] = j \,,\, \mathsf{PolyWgtReg}_{n;([x-2,x+3],0)}^{([y-2,y+3],1)}\big(2^{-2-1/4}S\big)\Big)$$
$$\leq \quad 2^{1+2(j-i_0+1)} \exp\big\{ -2^{-3-i_0-k/2+j}S^2\big\} .$$

Summing over $k \geq 1$, the two values of ℓ for each k, and the values $j \in \mathbb{N}$ that are at least i_0-1+k, we learn that

$$\mathbb{P}\Big(\sum_{\ell=0}^{\infty} W^*[\ell] \geq 2S \sum_{k=0}^{\infty} 2^{-i_0/2-k/4} \,,\, \mathsf{PolyWgtReg}_{n;([x-2,x+3],0)}^{([y-2,y+3],1)}\big(2^{-9/4}S\big)\Big)$$

$$\leq \quad 2 \sum_{\ell=0}^{\infty} \sum_{j=i_0-1+\lfloor \ell/2 \rfloor}^{\infty} 2^{1+2(j-i_0+1)} \exp\big\{ -2^{-3-i_0-\ell/4+j}S^2\big\}$$

$$\leq \quad 8 \sum_{\ell=0}^{\infty} 2^{2\lfloor \ell/2 \rfloor} \exp\big\{ -2^{-3-i_0-\ell/4+i_0-1+\lfloor \ell/2 \rfloor}S^2\big\}$$

$$\leq \quad 8 \sum_{\ell=0}^{\infty} 2^{\ell} \exp\big\{ -2^{-5+\ell/4}S^2\big\} . \tag{36}$$

The final inequality is trivial, but a brief explanation is needed to verify the second. Note that the ratio of the summands on this inequality's left-hand side indexed by $(\ell, j+1)$ and (ℓ, j) is at most one-half provided that $2^{-3-i_0-\ell/4+j}S^2 \geq 3\log 2$; since $j \geq i_0 - 1 + k$, while $k = \lfloor \ell/2 \rfloor$ and $\ell \geq 0$, the latter bound is seen to be valid when $S \geq 2^{5/2}3^{1/2}(\log 2)^{1/2}$. Since S equals $2^{9/2}$, the last condition holds; and thus the bound in question emerges because its right-hand side is the sum of the concerned geometric series.

Consider then the expression $2^{\ell} \exp\big\{ -2^{-5+\ell/4}S^2\big\}$. The ratio of each summand, indexed by $\ell \geq 1$, to its predecessor is at most $2\exp\big\{ -S^2 \cdot 2^{-5}(2^{1/4}-1)\big\}$ which when, as is presently supposed, $S \geq 2^{5/2+2} = 2^{9/2}$ is at most $2\exp\big\{ -2^{2\cdot2}(2^{1/4}-1)\big\} \leq 3/4$; so that (36) is at most

$$4 \cdot 8 \exp\big\{ -2^{-5}S^2\big\} = 32 \exp\big\{ -2^{-5}S^2\big\} .$$

By (35) and the definition of the sequence $\{W^*[\ell] : \ell \geq \mathbb{N}, \ell \geq 1\}$,

$$\sum_{\ell=1}^{\infty} W^*[\ell] \geq \sup_{(u,v)\in[x,x+\epsilon]\times[y,y+\epsilon]} \Big| \Delta^{\cup} \mathsf{Wgt}_{n;([x,u],0)}^{([y,v],1)} \Big| .$$

Thus,

$$\mathbb{P}\Big(\sup_{(u,v)\in[x,x+\epsilon]\times[y,y+\epsilon]} \Big| \Delta^{\cup} \mathsf{Wgt}_{n;([x,u],0)}^{([y,v],1)} \Big| \geq 2^{1-i_0/2}(1-2^{-1/4})^{-1}S \,,$$

$$\mathsf{PolyWgtReg}_{n;([x-2,x+3],0)}^{([y-2,y+3],1)}\big(2^{-9/4}S\big) \Big) \leq 32 \exp\big\{ -2^{-5}S^2\big\} .$$

We find that

$$\mathbb{P}\Big(\sup \Big| \Delta^{\cup} \mathsf{Wgt}_{n;([u_1,u_2],0)}^{([v_1,v_2],1)} \Big| \geq 2 \cdot 2^{1-i_0/2}(1-2^{-1/4})^{-1}S \,,$$

$$\mathsf{PolyWgtReg}_{n;([x-2,x+3],0)}^{([y-2,y+3],1)}\big(2^{-9/4}S\big) \Big) \leq 32 \exp\big\{ -2^{-5}S^2\big\} ,$$

where the supremum is over arbitrary $u_1, u_2 \in [x, x + \epsilon]$ and $v_1, v_2 \in [y, y + \epsilon]$.

Applying (32), and recalling that $\epsilon \geq 2^{-i_0}$,

$$\mathbb{P}\left(\sup\left|\Delta^\cup \mathsf{Wgt}_{n;([u_1,u_2],0)}^{([v_1,v_2],1)}\right| \geq \epsilon^{1/2} 2^2 (1 - 2^{-1/4})^{-1} S\right)$$
$$\leq \mathbb{P}\left(\sup\left|\Delta^\cup \mathsf{Wgt}_{n;([u_1,u_2],0)}^{([v_1,v_2],1)}\right| \geq 2^{2-i_0/2}(1 - 2^{-1/4})^{-1} S\right)$$
$$\leq 32 \exp\left\{-2^{-5}S^2\right\} + 5^2 \cdot 400\, C \exp\left\{-c_1 2^{-10} r_0^{3/2}\right\},$$

where $r_0 = 2^{-9/4} S$. The right-hand side is at most

$$32 \exp\left\{-2^{-5}S^2\right\} + 10000\, C \exp\left\{-c_1 2^{-10-27/8} S^{3/2}\right\}$$
$$\leq 10032\, C \exp\left\{-c_1 2^{-14} S^{3/2}\right\},$$

the displayed bound due to $S \geq 1$, $C \geq 1$ and $c_1 \leq 1$. Setting $R = 2^2(1 - 2^{-1/4})^{-1} S$ and noting that $2^{-14-3}(1 - 2^{-1/4})^{3/2} \geq 2^{-21}$ completes the proof of Theorem 1.1. □

6. PROFILE REGULARITY FOR GENERAL INITIAL DATA: PROVING THEOREMS 1.3 AND 1.4

For the proofs of these two theorems, it is needed that, typically, every f-rewarded polymer that ends in the interval $[-1, 1] \times \{1\}$ begins in a compact subset of the x-axis $\mathbb{R} \times \{0\}$. We first introduce a suitable regular fluctuation event $\mathsf{RegFluc}$, and then derive a result to this effect, Lemma 6.2.

Definition 6.1. Recall Definition 1.2 and the notational usage of $\rho_{n;(*:f,0)}^{(y,1)}$ from Subsection 2.4.3. Let $\overline{\Psi} \in (0,\infty)^3$ and $f \in \mathcal{I}_{\overline{\Psi}}$. For $R \geq 0$, define the event $\mathsf{RegFluc}_{n;(*:f,0)}^{(\{-1,1\},1)}(R)$ that any f-rewarded line-to-point polymer $\rho_{n;(*:f,0)}^{(-1,1)}$ that ends at $(-1,1)$ begins at a location $(x,0)$ where $x \geq -(R+1)$; and any such polymer $\rho_{n;(*:f,0)}^{(1,1)}$ that ends at $(1,1)$ begins at $(x,0)$, where $x \leq R+1$.

We remark that our $\mathsf{RegFluc}$ event entails that any f-rewarded line-to-point polymer that ends at a location $(y,1)$ with $y \in [-1,1]$ must begin at a location $(x,0)$, where $|x| \leq R+1$. Indeed, were such a polymer $\rho_{n;(*:f,0)}^{(y,1)}$ to begin at $(x,0)$, with $x < -(R+1)$, then, in the event $\mathsf{RegFluc}$, it would cross any example of $\rho_{n;(*:f,0)}^{(-1,1)}$. The rewiring of these two polymers described in Subsection 2.4.3 would then furnish an example of $\rho_{n;(x,0)}^{(-1,1)}$ with $x < -(R+1)$, in conflict with the occurrence of $\mathsf{RegFluc}$.

Lemma 6.2. Let $n \in \mathbb{N}$, $R > 0$ and $\overline{\Psi} \in (0,\infty)^3$ satisfy

$$n \geq c^{-18} \max\left\{(\Psi_2 + 1)^9, 10^{23}\Psi_1^9, 3^9\right\}, \quad R \geq \max\left\{39\Psi_1, 5, 3c^{-3}, 2((\Psi_2 + 1)^2 + \Psi_3)^{1/2}\right\},$$

and $R \leq 6^{-1} c n^{1/9}$. Then, for any $f \in \mathcal{I}_{\overline{\Psi}}$,

$$\mathbb{P}\left(\neg\, \mathsf{RegFluc}_{n;(*:f,0)}^{(\{-1,1\},1)}(R)\right) \leq 38RC \exp\left\{-2^{-6}cR^3(2^{-1/2} - 2^{-1})^{3/2}\right\}.$$

Proof. The event $\neg\, \mathsf{RegFluc}_{n;(*:f,0)}^{(\{-1,1\},1)}(R)$ equals $A_1 \cup A_2$, where A_1 is the event that $y \to \mathcal{L}_{n;0}^{\downarrow;(-1,1)}(1,y) + f(y)$ achieves its maximum for a value of y that is less than $-1 - R$, and A_2 is the event that $y \to \mathcal{L}_{n;0}^{\downarrow;(1,1)}(1,y) + f(y)$ achieves its maximum for a value of y that is at greater than $1 + R$.

Note the inclusion

$$\left\{ \sup_{x\in[-\Psi_2,\Psi_2]} \left(\mathcal{L}_{n;0}^{\downarrow;(-1,1)}(1,x) + f(x)\right) > -R^2/2 \,,\, \sup_{x\leq-1-R} \left(\mathcal{L}_{n;0}^{\downarrow;(-1,1)}(1,x) + f(x)\right) \leq -R^2/2 \right\} \subseteq A_1^c \,. \tag{37}$$

Upper bounds on the failure probability of the left-hand events will now be found. The first event will be shown to be probable because, in view of Definition 1.2, the function f is known to assume a not highly negative value somewhere in a compact interval about the origin. The second event is probable due to the at most linear growth of f far from the origin, combined with decay estimates on the curve $\mathcal{L}_{n;0}^{\downarrow;(-1,1)}(1,x)$ for large x. These estimates take two forms: when x is large, but less than order $n^{1/9}$, the curve hews to a parabola, in accordance with the No Big Max Proposition 3.3(2), applied to the normalized cousin of the ensemble in question; when x becomes even larger, the curve may escape the reaches of this parabola, but it continues to decay rapidly, in accordance with collapse near infinity Proposition 3.3(3).

Since $f \in \mathcal{I}_{\overline{\Psi}}$, there exists $x_0 \in [-\Psi_2, \Psi_2]$ such that $f(x_0) \geq -\Psi_3$. As such, the first left-hand event in (37) fails with a probability that satisfies

$$\mathbb{P}\left(\sup_{x\in[-\Psi_2,\Psi_2]} \left(\mathcal{L}_{n;0}^{\downarrow;(-1,1)}(1,x) + f(x)\right) \leq -R^2/2 \right) \leq \mathbb{P}\left(\mathcal{L}_{n;0}^{\downarrow;(-1,1)}(1,x_0) \leq -R^2/2 + \Psi_3\right)$$

$$= \mathbb{P}\left(\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(-1,1)}(1,x_0+1) \leq -R^2/2 + \Psi_3\right)$$

$$\leq \mathbb{P}\left(\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(-1,1)}(1,x_0+1) + 2^{-1/2}(x_0+1)^2 \leq -R^2/2 + 2^{-1/2}(\Psi_2+1)^2 + \Psi_3\right)$$

$$\leq \mathbb{P}\left(\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(-1,1)}(1,x_0+1) + 2^{-1/2}(x_0+1)^2 \leq -R^2/4\right) \leq C\exp\left\{-2^{-3}cR^3\right\},$$

where the penultimate inequality depends on $R^2/4 \geq 2^{-1/2}(\Psi_2+1)^2 + \Psi_3$. The final inequality was obtained by applying the one-point lower tail Reg(2) with parameter choices $\mathbf{z} = x_0$ and $\mathbf{s} = R^2/4$ to the (c,C)-regular ensemble $\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(-1,1)}$. Since the ensemble has $n+1$ curves, this application of Reg(2) may be made provided that $|x_0| + 1 \leq cn^{1/9}$ and $R^2/4 \in [1, n^{1/3}]$. The first of these conditions due to $n \geq c^{-9}(\Psi_2+1)^9$ alongside $|x_0| \leq \Psi_2$; the second we assume.

The failure probability of the second left-hand event in (37) may be gauged as follows: since $f(x)$ is at most $\Psi_1(1 + |x|)$ for any $x \in \mathbb{R}$, we may note that

$$\mathbb{P}\left(\sup_{x\leq-1-R} \left(\mathcal{L}_{n;0}^{\downarrow;(-1,1)}(1,x) + f(x)\right) > -R^2/2 \right)$$

$$= \mathbb{P}\left(\sup_{x\leq-R} \left(\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(-1,1)}(1,x) + f(x-1)\right) > -R^2/2 \right)$$

$$\leq \mathbb{P}\left(\sup_{x\leq-R} \left(\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(-1,1)}(1,x) + \Psi_1(2+|x|)\right) > -R^2/2 \right); \tag{38}$$

the latter term may then be bounded above by

$$\sum \mathbb{P}\left(\sup_{x\in-R[2^j,2^{j+1}]} \mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(-1,1)}(1,x) > -R^2/2 - \Psi_1\left(2 + 2^{j+1}R\right) \right) + E_1 + E_2\,,$$

where the first sum is indexed by a parameter j that varies over the integer interval $[\![0,k]\!]$ where $k \in \mathbb{N}$ to chosen to be maximal subject to $2^{k+1}R \leq 3^{-1}cn^{1/9}$. (Such a k exists because we suppose that $2R \leq 3^{-1}cn^{1/9}$.) The term E_1 corresponds to part of a dyadic scale that has been sliced in two by the value $-3^{-1}cn^{1/9}$: this term is specified by the expression in (38) when the supremum in

the variable x is chosen to be over the interval $\left[-3^{-1}cn^{1/9}, -2^{k+1}R\right]$. The remaining term E_2 is a long-range error term corresponding to the interval $\left[-z_{\mathcal{L}}, -3^{-1}cn^{1/9}\right)$. Since $R \leq 3^{-1}cn^{1/9}$,

$$E_2 \leq \mathbb{P}\Big(\sup\Big\{\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(-1,1)}(1,x) + \Psi_1(2+|x|) : x \in \left[-z_{\mathcal{L}}, -3^{-1}cn^{1/9}\right]\Big\} \geq -2^{-1}(c/3)^2 n^{2/9}\Big).$$

This right-hand side will be bounded above by applying collapse-near-infinity Proposition 3.3(3) to the $(n+1)$-curve (c,C)-regular ensemble $\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(-1,1)}$. We apply Proposition 3.3(3) with its parameter $\boldsymbol{\eta}$ chosen so that $\boldsymbol{\eta}(n+1)^{1/9} = -3^{-1}cn^{1/9}$. In order to make the application, we first claim that the affine function $x \to \ell(x)$ in the proposition lies below the function

$$x \to -2^{-1}(c/3)^2 n^{2/9} - \Psi_1\big(2+|x|\big) \tag{39}$$

whenever $x \leq -3^{-1}cn^{1/9}$. To verify this, note that, when $x = -3^{-1}cn^{1/9}$, the assertion takes the form $\Psi_1\big(2 + 3^{-1}cn^{1/9}\big) \leq \big(2^{-1/2} - 2^{-5/2} - 2^{-1}\big)(c/3)^2 n^{2/9}$, which holds due to the supposed $1 \leq 3^{-1}cn^{1/9}$ and $\Psi_1 \leq \big(2^{-1/2} - 2^{-5/2} - 2^{-1}\big)(c/9)n^{1/9}$. Confirming the claim is then a matter of checking that the gradient of ℓ exceeds that of the function (39), which holds due to $\Psi_1 \leq 5 \cdot 2^{3/2}c/3 \cdot n^{1/9}$.

We may thus apply Proposition 3.3(3) when $n + 1 \geq 2^{45/4}c^{-9}$, finding that

$$E_2 \leq 6C \exp\big\{-2^{-15/4}3^{-3}c^4 n^{1/3}\big\}.$$

Note that

$$\mathbb{P}\Big(\sup_{x \in -R[2^j, 2^{j+1}]} \mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(-1,1)}(1,x) > -R^2/2 - \Psi_1\big(2 + 2^{j+1}R\big)\Big) \tag{40}$$

$$\leq \mathbb{P}\Big(\sup_{x \in -R[2^j, 2^{j+1}]}\big(\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(-1,1)}(1,x) + 2^{-1/2}x^2\big) > R^2\big(2^{2j-1/2} - 1/2\big) - \Psi_1\big(2 + 2^{j+1}R\big)\Big)$$

$$\leq \mathbb{P}\Big(\sup_{x \in -R[2^j, 2^{j+1}]}\big(\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(-1,1)}(1,x) + 2^{-1/2}x^2\big) > 2^{-1}R^2\big(2^{2j-1/2} - 1/2\big)\Big)$$

$$\leq 6C(2^{j-1}R + 1)\exp\big\{-2^{-6}cR^3\big(2^{2j-1/2} - 2^{-1}\big)^{3/2}\big\},$$

where in the second inequality we used $R \geq 1 \vee 39\Psi_1$ in the form

$$2^{-1}R^2\big(2^{2j-1/2} - 1/2\big) \geq \Psi_1\big(2 + 2^{j+1}R\big) \quad \text{for each } j \geq 0.$$

The final inequality arises from an application of Proposition 3.3(2) to the ensemble $\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(-1,1)}$. The parameters of the application are set to be

$$\mathbf{y} = -2^{-1}R\big(2^j + 2^{j+1}\big), \ \mathbf{r} = 2^{-1}R\big(2^{j+1} - 2^j\big) \text{ and } \mathbf{t} = 2^{-1}R^2\big(2^{2j-1/2} - 2^{-1}\big).$$

The application's hypotheses are implied by

$$3 \cdot 2^j R \leq cn^{1/9}, \ 2^{j+1}R \leq cn^{1/9}, \ 2^{-1}R^2\big(2^{2j-1/2} - 1/2\big) \in \big[2^{7/2}, 2n^{1/3}\big] \text{ and } n \geq c^{-18}.$$

The first three of these conditions are valid when $j \in [\![0, k]\!]$ in light of the assumed bound $2^{k+1}R \leq 3^{-1}cn^{1/9}$ (and $c \leq 1$); indeed, they are also valid when $j = k+1$, a fact that we will use momentarily.

The term E_1 is bounded above by (40) with $j = k + 1$, so that the preceding argument shows that

$$E_1 \leq 6C(2^k R + 1)\exp\big\{-2^{-6}cR^3\big(2^{2k+3/2} - 2^{-1}\big)^{3/2}\big\}.$$

Thus,

$$\mathbb{P}\Big(\sup_{x \leq -1-R} \big(\mathcal{L}^{\downarrow;(-1,1)}_{n;0}(1,x) + f(x) \big) > -R^2/2 \Big)$$

$$\leq \; 6C \sum_{j=0}^{k+1} (2^j R + 1) \exp \big\{ -2^{-6} cR^3 \big(2^{2j-1/2} - 2^{-1} \big)^{3/2} \big\} + 6C \exp \big\{ -2^{-15/4} 3^{-3} c^4 n^{1/3} \big\}$$

$$\leq \; 12RC \exp \big\{ -2^{-6} cR^3 \big(2^{-1/2} - 2^{-1} \big)^{3/2} \big\} + 6C \exp \big\{ -2^{-15/4} 3^{-3} c^4 n^{1/3} \big\}$$

$$\leq \; 18RC \exp \big\{ -2^{-6} cR^3 \big(2^{-1/2} - 2^{-1} \big)^{3/2} \big\},$$

where the second inequality used $R \geq (\log 4)^{1/3} 2^2 c^{-3} \big((2^{3/2} - 2^{-1})^{3/2} - (2^{-1/2} - 2^{-1})^{3/2} \big)^{-1/3}$ in order to ensure that each ratio of consecutive summands in the sum is at most one-half; the third makes use of $1 \leq R \leq (2^{-1/2} - 2^{-1})^{-1/2} 2^{3/4} 3^{-1} c n^{1/9}$.

Thus,

$$\mathbb{P}(A_1) \;\leq\; C \exp \big\{ -2^{-3} cR^3 \big\} + 18RC \exp \big\{ -2^{-6} cR^3 \big(2^{-1/2} - 2^{-1} \big)^{3/2} \big\}$$

$$\leq\; 19RC \exp \big\{ -2^{-6} cR^3 \big(2^{-1/2} - 2^{-1} \big)^{3/2} \big\},$$

the latter inequality due to $R \geq 1$. The same argument yields that $\mathbb{P}(A_2)$ satisfies the same upper bound. Combining the estimates completes the proof of Lemma 6.2. $\qquad\square$

Theorem 1.3's proof will also harness estimates showing that a *local weight regularity* event is typical. For $x, y \in \mathbb{R}$ and $\epsilon, r > 0$, this event is defined by

$$\mathsf{LocWgtReg}^{([y,y+\epsilon],1)}_{n;([x,x+\epsilon],0)}(\epsilon, r) \;=\; \left\{ \sup_{\substack{x_1,x_2 \in [x,x+\epsilon] \\ y_1,y_2 \in [y,y+\epsilon]}} \Big| \mathsf{Wgt}^{(y_2,1)}_{n;(x_2,0)} - \mathsf{Wgt}^{(y_1,1)}_{n;(x_1,0)} \Big| \leq r\epsilon^{1/2} \right\}.$$

The relevant control is offered by Theorem 1.1, except that this theorem addresses parabolically adjusted weight. The next result is the one we will apply in proving Theorem 1.4: the main new hypothesis, $|x - y| \leq \epsilon^{-1/2}$, limits parabolic curvature.

Corollary 6.3. *Let $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$ satisfy $n \geq 10^{32} c^{-18}$ and $|x - y| \leq \epsilon^{-1/2} \wedge 2^{-2} 3^{-1} c n^{1/18}$. Let $\epsilon \in (0, 2^{-4}]$ and $R \in \big[2 \cdot 10^4, \, 10^3 n^{1/18} \big]$. Then*

$$\mathbb{P}\Big(\neg \, \mathsf{LocWgtReg}^{([y,y+\epsilon],1)}_{n;([x,x+\epsilon],0)}(\epsilon, R) \Big) \leq 10032 \, C \exp \big\{ -c_1 2^{-22-1/2} R^{3/2} \big\}. \tag{41}$$

Proof. Recall that we denote $Q : \mathbb{R} \to \mathbb{R}$, $Q(u) = 2^{-1/2} u^2$. For $u_1, u_2 \in [x, x+\epsilon]$ and $v_1, v_2 \in [y, y+\epsilon]$, note that, since $|x - y| \leq \epsilon^{-1/2}$ and $\epsilon \leq 1$,

$$\big| Q(v_2 - u_2) - Q(v_1 - u_1) \big| \leq 2\epsilon Q' \big(|x - y| + \epsilon \big) = 2\epsilon \cdot 2^{1/2} \big(|x - y| + \epsilon \big) \leq 2^{5/2} \epsilon^{1/2}.$$

Thus, when $R \geq 2 \cdot 2^{5/2}$, the left-hand side of (41) is at most (2) with $\mathbf{R} = R/2$. The corollary thus follows from Theorem 1.1. $\qquad\square$

6.1. **Deriving Theorem 1.3.** For $f \in \mathcal{I}_{\overline{\Psi}}$, $\epsilon \in (0,2)$ and $\rho > 0$, define the equicontinuity event

$$\mathsf{EquiCty}^{[-1,1]}_{n;(*:f,0)}(\rho, \epsilon) = \left\{ \omega_{[-1,1],\epsilon}\big(y \to \mathsf{Wgt}^{(y,1)}_{n;(*:f,0)}\big) < \rho \right\},$$

where the modulus of continuity of a function $h : [-1,1] \to \mathbb{R}$ is denoted by

$$\omega_{[-1,1],\epsilon}(h) = \sup\left\{ \big|h(x) - h(y)\big| : x, y \in [-1,1], \, |x - y| \le \epsilon \right\}.$$

Here is an equicontinuity claim.

Lemma 6.4. *Let $\overline{\Psi} \in (0,\infty)^3$. Set*

$$C_- = 2^{17} c_1^{-2/3} \quad and \quad C_+ = \max\left\{ 39\Psi_1 \,, \, 12c^{-3} \,, \, 2\big((\Psi_2 + 1)^2 + \Psi_3\big)^{1/2} \right\},$$

and define the function $g(\epsilon) = 2C_- \, \epsilon^{1/2}\big(\log \epsilon^{-1}\big)^{2/3}$. Let $\epsilon > 0$ satisfy $\epsilon \le C_+^{-6} \exp\left\{ -2^{3/2} 10^6 \right\} C^{-2}$. When $n \in \mathbb{N}$ verifies $n \ge 10^{32} c^{-18} C_+^{18} C_-^{18}\big(\log \epsilon^{-1}\big)^{12}$, we have that, for $f \in \mathcal{I}_{\overline{\Psi}}$,

$$\mathbb{P}\left(\neg\, \mathsf{EquiCty}^{[-1,1]}_{n;(*:f,0)}\big(g(\epsilon), \epsilon\big) \right) \le \epsilon. \tag{42}$$

Proof. We first argue that, whenever $R \ge 1$,

$$\mathsf{RegFluc}^{\big(\{-1,1\},1\big)}_{n;(*:f,0)}(R - 1) \cap \bigcap_{u \in \epsilon\mathbb{Z} \cap [-R,R], v \in \epsilon\mathbb{Z} \cap [-1,1]} \mathsf{LocWgtReg}^{([v,v+\epsilon],1)}_{n;([u,u+\epsilon],0)}\left(\epsilon, C_-\big(\log \epsilon^{-1}\big)^{2/3} \right)$$

$$\subseteq \; \mathsf{EquiCty}^{[-1,1]}_{n;(*:f,0)}\big(g(\epsilon), \epsilon\big). \tag{43}$$

To verify this inclusion, suppose that $\mathsf{RegFluc}^{\big(\{-1,1\},1\big)}_{n;(*:f,0)}(R - 1)$ occurs, and consider $y \in [-1,1]$. By the remark made after Definition 6.1, all f-rewarded line-to-point polymers that abut at time one on $[-1,1]$ must begin at time zero somewhere on $[-R,R]$, (where here of course the present parameter value $R-1$ is involved). Thus, the quantity $\mathsf{Wgt}^{(y,1)}_{n;(*:f,0)}$ is seen to equal $\mathsf{Wgt}^{(y,1)}_{n;(x,0)} + f(x)$ for some $x \in [-R,R]$. Note that the event $\mathsf{LocWgtReg}^{([y,y+\epsilon],1)}_{n;(x,0)}\left(\epsilon, 2C_-\big(\log \epsilon^{-1}\big)^{2/3} \right)$ occurs when the intersection of the $\mathsf{LocWgtReg}$ events displayed above occurs; in this circumstance, we thus see that, for any $\eta \in (0, \epsilon)$, $\mathsf{Wgt}^{(y+\eta,1)}_{n;(x,0)} \ge \mathsf{Wgt}^{(y,1)}_{n;(x,0)} + f(x) - \epsilon^{1/2} \cdot 2C_-\big(\log \epsilon^{-1}\big)^{2/3}$ and thus $\mathsf{Wgt}^{(y+\eta,1)}_{n;(*:f,0)} \ge \mathsf{Wgt}^{(y,1)}_{n;(*:f,0)} - \epsilon^{1/2} \cdot 2C_-\big(\log \epsilon^{-1}\big)^{2/3}$. Provided that we further suppose that $y + \eta \le 1$, the inequality with the roles of y and $y + \eta$ reversed is similarly obtained, so that

$$\left| \mathsf{Wgt}^{(y+\eta,1)}_{n;(*:f,0)} - \mathsf{Wgt}^{(y,1)}_{n;(*:f,0)} \right| \le \epsilon^{1/2} \cdot 2C_-\big(\log \epsilon^{-1}\big)^{2/3}.$$

Thus, (43) is obtained. Verifying the equicontinuity claim is now a matter of arguing that the $\mathsf{RegFluc}$ and the intersection of the $\mathsf{LocWgtReg}$ events on the left-hand side of (43) both have probability at least $1 - \epsilon/2$.

Treating the intersection $\mathsf{LocWgtReg}$ event first, we now set the value of R equal to $1 + C_+\big(\log \epsilon^{-1}\big)^{1/3}$ where C_+ is a further positive parameter on which we will impose certain lower bounds.

Let $u, v \in \mathbb{R}$ satisfying $|u| \le R$ and $|v| \le 1$ be given. We apply Corollary 6.3 with $\mathbf{x} = u$, $\mathbf{y} = v$, $\boldsymbol{\epsilon} = \epsilon$ and $\mathbf{R} = C_-\big(\log \epsilon^{-1}\big)^{2/3}$ to find that

$$\mathbb{P}\left(\neg\, \mathsf{LocWgtReg}^{([v,v+\epsilon],1)}_{n;([u,u+\epsilon],0)}\left(\epsilon, C_-\big(\log \epsilon^{-1}\big)^{2/3} \right) \le 10032\, C\epsilon^{c_1 2^{-22-1/2} C_-^{3/2}}.$$

Since $|u| \leq R = 1 + C_+\left(\log \epsilon^{-1}\right)^{1/3}$ and $|v| \leq 1$, this application may be made provided that $\epsilon \in (0, 2^{-4}]$, $n \geq 10^{32}c^{-18}$, $C_+\left(\log \epsilon^{-1}\right)^{1/3} + 2 \leq \epsilon^{-1/2} \wedge 2^{-2}3^{-1}cn^{1/18}$, and $C_-\left(\log \epsilon^{-1}\right)^{2/3} \in \left[2 \cdot 10^4, 10^3 n^{1/18}\right]$. Thus, it may be made for $\epsilon > 0$ sufficiently small, and with n exceeding an ϵ-determined level whose order is $\left(\log \epsilon^{-1}\right)^{12}$.

Allowing u and v to vary over $\epsilon\mathbb{Z} \cap [-R, R]$ and $\epsilon\mathbb{Z} \cap [-1, 1]$, the probability that any of the LocWgtReg events so indexed fails is seen to be at most

$$\left(2R\epsilon^{-1} + 1\right)\left(2\epsilon^{-1} + 1\right) \cdot 10032\, C\epsilon^{c_1 2^{-22-1/2}C_-^{3/2}} \tag{44}$$

and thus at most $\epsilon/2$ since C_- satisfies $c_1 2^{-22-1/2}C_-^{3/2} - 2 > 1$, and $\epsilon > 0$ is small enough.

Lemma 6.2 shows that the failure probability of the RegFluc event is governed by a similar bound. Indeed, setting $\mathbf{R} = R - 1$ in the lemma, we see that

$$\mathbb{P}\left(\neg\, \mathsf{RegFluc}_{n;(*:f,0)}^{(\{-1,1\},1)}(R-1)\right) \leq 38(R-1)C \exp\left\{-2^{-6}c(R-1)^3\left(2^{-1/2} - 2^{-1}\right)^{3/2}\right\}, \tag{45}$$

provided that $n \geq c^{-18}\max\left\{(\Psi_2 + 1)^9, 10^{23}\Psi_1^9, 3^9\right\}$,

$$R \geq 1 + \max\left\{39\Psi_1, 5, 3c^{-3}, 2\left((\Psi_2 + 1)^2 + \Psi_3\right)^{1/2}\right\},$$

and $R - 1 \leq 6^{-1}cn^{1/9}$.

Recalling that $R = 1 + C_+\left(\log \epsilon^{-1}\right)^{1/3}$, we see that, in essence since $2^{-6}cC_+^3\left(2^{-1/2} - 2^{-1}\right)^{3/2}$ exceeds one, and for $\epsilon > 0$ small enough,

$$\mathbb{P}\left(\neg\, \mathsf{RegFluc}_{n;(*:f,0)}^{(\{-1,1\},1)}(R)\right) \leq \epsilon/2. \tag{46}$$

We infer then from (43) that Lemma 6.4 holds. $\qquad\square$

Proof of Theorem 1.3. Set $X_n : [-1, 1] \to \mathbb{R}$, $X_n(y) = \mathsf{Wgt}_{n;(*:f,0)}^{(y,1)}$, and let $g : (0, 1) \to (0, \infty)$ be specified via Lemma 6.4. Setting $c' = 10^{-29/12}c^{-3/2}C_+^{3/2}C_-^{3/2}$, this lemma may be used to show that

$$\mathbb{P}\left(\omega_{[-1,1],\rho}\left(y \to X_n(y)\right) \leq g(2\rho)\ \forall \rho \in \left(e^{-c'n^{1/12}}, 2^{-j}\right)\right) \geq 1 - 2^{1-j} \tag{47}$$

whenever $j \in \mathbb{N}$ satisfies $2^{-j} \leq \epsilon_0$ where $\epsilon_0 = C_+^{-6}\exp\left\{-2^{3/2}10^6\right\}C^{-2}$ (which is the upper bound on ϵ in Lemma 6.4). Indeed, this bound is obtained by noting that g is increasing on the interval $\left(0, e^{-4/3}\right)$ and $\epsilon_0 \leq e^{-4/3}$, and applying Lemma 6.4 on decreasing dyadic scales $\epsilon = 2^{-j}, 2^{-j-1}, \cdots$ for as long as the lemma's hypothesis $\epsilon \geq \exp\left\{-c'n^{1/12}\right\}$ permits.

Define the random variable $\zeta = \zeta_n \in \left[e^{-c'n^{1/12}}, \epsilon_0\right]$ to be the maximal value on this interval such that, for all $\rho \in \left(e^{-c'n^{1/12}}, \zeta\right)$, $\omega_{[-1,1],\rho}\left(y \to X_n(y)\right) \leq g(2\rho)$; if no such value exists, set $\zeta = 0$. We see from (47) that $\mathbb{P}\left(\zeta \leq s\right) \leq 4s$ for $s \in (e^{-c'n^{1/12}}, \epsilon_0]$.

We now fix $y, z \in [-1, 1]$ with $y + 2e^{-c'n^{1/12}} < z \leq y + e^{-1}$. Define $K \in \mathbb{N}$ to be the random integer $\lceil(z - y)\zeta^{-1}\rceil$. Setting $h = (z - y)K^{-1}$, it is readily verified that $h \in (e^{-c'n^{1/12}}, \zeta]$, and so we find

$$\begin{aligned}\left|X_n(z) - X_n(y)\right| &\leq \sum_{k=0}^{K-1}\left|X_n\left(y + (k+1)h\right) - X_n\left(y + kh\right)\right| \leq Kg(2h) \\ &\leq 2^{3/2}KC_-h^{1/2}\left(\log h^{-1}\right)^{2/3}.\end{aligned} \tag{48}$$

In the case that $K = 1$, we may now apply $h \leq \zeta \leq e^{-4/3}$ in order to learn that $\left|X_n(z) - X_n(y)\right|$ is at most $2^{3/2} K C_- \zeta^{1/2} \left(\log \zeta^{-1}\right)^{2/3}$.

Now suppose that $K \geq 2$. Distinctive to this case is the bound $h \geq \zeta/2$, which follows from $(z - y)\zeta^{-1} > 1$. Before we use this bound, note that, since $Kh = (z - y)$, the quantity (48) equals

$$2^{3/2} C_- (z - y)^{1/2} h^{-1/2} \left(\frac{\log h^{-1}}{\log(z - y)^{-1}}\right)^{2/3} (z - y)^{1/2} \left(\log(z - y)^{-1}\right)^{2/3}$$

and thus, in view of $z - y \leq e^{-1}$ and $\zeta/2 \leq h \leq e^{-4/3}$, may be bounded above by

$$2^{3/2} C_- h^{-1/2} \left(\log h^{-1}\right)^{2/3} (z - y)^{1/2} \left(\log(z - y)^{-1}\right)^{2/3}$$

$$\leq \quad 2^{3/2} C_- (\zeta/2)^{-1/2} \left(\log(\zeta/2)^{-1}\right)^{2/3} (z - y)^{1/2} \left(\log(z - y)^{-1}\right)^{2/3} .$$

Further using $\zeta \leq 1/2$, we find that $2^{3/2 + 1/2 + 2/3} C_- \zeta^{-1/2} \left(\log \zeta^{-1}\right)^{2/3} (z - y)^{1/2} \left(\log(z - y)^{-1}\right)^{2/3}$ serves as an upper bound on $\left|X_n(z) - X_n(y)\right|$ in the case that $K \geq 2$. Whether this case applies, or rather $K = 1$, we see that the random variable

$$S_n := \sup \left\{ \left|X_n(z) - X_n(y)\right| (z - y)^{-1/2} \left(\log(z - y)^{-1}\right)^{-2/3} : -1 \leq y, z \leq 1, \, 2e^{-c'n^{1/12}} < z - y \leq e^{-1} \right\}$$

is bounded above by $2^{8/3} C_- \zeta^{-1/2} \left(\log \zeta^{-1}\right)^{2/3}$. Recalling that $\mathbb{P}(\zeta \leq s) \leq 4s$ for $s \in (e^{-c'n^{1/12}}, \epsilon_0]$, we see that, for such s,

$$\mathbb{P}\left(S_n \geq 2^{3/2} C_- s^{-1/2} \left(\log s^{-1}\right)^{2/3}\right) \leq 4s .$$

Set $r = 2^{8/3} C_- s^{-1/2} \left(\log s^{-1}\right)^{2/3}$. We *claim* that $r \geq 2^{10/3} C_- e$ implies that $s \leq d^{-2} r^{-2} \left(\log r\right)^{4/3}$. Indeed, setting $h = s^{-1/2}$ and $K = dr$, with $d = 2^{-10/3} C_-^{-1}$, we have $K = h(\log h)^{2/3}$. We also have $K \geq e$, and this implies $h \leq K$, whence $h \geq K(\log K)^{-2/3}$ and also $s \leq d^{-2} r^{-2} \left(\log(dr)\right)^{4/3}$. Since $d \leq 1$, we have our claim. Furthermore, the condition that $s \leq \epsilon_0$ is ensured when $d^{-2} r^2 (\log r)^{4/3} \leq \epsilon_0$, and two omitted lines of working imply that $r \geq 2^{1/2} d^{-3/2} \epsilon_0^{-3/4}$ is enough to ensure the latter.

That is, setting $r_0 = r_0(\overline{\Psi}) = 2^{10/3} C_- e \vee 2^{11/2} C_-^{3/2} C_+^{9/2} C^{3/2} \exp\left\{3 \cdot 2^{-1/2} \cdot 10^6\right\}$, we have found that $r \geq r_0$ implies that

$$\mathbb{P}(S_n \geq r) \leq 2^{26/3} C_-^2 r^{-2} \left(\log r\right)^{4/3} \vee 4e^{-c'n^{1/12}} .$$

Noting that $2^{26/3} C_-^2 \leq 2^{43} c_1^{-4/3}$, and $c_1 = 2^{-5/2} c \wedge 8^{-1} \geq 2^{-3} c$ in view of $c \geq 1$, completes the proof of Theorem 1.3. $\qquad\square$

6.2. Deriving Theorem 1.4.

Proof of Theorem 1.4(1). The result is Lemma 2.2(3).

Proof of Theorem 1.4(2). Let $f \in \mathcal{I}_{\overline{\Psi}}$ be given. From [Bil99, Theorem 8.2], the sequence of probability measures $\{\nu_{n;(*:f,0)}^{([-1,1],1)} : n \in \mathbb{N}\}$ is tight if, first, the one-point distribution is tight, in the sense that for all $\eta > 0$, there exists $K > 0$ such that, for all $n \in \mathbb{N}$,

$$\mathbb{P}\left(\left|\mathsf{Wgt}_{n;(*:f,0)}^{(0,1)}\right| \leq K\right) \geq 1 - \eta; \tag{49}$$

and, second, if, for each $\epsilon > 0$ and $\eta > 0$, there exist $\rho > 0$ and $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$,

$$\mathbb{P}\left(\mathsf{EquiCty}_{n;(*:f,0)}^{[-1,1]}(\rho, \epsilon)\right) \geq 1 - \eta . \tag{50}$$

Moreover, if a choice of $n_0 = n_0(\epsilon, \eta)$ such that (49) and (50) hold whenever $n \geq n_0$ may be made independently of $f \in \mathcal{I}_{\overline{\Psi}}$, then the collection of measures $\{\nu_{n;(*:f,0)}^{([-1,1],1)} : n \in \mathbb{N}\}$ is $\mathcal{I}_{\overline{\Psi}}$-uniformly tight, where the indexing variable is $f \in \mathcal{I}_{\overline{\Psi}}$. A little work is needed to use the proof of [Bil99, Theorem 8.2] to establish this last assertion. We need to understand that, if the two bounds (49) and (50) hold whenever $n \geq n_0(\epsilon, \eta)$, we are able to assert that the same bounds also hold whenever $n \geq n_0$ where the new selection of n_0 is made merely as a function of $\overline{\Psi}$. For this, what is needed is that, for a *given* value of n that exceeds an $\overline{\Psi}$-determined level, these two bounds may be asserted with the parameters K and η being selected independently of $f \in \mathcal{I}_{\overline{\Psi}}$. We omit this fact's proof, but mention that the essence of the derivation lies in the argument for Lemma 2.2(1) and (3).

Theorem 1.4(2) thus follows from equicontinuity Lemma 6.4 and the next *uniform boundedness* lemma. □

Let $f \in \mathcal{I}_{\overline{\Psi}}$. For $K > 0$, define the event

$$\mathsf{UnifBd}_{n;(*:f,0)}^{[-1,1]}(K) = \left\{ \sup_{y \in [-1,1]} \left| \mathsf{Wgt}_{n;(*:f,0)}^{(y,1)} \right| \leq K \right\}.$$

Lemma 6.5. *For any $\epsilon > 0$ small enough, there exists $K = K(\epsilon, \overline{\Psi}) > 0$ such that, for all $f \in \mathcal{I}_{\overline{\Psi}}$,*

$$\mathbb{P}\left(\neg\, \mathsf{UnifBd}_{n;(*:f,0)}^{[-1,1]}(K) \right) \leq \epsilon$$

whenever n exceeds a certain constant multiple of $\left(\log \epsilon^{-1} \right)^{12}$.

Proof. We begin by arguing that, for $R \geq \Psi_2$,

$$\mathsf{RegFluc}_{n;(*:f,0)}^{(\{-1,1\},1)}(R-1) \cap \mathsf{PolyWgtReg}_{n;([-R,R],0)}^{([-1,1],1)}(R^2) \tag{51}$$
$$\subseteq \mathsf{UnifBd}_{n;(*:f,0)}^{[-1,1]}\left(R^2 + 2^{-1/2}(R+1)^2 + \max\{\Psi_3, \Psi_1(1+R)\} \right).$$

Indeed, it was noted after (43) that $\mathsf{Wgt}_{n;(*:f,0)}^{(y,1)} = \sup \left\{ \mathsf{Wgt}_{n;(x,0)}^{(y,1)} + f(x) : x \in [-R,R] \right\}$ when the event $\mathsf{RegFluc}_{n;(*:f,0)}^{(\{-1,1\},1)}(R-1)$ occurs; since $R \geq \Psi_2$, $-\Psi_3 \leq \sup_{|x| \leq R} f(x) \leq \Psi_1(1+R)$. On the event $\mathsf{PolyWgtReg}_{n;([-R,R],0)}^{([-1,1],1)}(R^2)$, $\left| \mathsf{Wgt}_{n;(x,0)}^{(y,1)} \right| \leq R^2 + 2^{-1/2}(R+1)^2$ whenever $|x| \leq R$ and $|y| \leq 1$; this proves (51).

Set $R = 1 + C_+\left(\log \epsilon^{-1} \right)^{1/3}$ as in the proof of Lemma 6.4. We now apply Corollary 2.1 with parameter settings $\mathbf{t_1} = 0$, $\mathbf{t_2} = 1$, $\mathbf{x} = -R$, $\mathbf{y} = -1$, $\mathbf{a} = \lceil 2R \rceil$, $\mathbf{b} = 2$ and $\mathbf{r} = R^2$ to find that

$$\mathbb{P}\left(\neg\, \mathsf{PolyWgtReg}_{n;([-R,R],0)}^{([-1,1],1)}(R^2) \right) \leq (2R+1) \cdot 400C \exp\left\{ -c_1 2^{-10} R^3 \right\}$$

provided that n exceeds an ϵ-determined level (which is of the order $\left(\log \epsilon^{-1} \right)^{12}$, in order that the hypothesis $\mathbf{r} \leq 4n^{1/18}$ be satisfied). This upper bound is at most $\epsilon/2$ for small enough $\epsilon > 0$, since $c_1 2^{-10} C_+^3 > 1$ holds in view of $C_+ \geq 12c^{-3}$, $c_1 \geq 2^{-5/2}c$ and $c \leq 1/2$.

Set $K = R^2 + 2^{-1/2}(R+1)^2 + \max\{\Psi_3, \Psi_1(1+R)\}$. From (51), we combine the last inference with (46) to obtain Lemma 6.5. □

Proof of Theorem 1.4(3). Let $\nu \in \mathsf{WLP}_{\overline{\Psi}}$. For a sequence $\{f_n : n \in \mathbb{N}\}$ of elements of $\mathcal{I}_{\overline{\Psi}}$, set $\nu_n = \nu_{n;(*:f_n,0)}^{([-1,1],1)}$. Then, for some such sequence of functions, and along a certain subsequence of n, the sequence ν_n converges to ν weakly.

Recall that r is a positive parameter that is at least r_0. For $\epsilon > 0$ and $m \in \mathbb{N}$, the set

$$\left\{ h \in \mathcal{C} : \sup_{\substack{y,z \in [-1,1], \\ 2\exp\{-c'm^{1/12}\} < z-y < e^{-1}}} \frac{\left| h(z) - h(y) \right|}{(z-y)^{1/2}\left(\log(z-y)^{-1}\right)^{2/3}} > r - \epsilon \right\}$$

is open in $\mathcal{C}$. Recall the notation that X is ν-distributed. Applying the Portmanteau theorem along the convergent subsequence of ν_n, we learn that

$$\nu \left(\sup_{\substack{y,z \in [-1,1], \\ 2\exp\{-c'm^{1/12}\} < z-y < e^{-1}}} \frac{\left| X(z) - X(y) \right|}{(z-y)^{1/2}\left(\log(z-y)^{-1}\right)^{2/3}} > r - \epsilon \right)$$

is at most the limit infimum along the concerned subsequence of $n \in \mathbb{N}$ of the left-hand side of (3) when m replaces n in the subscripted lower bound on $z - y$; f_n replaces f; and $> r - \epsilon$ replaces $\geq r$. We then apply Theorem 1.3, consider $m \to \infty$ and then $\epsilon \searrow 0$ to obtain Theorem 1.4(3). $\qquad\square$

APPENDIX A. CALCULATIONAL DERIVATIONS

We mentioned in Subsection 2.1.2 that, during the proofs of our results, we have taken care to record the hypotheses needed in order to invoke results needed along the way. Clearly, in each proof, it is necessary that the hypotheses of the result being proved apply all the conditions so invoked. What we have not done during the course of the proofs is to justify that this is the case. In this appendix, we provide these justifications, which we call *calculational derivations.*

There are five results where this work needs to be done. We present our working for the results in the order in which the proofs have appeared. After a first section, about which more momentarily, we treat in the consecutive sections of this appendix Proposition 1.5, Proposition 5.1, Theorem 1.1, Lemma 6.2 and Lemma 6.4.

In the typical calculational derivation, we begin by recording all of the conditions invoked during the proof of the result in question. Some analysis then follows, which sometimes involves introducing further conditions, whose role is to imply several of the needed hyptoheses, with a view to producing a simplified list of conditions that imply the complete list of needed hypotheses. Sometimes after a little further simplification, we produce a final list of conditions, which coincide with the hypotheses set of the result in question.

There are a few notational devices that we will use in the derivations. Conditions are given square bracketed names such as [1] or [n4], shown on the left. These names may be recycled from one calculational derivation to the next. The notation [1, n4] means 'conditions [1] and [n4]'. Implication is denoted by a right arrow, so that [1, n4] → [r3] means 'conditions [1] and [n4] imply condition [r3]'.

The notation [n2, 4, 5] is used as a shorthand for [n2, n4, n5]. This meaning would be ambiguous, were there named conditions [4] or [5], but we employ this condition only when the meaning is unambiguous.

Before these calculational derivations are given, a first section gives details that were deferred from the proof of Proposition 1.5 in Section 4.

A.1. **Proposition 1.5: reducing the odd case to the even.** Here we give a detailed version of a derivation merely outlined in Section 4, showing Proposition 1.5 when $n \in \mathbb{N}$ is supposed to be odd. We handle this case by reducing to that where n is even, and it is with this aim in mind that we indicated at the start of the proof of Proposition 1.5 that the statement proved in the case where n is even is stronger than what is asserted by the proposition for general n. We now make two further comments about the strengthened form when $n \in \mathbb{N}$ is even.

First, if we are to prove the odd version of the proposition under the stated condition that $n \geq 10^{29} \vee 2(c/3)^{-18}$, then we need to prove the even version under the weaker condition that n be at least $\left(10^{29} \vee 2(c/3)^{-18}\right) - 1$. We do not labour this very minor point beyond noting that, in the calculational derivation for the even case given in the next section, it is readily verified that the weaker condition is indeed adequate.

Second, we will wish to make use of the even version of Proposition 1.5 when the variables u and v that appear in the supremum in (5) and the infimum in (6) vary not merely over $[0, 1]$ but over a slightly longer interval – the interval $[0, 1 + 10^{-3}]$ will do. Our proof of the even version of the proposition yields this conclusion without modification. This is because the changes in the proof that arise take the form of a slight increase in the term $2^{-1/3}$ in (19) and in the term $2^{-5/3}$ in (22); and the inequalities in these two lines absorb these changes as they stand.

Suppose then that $n \in \mathbb{N}$ is odd. The reduction of this case to the even one involves retreating to unscaled coordinates. The argument harnesses a trivial bound on unscaled energy, namely that

$$M^1_{(z_1,0)\to(z_2,m)} \leq M^1_{(z_1,0)\to(z_2,m+1)} \tag{52}$$

whenever $m \in \mathbb{N}$, and $z_1, z_2 \in \mathbb{R}$ satisfy $z_1 \leq z_2$. In the present application, as well as $n \in \mathbb{N}$ are given $x, y \in \mathbb{R}$; and $u, v \in [0, 1]$.

We begin by deriving (5). Set

$$S = \sup_{u,v\in[0,1]} \left(\mathsf{Wgt}^{(y+v,1)}_{n;(x+u,0)} + Q(y + v - x - u) \right),$$

so that the sought bound is an upper bound on $\mathbb{P}(S \geq t)$.

We introduce new variables (x', u', y', v') specified by

$$2n^{2/3}x = 2(n+1)^{2/3}x' \quad ; \quad 2n^{2/3}u = 2(n+1)^{2/3}u' \ ;$$
$$2n^{2/3}y = 1 + 2(n+1)^{2/3}y' \quad ; \quad \text{and} \ \ 2n^{2/3}v = 2(n+1)^{2/3}v' \ .$$

Note for later use that

$$y + v - x - u = \left(1 + n^{-1}\right)^{2/3}\left(y' + v' - x' - u'\right) + 2^{-1}n^{-2/3} . \tag{53}$$

We now express the weight and parabolic terms appearing in S in terms of these variables.

Lemma A.1.

(1) *The weight term* $\mathsf{Wgt}^{(y+v,1)}_{n;(x+u,0)}$ *is at most* $\left(1 + n^{-1}\right)^{1/3}\mathsf{Wgt}^{(y'+v',1)}_{n+1;(x'+u',0)} + 2^{-1/2}n^{-1/3}$.

(2) *The parabolic term* $Q(y + v - x - u)$ *is at most*

$\left(1 + n^{-1}\right)^{4/3}Q(y' + v' - x' - u') + 2^{-1/2}\left(1 + n^{-1}\right)^{2/3}(y' + v' - x' - u')n^{-2/3} + 2^{-5/2}n^{-4/3}$.

Proof. (1). We apply the unscaled inequality (52) with $m = n$, $z_1 = 2n^{2/3}(x + u)$ and $z_2 = n + 2n^{2/3}(y+v)$, and rewrite the left-hand side of the resulting bound by means of (1) with $\mathbf{x} = x + u$ and $\mathbf{y} = y + v$ to obtain

$$\mathsf{Wgt}_{n;(x+u,0)}^{(y+v,1)} \leq 2^{-1/2}n^{-1/3}\left(M^1_{\left(2n^{2/3}(x+u),0\right) \to \left(n+2n^{2/3}(y+v),n+1\right)} - 2n - 2n^{2/3}(y + v - x - u)\right). \quad (54)$$

The M^1 term on the right-hand side of (54) may then be written

$$M^1_{\left(2(n+1)^{2/3}(x'+u'),0\right) \to \left(n+1+2(n+1)^{2/3}(y'+v'),n+1\right)}$$

and thus, via (1) with $\mathbf{n} = n + 1$, $\mathbf{x} = x' + u'$ and $\mathbf{y} = y' + v'$, it may also be expressed in the form

$$2^{1/2}(n+1)^{1/3}\mathsf{Wgt}_{n+1;(x'+u',0)}^{(y'+v',1)} + 2(n+1) + 2(n+1)^{2/3}\left(y' + v' - x' - u'\right).$$

Making use of $2(n+1)^{2/3}(y'+v') - 2n^{2/3}(y+v) = -1$, we return to (54) to obtain Lemma A.1(1).

(2). This follows from $Q(z) = 2^{-1/2}z^2$ and (53). $\qquad\qquad\qquad\square$

By the definition of S, the two parts of Lemma A.1, and a little rearrangement, the occurrence of the event of interest $\{S \geq t\}$ entails that

$$\sup\left(\mathsf{Wgt}_{n+1;(x'+u',0)}^{(y'+v',1)} + Q(y' + v' - x' - u')\right) \geq (t + \eta)\alpha$$

where the supremum is taken by varying the original variables u, v over $[0, 1]$, where α is set equal to $\left(1 + n^{-1}\right)^{-1/3}$ and where η equals

$$-2^{-1/2}n^{-2/3}\left(1 + n^{-1}\right)^{2/3}(y' + v' - x' - u') - 2^{-1/2}n^{-1/3}$$
$$-2^{-5/2}n^{-4/3} - \left(1 + n^{-1}\right)^{1/3}n^{-1}Q(y' + v' - x' - u').$$

We now make a claim; use the claim to close out the derivation of (5); and prove the claim. The claim is that

$$t + \eta \geq 2^{-1/48}t \text{ and } \alpha \geq 2^{-1/48}. \quad (55)$$

By the claim, we see that $\mathbb{P}(S \geq t)$ is at most

$$\mathbb{P}\left(\sup\left(\mathsf{Wgt}_{n+1;(x'+u',0)}^{(y'+v',1)} + Q(y' + v' - x' - u')\right) \geq 2^{-1/24}t\right),$$

where the supremum is taken over $u, v \in [0, 1]$. However, the supremum may instead be taken over $u', v' \in [0, 1]$ without altering the validity of this assertion. This is because, in view of the specification of (u', v') in terms of (u, v), variation of (u, v) over $[0, 1]^2$ corresponds to variation of (u', v') over a subset of $[0, 1]^2$. When the supremum over (u', v') is considered, a probability is obtained to which we may apply the already derived version of (5) whose parameter $\mathbf{n}$ is even; recall that this form of the bound is strengthened in this case, as specified at the start of the proof of Proposition 1.5. We make this application with the choices $\mathbf{n} = n + 1$, $\mathbf{x} = \left(1 + n^{-1}\right)^{-2/3}x$, $\mathbf{y} = \left(1 + n^{-1}\right)^{-2/3}y - 2^{-1}(n + 1)^{-2/3}$ and $\mathbf{t} = 2^{-1/24}t$. By doing so, we obtain (5) in the present case of odd n. Note that, since we suppose that $t \geq 34$, the choice $\mathbf{t}$ is at least 33, as required by the hypotheses of the even case. The needed upper bound on $|\mathbf{y} - \mathbf{x}|$ follows from the hypothesised upper bound on $|x - y|$ as well as $n \geq 10^{29}$.

For the derivation of (5), it remains only to verify the two assertions of the claim (55). Regarding the latter, note that, since $\alpha = \left(1 + n^{-1}\right)^{-1/3}$, the claimed condition that $\alpha \geq 2^{-1/48}$ is equivalent to $n \geq (2^{1/16} - 1)^{-1}$, and the latter is implied by $n \geq 23$ and thus also by the hypothesis $n \geq 10^{29}$

of Proposition 1.5. Note that the former of the two claimed statements, that $t + \eta \geq 2^{-1/48}t$, is implied by $|\eta| \leq (1 - 2^{-1/48})t$ and thus also by $|\eta| \leq t/70$. Since the parameter t is supposed in Proposition 1.5 to be at least 34, the next lemma concludes the proof of the claim and thus of (5).

Lemma A.2. *When* $t \geq 34$, $|\eta| \leq t/70$.

Proof. It is enough to argue that $|\eta| \leq 1/3$. To derive this, we begin by arguing that

$$\left| y' + v' - x' - u' \right| \leq cn^{1/18} \,.$$

This bound is derived by noting that

$$\begin{aligned}
\left| y' + v' - x' - u' \right| &\leq \left| y + v - x - u \right| + 2^{-1}n^{-2/3} \leq 6^{-1}cn^{1/18} + 1 + 2^{-1}n^{-2/3} \\
&\leq 6^{-1}cn^{1/18} + 2 \leq (1/6 + 2/3)cn^{1/18} \leq cn^{1/18} \,,
\end{aligned}$$

where the successive displayed bounds are due to (53); the hypothesis that $|x - y| \leq 6^{-1}cn^{1/8}$, and $u, v \in [0, 1]$; $n \geq 1$; the hypothesis that $n \geq (3c^{-1})^{18}$ in the guise $2 \leq 2/3 \cdot cn^{1/18}$; and with the last bound being trivial.

We find then that the quantity η verifies

$$|\eta| \leq 2^{-1/2}n^{-2/3}(1 + n^{-1})^{2/3}cn^{1/18} + 2^{-1/2}n^{-1/3} + 2^{-5/2}n^{-4/3} + (1 + n^{-1})^{1/3}n^{-1}2^{-1/2}c^2n^{1/9} \,.$$

Since $c \leq 1$, each of the four right-hand terms is at most one-twelfth when n satisfies the hypothesised bound $n \geq 10^{29}$. This completes the proof of Lemma A.2. $\qquad\square$

We now turn to deriving (6) in the case that $n \in \mathbb{N}$ is odd. Recall that given are $x, y \in \mathbb{R}$ and $u, v \in [0, 1]$. We introduce variables $\tilde{x}$, $\tilde{u}$, $\tilde{y}$ and $\tilde{v}$ according to

$$\begin{aligned}
2n^{2/3}\tilde{x} &= 2(n + 1)^{2/3}x \quad ; \quad 2n^{2/3}\tilde{u} = 2(n + 1)^{2/3}u \ ; \\
2n^{2/3}\tilde{y} &= 1 + 2(n + 1)^{2/3}y \quad ; \quad \text{and} \ \ 2n^{2/3}\tilde{v} = 2(n + 1)^{2/3}v \,.
\end{aligned}$$

Replacing $x' \to x$ and $x \to \tilde{x}$ in Lemma A.1(1), and likewise for the variables labelled by u, y and v; replacing $n \to n - 1$; and rearranging, we obtain

$$\left(1 + (n - 1)^{-1}\right)^{-1/3}\left(\mathsf{Wgt}^{(\tilde{y}+\tilde{v},1)}_{n-1;(\tilde{x}+\tilde{u},0)} - 2^{-1/2}(n - 1)^{-1/3}\right) \leq \mathsf{Wgt}^{(y+v,1)}_{n;(x+u,0)} \,.$$

Set

$$I = \inf_{u,v \in [0,1]} \left(\mathsf{Wgt}^{(y+v,1)}_{n;(x+u,0)} + Q(y + v - x - u)\right),$$

so that the bound that we now seek is an upper bound on $\mathbb{P}(I \leq -t)$.

To express a lower bound on I using the tilde variables, note that

$$y + v - x - u = \left(1 + (n - 1)^{-1}\right)^{2/3}(\tilde{y} + \tilde{v} - \tilde{x} - \tilde{u}) - 2^{-1}n^{-2/3}$$

and

$$Q(y + v - x - u) = \left(1 + n^{-1}\right)^{4/3}Q(\tilde{y} + \tilde{v} - \tilde{x} - \tilde{u}) + 2^{-1/2}\left(1 - n^{-1}\right)^{2/3}(\tilde{y} + \tilde{v} - \tilde{x} - \tilde{u})n^{-2/3} + 2^{-5/2}n^{-4/3} \,.$$

Thus, I is bounded below by the infimum of

$$\left(1 + (n - 1)^{-1}\right)^{-1/3}\left(\mathsf{Wgt}^{(\tilde{y}+\tilde{v},1)}_{n-1;(\tilde{x}+\tilde{u},0)} - 2^{-1/2}(n - 1)^{-1/3}\right) + \left(1 + n^{-1}\right)^{4/3}Q(\tilde{y} + \tilde{v} - \tilde{x} - \tilde{u})$$

$$- 2^{-1/2}\left(1 - n^{-1}\right)^{2/3}(\tilde{y} + \tilde{v} - \tilde{x} - \tilde{u})n^{-2/3} + 2^{-5/2}n^{-4/3} \,,$$

where the infimum is taken by varying the original variables u, v over $[0, 1]$. We see then that the occurrence of $\{I \leq -t\}$ entails that

$$\inf_{u,v \in [0,1]} \left(\mathsf{Wgt}_{n-1;(\tilde{x}+\tilde{u},0)}^{(\tilde{y}+\tilde{v},1)} + Q(\tilde{y} + \tilde{v} - \tilde{x} - \tilde{u}) \right) \leq \left(-t + \zeta \right) \left(1 + (n-1)^{-1} \right)^{1/3}$$

where $\zeta = \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4$, with

$$\zeta_1 = 2^{-1/2}(n-1)^{-1/3}; \quad \zeta_2 = -2^{-1/2}\left(1 - n^{-1}\right)^{2/3}(\tilde{y} + \tilde{v} - \tilde{x} - \tilde{u}\,)n^{-2/3}; \quad \zeta_3 = 2^{-5/2}n^{-4/3};$$

$$\text{and } \zeta_4 = \left(\left(1 + n^{-1}\right)^{4/3} - \left(1 + (n-1)^{-1}\right)^{-1/3} \right) Q(\tilde{y} + \tilde{v} - \tilde{x} - \tilde{u})\left(1 + n^{-1}\right)^{1/3};$$

and, provided that $-t + \zeta > 0$, – a condition that we will see to be ensured – this event also entails that

$$\inf_{u,v \in [0,1]} \left(\mathsf{Wgt}_{n-1;(\tilde{x}+\tilde{u},0)}^{(\tilde{y}+\tilde{v},1)} + Q(\tilde{y} + \tilde{v} - \tilde{x} - \tilde{u}) \right) \leq -t + \zeta.$$

Lemma A.3. $|\zeta| \leq 10^{-3}t$.

Proof. We first show that

$$\left| \tilde{y} + \tilde{v} - \tilde{x} - \tilde{u} \right| \leq 3cn^{1/18}. \tag{56}$$

This bound is derived by noting that

$$\begin{aligned}
\left| \tilde{y} + \tilde{v} - \tilde{x} - \tilde{u} \right| &\leq 2^{-1}(n-1)^{-2/3} + \left(1 - n^{-1}\right)^{-2/3}\left| y + v - x - u \right| \\
&\leq 2^{-1}(n-1)^{-2/3} + 2^{2/3}\left(6^{-1}cn^{1/18} + 1\right) \leq 1 + 2^{5/3}3^{-1}cn^{1/18} \leq 3cn^{1/18},
\end{aligned}$$

where the second bound uses $|x - y| \leq 6^{-1}cn^{1/8}$, $|u - v| \leq 1$ and $n \geq 2$; the third uses $n \geq 1 + 2^{-2/3}$ and $n \geq (3c^{-1})^{18}$ in the guise $1 \leq 1/3 \cdot cn^{1/18}$; and the fourth uses $1 \leq cn^{1/18}$.

From (56), we see that $|\zeta_2| \leq 2^{-1/2}3cn^{1/18-2/3}$ and that

$$|\zeta_4| \leq \left| \left(1 + n^{-1}\right)^{4/3} - \left(1 + (n-1)^{-1}\right)^{-1/3} \right| \cdot 2^{-1/2}\left(3cn^{1/18}\right)^2.$$

To analyse the first term on this right-hand side (a term that we will label β), note that, by convexity,

$$1 \leq (1 + x)^a \leq 1 + (2^a - 1)x \qquad \text{when } a > 0 \text{ and } x \in [0, 1]; \text{ and}$$

$$1 \leq (1 - z)^{-b} \leq 1 + (2^b - 1)z \qquad \text{when } b > 0 \text{ and } z \in [0, 1/2].$$

By taking $a = 4/3$, $x = n^{-1}$, $b = 1/3$ and $z = (n-1)^{-1}$, we see that, when $n \geq 3$, $\beta \leq \max\left\{ \left(2^{4/3} - 1\right)n^{-1}, \left(2^{1/3} - 1\right)(n-1)^{-1} \right\}$. Using $n \geq 2$, we obtain $\beta \leq 4n^{-1}$. In summary,

$$\zeta_1 = 2^{-1/2}(n-1)^{-1/3}; \quad |\zeta_2| \leq 3 \cdot 2^{-1/2}cn^{-11/18}; \quad \zeta_3 = 2^{-5/2}n^{-4/3}; \quad \text{and } |\zeta_4| \leq 26n^{-8/9}.$$

Since $n \geq 10^{29}$ and $t \geq 34$, we find that each $|\zeta_i|$ is at most $10^{-4}t$. Thus, $|\zeta| \leq 10^{-3}t$, and the lemma is proved. $\qquad \square$

Combining Lemma A.3 with the inference that preceded it, we see that $\{I \leq -t\}$ entails that

$$\inf_{u,v \in [0,1]} \left(\mathsf{Wgt}_{n-1;(\tilde{x}+\tilde{u},0)}^{(\tilde{y}+\tilde{v},1)} + Q(\tilde{y} + \tilde{v} - \tilde{x} - \tilde{u}) \right) \leq -(1 - 10^{-3})t,$$

and thus also that

$$\inf_{\tilde{u},\tilde{v} \in [0,1+10^{-3}]} \left(\mathsf{Wgt}_{n-1;(\tilde{x}+\tilde{u},0)}^{(\tilde{y}+\tilde{v},1)} + Q(\tilde{y} + \tilde{v} - \tilde{x} - \tilde{u}) \right) \leq -(1 - 10^{-3})t,$$

where in the latter inference, we replaced the variables u and v in the infimum by their tilde counterparts at the expense of expanding the interval over which the variables range from $[0, 1]$ to

$[0, 1 + 10^{-3}]$. This step is permitted in view of the specification of $(\tilde{u}, \tilde{v})$ in terms of (u, v) and $(1 - n^{-1})^{2/3} \geq 1000/1001$, the latter implied by $n \geq 668$.

We now bound above the probability of the event that the preceding displayed inequality holds, by applying the even version of (6) with $\mathbf{n} = n - 1$; $\mathbf{t} = (1 - 10^{-3})t$; $\mathbf{x} = \tilde{x}$; and $\mathbf{y} = \tilde{y}$. For this application to be valid, we must confirm that

$$\left| \mathbf{x} - \mathbf{y} \right| \leq 3^{-1} 2^{-2/3} c(n-1)^{1/18}, \tag{57}$$

and this we now do. Note that $|\tilde{x} - \tilde{y}| \leq (1 - n^{-1})^{-2/3}|x - y| + 2^{-1}(n-1)^{-2/3}$. Thus,

$$
\begin{aligned}
\left| \mathbf{x} - \mathbf{y} \right| &\leq |\tilde{x} - \tilde{y}| + 10^{-1} \leq (1 - n^{-1})^{-2/3}|x - y| + 2^{-1}(n-1)^{-2/3} + c/30 \cdot n^{1/18} \\
&\leq (1 - n^{-1})^{-2/3} 6^{-1} cn^{1/18} + 11/10 \cdot c/30 \cdot n^{1/18} \leq 3^{-1} 2^{-2/3} c(n-1)^{1/18},
\end{aligned}
$$

where in the second inequality, we used $n \geq (c/3)^{-18}$; in the third, we used $|x - y| \leq 6^{-1} cn^{1/18}$, $n \geq (50)^{3/2} + 1$, and $1/100 \leq 1/100 \cdot c/3 \cdot n^{1/18}$; and in the fourth, we used $n \geq 10$. This confirms (57).

The bound on $\mathbb{P}(I \leq -t)$ that results from this application of the strengthened, even, version of (6) yields the bound (6) for odd n. This completes our discussion of the reduction of the odd version of Proposition 1.5 to its even counterpart.

A.2. **Proposition 1.5: derivation.** It is understood here that the principal argument for Proposition 1.5, is in which n is even, is being addressed. It is our task to verify that the slightly weaker hypotheses stated at the start of Section 4 are adequate for this derivation.

The proof of the proposition consists of the derivation of (6) and that of (5). The calculational derivation of the proposition treats the two pieces separately. In the first case, we follow the plan described at the beginning of Appendix A: recording of all needed hypotheses is followed by an analysis that reveals that all of these conditions are implied by the hypotheses of Proposition 1.5. In the second, we split the derivation again into two pieces, and analyse them separately.

In the derivation of (6), the following assertions are used:

$$[1] \ n/2 \geq (c/3)^{-1/18} \vee 6^{36}$$

$$[2] \ 2^{-1/3} \leq (n/2)^{1/18}$$

$$[3] \ 2^{-5/3} t = s \in [2^{3/2}, 2(n/2)^{1/18}]$$

and

$$[4] \ 2^{-1/3}(|x - y| + 1) \leq 2^{-1} c(n/2)^{1/18}$$

Thus, $[1, 2, 3, 4]$ is our list of needed conditions. Turning to analyse them, we first note that $[1] \rightarrow [2]$. Introducing a further condition

$$[5] \ n \geq 10^{29} \vee 2(c/3)^{-18},$$

we have $[5] \rightarrow [1]$.

$[3]$ is equivalent to

$$t \in [2^{19/6}, 2^{47/18} n^{1/18}]$$

and thus is implied by

$$[6] \ t \in [33, 2^{47/18} n^{1/18}]$$

since $33 \geq 2^{19/6}$.

[4] is equivalent to

$$|x - y| + 1 \leq 2^{-13/18}cn^{1/18}$$

Since [5] entails that $1 \leq 3^{-1}2^{-1/18}cn^{1/18}$, [4] is implied by [5] and

$$|x - y| \leq (2^{-13/18} - 3^{-1}2^{-1/18})cn^{1/18}$$

Since $2^{-13/18} - 3^{-1}2^{-1/18} = 0.28542\cdots \geq 0.20998\cdots = 3^{-1}2^{-2/3}$, we see that, if we write

$$[7] \quad |x - y| \leq 3^{-1}2^{-2/3}cn^{1/18}$$

then $[5, 7] \to [4]$.

In summary, $[5, 6, 7] \to [1, 2, 3, 4]$. Since $[5, 6, 7]$ are hypothesised in Proposition 1.5, we see that the hypotheses are adequate for deriving (6).

Derivation of (5).

This derivation involves a use of Proposition 3.3(1) and a use of Proposition 3.3(2). We structure the calculational derivational of (5) by analysing each of these applications in turn.

In the application of Proposition 3.3(1), the bounds needed are

$$n \geq 1 \vee (c/3)^{-18} \vee 6^{36}$$

$$1/2 \leq n^{1/18}$$

$$t/2 \in \left[2^{3/2}, 2n^{1/18}\right]$$

$$|x - z + 1/2| = |y - x + 1/2| \leq c/2 \cdot n^{1/18}$$

The first two follow directly from the hypotheses. The third follows from the hypothesised $t \in [33, 4n^{1/18}]$. For the fourth,

$$[a] \quad |x - y| + 1/2 \leq 2^{-1}cn^{1/18}$$

suffices. By hypothesis, we know $n/2 \geq (c/3)^{-18}$ or equivalently $c/6 \cdot 2^{-1/18}n^{1/18} \geq 1/2$. To show [a], it thus suffices to show that

$$|x - y| \leq c(2^{-1} - 6^{-1}2^{-1/18})n^{1/18}$$

Since we suppose that the left-hand side is at most $c \cdot 3^{-1}2^{-2/3}n^{1/18}$, this inequality holds in view of $3^{-1}2^{-2/3} \leq 2^{-1} - 6^{-1}2^{-1/18}$.

In the application of Proposition 3.3(2), we need

$$[b] \quad \left|2^{-2/3}(y - z) + 2^{-5/3}\right| = \left|2^{1/3}(y - x) + 2^{-5/3}\right| \leq c/2 \cdot (2n)^{1/9}$$

$$[c] \quad 2^{-5/3} \leq c/4 \cdot (2n)^{1/9}$$

$$[d] \quad 2^{-4/3}t - 5 \cdot 2^{-11/6} \in \left[2^{7/2}, 2(2n)^{1/3}\right]$$

$$[e] \quad 2n \geq c^{-18}$$

Note that [b] is equivalent to

$$[f] \quad |x - y| + 4^{-1} \leq 2^{-11/9}cn^{1/9}$$

When $|y - x| \geq 1$, [f] is implied by squaring [a] and using $c \leq 1$. When $|y - x| \leq 1$, [f] is implied by $5^9 2^{-7} c^{-9} \leq n$, a condition which follows from the hypothesised $n \geq 2(c/3)^{-18}$ and $c \leq 1$.

[c] is equivalent to $n \geq 4c^{-9}$ which follows by the hypothesised lower bound on n as well as $c \leq 1$.

$[d]$ follows from the hypothesised $t \in [33, 4n^{1/18}]$ via

$$2^{-4/3} \cdot 33 - 5 \cdot 2^{-11/6} = 11.69 \cdots \geq 11.31 \cdots = 2^{7/2}$$

$[e]$ follows from the hypothesised lower bound on n.

A.3. Proposition 5.1: derivation. In this proof, Proposition 3.5 is applied twice, and here we make use of $|y + u_1 - x - z| \leq 2^{-1}cn^{1/9}$ and $|x + u_1 - y - z| \leq 2^{-1}cn^{1/9}$. Since $u_1, z \in [0, 1]$, both of these are implied by

$$[1] \; |y - x| + 1 \leq 2^{-1}cn^{1/9} \, .$$

Note that $[1]$ is implied by the hypothesised inequalities that $|y - x| \leq 4^{-1}cn^{1/9}$ and $1 \leq 4^{-1}cn^{1/9}$.

We also use $n + 1 \geq 6^3 c^{-3}$, which follows from $n \geq (4/c)^9$ and $c \leq 1$.

A.4. Theorem 1.1: derivation. For this calculational derivation, in defiance of the guideline offered in the paragraphs that begin this appendix, we begin by labelling the actual hypotheses of Theorem 1.1.

These hypotheses concern parameters $n \in \mathbb{N}$, $\epsilon > 0$, $x, y \in \mathbb{R}$ and $R \geq 0$. The conditions in question are:

$$[h1] \; \epsilon \in (0, 2^{-4}]$$
$$[h2] \; n \geq 10^{32} c^{-18}$$

Let $x, y \in \mathbb{R}$ satisfy

$$[h3] \; |x - y| \leq 2^{-2} 3^{-1} cn^{1/18}$$
$$[h4] \; R \geq 10^4$$
$$[h5] \; R \leq 10^3 n^{1/18}$$

Naturally we must now argue that these conditions imply all conditions invoked during the proof of Theorem 1.1. With a few exceptions, discussed at the end of the derivation, these needed conditions are invoked in two contexts: an application of Corollary 2.1 and an application of Proposition 5.1. We discuss the two contexts in turn.

The first context is the use of Corollary 2.1 via (32). Here, the following conditions are needed:

$$[c1] \; n \geq 10^{29} \vee 2(c/3)^{-18} \, ,$$
$$[c2] \; |x - y| + 4 \leq 3^{-1} 2^{-1} cn^{1/18}$$
$$[c3] \; r_0 \geq 34$$
$$[c4] \; r_0 \leq 4n^{1/18}$$

where

$$r_0 = 2^{-9/4} S$$

and

$$R = 2^2 (1 - 2^{-1/4})^{-1} S$$

Note that $[h2, h3] \to [c1]$.

$[c2]$ is implied by

$$|x - y| \leq 2^{-1} 3^{-1} 2^{-1} cn^{1/18} \text{ and } n \geq (2^4 3c^{-1})^{18} = 2^{72} 3^{18} c^{-18}$$

the first of which is $[h3]$ and the second of which is implied by $[h2]$ since $2^{72} 3^{18} \in [10^{30}, 10^{31}]$.

[c3] expressed in terms of R is

$$[c3] \ R \geq 34 \cdot 2^{9/4} \cdot 2^2 (1 - 2^{-1/4})^{-1}$$

[c4] expressed in terms of R is

$$[c4] \ R \leq 4n^{1/18} \cdot 2^{9/4} \cdot 2^2 (1 - 2^{-1/4})^{-1}$$

The second of the two contexts is the application of Proposition 5.1. For this purpose, we must have:

$$[p1] \ n \geq (4/c)^9$$
$$[p2] \ |x - y| \leq c/4 \cdot n^{1/9}$$
$$[p3] \ K_0 \geq 4$$

Note that $[h2] \to [p1]$ since $c \leq 1$. Also $[h3] \to [p2]$.

In $[p3]$, the value of K_0 ranges over infinitely many values as Proposition 5.1 is repeatedly applied, but in each case, this value satisfies

$$K_0 \geq 2^{k/4 - 1/2} S$$

where $k \in \mathbb{N}$. Thus, it is enough to ensure that $[p3]$ is satisfied to impose that

$$S \geq 2^{2 + 1/2}$$

or equivalently

$$[*] \ R \geq 2^{5/2} \cdot 2^2 (1 - 2^{-1/4})^{-1}$$

which is implied by $[c3]$.

Returning to $[c3, c4]$, note that $(1 - 2^{-1/4})^{-1} = 6.285 \cdots$ and $(1 - 2^{-1/4})^{-1} \cdot 33 \cdot 2^{9/4 + 2} = 3946. \cdots$, so that $[c3]$ is implied by

$$R \geq 10^4$$

which is $[h4]$, while $[c4]$ is implied by

$$R \leq 10^3 n^{1/18}$$

which is $[h5]$.

During the proof of the theorem, we also use that S is bounded below by $2^{7/2}$, $2^{5/2 + 2} = 2^{9/2}$ and 1. Thus, we set $S = 2^{9/2}$. All of these bounds follow from the variant of $[*]$ where the right-hand side is multiplied by four, and this bound follows from $[c3]$.

A.5. **Lemma 6.2: derivation.** We begin this longer derivation by gathering together the collection of conditions that were used during the proof of this lemma.

First recall that the parameters on which conditions are imposed are $n \in \mathbb{N}$, $R > 0$, and the three positive components of $\overline{\Psi}$.

The penultimate inequality of the first displayed set of the equations in the proof makes use of

$$R^2/4 \geq 2^{-1/2} (\Psi_2 + 1)^2 + \Psi_3 .$$

When $\text{Reg}(2)$ with parameter choices $\mathbf{z} = x_0$ and $\mathbf{s} = R^2/4$ is applied to the ensemble $\text{Nr}\mathcal{L}_{n;0}^{\downarrow;(-1,1)}$, the hypotheses

$$n \geq c^{-9} (\Psi_2 + 1)^9$$

and

$$R^2/4 \in [1, n^{1/3}]$$

are used.

Next, it is supposed that
$$2R \leq 3^{-1}cn^{1/9}.$$

When Proposition 3.3(3) is applied to $\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(-1,1)}$, use is made of the conditions:
$$1 \leq 3^{-1}cn^{1/9}$$
$$\Psi_1 \leq \left(2^{-1/2} - 2^{-5/2} - 2^{-1}\right)(c/9)n^{1/9}$$
$$\Psi_1 \leq 5 \cdot 2^{3/2}c/3 \cdot n^{1/9}$$
$$n + 1 \geq 2^{45/4}c^{-9}$$

Next, the condition
$$R \geq 1 \vee 39\Psi_1$$
is used to assert that
$$2^{-1}R^2\left(2^{2j-1/2} - 1/2\right) \geq \Psi_1\left(2 + 2^{j+1}R\right)$$
for each $j \geq 0$.

When Proposition 3.3(2) is applied to the ensemble $\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(-1,1)}$, the conditions
$$3 \cdot 2^j R \leq cn^{1/9},$$
$$2^{j+1}R \leq cn^{1/9},$$
$$2^{-1}R^2\left(2^{2j-1/2} - 1/2\right) \leq 2n^{1/3}$$
$$2^{-1}R^2\left(2^{2j-1/2} - 1/2\right) \geq 2^{7/2}$$
and
$$n \geq c^{-18}$$
are used. The first three of these conditions are valid when $j \in [\![0, k]\!]$ in light of the condition $2^{k+1}R \leq 3^{-1}cn^{1/9}$ that defines k (and $c \leq 1$ is used as well); indeed, they are also valid when $j = k+1$, a fact which is used immediately after in this proof. For this reason, none of these three conditions impose any additional requirements, and they will be omitted henceforth in our analysis.

Finally, the following bounds are needed:
$$R \geq (\log 4)^{1/3}2^2c^{-3}\left(\left(2^{3/2} - 2^{-1}\right)^{3/2} - \left(2^{-1/2} - 2^{-1}\right)^{3/2}\right)^{-1/3}$$
$$R \leq \left(2^{-1/2} - 2^{-1}\right)^{-1/2}2^{3/4}3^{-1}cn^{1/9}$$
and
$$R \geq 1.$$

Our analysis begins by confirming that the condition $R \geq 1 \vee 39\Psi_1$ indeed suffices for its role above.

Claim. The condition
$$R \geq 1 \vee 39\Psi_1$$
implies that
$$2^{-1}R^2\left(2^{2j-1/2} - 1/2\right) \geq \Psi_1\left(2 + 2^{j+1}R\right)$$
for each $j \geq 0$.

Proof. We claim that $R \geq 1 \vee 39\Psi_1$ implies that
$$2^{-1}R^2\left(2^{2j-1/2} - 1/2\right) \geq \Psi_1\left(2 + 2^{j+1}R\right) \qquad \text{for each } j \geq 0. \tag{58}$$

The displayed inequality is equivalent to

$$R^2\big(2^{2j+1/2} - 1\big) \geq 2^3 \Psi_1\big(1 + 2^j R\big)$$

for $j \geq 0$. Since $2^{2j+1/2} - 1 \geq 2^{2j}\big(2^{1/2} - 1\big)$ for $j \geq 0$ and $1 + 2^j R \leq 2^{j+1}R$ (this due to $R \geq 1$), the last displayed bound is implied by

$$R^2 2^j \big(2^{1/2} - 1\big) \geq 2^4 \Psi_1 R.$$

This holds for all $j \geq 0$ provided that $R \geq 2^4\big(2^{1/2} - 1\big)^{-1}\Psi_1$. Since $2^4\big(2^{1/2} - 1\big)^{-1} = 38.62\cdots$, it is also implied by $R \geq 39\Psi_1$.

Thus, $R \geq 1 \vee 39\Psi_1$ implies (58), as we claimed. □

Next we write the set of needed conditions in a list:

$$R^2/4 \geq 2^{-1/2}(\Psi_2 + 1)^2 + \Psi_3\,.$$

$$n \geq c^{-9}(\Psi_2 + 1)^9$$

$$R^2 \geq 4$$

$$R^2 \leq 4n^{1/3}$$

$$2R \leq 3^{-1}cn^{1/9}\,.$$

$$1 \leq 3^{-1}cn^{1/9}$$

$$\Psi_1 \leq \big(2^{-1/2} - 2^{-5/2} - 2^{-1}\big)(c/9)n^{1/9}$$

$$\Psi_1 \leq 5 \cdot 2^{3/2}c/3 \cdot n^{1/9}$$

$$n + 1 \geq 2^{45/4}c^{-9}$$

$$R \geq 1 \vee 39\Psi_1$$

$$2^{-1}R^2\big(2^{1/2} - 2^{-1}\big) \geq 2^{7/2}$$

$$n \geq c^{-18}\,.$$

$$R \geq (\log 4)^{1/3}2^2 c^{-3}\big((2^{3/2} - 2^{-1})^{3/2} - (2^{-1/2} - 2^{-1})^{3/2}\big)^{-1/3}$$

$$R \leq (2^{-1/2} - 2^{-1})^{-1/2}2^{3/4}3^{-1}cn^{1/9}$$

$$R \geq 1\,.$$

We now split this list into categories:

- Lower bounds on R.

- Upper bounds on R.

- Lower bounds on n without dependence on R.

Lower bounds on R.

$$[L1] \; R^2/4 \geq 2^{-1/2}(\Psi_2 + 1)^2 + \Psi_3 \,.$$
$$[L2] \; R^2 \geq 4$$
$$[L3] \; R \geq 39\Psi_1$$
$$[L4] \; 2^{-1}R^2(2^{1/2} - 2^{-1}) \geq 2^{7/2}$$
$$[L5] \; R \geq (\log 4)^{1/3}2^2 c^{-3}\big((2^{3/2} - 2^{-1})^{3/2} - (2^{-1/2} - 2^{-1})^{3/2}\big)^{-1/3}$$
$$[L6] \; R \geq 1 \,.$$

Upper bounds on R.

$$[U1] \; R^2 \leq 4n^{1/3}$$
$$[U2] \; 2R \leq 3^{-1}cn^{1/9} \,.$$
$$[U3] \; R \leq (2^{-1/2} - 2^{-1})^{-1/2}2^{3/4}3^{-1}cn^{1/9}$$

Lower bounds on n.

$$[n1] \; n \geq c^{-9}(\Psi_2 + 1)^9$$
$$[n2] \; 1 \leq 3^{-1}cn^{1/9}$$
$$[n3] \; \Psi_1 \leq \big(2^{-1/2} - 2^{-5/2} - 2^{-1}\big)9^{-1} \cdot cn^{1/9}$$
$$[n4] \; \Psi_1 \leq 5 \cdot 2^{3/2}3^{-1} \cdot cn^{1/9}$$
$$[n5] \; n + 1 \geq 2^{45/4}c^{-9}$$
$$[n6] \; n \geq c^{-18} \,.$$

Simplifying the lists.

For each of the three lists, we now find a simpler set of conditions that imply the conditions listed.

Simplifying the list of lower bounds on R. We consider three further conditions:

$$[L7] \; R \geq 5$$
$$[L8] \; R \geq 3c^{-3}$$

and

$$[L9] \; R \geq 2\big((\Psi_2 + 1)^2 + \Psi_3\big)^{1/2} \,.$$

[L4] takes the form $R \geq (2^{1/2} - 2^{-1})^{-1/2}2^{9/4}$. Since $(2^{1/2} - 2^{-1})^{-1/2}2^{9/4} = 4.975\cdots$, $[L7] \to [L4]$. Note also that $[L7] \to [L2, 6]$.

Since

$$(\log 4)^{1/3}2^2\big((2^{3/2} - 2^{-1})^{3/2} - (2^{-1/2} - 2^{-1})^{3/2}\big)^{-1/3}$$
$$= \;\; 3.37758\cdots \times \big(3.55298\cdots - 0.09425\cdots\big)^{-1/3} = 3.37758\cdots \times 0.66124\cdots = 2.23\cdots \,,$$

we see that $[L8] \to [L5]$.

Note that $[L9] \to [L1]$.

Thus $[L7, 8, 9] \to [L1, 2, 4, 5, 6]$.

We see then that $[L3, 7, 8, 9]$ is a simplified list, implying the entire set of conditions $[L1, 2, 3, 4, 5, 6]$.

Simplifying the list of upper bounds on R.

Note that the conditions $[U1]$ and $[U2]$ may be rewritten:

$$[U1]\ R \leq 2n^{1/9}$$

and

$$[U2]\ R \leq 6^{-1}cn^{1/9}\,.$$

Since $(2^{-1/2} - 2^{-1})^{-1/2}2^{3/4}3^{-1} = 1.231\cdots$, $[U2] \to [U3]$. That $[U2] \to [U1]$ follows from $c \leq 1$. We see that the simplified list for the upper bounds on R may be chosen to consist of the condition $[U2]$.

Simplifying the list of lower bounds on n. Note that $[n2, 3, 4]$ may be rewritten:

$$[n2]\ n \geq 3^9 c^{-9}$$

$$[n3]\ n \geq \Psi_1^9 \big(2^{-1/2} - 2^{-5/2} - 2^{-1}\big)^{-9}9^9 c^{-9}$$

and

$$[n4]\ n \geq \Psi_1^9 5^{-9} 2^{-27/2} 3^9 c^{-9}$$

Noting that

$$\big(2^{-1/2} - 2^{-5/2} - 2^{-1}\big)^{-9}9^9 = (0.03033\cdots)^{-9}9^9 = 1.783\cdots \times 10^{22}\,,$$

we see that, if we write

$$[n7]\ n \geq \Psi_1^9 10^{23} c^{-9}\,,$$

then $[n7] \to [n3, 4]$.

We also introduce

$$[n8]\ n \geq 3^9 c^{-18}$$

Since $c \leq 1$, $[n8] \to [n2, 5]$. Also $[n8] \to [n6]$.

Thus, $[n7, 8] \to [n2, 3, 4, 5, 6]$. We see that $[n1, 7, 8] \to [n1, 2, 3, 4, 5, 6]$. The list $[n1, 7, 8]$ may be written:

$$n \geq \max\left\{ c^{-9}(\Psi_2 + 1)^9\,,\ \Psi_1^9 10^{23} c^{-9}\,,\ 3^9 c^{-18}\right\}$$

which since $c \leq 1$ is implied by

$$[n9]\ n \geq c^{-18}\max\left\{(\Psi_2 + 1)^9\,,\ 10^{23}\Psi_1^9\,,\ 3^9\right\}\,.$$

The condition $[n9]$ may thus be chosen as the simplified list of lower bounds on n.

Summary. Drawing together the three simplified lists, we see that the set of conditions

$$[L3, L7, L8, L9, U2, n9]$$

are sufficient for all the conditions that are applied during the proof of Lemma 6.2. These conditions are:

$$R \geq \max\left\{ 39\Psi_1\,,\ 5\,,\ 3c^{-3}\,,\ 2\big((\Psi_2 + 1)^2 + \Psi_3\big)^{1/2}\right\}\,,$$

$$R \leq 6^{-1}cn^{1/9}\,,$$

and

$$n \geq c^{-18}\max\left\{(\Psi_2 + 1)^9\,,\ 10^{23}\Psi_1^9\,,\ 3^9\right\}\,.$$

Since these are the set of hypotheses of Lemma 6.2, the calculational derivation of this result is complete. $\qquad\square$

A.6. **Lemma 6.4 derivation.** The calculational derivation of this result is also long, and we first mention that conditions invoked during its proof fall under four categories. They are hypotheses needed in order to

- apply Corollary 6.3;

- find that the quantity (44) is at most $\epsilon/2$;

- apply Lemma 6.2;

- and find that the right-hand side of (45) is also at most $\epsilon/2$.

We begin with four subsections, labelled A, B, C and D, whose respective roles are to gather these hypotheses and analyse them, so that a simplified collection of conditions is provided at the end of each subsection. When this is done, we will record all the resulting conditions, split them according to type, and further simplify them.

We mention also that the values of the two positive constants C_- and C_+ have been set in the statement of Lemma 6.4. We will however treat them for now as free parameters, and will later set them, in accordance with their stated values, while showing that these choices are compatible with all conditions that concern them.

A: Hypotheses and analysis for application of Corollary 6.3. Here are the hypotheses:

$$[a1] \ \epsilon \in (0, 2^{-4}]$$
$$[a2] \ n \geq 10^{32} c^{-18}$$
$$[a3] \ C_+ \big(\log \epsilon^{-1} \big)^{1/3} + 2 \leq \epsilon^{-1/2}$$
$$[a4] \ C_+ \big(\log \epsilon^{-1} \big)^{1/3} + 2 \leq 2^{-2} 3^{-1} c n^{1/18}$$
$$[a5] \ C_- \big(\log \epsilon^{-1} \big)^{2/3} \geq 2 \cdot 10^4$$
$$[a6] \ C_- \big(\log \epsilon^{-1} \big)^{2/3} \leq 10^3 n^{1/18}$$
$$[a7] \ n \in \mathbb{N}$$

We introduce the conditions

$$[a8] \ 2 \leq C_+ \big(\log \epsilon^{-1} \big)^{1/3}$$
$$[a9] \ 2C_+ \big(\log \epsilon^{-1} \big)^{1/3} \leq \epsilon^{-1/2}$$
$$[a10] \ 2C_+ \big(\log \epsilon^{-1} \big)^{1/3} \leq 2^{-2} 3^{-1} c n^{1/18}$$

Note that $[a8, 9] \to [a3]$ and $[a8, 10] \to [a4]$.

Since $[a9]$ is equivalent to

$$\epsilon^{-1} \leq \exp \big\{ (2C_+)^{-3} \epsilon^{-3/2} \big\},$$

it is implied by

$$\epsilon^{-1} \leq (2C_+)^{-3} \epsilon^{-3/2}$$

or equivalently by

$$[a11] \ \epsilon \leq (2C_+)^{-6}$$

$[a10]$ is equivalent to

$$[a12] \ n \geq 3^{18} 2^{48} c^{-18} C_+^{18} \big(\log \epsilon^{-1} \big)^6$$

$[a5]$ is equivalent to

$$[a13] \ \epsilon \leq \exp \big\{ - C_-^{-3/2} 2^{3/2} 10^6 \big\}$$

[a6] is equivalent to

$$[a14]\ n \geq 10^{-54} C_-^{18} \left(\log \epsilon^{-1} \right)^{12}$$

[a8] is equivalent to

$$[a15]\ \epsilon \leq \exp\left\{ -8C_+^{-3} \right\}$$

With

$$[a16]\ C_+ \geq 2$$

then $[a11, 16] \to [a15]$.

In summary, $[a1, 2, 7, 11, 12, 13, 14, 16] \to [a1 - 10]$.

B: Hypotheses and analysis to obtain the bound

$$\left(2R\epsilon^{-1} + 1 \right)\left(2\epsilon^{-1} + 1 \right) \cdot 10032\, C\epsilon^{c_1 2^{-22-1/2} C_-^{3/2}} \leq \epsilon/2 \,. \tag{59}$$

Recall that $R = 1 + C_+ \left(\log \epsilon^{-1} \right)^{1/3}$. The left-hand side here is at most $16 \cdot 10032 C \epsilon^{c_1 2^{-22-1/2} C_-^{3/2}}$ provided that

$$[b1]\ 2R\epsilon^{-1} \geq 1$$

and

$$[b2]\ 2\epsilon^{-1} \geq 1$$

Imposing

$$[b3]\ c_1 2^{-22-1/2} C_-^{3/2} \geq 4 \,,$$

the left-hand side is bounded above by $16 \cdot 10032 C R \epsilon^2$. [a3] implies that $R \leq \epsilon^{-1/2}$, so that the bound may become $16 \cdot 10032 C \epsilon^{3/2}$. This latter quantity is seen to be at most $\epsilon/2$ precisely when $\epsilon \leq C^{-2} 2^{-2} (16 \cdot 10032)^{-2}$. Since $2^2 (16 \cdot 10032)^2 = 103056408576$, this upper bound on ϵ is implied by

$$[b4]\ \epsilon \leq 10^{-12} C^{-2}$$

Note that $[b2] \to [b1]$ since $R \geq 1$, and that $[a1] \to [b2]$.

Note that $[b3]$ is equivalent to

$$[b5]\ C_- \geq 2^{49/3} c_1^{-2/3}$$

In summary, (59) is implied by $[a1, a3, b4, b5]$.

C: Hypotheses and analysis for the application of Lemma 6.2.
In this case, our hypotheses are:

$$[d1]\ n \geq c^{-18} \max\left\{ (\Psi_2 + 1)^9 \,, 10^{23} \Psi_1^9 \,, 3^9 \right\},$$

$$[d2]\ R \geq 1 + \max\left\{ 39\Psi_1 \,, 5 \,, 3c^{-3} \,, 2\left((\Psi_2 + 1)^2 + \Psi_3 \right)^{1/2} \right\},$$

and

$$[d3]\ R - 1 \leq 6^{-1} c n^{1/9}$$

Recalling that $R = 1 + C_+ \left(\log \epsilon^{-1} \right)^{1/3}$, we see that $[d3]$ is equivalent to

$$C_+ \left(\log \epsilon^{-1} \right)^{1/3} \leq 6^{-1} c n^{1/9}$$

or

$$[d4]\ n \geq C_+^9 \left(\log \epsilon^{-1} \right)^3 6^9 c^{-9}$$

Since $[a1]$ implies that $\epsilon \leq e^{-1}$, we see that $[a1, d5] \to [d2]$, where

$$[d5]\ C_+ \geq \max\left\{ 39\Psi_1 \,, 5 \,, 3c^{-3} \,, 2\left((\Psi_2 + 1)^2 + \Psi_3 \right)^{1/2} \right\}$$

In summary, $[a1, d1, d4, d5] \rightarrow [d1, 2, 3]$.

D: Hypotheses and analysis for the bound

$$38(R-1)C \exp\left\{ -2^{-6}c(R-1)^3\left(2^{-1/2} - 2^{-1}\right)^{3/2}\right\} \leq \epsilon/2. \tag{60}$$

Since $R = 1 + C_+\left(\log \epsilon^{-1}\right)^{1/3}$, this bound is implied by

$$[e1]\ 2^{-6}cC_+^3\left(2^{-1/2} - 2^{-1}\right)^{3/2} \geq 2$$

and

$$[e2]\ 38 \cdot 2C_+\left(\log \epsilon^{-1}\right)^{1/3}C \leq \epsilon^{-1}$$

$[e2]$ is equivalent to

$$\epsilon^{-1} \leq \exp\left\{(76)^{-3}C_+^{-3}C^{-3}\epsilon^{-3}\right\}$$

and is thus implied by

$$\epsilon^{-1} \leq (76)^{-3}C_+^{-3}C^{-3}\epsilon^{-3}$$

or

$$[e3]\ \epsilon \leq (76)^{-3/2}C_+^{-3/2}C^{-3/2}$$

In summary, the bound (60) is implied by $[e1, 3]$.

We continue by collecting all the new conditions, and splitting them according to the form of the inequality on parameters.

E: Gathering and splitting the obtained conditions. For the calculational derivation of Lemma 6.4, a sufficient list of hypotheses has been found to be:

$[a1, 2, 7, 11, 12, 13, 14, 16]; [a1, a3(\text{implied by other } a\text{-conditions}), b4, b5]; [a1, d1, d4, d5]; \text{ and } [e1, 3]$.

We now recall these conditions:

$$[a1]\ \epsilon \in (0, 2^{-4}]$$
$$[a2]\ n \geq 10^{32}c^{-18}$$
$$[a7]\ n \in \mathbb{N}$$
$$[a11]\ \epsilon \leq (2C_+)^{-6}$$
$$[a12]\ n \geq 3^{18}2^{48}c^{-18}C_+^{18}\left(\log \epsilon^{-1}\right)^6$$
$$[a13]\ \epsilon \leq \exp\left\{ -C_-^{-3/2}2^{3/2}10^6\right\}$$
$$[a14]\ n \geq 10^{-54}C_-^{18}\left(\log \epsilon^{-1}\right)^{12}$$
$$[a16]\ C_+ \geq 2$$
$$[b4]\ \epsilon \leq 10^{-12}C^{-2}$$
$$[b5]\ C_- \geq 2^{49/3}c_1^{-2/3}$$
$$[d1]\ n \geq c^{-18}\max\left\{(\Psi_2 + 1)^9,\ 10^{23}\Psi_1^9,\ 3^9\right\},$$
$$[d4]\ n \geq C_+^9\left(\log \epsilon^{-1}\right)^3 6^9 c^{-9}$$
$$[d5]\ C_+ \geq \max\left\{ 39\Psi_1,\ 5,\ 3c^{-3},\ 2\left((\Psi_2 + 1)^2 + \Psi_3\right)^{1/2}\right\}$$
$$[e1]\ 2^{-6}cC_+^3\left(2^{-1/2} - 2^{-1}\right)^{3/2} \geq 2$$
$$[e3]\ \epsilon \leq (76)^{-3/2}C_+^{-3/2}C^{-3/2}$$

We reexpress this list in four categories:

Upper bounds on ϵ:

$$[a1] \ \epsilon \in (0, 2^{-4}]$$
$$[a11] \ \epsilon \leq (2C_+)^{-6}$$
$$[a13] \ \epsilon \leq \exp\{-C_-^{-3/2} 2^{3/2} 10^6\}$$
$$[b4] \ \epsilon \leq 10^{-12} C^{-2}$$
$$[e3] \ \epsilon \leq (76)^{-3/2} C_+^{-3/2} C^{-3/2}$$

Lower bounds on C_+:

$$[a16] \ C_+ \geq 2$$
$$[d5] \ C_+ \geq \max\left\{39\Psi_1\,,\, 5\,,\, 3c^{-3}\,,\, 2\big((\Psi_2+1)^2 + \Psi_3\big)^{1/2}\right\}$$
$$[e1] \ 2^{-6} c C_+^3 \big(2^{-1/2} - 2^{-1}\big)^{3/2} \geq 2$$

Lower bound on C_-:

$$[b5] \ C_- \geq 2^{49/3} c_1^{-2/3}$$

Conditions on n:

$$[a2] \ n \geq 10^{32} c^{-18}$$
$$[a7] \ n \in \mathbb{N}$$
$$[a12] \ n \geq 3^{18} 2^{48} c^{-18} C_+^{18} \big(\log \epsilon^{-1}\big)^6$$
$$[a14] \ n \geq 10^{-54} C_-^{18} \big(\log \epsilon^{-1}\big)^{12}$$
$$[d1] \ n \geq c^{-18} \max\left\{(\Psi_2+1)^9\,,\, 10^{23}\Psi_1^9\,,\, 3^9\right\},$$
$$[d4] \ n \geq C_+^9 \big(\log \epsilon^{-1}\big)^3 6^9 c^{-9}$$

In the next three subsections, we simplify three of these four lists.

F: Simplifying lower bounds on C_+: $[e1]$ is equivalent to

$$C_+^3 \geq 2^7 \big(2^{-1/2} - 2^{-1}\big)^{-3/2} c^{-1}$$

which, since $\big(2^{-1/2} - 2^{-1}\big)^{-3/2} = 10.6098\cdots$, is implied by $C_+^3 \geq 1359 c^{-1}$ or

$$[f1] \ C_+ \geq 12 c^{-1/3}$$

Since $c \leq 1$, the condition

$$[f2] \ C_+ \geq \max\left\{39\Psi_1\,,\, 12c^{-3}\,,\, 2\big((\Psi_2+1)^2 + \Psi_3\big)^{1/2}\right\}$$

is seen to imply all lowers bound $[a16, d5, e1]$ on C_+.

In order to simplify other bounds, we now set the values of C_+ and C_-. We set

$$C_+ = \max\left\{39\Psi_1\,,\, 12c^{-3}\,,\, 2\big((\Psi_2+1)^2 + \Psi_3\big)^{1/2}\right\}$$

and

$$C_- = 2^{17} c_1^{-2/3}.$$

These choices verify the bounds demanded on these two constants.

G: Simplifying conditions on n.

Since $\epsilon \leq e^{-1}$, $c \leq 1$, $C_+ \geq 1$, $C_- \geq 1$ and $3^{18}2^{48} \in [10^{23}, 10^{24}]$, we have $[g1] \to [a2, a12, a14, d4]$, where

$$[g1] \ n \geq 10^{32} c^{-18} C_+^{18} C_-^{18} \big(\log \epsilon^{-1}\big)^{12}$$

In fact, we also have $[g1, d5] \to [d1]$, because $10^{32}(39)^{18}\Psi_1^{18} \geq 10^{23}\Psi_1^9$ when $\Psi_1 \geq 1$; and, if $\Psi_1 \in (0, 1)$, then we have $10^{32} \geq 10^{23}$.

That is, all conditions $[a2, a7, a12, a14, d1, d4]$ on n are implied by $[a7, g1]$.

H: Simplifying upper bounds on ϵ.

Since $C_- \geq 1$, $C_+ \geq 1$ and $C \geq 1$, the condition

$$[h1] \ \epsilon \leq C_+^{-6} \exp\big\{-2^{3/2}10^6\big\}C^{-2}$$

satisfies $[h1] \to [a1, a11, a13, b4, e3]$, so that it implies all needed upper bounds on ϵ.

We are ready to conclude this calculational derivation.

I: Conclusion. In summary of these simplifications, all hypotheses needed for the proof of Lemma 6.4 are implied by $[a7, g1, h1]$ provided that the constants C_+ and C_- are set as we have specified.

That is, with the choices

$$C_+ = \max\left\{39\Psi_1\,,\, 12c^{-3}\,,\, 2\big((\Psi_2 + 1)^2 + \Psi_3\big)^{1/2}\right\}$$

and $C_- = 2^{17}c_1^{-2/3}$, a set of conditions sufficient to permit the derivation of Lemma 6.4 is: $[a7] \ n \in \mathbb{N}$, $[g1] \ n \geq 10^{32} c^{-18} C_+^{18} C_-^{18} \big(\log \epsilon^{-1}\big)^{12}$ and $[h1] \ \epsilon \leq C_+^{-6} \exp\big\{-2^{3/2}10^6\big\}C^{-2}$. Since these conditions are exactly the hypotheses of Lemma 6.4, we have completed the calculational derivation of this lemma.

References

[Aub05] Guillaume Aubrun. A sharp small deviation inequality for the largest eigenvalue of a random matrix. In *Séminaire de Probabilités XXXVIII*, volume 1857 of *Lecture Notes in Math.*, pages 320–337. Springer, Berlin, 2005.

[BDJ99] Jinho Baik, Percy Deift, and Kurt Johansson. On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.*, 12(4):1119–1178, 1999.

[BFPS07] Alexei Borodin, Patrik L. Ferrari, Michael Prähofer, and Tomohiro Sasamoto. Fluctuation properties of the TASEP with periodic initial configuration. *J. Stat. Phys.*, 129(5-6):1055–1080, 2007.

[Bil99] Patrick Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[BSS16] Riddhipratim Basu, Vladas Sidoravicius, and Allan Sly. Last passage percolation with a defect line and the solution of the slow bond problem. *arXiv:1408.3464*, 2016.

[CH14]   Ivan Corwin and Alan Hammond. Brownian Gibbs property for Airy line ensembles. *Invent. Math.*, 195(2):441–508, 2014.

[Dur10] Rick Durrett. *Probability: theory and examples*, volume 31 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, fourth edition, 2010.

[GW91]   Peter W. Glynn and Ward Whitt. Departures from many queues in series. *Ann. Appl. Probab.*, 1(4):546–572, 1991.

[Häg08] Jonas Hägg. Local Gaussian fluctuations in the Airy and discrete PNG processes. *Ann. Probab.*, 36(3):1059–1092, 2008.

[Ham17a] Alan Hammond. Brownian regularity for the Airy line ensemble, and multi-polymer watermelons in Brownian last passage percolation. 2017. https://arxiv.org/abs/1609.02971.

[Ham17b] Alan Hammond. Exponents governing the rarity of disjoint polymers in Brownian last passage percolation. 2017. `https://arxiv.org/abs/1709.04110`.

[Ham17c] Alan Hammond. A patchwork quilt sewn from Brownian fabric: regularity of polymer weight profiles in Brownian last passage percolation. 2017. `https://arxiv.org/abs/1709.04113`.

[Joh00] Kurt Johansson. Transversal fluctuations for increasing subsequences on the plane. *Probab. Theory Related Fields*, 116(4):445–456, 2000.

[Led07] M. Ledoux. Deviation inequalities on largest eigenvalues. In *Geometric aspects of functional analysis*, volume 1910 of *Lecture Notes in Math.*, pages 167–219. Springer, Berlin, 2007.

[MQR17] Konstantin Matetski, Jeremy Quastel, and Daniel Remenik. The KPZ fixed point. *arXiv:1701.00018*, 2017.

[OY02] Neil O'Connell and Marc Yor. A representation for non-colliding random walks. *Electron. Comm. Probab.*, 7:1–12 (electronic), 2002.

[Pim17] Leandro Pimentel. Local behavior of Airy processes. *arXiv:1704.01903*, 2017.

[PS02] Michael Prähofer and Herbert Spohn. Scale invariance of the PNG droplet and the Airy process. *J. Statist. Phys.*, 108(5-6):1071–1106, 2002. Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays.

[QR13] Jeremy Quastel and Daniel Remenik. Local behavior and hitting probabilities of the Airy_1 process. *Probab. Theory Related Fields*, 157(3-4):605–634, 2013.

[Sas05] T. Sasamoto. Spatial correlations of the 1D KPZ surface on a flat substrate. *J. Phys. A*, 38(33):L549–L556, 2005.

[Wil91] David Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991.

A. HAMMOND, DEPARTMENT OF MATHEMATICS AND STATISTICS, U.C. BERKELEY, 899 EVANS HALL, BERKELEY, CA, 94720-3840, U.S.A.

E-mail address: `alanmh@berkeley.edu`