MODULUS OF CONTINUITY OF POLYMER WEIGHT PROFILES IN BROWNIAN LAST PASSAGE PERCOLATION

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Abstract. In last passage percolation models lying in the KPZ universality class, the energy of long energy-maximizing paths may be studied as a function of the paths' pair of endpoint locations. Scaled coordinates may be introduced, so that these maximizing paths, or polymers, now cross unit distances with unit-order fluctuations, and have scaled energy, or weight, of unit order. In this article, we consider Brownian last passage percolation in these scaled coordinates. In the narrow wedge case, one endpoint of such polymers is fixed, say at $(0,0) \in \mathbb{R}^2$, and the other is varied horizontally, over $(z,1)$, $z \in \mathbb{R}$, so that the polymer weight profile may be studied as a function of $z \in \mathbb{R}$. This profile is known to manifest a one-half power law, having $1/2-$-Hölder continuity. The polymer weight profile may be defined beginning from a much more general initial condition. In this article, we present a more general assertion of this one-half power law, as well as a bound on the poly-logarithmic correction. For a very broad class of initial data, the polymer weight profile has a modulus of continuity of the order of $x^{1/2}\big(\log x^{-1}\big)^{2/3}$, uniformly in the scaling parameter and the initial condition.

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1. Introduction

The $1+1$ dimensional Kardar-Parisi-Zhang (KPZ) universality class includes a wide range of interface models suspended over a one-dimensional domain, in which growth in a direction normal to the surface competes with a smoothening surface tension in the presence of a local randomizing force that roughens the surface. Such surfaces typically grow linearly, with fluctuations after that linear growth is subtracted being described by scaling exponents: if linear growth has order n, then interface height above a given point has typical deviation from the mean of order $n^{1/3}$, while non-trivial correlations in this height as the spatial coordinate is varied are encountered on scale $n^{2/3}$. Moreover, an exponent of one-half dictates the interface's regularity, with the interface height being expected to vary between a pair of locations at distance of order at most $n^{2/3}$ on the order of the square root of the distance between these locations.

Such growth models may be initiated at time zero with a given interface profile. In the narrow wedge case, when growth is initiated from a unique point, a limiting description of the late time interface, suitably scaled in light of the one-third and two-thirds powers and up to the subtraction of a parabola, is offered by the Airy_2 process, which is a random function $\mathcal{A} : \mathbb{R} \to \mathbb{R}$, whose finite dimensional distributions are specified by Fredholm determinants, that was introduced by [PS02]. Another well-known initial condition is the flat case, when growth begins from a zero initial condition. Here, the Airy_1 process describes the interface at late time. The one-half power law for interface regularity is expressed by the Hölder-$1/2-$-continuity of the processes Airy_1 and Airy_2, which was proved in [QR13].

Growth may be initiated from a much more general initial condition than in these narrow wedge or flat cases. For initial conditions that grow at most linearly, it has been anticipated that a limiting description of the suitably scaled late-time interface should exist in these cases also. Indeed, in a recent preprint [MQR17], Matetski, Quastel and Remenik have utilized a biorthogonal ensemble representation found by [Sas05, BFPS07] associated to the totally asymmetric exclusion process in order to find Fredholm determinant formulas for the multi-point distribution of the height function of this growth process begun from an arbitrary initial condition. Using these formulas to take the KPZ scaling limit, the authors construct a scale invariant Markov process that lies at the heart of the KPZ universality class. The time-one evolution of this Markov process may be applied to very general initial data, and the result is the scaled profile begun from such data, which generalizes the Airy_1 and Airy_2 processes seen in the flat and narrow wedge cases. These more general limiting processes also enjoy Hölder-$1/2-$-continuity: see [MQR17, Theorem 4.4].

The broad range of interface models that are rigorously known or expected to lie in the KPZ universality class includes many last passage percolation models. Such an LPP model comes equipped with a planar random environment, which is independent in disjoint regions. Directed paths, that are permitted say to move only in a direction in the first quadrant, are then assigned energy via this randomness, by say integrating the environment's value along the path. For a given pair of planar points, the path attaining the maximum energy over directed paths with such endpoints is called a geodesic. The random interface model that we alluded to at the outset is then specified as the maximum geodesic energy when one geodesic endpoint is varied and the other held fixed, in the narrow wedge case, or when the other is free to vary and is rewarded according to the initial condition, in the more general case. The one-third and two-thirds power laws for typical deviation of maximum energy and for lateral correlation have been rigorously demonstrated for only a few LPP models, each of which enjoys an integrable structure: the seminal work of Baik, Deift and Johansson [BDJ99] rigorously established the one-third exponent, and moreover obtained the GUE Tracy-Widom distributional limit, for the case of Poissonian last passage percolation, while the two-thirds power law for transversal fluctuation was derived for this model by Johansson [Joh00]. For models in which these two exponents have been rigorously identified, the exponent pair dictates a system of scaled coordinates in which the concerned maximizing paths and their weights are unit-order, random, quantities: the scaled geodesics may be called polymers, and their scaled energies, weights.

Brownian last passage percolation is an LPP model with attractive integrable and probabilistic features. In this article, we study the scaled interface profile (that is, the polymer weight profile) in Brownian LPP begun from a very general initial condition. We present results proving a more precise version of the one-half power law for interface regularity than has been established hitherto. There are two principal conclusions:

- In Theorem 1.2, we prove that any weak limit point of the scaled interface profiles, as the scaling parameter tends to infinity, has sample paths with a modulus of continuity of the order of $x^{1/2}\big(\log x^{-1}\big)^{2/3}$. This assertion is proved uniformly over large classes of the data that initiates the random growth.
- In Theorem 1.3, we prove that the maximum difference in the weight of two polymers whose endpoints differ by at most a small scaled quantity ϵ exceeds $\epsilon^{1/2}R$ with probability at most $\exp\big\{-O(1)R^{3/2}\big\}$ for a very broad range of values of R, uniformly in the scaling parameter for Brownian LPP.

For a given choice of initial condition, the weak limit point in Theorem 1.2 may be expected to be unique and to coincide with the interface profile obtained from this initial data by evolving for a given duration the Markov operator constructed in [MQR17]. However, this Markov operator has been constructed as a limit of totally asymmetric exclusion, so at present this assertion is only expected, not proved. Were the techniques of [MQR17] to be adapted to hold for Brownian last passage percolation, it would then presumably be possible to assert the order $x^{1/2}\big(\log x^{-1}\big)^{2/3}$ modulus of continuity for general initial condition interface profiles under the KPZ fixed point.

The strongly on-scale assertion of the one-half power law for interface regularity in Theorem 1.2 plays a significant role in two companion papers. In [Ham17b], it is harnessed in order to prove that, in Brownian last passage percolation, it is a superpolynomial rarity that a large number of disjoint polymers coexist in a unit-order scaled region. This understanding is then employed in [Ham17c] to make a strong unit-order Brownian comparison for polymer weight profiles (about which more momentarily).

Beyond the two conclusions just discussed, the present article also presents a useful tool, Proposition 1.4. Although the weight of a polymer is random, this weight is dictated in the large by parabolic curvature, with the randomness playing a unit-order role once this curvature is accounted for. The proposition shows that the discrepancy between polymer weight and parabola is controlled uniformly as the polymer's endpoints are varied over compact intervals lying in a very broad region. This tool is needed in the present article and in [Ham17b]. For exponential or Poissonian last passage percolation, a similar tool has been developed, in [BSS17, Propositions 10.1 and 10.5].

We also mention that an alternative expression of the one-half power law for interface regularity is the assertion that Airy processes such as Airy_1 and Airy_2, or scaled interface models in the last passage percolation setting, locally resemble Brownian motion. Such statements may be understood in a local limit, when Gaussianity of a process $\mathcal{A}$ is proved for the low ϵ limit for the random variable $\epsilon^{-1/2}\big(\mathcal{A}(x+\epsilon)-\mathcal{A}(x)\big)$ associated to any given $x \in \mathbb{R}$. Finite dimensional distributional convergence to Brownian motion (of diffusion rate two) in this limit has been proved for the Airy_2 process in [Häg08], for the Airy_1 process in [QR13], and for the more general versions of these Airy processes constructed in [MQR17] in Theorem 4.4 of that paper; in [Pim17], similar local limit results for general initial condition profiles have been obtained for geometric last passage percolation models. Comparison to Brownian motion may also be made without taking such a local limit. In [CH14], the Airy_2 process was understood to be absolutely continuous with respect to

Brownian motion on a unit-order interval, by a Brownian Gibbs technique in which this process is embedded as the uppermost curve in a random ensemble of, in effect, mutually avoiding Brownian motions. This inference was improved in [Ham17a], where the implied Radon-Nikodym derivative is shown to lie in all L^p-spaces for $p \in (1,\infty)$, albeit after an affine shift is applied to the Airy_2 process, so that comparison is made not to Brownian motion but to Brownian bridge. In a companion paper to the present article [Ham17c], the problem of unit-order scale Brownian comparison is made for the class of Brownian LPP polymer weight profiles, begun from general initial data, that are the subject of the present article. It is in essence shown there that a given unit-order interval may be split into a random but controlled number of intervals in such a way that the profile when restricted to the smaller intervals has, after affine adjustment, a Radon-Nikodym derivative with respect to Brownian bridge that lies in L^p for $p \in (1,3)$.

1.1. **Brownian last passage percolation.** This model was introduced in [OY02]; we call it Brownian LPP. On a probability space carrying a law labelled $\mathbb{P}$, let $B : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ denote an ensemble of independent two-sided standard Brownian motions $B(k,\cdot) : \mathbb{R} \to \mathbb{R}$, $k \in \mathbb{Z}$.

Let $i,j \in \mathbb{Z}$ with $i \leq j$. We denote the integer interval $\{i,\cdots,j\}$ by $[\![i,j]\!]$. Further let $x,y \in \mathbb{R}$ with $x \leq y$. With these parameters given, we consider the collection of non-decreasing lists $\big\{z_k : k \in [\![i+1,j]\!]\big\}$ of values $z_k \in [x,y]$. With the convention that $z_i = x$ and $z_{j+1} = y$, we associate an energy to any such list, namely $\sum_{k=i}^{j}\big(B(k,z_{k+1}) - B(k,z_k)\big)$. We may then define the maximum energy, $M^1_{(x,i)\to(y,j)}$, to be the supremum of the energies of all such lists.

The one-third and two-thirds KPZ scaling considerations that we outlined earlier in the introduction are manifest in Brownian LPP. When the ending height j exceeds the starting height i by a large quantity $n \in \mathbb{N}$, and the location y exceeds x also by n, then the maximum energy grows linearly, at rate $2n$, and has a fluctuation about this mean of order $n^{1/3}$. Moreover, if y is permitted to vary from this location, then it is changes of $n^{2/3}$ in its value that result in a non-trivial correlation of the maximum energy from its original value.

These facts prompt us to introduce scaled coordinates to describe the two endpoint locations, and a notion of scaled maximum energy, which we will refer to as weight. Let $n \in \mathbb{N}$, and suppose that $x,y \in \mathbb{R}$ satisfy $y \geq x - 2^{-1}n^{1/3}$. Define

$$\mathsf{Wgt}_{n;(x,0)}^{(y,1)} \, = \, 2^{-1/2} n^{-1/3} \Big(M^1_{(2n^{2/3}x,0)\to(n+2n^{2/3}y,n)} - 2n - 2n^{2/3}(y-x) \Big) \, . \tag{1}$$

(Clearly, n must be positive. In fact, $\mathbb{N}$ will denote $\{1,2,\cdots\}$ throughout.)

Consistently with the facts just mentioned, the quantity $\mathsf{Wgt}_{n;(x,0)}^{(y,1)}$ may be expected to be, for given real choices of x and y, a unit-order random quantity, whose law is tight in the scaling parameter $n \in \mathbb{N}$. The quantity describes, in units chosen to achieve this tightness, the maximum possible energy associated to journeys which in the original coordinates occur between $(2n^{2/3}x,0)$ and $(n+2n^{2/3}y,n)$. In scaled coordinates, this is a journey between $(x,0)$ and $(y,1)$. We view the first coordinate as space and the second as time, so this journey is between x and y over the unit time interval $[0,1]$.

Underlying this definition is a geometric picture of scaled maximizing paths, or polymers, that achieve these weight values. We will defer explicitly defining these polymers, but it is useful to bear in mind that $\mathsf{Wgt}_{n;(x,0)}^{(y,1)}$ equals the weight of a polymer that travels between locations that in scaled coordinates equal $(x,0)$ and $(y,1)$.

1.2. **Main results.** The article's two principal conclusions, on the modulus of continuity of polymer weight profiles from general initial condition, and on polymer weight difference under horizontal perturbation of endpoints, and a general tool, on the rarity of deviation from parabolic curvature by polymer weights, will be stated in the ensuing three subsections.

1.2.1. *Modulus of continuity of polymer weight profiles from general initial data.* What do we mean by such polymer weight profiles? The random function $y \to \mathsf{Wgt}_{n;(0,0)}^{(y,1)}$ may be viewed as the weight profile obtained by scaled maximizing paths that travel from the origin at time zero to the variable location y at time one. This insistence that the paths must begin at the origin, called the narrow wedge by physicists, is of course rather special. We now make a more general definition, of the f-rewarded line-to-point polymer weight $\mathsf{Wgt}_{n;(*:f,0)}^{(y,1)}$. Here, f is an initial condition, defined on the real line. Paths may begin anywhere on the real line at time zero; they travel to $y \in \mathbb{R}$ at time one. (Because they are free at the beginning and fixed at the end, we refer to these paths as 'line-to-point'.) They begin with a reward given by evaluating f at the starting location, and then gain the weight associated to the journey they make. The value $\mathsf{Wgt}_{n;(*:f,0)}^{(y,1)}$, which we will define momentarily, denotes the maximum f-rewarded weight of all such paths. In the notation $\mathsf{Wgt}_{n;(*:f,0)}^{(y,1)}$, we again use subscript and superscript expressions to refer to space-time pairs of starting and ending locations. The starting spatial location is being denoted $* : f$. The star is intended to refer to the free time-zero endpoint, which may be varied, and the $: f$ to the reward offered according to where this endpoint is placed.

The next definition specifies essentially the broadest class of f suitable for a study of the weight profiles $y \to \mathsf{Wgt}_{n;(*:f,0)}^{(y,1)}$ for all sufficiently high $n \in \mathbb{N}$.

Definition 1.1. Writing $\overline{\Psi} = \big(\Psi_1, \Psi_2, \Psi_3\big) \in (0,\infty)^3$ for a triple of positive reals, we let $\mathcal{I}_{\overline{\Psi}}$ denote the set of measurable functions $f : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ such that $f(x) \leq \Psi_1\big(1 + |x|\big)$ and $\sup_{x \in [-\Psi_2, \Psi_2]} f(x) > -\Psi_3$.

For f lying in one of the function spaces $\mathcal{I}_{\overline{\Psi}}$, we now formally define the f-rewarded line-to-point polymer weight $\mathsf{Wgt}_{n;(*:f,0)}^{(y,1)}$ to be

$$\sup_{x \in (-\infty, 2^{-1}n^{1/3}+y]} \Big(\mathsf{Wgt}_{n;(x,0)}^{(y,1)} + f(x) \Big) .$$

Let $n \in \mathbb{N}$, $\overline{\Psi} \in (0,\infty)^3$ and $f \in \mathcal{I}_{\overline{\Psi}}$. Let $\nu_{n;(*:f,0)}^{([-1,1],1)}$ denote the law of the random function

$$[-1,1] \to \mathbb{R} : y \to \mathsf{Wgt}_{n;(*:f,0)}^{(y,1)} .$$

Let $\mathcal{A}$ be an arbitrary index set, and let $\big\{\nu_{n,\alpha} : n \in \mathbb{N}\big\}$, $\alpha \in \mathcal{A}$, be an $\mathcal{A}$-indexed collection of sequences of probability measures on a given measurable space. The collection is here called uniformly tight if, for each $\epsilon > 0$, there exist $n_0 \in \mathbb{N}$ and a compact set K such that $\nu_{n,\alpha}(K) \geq 1 - \epsilon$ whenever $n \geq n_0$ and $\alpha \in \mathcal{A}$.

Theorem 1.2. *Let $\overline{\Psi} \in (0,\infty)^3$ denote a triple of positive reals.*

(1) *Suppose that $n \in \mathbb{N}$ satisfies $n > 2^{-3/2}\Psi_1^3 \vee 8(\Psi_2+1)^3$. Let $f \in \mathcal{I}_{\overline{\Psi}}$. Then the measure $\nu_{n;(*:f,0)}^{([-1,1],1)}$ is supported on the space $\mathcal{C}$ of continuous real-valued functions on $[-1,1]$.*

(2) *The collection of sequences of probability measures* $\big\{ \nu^{([-1,1],1)}_{n;(*:f,0)} : n \in \mathbb{N} \big\}$ *indexed by* $f \in \mathcal{I}_{\overline{\Psi}}$ *is uniformly tight. Here, the space* $\mathcal{C}$ *is endowed with the topology of uniform convergence.*

(3) *A law on* $\mathcal{C}$ *is said to belong to the* weak limit point *set* $\mathrm{WLP}_{\overline{\Psi}}$ *if, for some sequence* $f_n \in \mathcal{I}_{\overline{\Psi}}$, $n \in \mathbb{N}$, *it equals the weak limit of the measures* $\nu^{([-1,1],1)}_{n;(*:f_n,0)}$ *along some subsequence of* $n \in \mathbb{N}$. *By* (2), $\mathrm{WLP}_{\overline{\Psi}}$ *is non-empty. For any* $\nu \in \mathrm{WLP}_{\overline{\Psi}}$, *we write* X *for a random curve whose law is* ν. *Consider the random variable* S *given by the supremum of the ratio of* $\big\vert X(y) - X(x) \big\vert$ *and* $(y-x)^{1/2}\big(\log(y-x)^{-1}\big)^{2/3}$ *as the variables* x *and* y *vary over* $[-1,1]$ *subject to* $x < y$. *Then* S *is almost surely finite. Indeed, its probability of exceeding any* $y > 1$ *is at most* $Dy^{-2}\big(\log y\big)^{4/3}$, *where* $D \in (0,\infty)$ *is a constant that may be chosen independently of* $\nu \in \mathrm{WLP}_{\overline{\Psi}}$.

1.2.2. *Polymer weight regularity.* Let I and J denote two bounded closed intervals in the real line. We define the *maximum weight difference* $\mathsf{Max\Delta Wgt}^{(J,1)}_{n;(I,0)}$ to equal the supremum over $x_1, x_2 \in I$ and $y_1, y_2 \in J$ of $\big\vert \mathsf{Wgt}^{(y_1,1)}_{n;(x_1,0)} - \mathsf{Wgt}^{(y_2,1)}_{n;(x_2,0)} \big\vert$. We may then define the *local weight regularity* event by setting

$$\mathsf{LocWgtReg}^{(J,1)}_{n;(I,0)}\big(\epsilon, r\big) \, = \, \Big\{ \mathsf{Max\Delta Wgt}^{(J,1)}_{n;(I,0)} \leq r\epsilon^{1/2} \Big\} \, ,$$

where here I and J are supposed to be intervals whose length equals the parameter value ϵ.

In the next result and throughout, $c > 0$ denotes a certain small constant; it is fixed alongside a large constant $C > 0$ in the upcoming Proposition 1.8. We also set $c_1 = 2^{-5/2}c \wedge 1/8$, where $\wedge$ denotes minimum. We use the notation $\neg A$ to denote the complement of the event A.

Theorem 1.3. *Let* $\epsilon \in (0, 2^{-4}]$. *Let* $n \in \mathbb{N}$ *be an even integer that satisfies* $n \geq 10^{29}c^{-18}$ *and let* $x, y \in \mathbb{R}$ *satisfy* $\big\vert x - y \big\vert \leq 2^{-5/3}3^{-1}cn^{1/18}$. *Let* $R \in \big[10^6 \, , \, 10^4 n^{1/18}\big]$. *Then*

$$\mathbb{P}\Big(\neg\, \mathsf{LocWgtReg}^{([y,y+\epsilon],1)}_{n;([x,x+\epsilon],0)}\big(\epsilon, R\big) \Big) \leq 25064\, C \exp\big\{ - c_1 2^{-31} R^{3/2} \big\} \, .$$

In this result, an upper bound is being found on the probability that the maximum weight difference witnessed by variation of the starting or ending polymer endpoint on given intervals of length ϵ exceeds $\epsilon^{1/2}R$. We may take x and y to be fixed locations, in a bounded interval, so that the hypothesis $n \geq \Theta(1)\vert x - y\vert^{18}$ is rather mild, given that our aim is to understand these systems uniformly in high choices of n. The imposition that $R \in \big[10^6 \, , \, 10^4 n^{1/18}\big]$ is also weak, given that we in essence consider R to be a fixed parameter; and indeed, the decay rate asserted by the theorem is already very fast when R is of order $n^{1/18}$.

1.2.3. *Tail behaviour of polymer weight suprema and infima.* In this third result, we see that the point-to-point polymer weight profile hews to a given parabola. The regime where this is verified is that in which the polymer endpoints differ by at most an order of $n^{1/18}$. Within this zone, the inference is made uniformly as the endpoints vary over any given unit-order region.

Proposition 1.4. *Let* $n \in \mathbb{N}$ *be even and satisfy* $n \geq 10^{29} \vee 2(c/3)^{-18}$. *Let* $x, y \in \mathbb{R}$ *satisfy* $\big\vert x - y \big\vert \leq 3^{-1}2^{-2/3}cn^{1/18}$. *Let* $t \in \big[33 \, , \, 4n^{1/18}\big]$. *Then*

$$\mathbb{P}\bigg(\sup_{u,v \in [0,1]} \Big(\mathsf{Wgt}^{(y+v,1)}_{n;(x+u,0)} + 2^{-1/2}(y+v-x-u)^2 \Big) \geq t \bigg) \leq 139 C \exp\big\{ - c_1 2^{-8} t^{3/2} \big\} \qquad (2)$$

and

$$\mathbb{P}\bigg(\inf_{u,v\in[0,1]} \Big(\mathsf{Wgt}_{n;(x+u,0)}^{(y+v,1)} + 2^{-1/2}(y+v-x-u)^2 \Big) \leq -t \bigg) \leq 261C \exp\big\{ -c_1 2^{-5/2} t^{3/2} \big\} \,. \qquad (3)$$

In [BSS17, Propositions 10.1 and 10.5], comparable bounds are proved for exponential and Poissonian LPP, with bounds on the failure probabilities of the form $\exp\big\{-O(1)t\big\}$. These propositions also have the flexibility of treating extremal weights of polymers whose endpoints are permitted to vary over compact regions in space as well as time, rather than merely time, as it is the case for Proposition 1.4.

The introduction will continue by recalling apparatus needed for the proofs of the main results.

- Section 1.3 presents some generalities, including explanation of the role of polymers as scaled maximizing paths, the use of scaled coordinates, and the polymer operations of concatenation and splitting.
- Section 1.4 introduces notation for writing the polymer weight profile, begun at a given location, as the uppermost curve in a certain system of ordered random continuous curves called a line ensemble.
- A suitably normalized version of any such line ensemble has the Brownian Gibbs property, which in essence means it is a system of mutually avoiding Brownian bridges. In Section 1.5, the theory of Brownian Gibbs ensembles is recalled, a certain regularity property of such ensembles is defined, and the crucial Proposition 1.8 is cited. This result states that the Brownian Gibbs ensembles into which our polymer weight profiles are embedded as the uppermost curves enjoy this regularity property. The result is quoted from [Ham17b], but the fundamental ideas for the proof actually appear in [Ham17a], which is the paper in which all the necessary Brownian Gibbs results for this article have been developed. In order to prove our three main results, we will be making use of four properties of regular Brownian Gibbs ensembles, beyond their defining ones. These four properties are recalled from [Ham17a] in Proposition 1.9, at the end of Section 1.5.

A short further comment on this article's companion papers [Ham17a], [Ham17b] and [Ham17c] before we continue. This article draws on Brownian Gibbs results developed in [Ham17a], and its main conclusions about scaled Brownian LPP are then applied in the other two papers. The article has been written in order that it may be read independently of the other three. It may also be understood to be part of this four-paper study, an overview of which appears in [Ham17a, Section 1.2].

1.3. **Some basic notation, tools and remarks.** We start by presenting

1.3.1. *Notation.* Let $i,j \in \mathbb{Z}$ with $i \leq j$. Recall that we are denoting the integer interval $\{i,\cdots,j\}$ by $[\![i,j]\!]$.

For $k \geq 1$, we write $\mathbb{R}^k_{\leq}$ for the subset of $\mathbb{R}^k$ whose elements $(z_1,\cdots,z_k)$ are non-decreasing sequences. When the sequences are increasing, we instead write $\mathbb{R}^k_{<}$. We also use the notation $A^k_{\leq}$ and $A^k_{<}$. Here, $A \subset \mathbb{R}$ and the sequence elements are supposed to belong to A. We will typically use this notation when $k=2$.

A bar over a symbol is used to indicate a vector, as in the notation $\overline{\Psi} = \big(\Psi_1,\Psi_2,\Psi_3\big) \in (0,\infty)^3$ used in Theorem 1.2.

1.3.2. *Staircases.* Taking $i, j \in \mathbb{N}$ with $i \leq j$, and $x, y \in \mathbb{R}^2_\leq$, we have ascribed in Section 1.1 an energy to any non-decreasing list $\big\{ z_k : k \in [\![i+1, j]\!] \big\}$ of values $z_k \in [x, y]$. In order to emphasise the geometric aspects of this definition, and in the hope that it may aid the visualization of the concerned concepts, we associate to each list a subset of $[x, y] \times [i, j] \subset \mathbb{R}^2$, which will be the range of a piecewise affine path, that we call a staircase.

To define the staircase associated to $\big\{ z_k : k \in [\![i+1, j]\!] \big\}$, we again adopt the convention that $z_i = x$ and $z_{j+1} = y$. The staircase is specified as the union of certain horizontal planar line segments, and certain vertical ones. The horizontal segments take the form $[z_k, z_{k+1}] \times \{k\}$ for $k \in [\![i, j]\!]$. The right and left endpoints of each consecutive pair of horizontal segments are interpolated by a vertical planar line segment of unit length. It is this collection of vertical line segments that form the vertical segments of the staircase.

The resulting staircase may be depicted as the range of an alternately rightward and upward moving path from starting point (x, i) to ending point (y, j). The set of staircases with these starting and ending points will be denoted by $SC_{(x,i) \to (y,j)}$. Such staircases are in bijection with the collection of non-decreasing lists already considered. Thus, any staircase $\phi \in SC_{(x,i) \to (y,j)}$ is assigned an energy $E(\phi) = \sum_{k=i}^{j} \big(B(k, z_{k+1}) - B(k, z_k) \big)$ via the associated z-list.

1.3.3. *Energy maximizing staircases are called geodesics.* A staircase $\phi \in SC_{(x,i) \to (y,j)}$ whose energy attains the maximum value $M^1_{(x,i) \to (y,j)}$ is called a geodesic from (x, i) to (y, j). It is a simple consequence of the continuity of the constituent Brownian paths $B(k, \cdot)$ that this geodesic exists for all choices of $(x, y) \in \mathbb{R}^2_\leq$. It is also true, and is proved in [Ham17c, Lemma A.1], that, for any given such choice of the pair (x, y), there is an almost surely unique geodesic from (x, i) to (y, j). However, this uniqueness will not be needed in the present article.

1.3.4. *The scaling map.* For $n \in \mathbb{N}$, consider the n-indexed *scaling* map $R_n : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$R_n(v_1, v_2) = \Big(2^{-1} n^{-2/3} (v_1 - v_2) \, , \, v_2 / n \Big) \, .$$

The scaling map acts on subsets C of $\mathbb{R}^2$ by setting $R_n(C) = \big\{ R_n(x) : x \in C \big\}$.

1.3.5. *Scaling transforms staircases to zigzags.* The image of any staircase under R_n will be called an n-zigzag. The starting and ending points of an n-zigzag Z are defined to be the image under R_n of such points for the staircase S for which $Z = R_n(S)$.

Note that the set of horizontal lines is invariant under R_n, while vertical lines are mapped to lines of gradient $-2n^{-1/3}$. As such, an n-zigzag is the range of a piecewise affine path from the starting point to the ending point which alternately moves rightwards along horizontal line segments and northwesterly along sloping line segments, where each sloping line segment has gradient $-2n^{-1/3}$; the first and last segment in this journey may be either horizontal or sloping.

1.3.6. *Compatible triples.* Let $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}^2_<$, which is to say that $n \in \mathbb{N}$ and $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$. Taking $x, y \in \mathbb{R}$, does there exist an n-zigzag from (x, t_1) and (y, t_2)? As far as the data (n, t_1, t_2) is concerned, such an n-zigzag may exist only if

$$t_1 \text{ and } t_2 \text{ are integer multiples of } n^{-1} \, . \tag{4}$$

We say that data $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}^2_<$ is a *compatible triple* if it verifies the last condition. We will consistently impose this condition, whenever we seek to study n-zigzags whose lifetime is $[t_1, t_2]$.

The use of compatible triples should be considered to be a fairly minor, microscopic, detail. As the index n increases, the n^{-1}-mesh becomes finer, so that the space of n-zigzags better approximates a field of functions, defined on arbitrary finite intervals of the vertical coordinate, and taking values in the horizontal coordinate.

An important piece of notation associated to a compatible triple is $t_{1,2}$, which we will use to denote the difference $t_2 - t_1$. The law of the underlying Brownian ensemble $B : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is invariant under integer shifts in the first, curve indexing, coordinate. This translates to a distributional invariance of scaled objects under vertical shifts by multiples of n^{-1}, something that makes the parameter $t_{1,2}$ of far greater relevance than t_1 or t_2.

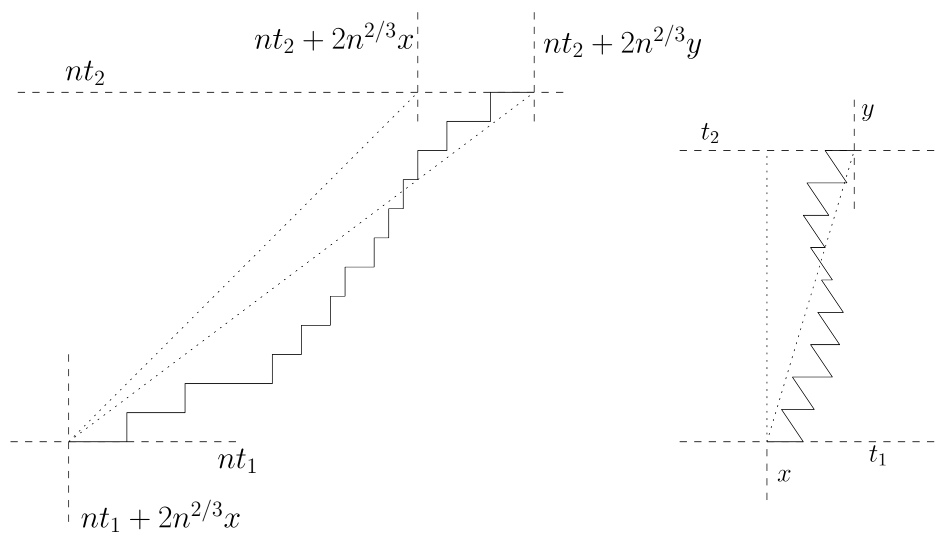

FIGURE 1. Let (n, t_1, t_2) be a compatible triple and let $x, y \in \mathbb{R}$. The endpoints of the geodesic in the left sketch have been selected so that, when the scaling map R_n is applied to produce the right sketch, the result is an n-polymer from (x, t_1) and (y, t_1).

1.3.7. *Staircase energy scales to zigzag weight.* Let $n \in \mathbb{N}$ and $(i,j) \in \mathbb{N}^2_<$. Any n-zigzag Z from $(x, i/n)$ to $(y, j/n)$ is ascribed a scaled energy, which we will refer to as its weight, $\mathsf{Wgt}(Z) = \mathsf{Wgt}_n(Z)$, given by

$$\mathsf{Wgt}(Z) = 2^{-1/2} n^{-1/3} \Big(E(S) - 2(j-i) - 2n^{2/3}(y-x) \Big) \tag{5}$$

where Z is the image under R_n of the staircase S.

1.3.8. *Maximum weight.* Let $n \in \mathbb{N}$. The quantity $\mathsf{Wgt}_{n;(x,0)}^{(y,1)}$ specified in (1) is nothing other than the maximum weight ascribed to any n-zigzag from $(x,0)$ to $(y,1)$.

Let $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}^2_<$ be a compatible triple. Suppose that $x, y \in \mathbb{R}$ satisfy $y \geq x - 2^{-1} n^{1/3} t_{1,2}$. We now offer a definition of $\mathsf{Wgt}_{n;(x,t_1)}^{(y,t_2)}$ such that this quantity equals maximum weight of any n-zigzag from (x, t_1) to (y, t_2). We must set

$$\mathsf{Wgt}_{n;(x,t_1)}^{(y,t_2)} = 2^{-1/2} n^{-1/3} \Big(M^1_{(nt_1 + 2n^{2/3}x, nt_1) \to (nt_2 + 2n^{2/3}y, nt_2)} - 2nt_{1,2} - 2n^{2/3}(y-x) \Big) . \tag{6}$$

1.3.9. *Highest weight zigzags are called polymers.* An n-zigzag that attains this maximum will be called an n-polymer, or usually, simply a polymer. Thus, geodesics map to polymers under the scaling map. For given endpoints, almost sure uniqueness of the polymer is known by [Ham17c, Lemma 4.6(1)]. We would thus be at liberty to denote the unique polymer from (x,t_1) to (y,t_2) by $\rho_{n;(x,t_1)}^{(y,t_2)}$. This polymer is depicted in Figure 1. This uniqueness is not needed in the present article; however, we will have cause (on just a few occasions) to allude to polymers. Preferring to leave polymer uniqueness unaddressed, our convention here will be to label *any* n-polymer (x,t_1) to (y,t_2) by $\rho_{n;(x,t_1)}^{(y,t_2)}$.

1.3.10. *The scaling principle.* Let $(n,t_1,t_2) \in \mathbb{N} \times \mathbb{R}^2_<$ be a compatible triple. The quantity $nt_{1,2}$ is a positive integer, in view of the defining property (4). The scaling map $R_k : \mathbb{R}^2 \to \mathbb{R}^2$ has been defined whenever $k \in \mathbb{N}$, and thus we may speak of R_n and $R_{nt_{1,2}}$. The map R_n is the composition of $R_{nt_{1,2}}$ and the transform $S_{t_{1,2}^{-1}}$ given by $\mathbb{R}^2 \to \mathbb{R}^2 : (a,b) \to \big(at_{1,2}^{-2/3}, bt_{1,2}^{-1}\big)$. That is, the system of $nt_{1,2}$-zigzags is transformed into the system of n-zigzags by an application of $S_{t_{1,2}^{-1}}$. Note from (6) that $\mathsf{Wgt}_{n;(x,t_1)}^{(y,t_2)} = t_{1,2}^{1/3} \mathsf{Wgt}_{nt_{1,2};(x,\kappa)}^{(y,\kappa+1)}$, where $\kappa = t_1 t_{1,2}^{-1}$; indeed this weight transformation law is valid for all zigzags, rather than just polymers, in view of (5).

We may summarise these inferences by saying that the system of $nt_{1,2}$-zigzags, including their weight data, is transformed into the n-zigzag system, and its accompanying weight data, by the transformation $\big(a,b,c\big) \to \big(at_{1,2}^{-1/3}, bt_{1,2}^{-2/3}, ct_{1,2}^{-1}\big)$, where the coordinates refer to the changes suffered in weight, horizontal and vertical coordinates. This fact leads us to what we call the *scaling principle.*

The scaling principle. Let $(n,t_1,t_2) \in \mathbb{N} \times \mathbb{R}^2_<$ be a compatible triple. Any statement concerning the system of n-zigzags, including weight information, is equivalent to the corresponding statement concerning the system of $nt_{1,2}$-zigzags, provided that the following changes are made:

- the index n is replaced by $nt_{1,2}$;
- any weight is multiplied by $t_{1,2}^{-1/3}$;
- and any horizontal distance is multiplied by $t_{1,2}^{-2/3}$.

1.3.11. *The scaling principle applied: uniform control on polymer weight for a general time-pair.* Proposition 1.4 provides a uniform control on polymer weights whose starting and ending points lie in $[x,x+1] \times \{0\}$ and $[y,y+1] \times \{1\}$. We now illustrate the scaling principle by using it to extend the proposition to treat the case where these intervals are replaced by $[x,x+t_{1,2}^{2/3}] \times \{t_1\}$ and $[y,y+t_{1,2}^{2/3}] \times \{t_2\}$ for a general time pair (t_1,t_2).

We phrase this more general result as an upper bound on the probability of a *polymer weight regularity* event. To define the new event, we again consider a compatible triple $(n,t_1,t_2) \in \mathbb{N} \times \mathbb{R}^2_<$. For $x,y \in \mathbb{R}$, $w_1,w_2 \geq 0$ and $r > 0$, let $\mathsf{PolyWgtReg}_{n;([x,x+w_1],t_1)}^{([y,y+w_2],t_2)}(r)$ denote the event that, for all $(u,v) \in [0,w_1] \times [0,w_2]$,

$$\Big| t_{1,2}^{-1/3} \mathsf{Wgt}_{n;(x+u,t_1)}^{(y+v,t_2)} + 2^{-1/2} t_{1,2}^{-4/3} \big(y+v-x-u\big)^2 \Big| \leq r \, .$$

Corollary 1.5. *Let $(n,t_1,t_2) \in \mathbb{N} \times \mathbb{R}^2_<$ be a compatible triple. Suppose further that the integer $nt_{1,2}$ is even and that it is at least $10^{29} \vee 2(c/3)^{-18}$. Let $x,y \in \mathbb{R}$, and let $a,b \in \mathbb{N}$ be positive, such* *that* $\big|x-y\big|t_{1,2}^{-2/3}+\max\{a,b\}-1\le 3^{-1}2^{-2/3}c(nt_{1,2})^{1/18}$. *Let* $r\in\big[33\,,\,4(nt_{1,2})^{1/18}\big]$. *Then*

$$\mathbb{P}\Big(\neg\,\mathsf{PolyWgtReg}_{n;([x,x+at_{1,2}^{2/3}],t_1)}^{([y,y+bt_{1,2}^{2/3}],t_2)}(r)\Big)\le ab\cdot 400C\exp\big\{-c_1 2^{-8}r^{3/2}\big\}\,.$$

Proof. It is immediate from Proposition 1.4 that

$$\mathbb{P}\Big(\neg\,\mathsf{PolyWgtReg}_{n;([x,x+1],0)}^{([y,y+1],1)}(r)\Big)\le 400C\exp\big\{-c_1 2^{-8}r^{3/2}\big\}\,. \tag{7}$$

when $n\ge 10^{29}\vee 2(c/3)^{-18}$, $\big|x-y\big|\le 3^{-1}2^{-2/3}cn^{1/18}$ and $r\in\big[33\,,\,4n^{1/18}\big]$.

By the scaling principle, we know that

$$\mathbb{P}\Big(\mathsf{PolyWgtReg}_{n;([x,x+t_{1,2}^{2/3}],t_1)}^{([y,y+t_{1,2}^{2/3}],t_2)}(r)\Big)\;=\;\mathbb{P}\bigg(\mathsf{PolyWgtReg}_{nt_{1,2};([0,1],0)}^{\big(\big[(y-x)t_{1,2}^{-2/3},(y-x)t_{1,2}^{-2/3}+1\big],1\big)}(r)\bigg)\,.$$

Consider (7) with parameter settings $\mathbf{n}=nt_{1,2}$, $\mathbf{x}=0$, $\mathbf{y}=(y-x)t_{1,2}^{-2/3}$ and $\mathbf{r}=r$. (The use of boldface in this notation is discussed in the next subsection, after this proof closes.) We find then that

$$\mathbb{P}\Big(\mathsf{PolyWgtReg}_{n;([x,x+t_{1,2}^{2/3}],t_1)}^{([y,y+t_{1,2}^{2/3}],t_2)}(r)\Big)\le 400C\exp\big\{-c_1 2^{-8}r^{3/2}\big\} \tag{8}$$

provided that $nt_{1,2}\ge 10^{29}\vee 2(c/3)^{-18}$, $\big|x-y\big|t_{1,2}^{-2/3}\le 3^{-1}2^{-2/3}c(nt_{1,2})^{1/18}$ and $r\in\big[33\,,\,4(nt_{1,2})^{1/18}\big]$.

This last bound is then summed over the ab choices of pairs

$$\big(\mathbf{x},\mathbf{y}\big)\in\big\{x,x+t_{1,2}^{2/3},\cdots,x+(a-1)t_{1,2}^{2/3}\big\}\times\big\{y,y+t_{1,2}^{2/3},\cdots,y+(b-1)t_{1,2}^{2/3}\big\}$$

in order to obtain the corollary. Note that we hypothesise that $\big|x-y\big|t_{1,2}^{-2/3}+\max\{a,b\}-1$ be at most $3^{-1}2^{-2/3}c(nt_{1,2})^{1/18}$ in order that the bound (8) be valid for these parameter choices. □

Actually, when we apply this corollary in the present article, it will be in the case that $t_1=0$ and $t_1=1$, so in fact the use of the scaling principle is unnecessary in this regard. It is useful, however, to have a general form for the corollary: for example, it is used in [Ham17b].

1.3.12. *Parameter settings in applications of results will be indicated in boldface.* We will often apply results involving several parameters. Typically these include the index n, times t_1 and t_2, and spatial locations such as x and y. Whenever we apply results, we will always state what these parameter settings are. When we do so, we will use boldface to indicate the variables in the result being applied, expressing these in terms of non-boldface variables, which assume their values from the current context. This device permits notational conflicts to be deescalated (so that notational choices need not proliferate, as they would were these conflicts to be eliminated by other means): such statements as $\mathbf{n}=nt_{1,2}$ in the preceding proof may be made without any ambiguity in meaning.

1.3.13. *Polymer concatenation.* Let $n\in\mathbb{N}$ and $(t_1,t_2,t_3)\in\mathbb{R}^3_\le$ be such that (n,t_1,t_2) and (n,t_2,t_3) be compatible triples. Let $x,y,z\in\mathbb{R}$. In accordance with the convention stated in Subsection 1.3.9, we may consider n-polymers $\rho_{n;(x,t_1)}^{(y,t_2)}$ and $\rho_{n;(y,t_2)}^{(z,t_3)}$. The union of these two subsets of $\mathbb{R}^2$ is clearly an n-zigzag from (x,t_1) and (z,t_3). In the union, the journey over the latter polymer follows that over the former. For this reason, we use regard the union polymer as the concatenation of the two given polymers, and denote it by $\rho_{n;(x,t_1)}^{(y,t_2)}\circ\rho_{n;(y,t_2)}^{(z,t_3)}$. The new polymer's weight equals $\mathsf{Wgt}_{n;(x,t_1)}^{(y,t_2)}+\mathsf{Wgt}_{n;(y,t_2)}^{(z,t_3)}$.

1.3.14. *Polymer splitting.* Opposite to the operation of polymer concatenation is the splitting of a given polymer into two pieces. Let $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}^2_\leq$ is a compatible triple, and let $(x, y) \in \mathbb{R}^2$ satisfy $y \geq x - 2^{-1}n^{1/3}t_{1,2}$. Let $t \in (t_1, t_2)$ be such that (n, t_1, t) and (n, t, t_2) are also compatible triples. For any polymer $\rho_{n;(x,t_1)}^{(y,t_2)}$, we may select an element $(z, t) \in \rho_{n;(x,t_1)}^{(y,t_2)}$. The removal of (z, t) from $\rho_{n;(x,t_1)}^{(y,t_2)}$ creates two connected components. Taking the closure of each of these amounts to adding the point (z, t) to each of them. The resulting sets are n-zigzags from (x, t_1) to (z, t), and from (z, t) to (y, t_2); indeed, it is straightforward to see that they are n-polymers.

1.3.15. *Polymer crossing and rewiring.* We now make some comments about the implications of the event that two polymers cross. We do so for line-to-point polymers. For $\overline{\Psi} \in (0, \infty)^3$, let $f \in \mathcal{I}_{\overline{\Psi}}$. Let $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}^2_\leq$ be a compatible triple, and let $y \in \mathbb{R}$. An n-zigzag ϕ from (x, t_1), where $x \in \mathbb{R}$, to (y, t_2), whose f-rewarded weight $\mathsf{Wgt}(\phi) + f(x)$ attains the maximum value $\mathsf{Wgt}_{n;(*:f,t_1)}^{(y,t_2)}$, is called an f-rewarded line-to-point polymer. Such polymers are born free, but not equal: an endowment of f is bestowed according to the place of birth. Pursuing a similar convention to that used in the point-to-point case, any such polymer will be denoted by $\rho_{n;(*:f,t_1)}^{(y,t_2)}$. The almost sure uniqueness of this polymer for given $y \in \mathbb{R}$ is assured by [Ham17c, Lemma 4.6(2)], but this information is irrelevant for our present study.

Suppose that two f-rewarded line-to-point polymers cross. That is, suppose that $(x_1, x_2) \in \mathbb{R}^2_<$ and $(y_1, y_2) \in \mathbb{R}^2_<$ are such that there exist such polymers, labelled $\rho_{n;(*:f,t_1)}^{(y_1,t_2)}$ and $\rho_{n;(*:f,t_1)}^{(y_2,t_2)}$ by our convention, whose journeys are $(x_2, t_1) \to (y_1, t_2)$ and $(x_1, t_1) \to (y_2, t_2)$. The two polymers necessarily meet, and indeed the union of the horizontal segments of the two polymers also meet. If (z, t) is such a point of intersection, the operation of polymer splitting at (z, t) may be applied to the two polymers, resulting in decompositions that may be respectively labelled $\rho_1 \circ \rho_2$ and $\rho_3 \circ \rho_4$. The zigzags ρ_1 and ρ_3 share their f-rewarded weights, $\mathsf{Wgt}(\rho_1) + f(x_2)$ and $\mathsf{Wgt}(\rho_3) + f(x_1)$, because the weight maximality of $\rho_1 \circ \rho_2$ and $\rho_3 \circ \rho_4$ each enforce one of the two inequalities between these quantities. Thus, $\rho_1 \circ \rho_2$ and $\rho_3 \circ \rho_2$ share their f-rewarded weight, and so do $\rho_3 \circ \rho_4$ and $\rho_1 \circ \rho_4$. The new, rewired, zigzags $\rho_3 \circ \rho_2$ and $\rho_1 \circ \rho_4$ are thus seen to be f-rewarded line-to-point polymers.

In summary, when two f-rewarded line-to-point polymers cross, the rewiring just undertaken results in an alternative pair of such polymers so that the old pair and the new share their set of starting and ending points. When the almost sure uniqueness of f-rewarded polymers ending at a given point is known, this observation may be developed to prove the absence of such crossing among these polymers, and to build a picture of a tree of coalescing f-rewarded polymers. We do not develop these ideas here, but we will make use of the rewiring of crossing polymers on one occasion.

1.3.16. *Some basic remarks concerning polymer weight.*

Lemma 1.6. *Let $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}^2_<$ be a compatible triple.*

(1) *The random function $(x, y) \to \mathsf{Wgt}_{n;(x,t_1)}^{(y,t_2)}$, which is defined on the set of $(x, y) \in \mathbb{R}^2$ satisfying $y \geq x - 2^{-1}n^{1/3}t_{1,2}$, is continuous almost surely.*

(2) *Further consider an intermediate time $t \in (t_1, t_2)$ such that (n, t_1, t) and (n, t, t_2) are compatible triples. Let $x, y, z \in \mathbb{R}$. Then*

$$\mathsf{Wgt}_{n;(x,t_1)}^{(y,t_2)} \geq \mathsf{Wgt}_{n;(x,t_1)}^{(z,t)} + \mathsf{Wgt}_{n;(z,t)}^{(y,t_2)} \,,$$

provided that these three weights are well defined. (The explicit conditions that ensure that the definitions make sense are $y \geq x - 2^{-1}n^{1/3}t_{1,2}$*,* $z \geq x - 2^{-1}n^{1/3}(t-t_1)$ *and* $y \geq z - 2^{-1}n^{1/3}(t_2 - t)$*.)*

(3) *Let* $\overline{\Psi} \in (0,\infty)^3$ *and* $f \in \mathcal{I}_{\overline{\Psi}}$*. Suppose that* $n \in \mathbb{N}$ *satisfies* $n > 2^{-3/2}\Psi_1^3 \vee 8(\Psi_2+1)^3$*. Then* $[-1,1] \to \mathbb{R} : y \to \mathsf{Wgt}_{n;(*:f,0)}^{(y,1)}$ *is almost surely finite and continuous.*

Proof. **(1):** In light of the definition (6) of polymer weight in terms of maximum energy, it is enough to prove, for each $i,j \in \mathbb{N}$, $i \leq j$, that the maximum energy $M^1_{(x,i)\to(y,j)}$ is a continuous function of the pair $(x,y) \in \mathbb{R}^2_{\leq}$. To treat continuity in the first variable, let x_1 and x_2 satisfy $x_1 \leq x_2 \leq y$. It is a simple matter to verify that

$$B(i,x_2) - B(i,x_1) \leq M^1_{(x_1,i)\to(y,j)} - M^1_{(x_2,i)\to(y,j)} \leq \sup_{k \in [\![i,j]\!]} M^1_{(x_1,i)\to(x_2,k)}\,;$$

the latter quantity may then be bounded above by noting that

$$\sup_{k\in[\![i,j]\!]} M^1_{(x_1,i)\to(x_2,k)} \leq \sum_{k=i}^{j} \Big(\sup_{z\in[x_1,x_2]} B(k,z) \, - \inf_{z\in[x_1,x_2]} B(k,z)\Big)\,.$$

In this way, we see that the continuity of $M^1_{(x,i)\to(y,j)}$ in the x-variable follows from the continuity of the two-sided Brownian motions $B(k,\cdot)$. Similar considerations dictate this continuity also in the y-variable, and these two continuities demonstrate the sought joint continuity of the energy maximum.

(2): Taking two polymers $\rho_{n;(x,t_1)}^{(z,t)}$ and $\rho_{n;(z,t)}^{(y,t_3)}$, whose endpoints are dictated by our convention, we may note that the weight of the two polymers' concatenation, which is $\mathsf{Wgt}_{n;(x,t_1)}^{(z,t)} + \mathsf{Wgt}_{n;(z,t)}^{(y,t_3)}$, offers a lower bound on $\mathsf{Wgt}_{n;(x,t_1)}^{(y,t_2)}$.

(3): In the proof of [Ham17c, Lemma 4.6(2)], which appears in [Ham17c, Appendix A], it is noted that $\mathsf{W}_{n;(*:f,0)}^{(y,1)}$ equals

$$2^{-1/2}n^{-1/3} \sup_{u\in(-\infty,n+2n^{2/3}y]} \Big(M^1_{(u,0)\to(n+2n^{2/3}y,n)} - n - 2n^{2/3}y + u + h(u)\Big)\,.$$

Here, $h:\mathbb{R}\to\mathbb{R}\cup\{-\infty\}$ is given by $h(x) = 2^{1/2}n^{1/3}f\big(n^{-2/3}x/2\big)$. Using a notation for unscaled line-to-point energy, namely

$$M^1_{(*:g,0)\to(y,n)} := \sup\Big\{ E(\phi) + g(x) : \phi \in D^1_{(x,0)\to(y,n)}\,,\, x \leq y \Big\}\,,$$

the quantity $\mathsf{W}_{n;(*:f,0)}^{(y,1)}$ is seen to equal $2^{-1/2}n^{-1/3}M^1_{(*:g,0)\to(n+2n^{2/3}y,n)}$ where the function g is given by $g(u) = -n - 2n^{2/3}y + u + h(u)$. It is further noted in the same proof that $\limsup_{u\to-\infty} g(u)/|u| < 0$ is satisfied when $n > 2^{-3/2}\Psi_1^3$; and that, since $f \in \mathcal{I}_{\overline{\Psi}}$, the condition that $g(u) > -\infty$ for some $u \leq n + 2n^{2/3}y$ (where $y \in [-1,1]$ is given) is verified when $n \geq 8(\Psi_2 - y)^3$. We may thus apply [Ham17c, Lemma A.2] to learn that $\mathsf{W}_{n;(*:f,0)}^{(y,1)}$ almost surely assumes finite real values whenever $y \in [-1,1]$ under our present hypotheses. Regarding the continuity of this function of $y \in [-1,1]$, note first that, for n given satisfying these hypotheses, the location of the maximizer '$*:f$' is tight as f varies over $\mathcal{I}_{\overline{\Psi}}$: this is a consequence of the square-root growth of $M^1_{(x,0)\to(y,n)}$ in the variable $y-x$, which is explained in and after equation (2.7) of [Ham17c], which growth cannot compete with the linear decrease in the function g. The question of the continuity of $y \to \mathsf{W}^{(y,1)}_{n;(*:f,0)}$ has in essence been reduced to the first part of the present lemma; we omit the details of this reduction. $\square$

1.4. **Polymer weight profiles as the uppermost curves in line ensembles.** Let $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}^2_\leq$ be a compatible triple and let $x \in \mathbb{R}$. Define the forward polymer weight profile

$$\mathcal{L}^{\uparrow;t_2}_{(x,t_1)}(1,\cdot) : \big[x - 2^{-1}n^{1/3}t_{1,2}, \infty\big) \to \mathbb{R}$$

with base-point (x, t_1) and end height t_2 by setting $\mathcal{L}^{\uparrow;t_2}_{(x,t_1)}(1,y) = \mathsf{Wgt}^{(y,t_2)}_{(x,t_1)}$ for $y \geq x - 2^{-1}n^{1/3}t_{1,2}$. We call this weight profile 'forward', and adorn the notation with the symbol $\uparrow$, to reflect that it is the spatial location y associated to the more advanced time t_2 that is treated as the variable: we stand at (x, t_1) and look forward in time to witness the weight profile as a function of $(\cdot, t_2)$.

Retaining the triple $(n, t_1, t_2) \in \mathbb{N} \times \mathbb{R}^2_\leq$ but now fixing $y \in \mathbb{R}$ (and treating $x \in \mathbb{R}$ as a variable), we also introduce the *backward* polymer weight profile

$$\mathcal{L}^{\downarrow;(y,t_2)}_{n;t_1}(1,\cdot) : \big(-\infty, y + 2^{-1}n^{1/3}t_{1,2}\big] \to \mathbb{R}$$

with base-point (y, t_2) and end height t_1 by setting $\mathcal{L}^{\downarrow;(x,t_2)}_{n;t_1}(1,x) = \mathsf{Wgt}^{(y,t_2)}_{(x,t_1)}$ for each $x \leq y + 2^{-1}n^{1/3}t_{1,2}$. In other words, we now stand at (y, t_2) and look backwards in time at those polymers, ending at our location, which begin at time (and height) t_1; it is the weight profile of these polymers that is being recorded.

We have just made two elaborate looking definitions that seem merely to introduce further notation to describe an already denoted object, the weight profile $\mathsf{Wgt}^{(y,t_2)}_{(x,t_1)}$ viewed as a function either of x or y. The conceptual significance of the new notation is suggested by our insistence on calling the argument of either profile '$(1,\cdot)$' rather than simply '$(\cdot)$'. Indeed, as we will see shortly, we will view $\mathcal{L}^{\uparrow;t_2}_{(x,t_1)}$ as an ensemble of $nt_{1,2}+1$ curves of which the lowest indexed curve, just defined, is the uppermost; and the backward object is just the same. Either ensemble collectively has a resampling property – the Brownian Gibbs property – which is a fundamental tool for the analysis of the weight profile that is its uppermost curve.

It is useful to retain a vivid picture of both of the processes $\mathcal{L}^{\uparrow;t_2}_{(x,t_1)}(1,\cdot)$ and $\mathcal{L}^{\downarrow;(x,t_2)}_{n;t_1}(1,\cdot)$ as random curves that locally resemble Brownian motion but that globally follow the shape of a parabola. The parabola in question is $-2^{-1/2}(y-x)^2 t_{1,2}^{-4/3}$. The forward process adopts values of order $t_{1,2}^{1/3}$ for argument values y that differ from x by order $t_{1,2}^{2/3}$, and is forced downwards rapidly by parabolic curvature outside this region. When $t_{1,2}$ is small, for example, the weight profile is sharply peaked, and it broadens out as $t_{1,2}$ rises.

(This description neglects the role of the index n, but roughly it develops accuracy as n rises. We will discuss this matter a little further soon.)

It is valuable to bring the forward and backward weight profiles for differing values of $t_{1,2}$ on to the same footing, by using a parabolic change of coordinates that, for example, flattens out the sharp peak witnessed when $t_{1,2}$ is small. The coordinate change will also bring the peak centre to the origin (from x or y, according to the forward or backward case). The above weight profiles are already scaled objects, and so we introduce the term *normalized* to refer to the profiles viewed after this new, $t_{1,2}$-determined, change of coordinates.

Indeed, we define the normalized forward polymer weight profile

$$\mathsf{Nr}\mathcal{L}^{\uparrow;t_2}_{n;(x,t_1)}(1,\cdot) : \big[-2^{-1}(nt_{1,2})^{1/3},\infty\big) \to \mathbb{R}\,,$$

by setting

$$\mathsf{Nr}\mathcal{L}^{\uparrow;t_2}_{n;(x,t_1)}\big(1,z\big) = t_{1,2}^{-1/3}\mathcal{L}^{\uparrow;t_2}_{n;(x,t_1)}\big(x + t_{1,2}^{2/3}z\big)\,.$$

Its backwards counterpart

$$\mathsf{Nr}\mathcal{L}^{\downarrow;(y,t_2)}_{n;t_1}(1,\cdot) : \big(-\infty, 2^{-1}(nt_{1,2})^{1/3}\big] \to \mathbb{R}$$

is obtained by setting $\mathsf{Nr}\mathcal{L}^{\downarrow;(y,t_2)}_{n;t_1}(1,z) = t_{1,2}^{-1/3}\mathcal{L}^{\downarrow;(y,t_2)}_{n;t_1}\big(1, y + t_{1,2}^{2/3}z\big)$.

Brownian motion is invariant under the parabolic change of coordinates, while the parabola $x \to -2^{-1/2}(y-x)^2 t_{1,2}^{-4/3}$ maps to $x \to -2^{1/2}x^2$. Thus, our normalized processes should be pictured as locally Brownian as before, but with curvature dictated by the curve $-2^{-1/2}x^2$. This picture in fact expands its domain of validity as the index increases, encompassing an expanding region about the origin, where the relevant indexing variable is now $nt_{1,2}$, rather than n. These heuristic comments find rigorously expressed counterparts in the next section.

1.5. **Brownian Gibbs ensembles and a regularity property.** Our weight profiles $\mathcal{L}^{\uparrow;t_2}_{(x,t_1)}(1,\cdot)$, $\mathsf{Nr}\mathcal{L}^{\uparrow;t_2}_{n;(x,t_1)}(1,\cdot)$, and their backward counterparts, may be embedded as uppermost curves in systems (or 'ensembles') of random curves. (In [Ham17b, Figure 3], such a scaled and a normalized forward ensemble are illustrated.) The normalized forward and backward ensembles satisfy the Brownian Gibbs property; moreover, these objects adhere well enough to the informal description of being locally Brownian and globally parabolic that we describe them as *regular* ensembles.

In the next few paragraphs:

- we specify in Definition 1.7 what it means for a Brownian Gibbs ensemble to be regular;
- in Proposition 1.8, we cite the result that shows that our normalized ensembles are regular;
- and in Proposition 1.9, we cite four further properties, needed for our proofs, that regular ensembles enjoy.

Note that two natural questions do not appear here:

- What is the definition of the curves, indexed by higher values $k > 1$, in ensembles such as $\mathcal{L}^{\uparrow;t_2}_{(x,t_1)}(k,\cdot)$ and $\mathsf{Nr}\mathcal{L}^{\uparrow;t_2}_{n;(x,t_1)}(k,\cdot)$?
- What does it mean to call an ensemble Brownian Gibbs?

Logically, it is not necessary to answer these questions here. In the first case, this is because in this article we only use results concerning regular ensembles that concern their lowest indexed curve. In the second, it is because Definition 1.7 and Proposition 1.9 provide all the information we will use about regular ensembles. However, it may be useful for the reader to have a sense in overview of these important ideas. Before turning to a precise discussion of what is really needed, we thus offer informal answers to the above two questions.

Embedding weight profiles into ensembles as the uppermost curve. For $(i,j) \in \mathbb{N}^2_{\leq}$ and $(x,y) \in \mathbb{R}^2_{\leq}$, recall that $M^1_{(x,i)\to(y,j)}$ is the energy maximum over staircases from (x,i) to (y,j). It has a counterpart $M^k_{(x,i)\to(y,j)}$, the maximum collective energy of a k-tuple of staircases with these endpoints satisfying a natual disjointness condition. The scaling relation (6) is extended to define the weight associated to the maximizing k-tuple after scaling. This weight is then defined to equal the sum (over i) of the k lowest indexed ensemble curves $\mathcal{L}^{\uparrow;t_2}_{(x,t_1)}(i,\cdot)$ evaluated at $\cdot = y$.

Brownian Gibbs line ensembles in overview. Let $n \in \mathbb{N}$ and let I be a closed interval in the real line. A $[\![1,n]\!]$-indexed line ensemble defined on I is a random collection of continuous curves $\mathcal{L} : [\![1,n]\!] \times I \to \mathbb{R}$ specified under a probability measure $\mathbb{P}$. The i^{th} curve is thus $\mathcal{L}(i,\cdot) : I \to \mathbb{R}$. (The adjective 'line' has been applied to these systems perhaps because of their origin in such models as Poissonian LPP, where the counterpart object has piecewise constant curves. We will omit it henceforth.) An ensemble is called *ordered* if $\mathcal{L}(i,x) > \mathcal{L}(i+1,x)$ whenever $i \in [\![1,n-1]\!]$ and x lies in the interior of I. The curves may thus assume a common value at any finite endpoint of I. We will consider ordered ensembles that satisfy a condition called the Brownian Gibbs property. Colloquially, we may say that an ordered ensemble is called Brownian Gibbs if it arises from a system of Brownian bridges or Brownian motions defined on I by conditioning on the mutual avoidance of the curves at all times in I.

1.5.1. *Defining (c,C)-regular ensembles.* The next definition specifies a $(\bar{\phi},c,C)$-regular ensemble from [Ham17a, Definition 2.4], in the special case where the vector $\bar{\phi}$ equals $(1/3,1/9,1/3)$.

Definition 1.7. Consider a Brownian Gibbs ensemble of the form

$$\mathcal{L} : [\![1,N]\!] \times \big[-z_{\mathcal{L}},\infty\big) \to \mathbb{R}\,,$$

and which is defined on a probability space under the law $\mathbb{P}$. The number $N = N(\mathcal{L})$ of ensemble curves and the absolute value $z_{\mathcal{L}}$ of the finite endpoint may take any values in $\mathbb{N}$ and $[0,\infty)$.

Let C and c be two positive constants. The ensemble $\mathcal{L}$ is said to be (c,C)-regular if the following conditions are satisfied.

(1) **Endpoint escape.** $z_{\mathcal{L}} \geq cN^{1/3}$.

(2) **One-point lower tail.** If $z \in [-z_{\mathcal{L}},\infty)$ satisfies $|z| \leq cN^{1/9}$, then

$$\mathbb{P}\Big(\mathcal{L}\big(1,z\big) + 2^{-1/2}z^2 \leq -s\Big) \leq C\exp\big\{-cs^{3/2}\big\}$$

for all $s \in \big[1,N^{1/3}\big]$.

(3) **One-point upper tail.** If $z \in [-z_{\mathcal{L}},\infty)$ satisfies $|z| \leq cN^{1/9}$, then

$$\mathbb{P}\Big(\mathcal{L}\big(1,z\big) + 2^{-1/2}z^2 \geq s\Big) \leq C\exp\big\{-cs^{3/2}\big\}$$

for all $s \in [1,\infty)$.

A Brownian Gibbs ensemble of the form

$$\mathcal{L} : [\![1,N]\!] \times \big(-\infty,z_{\mathcal{L}}\big] \to \mathbb{R}$$

is also said to be (c,C)-regular if the reflected ensemble $\mathcal{L}(\cdot,-\cdot)$ is. This is equivalent to the above conditions when instances of $[-z_{\mathcal{L}},\infty)$ are replaced by $(-\infty,z_{\mathcal{L}}]$.

We will refer to these three regular ensemble conditions as Reg(1), Reg(2) and Reg(3).

1.5.2. *The normalized forward and backward ensembles are (c,C)-regular.* Our assertion to this effect is [Ham17b, Proposition 4.2].

Proposition 1.8. *Let $(n,t_1,t_2) \in \mathbb{N} \times \mathbb{R}^2_<$ be a compatible triple, and let $x \in \mathbb{R}$. The normalized forward weight profile $\mathsf{Nr}\mathcal{L}^{\uparrow;t_2}_{n;(x,t_1)}(1,\cdot)$, defined on $\big[-2^{-1}(nt_{1,2})^{1/3},\infty\big)$, may be represented as the lowest indexed curve in an ensemble*

$$\mathsf{Nr}\mathcal{L}^{\uparrow;t_2}_{n;(x,t_1)} : [\![1, nt_{1,2}+1]\!] \times \big[-2^{-1}(nt_{1,2})^{1/3},\infty\big) \to \mathbb{R}$$

that enjoys the Brownian Gibbs property. Denoting this ensemble by $\mathcal{L}$, we naturally have $N(\mathcal{L}) = nt_{1,2}+1$ and $z_{\mathcal{L}} = 2^{-1}(nt_{1,2})^{1/3}$.

There exist positive constants C and c, which may be chosen independently of all such choices of the parameters t_1, t_2, x and n, such that the ensemble $\mathcal{L}$ is (c,C)-regular.

Similarly, the backward weight profile $\mathsf{Nr}\mathcal{L}^{\downarrow;(y,t_2)}_{n;t_1}(1,\cdot) : \big(-\infty, 2^{-1}(nt_{1,2})^{1/3}\big] \to \mathbb{R}$ may be embedded in an ensemble

$$\mathsf{Nr}\mathcal{L}^{\downarrow;(y,t_2)}_{n;t_1}(1,\cdot) : [\![1, nt_{1,2}+1]\!] \times \big(-\infty, 2^{-1}(nt_{1,2})^{1/3}\big] \to \mathbb{R}\,.$$

This new ensemble also enjoys the properties just described for its forward counterpart, uniformly in the concerned parameters.

The values of the positive constants c and C are fixed throughout in the role specified by this proposition.

1.5.3. *Non-trivial properties of (c,C)-regular ensembles.* Recall from Subsection 1.2.2 that $c_1 = 2^{-5/2}c \wedge 2^{-3}$.

Proposition 1.9. *Suppose that $\mathcal{L} = \mathcal{L}_N$, mapping either $[\![1,N]\!] \times [-z_{\mathcal{L}},\infty)$ or $[\![1,N]\!] \times (-\infty, z_{\mathcal{L}}]$, to $\mathbb{R}$, is a (c,C)-regular ensemble, where $N \in \mathbb{N}$ and $z_{\mathcal{L}} \geq 0$.*

(1) *(Uniform curve lower bound) Whenever $(t,r,y) \in \mathbb{R}$ satisfy $N \geq (c/3)^{-18} \vee 6^{36}$, $t \in \big[0, N^{1/18}\big]$, $r \in \big[2^{3/2}\,, 2N^{1/18}\big]$ and $|y| \leq 2^{-1}cN^{1/18}$,*

$$\mathbb{P}\Big(\inf_{x \in [y-t,y+t]} \big(\mathcal{L}_N(1,x) + 2^{-1/2}x^2\big) \leq -r \Big) \;\leq\; \Big(t \vee (3-2^{3/2})^{-1}5^{1/2}\Big) \cdot 10C \exp\big\{-c_1 r^{3/2}\big\}\,.$$

(2) *(No Big Max) For $|y| \leq 2^{-1}cN^{1/9}$, $r \in \big[0, 4^{-1}cN^{1/9}\big]$, $t \in \big[2^{7/2}, 2N^{1/3}\big]$ and $N \geq c^{-18}$,*

$$\mathbb{P}\Big(\sup_{x \in [y-r,y+r]} \big(\mathcal{L}_N(1,x) + 2^{-1/2}x^2\big) \geq t \Big) \leq (r+1) \cdot 6C \exp\big\{-2^{-9/2}ct^{3/2}\big\}\,.$$

(3) *(Local curve regularity) For $[x,x+\delta]$ a subset of $[-z_{\mathcal{L}},\infty)$, or of $(-\infty,z_{\mathcal{L}}]$, define the uppermost curve modulus of continuity*

$$\omega_{1,[x,x+\delta]}\big(\mathcal{L}_N,\delta\big) \;=\; \sup\Big\{ \big|\mathcal{L}_N(1,x+s) - \mathcal{L}_N(1,x)\big| : s \in [0,\delta] \Big\}\,.$$

For $x \geq -z_{\mathcal{L}} + 2$ and $t > 0$, define

$$\mathsf{G}_t(x) = \bigcap_{x-2 \leq y \leq x+2} \Big\{ \mathcal{L}_N(1,y) + 2^{-1/2}y^2 \leq t\,,\; \mathcal{L}_N(k+1,y) + 2^{-1/2}y^2 \geq -t \Big\}\,.$$

If $|x| \leq 2^{-1}cN^{1/9}$, $\epsilon \in (0,1/2)$, $K \geq 3 \cdot 2^{19/2}$ and $N \geq 2$, then

$$\mathbb{P}\Big(\omega_{1,[x,x+\epsilon]}\big(\mathcal{L}_N,\epsilon\big) \geq K\epsilon^{1/2}\,,\; \mathsf{G}_t(x) \Big) \leq 2^{3/2}\pi \cdot 60K^{-1} \exp\big\{-2^{-12}K^2\big\}$$

where here we set $t = 2^{-8}3^{-1}K$.

(4) *(Collapse near infinity) For* $\eta \in (0, c]$, *let* $\ell = \ell_\eta : \mathbb{R} \to \mathbb{R}$ *denote the even function which is affine on* $[0, \infty)$ *and has gradient* $-5 \cdot 2^{-3/2}\eta N^{1/9}$ *on this interval, and which satisfies* $\ell(\eta N^{1/9}) = \big(- 2^{-1/2} + 2^{-5/2}\big)\eta^2 N^{2/9}$. *If* $N \geq 2^{45/4}c^{-9}$, *then*

$$\begin{aligned} & \mathbb{P}\Big(\mathcal{L}_N\big(1, z\big) > \ell(z) \text{ for some } z \in D \setminus \big[- \eta N^{1/9}, \eta N^{1/9}\big] \Big) \\ \leq \;\; & 6C \exp\Big\{ - c\eta^3 2^{-15/4} N^{1/3} \Big\} . \end{aligned}$$

The set D *is the spatial domain of* $\mathcal{L}$, *either* $[-z_{\mathcal{L}}, \infty)$ *or* $(-\infty, z_{\mathcal{L}}]$.

These four assertions are proved in [Ham17a]. Respectively, they appear as, or are special cases of, the following results in that article: Proposition $A.2$, Proposition 2.29, Proposition 2.16 and Proposition 2.31.

A few words about the meaning of the four parts of this proposition: the first part asserts a lower bound for the minimum value of the lowest indexed curve on a compact interval. The result is a strengthening of the defining property Reg(2), which treats the one-point case. (In fact, this first part may also apply to a curve of any given index, but we have no use for information concerning these other curves in the present article.) The second part is a similar strengthening of the one-point upper tail Reg(3). In regard to the fourth, note that Reg(2) and Reg(3) do not assert that the lowest indexed curve hews to the parabola $-2^{-1/2}z^2$ globally, but only in an expanding region about the origin, of width $2cN^{1/9}$ centred at the origin, where N is the ensemble curve cardinality. Proposition 1.9(4) offers a substitute control on curves far from the origin, showing them to decay at a rapid but nonetheless linear rate in the region beyond scale $N^{1/9}$.

Finally, the modulus of continuity Proposition 1.9(3) will be a key input for the proofs of Theorems 1.2 and 1.3. Indeed, all three results give expression to the one-half power law that governs polymer weight: when the endpoints of polymers are varied by short horizontal displacements of order ϵ, the change in polymer weight has an order of $\epsilon^{1/2}$. Proposition 1.9(3) is notably flexible, in that the parameters for horizontal scale, ϵ, and scaled fluctuation, K, may be selected without imposing any dependence on the lower bound demanded on the ensemble curve cardinality N. This favourable feature comes at the price that the result gauges the small probability of high modulus of continuity only when we impose a global boundedness event $\mathsf{G}_t(x)$ on the ensemble $\mathcal{L}_N$. We will have more to say about the role of Proposition 1.9(3) and the implications of its strengths and weaknesses early in Section 3, when Theorem 1.3 is proved.

1.6. **Organization of the remainder of the paper.** This article's principal results are Proposition 1.4, Theorem 1.3 and Theorem 1.2. The article has three further sections and they are devoted respectively to the proofs of these results.

1.6.1. *The role of hypotheses invoked during proofs.* Our proofs invoke several inputs, notably the three Reg conditions and the four parts of Proposition 1.9. Whenever such results are invoked, certain conditions on the concerned hypotheses will be needed. We will always note explicitly what these conditions are, whenever such an application is made. Clearly, it is necessary that the hypotheses of the result that is being proved collectively imply all the conditions that are invoked during its proof. The work needed to do this for a given result may be called the *calculational derivation* of that result. These derivations have almost no conceptual content, reach conclusions that in their overall form are plausible, consist of largely trivial steps, and will be of interest to only the most committed of readers (perhaps only those who are actually applying the results). In some cases, however, the derivations occupy a fair amount of space. We have chosen to separate the principal calculational derivations from the body of the proofs in this article. The concerned results are Theorem 1.3, Proposition 1.4, Proposition 3.1 and Lemma 4.1. Their calculational derivations are presented in an appendix to this paper.

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2. Collective control on polymer weights: the proof of Proposition 1.4

The planar line segment with endpoints $(x,0)$ and $(y,1)$ crosses the horizontal line whose height is one-half at the location $(x+y)/2$. We record this location in the form $z=(x+y)/2$. We will prove the lower bound (3) by bounding the polymer weight $\mathsf{Wgt}_{n;(x+u,0)}^{(y+v,1)}$ for any given $(u,v) \in [0,1]^2$ below by considering routes from $(x+u,0)$ to $(y+v,1)$ that pass via $(z,1/2)$. The two polymer weights in the lower bound concern journeys from $(z,1/2)$ to $(x+u,0)$ (after time reversal) and from $(z,1/2)$ to $(y+v,1)$. Rooting in this way at $(z,1/2)$, we may gauge the probability of low weight values for polymers emanating from $(z,1/2)$ and ending in a compact interval at time zero or one by applying Proposition 1.9(1) to the duration one-half normalized ensembles rooted at $(z,1/2)$, of forward or backward type according to whether a time one or time zero endpoint is being considered.

Thus, we let $u,v \in [0,1]$. Note that

$$\begin{aligned}
& 2^{-1/2}\big(2^{2/3}(z-x-u)\big)^2 + 2^{-1/2}\big(2^{2/3}(y+v-z)\big)^2 \\
= \; & 2^{-1/2}\Big(2^{2/3}\big((y-x)/2-u\big)\Big)^2 + 2^{-1/2}\Big(2^{2/3}\big((y-x)/2+v\big)\Big)^2 \\
= \; & 2^{5/6}2^{-1}(y-x)^2 + 2^{5/6}(y-x)(v-u) + 2^{5/6}\big(u^2+v^2\big) \\
= \; & 2^{-1/6}(y+v-x-u)^2 + 2^{-1/6}(u+v)^2 \, .
\end{aligned}$$

Note further that

$$\begin{aligned}
\mathsf{Wgt}_{n;(x+u,0)}^{(y+v,1)} \; \geq \; & \mathsf{Wgt}_{n;(x+u,0)}^{(z,1/2)} + \mathsf{Wgt}_{n;(z,1/2)}^{(y+v,1)} = \mathcal{L}_{n;0}^{\downarrow;(z,1/2)}(1,x+u) + \mathcal{L}_{n;(z,1/2)}^{\uparrow;1}(1,y+v) \\
= \; & 2^{-1/3}\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(z,1/2)}\big(1,2^{2/3}(x+u-z)\big) + 2^{-1/3}\mathsf{Nr}\mathcal{L}_{n;(z,1/2)}^{\uparrow;1}\big(1,2^{2/3}(y+v-z)\big) \, .
\end{aligned}$$

The inequality here invokes Lemma 1.6(2) with $\mathbf{t_1}=0$, $\mathbf{t_2}=1$, $\mathbf{t}=2^{-1}$, $\mathbf{x}=x+u$, $\mathbf{y}=y+v$ and $\mathbf{z}=z$. The two equalities invoke the definitions of the four ensembles whose top curves are being evaluated. Regarding the inequality, we may note that the use of Lemma 1.6(2) entails that certain bounds on $\mathbf{z}$ be satisfied. We omit reference to these bounds now because they are anyway implicated later in the argument, when we come to analyse the above two right-hand terms. Finally, note that, in order to enable our use of Lemma 1.6(2) with the choice $\mathbf{t}=2^{-1}$, we have imposed in Proposition 1.4 the assumption that $n \in \mathbb{N}$ be even, in order that the triples $(n,0,2^{-1})$ and $(n,2^{-1},1)$ be compatible.

Adding to the above inequality the $2^{-1/3}$-rd multiple of the display that preceded it, we find that

$$\begin{aligned} & \mathsf{Wgt}_{n;(x+u,0)}^{(y+v,1)} + 2^{-1/2}(y+v-x-u)^2 + 2^{-1/2}(u+v)^2 \\ \geq \; & 2^{-1/3}\Big(\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(z,1/2)}\big(1, 2^{2/3}(x+u-z)\big) + 2^{-1/2}\big(2^{2/3}(z-x-u)\big)^2\Big) \\ & \quad + 2^{-1/3}\Big(\mathsf{Nr}\mathcal{L}_{n;(z,1/2)}^{\uparrow;1}\big(1, 2^{2/3}(y+v-z)\big) + 2^{-1/2}\big(2^{2/3}(y+v-z)\big)^2\Big) \, . \end{aligned} \tag{9}$$

Note that

$$\begin{aligned} & \mathbb{P}\bigg(\inf_{u\in[0,1]} \Big(\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(z,1/2)}\big(1, 2^{2/3}(x+u-z)\big) + 2^{-1/2}\big(2^{2/3}(z-x-u)\big)^2\Big) \leq -s\bigg) \qquad (10) \\ = \; & \mathbb{P}\bigg(\inf_{u\in[0,2^{2/3}]} \Big(\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(z,1/2)}\big(1, 2^{2/3}(x-z)+u\big) + 2^{-1/2}\big(2^{2/3}(x-z)+u\big)^2\Big) \leq -s\bigg) \, . \end{aligned}$$

The latter term equals

$$\mathbb{P}\Big(\inf_{x'\in[y-t,y+t]} \big(\mathcal{L}_n(1,x') + 2^{-1/2}(x')^2\big) \leq -r\Big)$$

when $t = 2^{-1/3}$; when $t \geq 2^{-1/3}$, the new expression is an upper bound. Here, we take $\mathcal{L}_n = \mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(z,1/2)}$, $y = 2^{2/3}(x-z) + 2^{-1/3}$ and $r = s$.

We seek then to apply Proposition 1.9(1) to (c,C)-regular ensemble $\mathcal{L}_N = \mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(z,1/2)}$, doing so with this choice of (r,y,t). It is Proposition 1.8 that permits this choice of ensemble. If we set $t_1 = 0$ and $t_2 = 1/2$, so that $t_{1,2} = 1/2$, the number of curves in the ensemble $\mathcal{L}_N = \mathsf{Nr}\mathcal{L}_{N;t_1}^{\downarrow;(z,t_2)}$ equals $\lfloor nt_{1,2}\rfloor + 1$ and thus is at least $nt_{1,2}$. For the application to be valid, our parameters must thus satisfy

$$t_{1,2}n \geq 1 \vee (c/3)^{-18} \vee 6^{36} \, , \; 2^{-1/3} \leq (nt_{1,2})^{1/18} \, , \; s \in \big[2^{3/2} \, , \, 2(nt_{1,2})^{1/18}\big] \, , \tag{11}$$

and $\big|2^{2/3}(x-z) + 2^{-1/3}\big| = \big|2^{2/3}(x-y)/2 + 2^{-1/3}\big| \leq c/2 \cdot (nt_{1,2})^{1/18}$. From this application of Proposition 1.9(1), we find that the probability in (10) is at most

$$\Big(2^{-1/3} \vee (3-2^{3/2})^{-1}5^{1/2}\Big) \cdot 10C \exp\big\{-c_1 s^{3/2}\big\} \, . \tag{12}$$

By applying Proposition 1.9(1) to the ensemble $\mathcal{L}_N = \mathsf{Nr}\mathcal{L}_{n;(z,1/2)}^{\uparrow;1}$, with $\mathbf{y}$ now chosen equal to $2^{2/3}(y-z) + 2^{-1/3}$, and with (r,t) again set to be $\big(s, 2^{-1/3}\big)$, we find that the quantity

$$\mathbb{P}\bigg(\inf_{v\in[0,1]} \Big(\mathsf{Nr}\mathcal{L}_{n;(z,1/2)}^{\uparrow;1}\big(1, 2^{2/3}(y+v-z)\big) + 2^{-1/2}\big(2^{2/3}(y+v-z)\big)^2\Big) \leq -s\bigg)$$

is also bounded above by (12). This application of the proposition requires in addition to the bounds in (11) that $\big|2^{2/3}(y-z) + 2^{-1/3}\big| = \big|2^{2/3}(y-x)/2 + 2^{-1/3}\big| \leq c/2 \cdot (nt_{1,2})^{1/18}$.

Since $2^{-1/2}(u+v)^2 \leq 2^{3/2}$ whenever $u,v \in [0,1]$, we find from the inequality (9) and the upper bound by (12) on the two probabilities that

$$\begin{aligned} & \mathbb{P}\bigg(\inf_{u,v\in[0,1]} \Big(\mathsf{Wgt}_{n;(x+u,0)}^{(y+v,1)} + 2^{-1/2}(y+v-x-u)^2\Big) \leq -2\cdot 2^{-1/3}s \, - \, 2^{3/2}\bigg) \\ \leq \; & 2\Big(2^{-1/3} \vee (3-2^{3/2})^{-1}5^{1/2}\Big) \cdot 10C \exp\big\{-c_1 s^{3/2}\big\} \, . \end{aligned}$$

Setting $t = 2 \cdot 2^{2/3} s$ and using $s \geq 2^{5/6}$ (so that $t \geq 2^{2/3}s + 2^{3/2}$), and noting $20\Big(2^{-1/3} \vee 5^{1/2}(3 - 2^{3/2})^{-1}\Big) \leq 261$, we obtain (3).

Let $\mathrm{High}_{n;([x,x+1],0)}^{([y,y+1],1)}(t)$ denote the event that $\sup_{u,v\in[0,1]} \Big(\mathsf{Wgt}_{n;(x+u,0)}^{(y+v,1)} + 2^{-1/2}(y+v-x-u)^2\Big) \geq t$. Clearly it is this event whose probability we must bound above as we turn to derive (2). The event entails the presence of a high weight polymer that crosses a square, but both of its endpoints may have exceptional locations. The derivation of (2) will proceed by noting that, typically, one of the endpoints can be made typical. Indeed, when the High event occurs, so that a high weight polymer runs between say $(x+U,0)$ and $(y+V,1)$, where the pair $(U,V) \in [0,1]^2$ is random, a fairly high weight polymer will also typically exist between the deterministic location $(2x-y,-1)$ and $(V,1)$. This is because a rather high lower bound on the weight of such a polymer is obtained by considering the pair of polymers, from $(2x-y,-1)$ to $(x+U,0)$, and from $(x+U,0)$ to $(y+V,1)$, whose weights are typically not too low, and high. The probability of the presence of this fairly high weight polymer may then be gauged by the No Big Max Proposition 1.9(2), because this event entails that the duration-two normalized forward ensemble rooted at $(2x-y,-1)$ assumes a high value within a compact interval.

To begin implementing this approach, we consider the event $\mathrm{High}_{n;([x,x+1],0)}^{([y,y+1],1)}(t)$, and let $(U,V) \in [0,1]^2$ be the lexicographically minimal pair of $(u,v) \in [0,1]^2$ that realize this event (a definition which makes sense because, by Lemma 1.6(1), the set of such pairs is closed).

Now reset the value of z to be $2x-y$, so that the planar line segment interpolating $(z,-1)$ and $(y,1)$ passes through $(x,0)$. Note that

$$
\begin{aligned}
& 2^{-1/2}(z-x-U)^2 + 2^{-1/2}(x+U-y-V)^2 \qquad (13)\\
= \; & 2^{-1/2}(x-y-U)^2 + 2^{-1/2}(x+U-y-V)^2 \\
= \; & 2^{1/2}(x-y)^2 - 2^{1/2}(x-y)V + 2^{-1/2}\big(2U^2 + V^2 - 2UV\big) \\
= \; & 2^{1/2}(x-y-V/2)^2 + 2^{-1/2}\big(2U^2 + V^2/2 - 2UV\big) \\
= \; & 2^{-3/2}(z-y-V)^2 + 2^{-1/2}\big(2U^2 + V^2/2 - 2UV\big) \, .
\end{aligned}
$$

Let $\mathrm{NotLow}_{n;(z,-1)}^{([x,x+1],0)}(t/2)$ denote the event that $\inf_{u\in[0,1]} \Big(\mathsf{Wgt}_{n;(z,-1)}^{(x+u,0)} + 2^{-1/2}(x+u-z)^2\Big) \geq -t/2$. Note that

$$
\neg \, \mathrm{NotLow}_{n;(z,-1)}^{([x,x+1],0)}(t/2) \, = \, \bigg\{ \inf_{u\in[0,1]} \Big(\mathsf{Nr}\mathcal{L}_{n;(z,-1)}^{\uparrow;0}\big(1, x+u-z\big) + 2^{-1/2}(x+u-z)^2 \Big) \, < \, -t/2 \bigg\} \, .
$$

We apply Proposition 1.9(1) to the ensemble $\mathsf{Nr}\mathcal{L}_{n;(z,-1)}^{\uparrow;0}$, the choice admissible by Proposition 1.8, in order to find an upper bound on the probability of the displayed event. The application is made with $\mathbf{y} = x-z+1/2$, $\mathbf{t} = 1/2$ and $\mathbf{r} = t/2$. Since the ensemble $\mathsf{Nr}\mathcal{L}_{n;(z,-1)}^{\uparrow;0}$ has $n+1$, and therefore at least n, curves, we see that the next bounds suffice for the application to be made: $n \geq (c/3)^{-18} \vee 6^{36}$, $1/2 \leq n^{1/18}$, $t/2 \in \big[2^{3/2}\, , \, 2n^{1/18}\big]$ and $\big|x-z+1/2\big| = \big|y-x+1/2\big| \leq 2^{-1}cn^{1/18}$. We learn from the application that

$$
\mathbb{P}\Big(\neg \, \mathrm{NotLow}_{n;(z,-1)}^{([x,x+1],0)}(t/2) \Big) \, \leq \, (3-2^{3/2})^{-1} 5^{1/2} \cdot 10 C \exp\big\{ -c_1 2^{-3/2} t^{3/2} \big\} \, .
$$

When $\mathrm{NotLow}_{n;(z,-1)}^{([x,x+1],0)}(t/2) \cap \mathrm{High}_{n;([x,x+1],0)}^{([y,y+1],1)}(t)$ occurs, consider the concatenation $\rho_{n;(z,-1)}^{(x+U,0)} \circ \rho_{n;(x+U,0)}^{(y+V,1)}$ of any pair of polymers with the endpoints implied by our convention governing this notation. The concatenation has weight at least

$$\begin{aligned} & \Big(-t/2 - 2^{-1/2}(x+U-z)^2 \Big) \, + \, \Big(t - 2^{-1/2}(y+V-x-U)^2 \Big) \\ \geq \; & t/2 - 2^{-3/2}(z-y-V)^2 - 2^{-1/2}\big(2U^2 + V^2/2 - 2UV\big) \\ \geq \; & t/2 - 2^{-3/2}(z-y-V)^2 - 5 \cdot 2^{-3/2} \, , \end{aligned}$$

the displayed inequalities due to (13) and $U,V \in [0,1]$. Thus,

$$\mathrm{High}_{n;([x,x+1],0)}^{([y,y+1],1)}(t) \cap \mathrm{NotLow}_{n;(z,-1)}^{([x,x+1],0)}(t/2) \subseteq \bigg\{ \sup_{v \in [0,1]} \Big(\mathsf{Wgt}_{n;(z,-1)}^{(y+v,1)} + 2^{-3/2}(z-y-v)^2 \Big) \geq t/2 - 5 \cdot 2^{-3/2} \bigg\} \, .$$

The right-hand event equals

$$\begin{aligned} & \bigg\{ \sup_{v \in [0,1]} \Big(\mathcal{L}_{n;(z,-1)}^{\uparrow;1}(1, y+v) + 2^{-3/2}(z-y-v)^2 \Big) \geq t/2 - 5 \cdot 2^{-3/2} \bigg\} \qquad (14) \\ = \; & \bigg\{ \sup_{v \in [0,1]} \Big(\mathsf{Nr}\mathcal{L}_{n;(z,-1)}^{\uparrow;1}\big(1, 2^{-2/3}(y+v-z)\big) + 2^{-1/2}\big(2^{-2/3}(z-y-v)\big)^2 \Big) \geq 2^{-4/3}t - 5 \cdot 2^{-11/6} \bigg\} \\ = \; & \bigg\{ \sup_{v \in [0,2^{-2/3}]} \Big(\mathsf{Nr}\mathcal{L}_{n;(z,-1)}^{\uparrow;1}\big(1, 2^{-2/3}(y-z)+v\big) + 2^{-1/2}\big(2^{-2/3}(z-y)-v\big)^2 \Big) \geq 2^{-4/3}t - 5 \cdot 2^{-11/6} \bigg\} \, . \end{aligned}$$

We now apply the No Big Max Proposition 1.9(2) to the ensemble $\mathcal{L} = \mathsf{Nr}\mathcal{L}_{n;(z,-1)}^{\uparrow;1}$, the application permitted by Proposition 1.8. In the application, we set $\mathbf{y} = 2^{-2/3}(y-z) + 2^{-5/3}$, $\mathbf{r} = 2^{-5/3}$ and $\mathbf{t} = 2^{-4/3}t - 5 \cdot 2^{-11/6}$. The curve cardinality of the ensemble in queston is $2n+1 \geq 2n$. As such, it is sufficient for the application to be valid that

$$\big| 2^{-2/3}(y-z) + 2^{-5/3} \big| = \big| 2^{1/3}(y-x) + 2^{-5/3} \big| \leq c/2 \cdot (nt_{1,2})^{1/9} \, , \; 2^{-5/3} \leq c/4 \cdot (nt_{1,2})^{1/9}$$

and $2^{-4/3}t - 5 \cdot 2^{-11/6} \in \big[2^{7/2}, 2(nt_{1,2})^{1/3} \big]$ as well as $t_{1,2}n \geq c^{-18}$ where here $t_{1,2}$ equals 2 (in accordance with the time-pair $t_1 = -1$ and $t_2 = 1$ being considered). This application tells us that the $\mathbb{P}$-probability of the event (14) is at most

$$\big(2^{-5/3} + 1 \big) \cdot 6C \exp\big\{ -2^{-9/2}c\big(2^{-4/3}t - 5 \cdot 2^{-11/6}\big)^{3/2} \big\} \leq 8C \exp\big\{ -2^{-8}ct^{3/2} \big\} \, ,$$

where we used $2^{-4/3}t - 5 \cdot 2^{-11/6} \geq 2^{-7/3}t$ when $t \geq 5 \cdot 2^{1/2}$.

We find then that

$$\mathbb{P}\Big(\mathrm{High}_{n;([x,x+1],0)}^{([y,y+1],1)}(t) \Big) \, \leq \, \mathbb{P}\Big(\mathrm{High}_{n;([x,x+1],0)}^{([y,y+1],1)}(t) \cap \mathrm{NotLow}_{n;(z,-1)}^{([x,x+1],0)}(t/2) \Big) + \mathbb{P}\Big(\neg \, \mathrm{NotLow}_{n;(z,-1)}^{([x,x+1],0)}(t/2) \Big)$$

is bounded above by

$$8C \exp\big\{ -2^{-8}ct^{3/2} \big\} \, + \, 10(3-2^{3/2})^{-1}5^{1/2}C \exp\big\{ -c_1 2^{-3/2}t^{3/2} \big\} \, .$$

We now use $c_1 \leq c$ and $8 + 10(3-2^{3/2})^{-1}5^{1/2} \leq 139$ to obtain (2). This completes the proof of Proposition 1.4. □

The reader may have noticed that every application of a result concerning a (c,C)-regular ensemble in the preceding proof invoked Proposition 1.8 in order to justify that the ensemble in question indeed enjoys this property. Every subsequent such application is no different, and henceforth we feel free to omit mention of Proposition 1.8's role.

3. Polymer weight regularity: proving Theorem 1.3

We begin this section by introducing a weight difference notation $\Delta\mathsf{Wgt}$ to denote the difference in weight of two polymers crossing between the opposite endpoints of two intervals. When (x_1, x_2) and (y_1, y_2) are elements of $\mathbb{R}^2_{\leq}$, we write $\Delta\mathsf{Wgt}^{([y_1,y_2],1)}_{n;([x_1,x_2],0)}$ for $\mathsf{Wgt}^{(y_2,1)}_{n;(x_2,0)} - \mathsf{Wgt}^{(y_1,1)}_{n;(x_1,0)}$. We will abuse this notation when one of the concerned intervals collapses a point, writing for example $\Delta\mathsf{Wgt}^{([y_1,y_2],1)}_{n;(x_1,0)} = \mathsf{Wgt}^{(y_2,1)}_{n;(x_1,0)} - \mathsf{Wgt}^{(y_1,1)}_{n;(x_1,0)}$.

We also write $y + U = \{y + u : u \in U\}$ when $y \in \mathbb{R}$ and $U \subset \mathbb{R}$.

Any integer is called a dyadic rational of scale zero. A dyadic rational of scale $i \in \mathbb{N}$, $i \geq 1$, has for the form $p2^{-i}$ where $p \in \mathbb{Z}$ is odd. A dyadic interval of scale $i \in \mathbb{N}$ is a closed interval of length 2^{-i} that has an endpoint which is a dyadic rational of scale i.

Recall from Subsection 1.3.11 the polymer weight regularity events $\mathsf{PolyWgtReg}$.

Proposition 3.1. *Let $n \in \mathbb{N}$, $n \geq (4/c)^9$, let $x, y \in \mathbb{R}$ satisfy $\big|x - y\big| \leq c/4 \cdot n^{1/9}$, and let K_0 be a real parameter satisfying $K_0 \geq 3 \cdot 2^{19/2}$. Let $i, k_0 \in \mathbb{N}$ with $i \geq k_0 \geq 2$. Set $t_0 = 2^{-8}K_0/3$. Consider the quantities*

$$\mathbb{P}\Big(\sup \Big|\Delta\mathsf{Wgt}^{(y+U,1)}_{n;(x+z,0)}\Big| \geq K_0 2^{-i/2} \, , \, \mathsf{PolyWgtReg}^{([y-2,y+3],1)}_{n;([x,x+1],0)}(t_0) \Big) \tag{15}$$

and

$$\mathbb{P}\Big(\sup \Big|\Delta\mathsf{Wgt}^{(y+z,1)}_{n;(x+U,0)}\Big| \geq K_0 2^{-i/2} \, , \, \mathsf{PolyWgtReg}^{([y,y+1],1)}_{n;([x-2,x+3],0)}(t_0) \Big) \, , \tag{16}$$

where in each case the supremum is taken over all dyadic rationals $z \in [0, 2^{-k_0}]$ of scale i and dyadic intervals $U \subseteq [0, 2^{-k_0}]$ of scale at least i. Each of these probabilities is at most

$$2^{2(i-k_0)} \cdot 4266 \exp\big\{ -2^{-12}K_0^2 \big\} \, .$$

This result is naturally a key technical ingredient in the proof of Theorem 1.3. The theorem concerns polymer weight differences when small changes are made in endpoint locations. The proposition is similar, but restricts to endpoint locations that are dyadic rationals. The theorem will follow from the proposition by noting that one may skip between the two nearby endpoint locations x and $x+\epsilon$ (and similarly for y and $y+\epsilon$) by jumping through a possibly infinite sequence of intermediate dyadic rational locations where no dyadic scale need be visited more than twice and the minimal dyadic scale is of the order of the difference ϵ. This property of this 'stepping stone' sequence makes the union bounds over the estimate in Proposition 3.1 manageable. This inference of the theorem from the proposition is similar to the derivation of the Kolmogorov continuity criterion, in which moment bounds on the difference of a stochastic process between a generic pair of times imply Hölder continuity of the process: see [Dur10, Thoerem 8.13].

One further aspect of the plan for proving Theorem 1.3 deserves mention before we proceed. Note that in the theorem the parameter R, which measures the degree of polymer weight fluctuation, must verify an n-dependent upper bound. Although this bound in a sense is insignificant for the purpose of analysing high n behaviour, Proposition 3.1 has been stated so that there is no counterpart to this hypothesis: the quantities K_0 and 2^{-i} are counterparts to R and ϵ, and the proposition holds for all high n, where the lower bound on n deteriorates neither as the dyadic scale 2^{-i} decreases, nor as the parameter K_0 increases. Now this is a valuable property, but it comes at a certain price, about which more in a moment. The reason that the property is valuable is that, for a given high value of n, Proposition 3.1 may be applied as i increases to infinity, while in the meantime,

K_0 also increases; indeed, this is how we will derive Theorem 1.3, with a union bound over the infinite number of applications of the proposition indexed by i being controllable due to the ongoing increase in K_0. As for the price to be paid, we mention that, in order that the property obtains, it has been necessary in the events whose probabilities are gauged in (15) and (16) to include the global polymer weight regularity events $\mathsf{PolyWgtReg}$; this in turn is because the proof of the proposition will invoke the local curve regularity Theorem 1.9(3), in which the spatial scale $\epsilon > 0$ is permitted to be arbitrarily small without forcing the ensemble curve cardinality N to be higher in an ϵ-dependent way, a favourable circumstance which is only possible at the expense of introducing the global regularity event G_t, counterpart to the above $\mathsf{PolyWgtReg}$ events, into the probability upper bound in that result. In any case, it would seem that we can derive Theorem 1.3 from Proposition 3.1 only by invoking the $\mathsf{PolyWgtReg}$ event. In fact, we do impose this event, but, at the very end of the derivation, we gauge the probability of the complementary event $\neg\,\mathsf{PolyWgtReg}$ via Corollary 1.5; thus, this probability cost is paid only once, rather than with each of the infinitely many applications of Proposition 3.1.

Proof of Proposition 3.1. Consider a given dyadic interval $U \subseteq [0, 2^{-k_0}]$ of a scale j such that $j \geq i$ and write $U = [u_1, u_2]$. For a given dyadic rational $z \in [0, 2^{-k_0}]$, note that

$$\Delta\mathsf{Wgt}_{n;(x+z,0)}^{(y+U,1)} = \mathcal{L}_{n;(x+z,0)}^{\uparrow;1}\big(1, y+u_2\big) - \mathcal{L}_{n;(x+z,0)}^{\uparrow;1}\big(1, y+u_1\big)$$

and that

$$\Delta\mathsf{Wgt}_{n;(x+U,0)}^{(y+z,1)} = \mathcal{L}_{n;0}^{\downarrow;(y+z,1)}\big(1, x+u_2\big) - \mathcal{L}_{n;0}^{\downarrow;(y+z,1)}\big(1, x+u_1\big)\,.$$

Recall that $U = [u_1, u_2]$ is a given dyadic interval of scale $j \geq i \geq 2$ with $U \subset [0,1]$. Let $K > 0$ be a parameter whose value we will later specify. We now seek to apply Proposition 1.9(3) in order to bound the $\mathbb{P}$-probability that the quantities

$$\omega_{1,[y+u_1,y+u_2]}\big(\mathcal{L}_{n;(x+z,0)}^{\uparrow;1}, 2^{-j}\big) \ \text{ and } \ \omega_{1,[x+u_1,x+u_2]}\big(\mathcal{L}_{n;0}^{\downarrow;(y+z,1)}, 2^{-j}\big) \tag{17}$$

exceed $K|U|^{1/2} = K2^{-j/2}$. In order to do so for the first quantity, the proposition will be applied to the ensemble $\mathcal{L}_n = \mathsf{Nr}\mathcal{L}_{n;(x+z,0)}^{\uparrow;1}$, with Proposition 1.8 showing that this choice is admissible. Proposition 1.9(3)'s parameters in this case are set $\mathbf{x} = y + u_1 - x - z$, $\boldsymbol{\epsilon} = 2^{-j}$ and $\mathbf{K} = K$; this choice forces $\mathbf{t} = 2^{-8}K/3$. Note that the event whose probability is bounded above by the proposition is a subset of $\mathsf{G}_t(\mathbf{x})$. With the present choice of ensemble $\mathcal{L}_n$, the event $\mathsf{G}_t(\mathbf{x})$ equals $\mathsf{PolyWgtReg}_{n;(\{x+z\},0)}^{([y+u_1-2,y+u_1+2],1)}(t)$. (Note that here $\{x+z\}$ is a singleton set, so that a space of polymers emanating from the point $(x+z,0)$ is at stake.) Thus, $\mathsf{PolyWgtReg}_{n;([x,x+1],0)}^{([y-2,y+3],1)}(t) \subseteq \mathsf{G}_t(\mathbf{x})$. As such, this application of Proposition 1.9(3) implies that

$$\begin{aligned}
&\mathbb{P}\bigg(\Big|\mathcal{L}_{n;(x+z,0)}^{\uparrow;1}\big(1, y+u_2\big) - \mathcal{L}_{n;(x+z,0)}^{\uparrow;1}\big(1, y+u_1\big)\Big| \geq K2^{-j/2}\,,\ \mathsf{PolyWgtReg}_{n;([x,x+1],0)}^{([y-2,y+3],1)}(t)\bigg) \\
\leq\ \ &\mathbb{P}\Big(\omega_{1,[y+u_1,y+u_2]}\big(\mathcal{L}_{n;(x+z,0)}^{\uparrow;1}, 2^{-j}\big) \geq K2^{-j/2}\,,\ \mathsf{G}_t(\mathbf{x})\Big) \leq 2^{3/2}\pi \cdot 60K^{-1}\exp\big\{-2^{-12}K^2\big\}\,.
\end{aligned} \tag{18}$$

The hypotheses of Proposition 1.9(3) that are invoked to obtain this bound are

$$n \geq 2\,,\ |y+u_1-x-z| \leq 2^{-1}cn^{1/9}\,,\ 2^{-j} \in (0,1/2) \text{ and } K \geq 3\cdot 2^{19/2}\,;$$

note that $j \geq 2$ is used to validate the third of these.

Another application of Proposition 1.9(3) is made in regard to the second quantity in (17). On this occasion, the ensemble $\mathcal{L}_n$ is set equal to $\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(y+z,1)}$, and the parameters are set: $\mathbf{x} = x + u_1 - y - z$, $\boldsymbol{\epsilon} = 2^{-j}$, $\mathbf{K} = K$ and $\mathbf{t} = 2^{-8}K/3$. In this instance, the event $\mathsf{G}_t(\mathbf{x})$ equals

$\mathsf{PolyWgtReg}^{(y+z,1)}_{n;([x+u_1-2,x+u_1+2],0)}(t)$, so that $\mathsf{PolyWgtReg}^{([y,y+1],1)}_{n;([x-2,x+3],0)}(t) \subseteq \mathsf{G}_t(\mathbf{x})$. The outcome of the application in this case is the conclusion that

$$\begin{aligned} &\mathbb{P}\Big(\Big| \mathcal{L}^{\downarrow;(y+z,1)}_{n;0}\big(1, x+u_2\big) - \mathcal{L}^{\downarrow;(y+z,1)}_{n;0}\big(1, x+u_1\big) \Big| \geq K 2^{-j/2} \, , \, \mathsf{PolyWgtReg}^{([y,y+1],1)}_{n;([x-2,x+3],0)}(t) \Big) \\ &\leq \; 2^{3/2}\pi \cdot 60 K^{-1} \exp\big\{ -2^{-12} K^2 \big\} \, , \end{aligned} \tag{19}$$

while the hypothesis $|x+u_1-y-z| \leq 2^{-1} c n^{1/9}$ is used to make the application.

The dyadic scale j is at least the scale i by assumption, and we now denote $\ell = j - i \geq 0$. We recall from the statement of Proposition 3.1 that we consider a positive parameter K_0 that is supposed to satisfy $K_0 \geq 3 \cdot 2^{19/2}$, and also that we are setting $t_0 = 2^{-8} K_0/3$. We now set the value of our parameter K, choosing it to be $K_0 2^{(j-i)/2}$. Our conclusions (18) and (19) tell us that

$$\mathbb{P}\Big(\Big| \mathcal{L}^{\uparrow;1}_{n;(x+z,0)}\big(1, y+u_2\big) - \mathcal{L}^{\uparrow;1}_{n;(x+z,0)}\big(1, y+u_1\big) \Big| \geq K_0 2^{-i/2} \, , \, \mathsf{PolyWgtReg}^{([y-2,y+3],1)}_{n;([x,x+1],0)}(t_0) \Big)$$

and

$$\mathbb{P}\Big(\Big| \mathcal{L}^{\downarrow;(y+z,1)}_{n;0}\big(1, x+u_2\big) - \mathcal{L}^{\downarrow;(y+z,1)}_{n;0}\big(1, x+u_1\big) \Big| \geq K_0 2^{-i/2} \, , \, \mathsf{PolyWgtReg}^{([y,y+1],1)}_{n;([x-2,x+3],0)}(t_0) \Big) \tag{20}$$

are both at most $2^{3/2}\pi \cdot 60 K_0^{-1} \exp\big\{ -2^{-12} \cdot 2^{\ell} K_0^2 \big\}$, where we used $K \geq K_0$ in the form $t \geq t_0$.

We will now sum the stated bound on quantities of the form (20) in order to find an upper bound on the expression (16), namely

$$\mathbb{P}\Big(\sup \Big| \mathcal{L}^{\downarrow;(y+z,1)}_{n;0}\big(1, x+u_2\big) - \mathcal{L}^{\downarrow;(y+z,1)}_{n;0}\big(1, x+u_1\big) \Big| \geq K_0 2^{-i/2} \, , \, \mathsf{PolyWgtReg}^{([y,y+1],1)}_{n;([x-2,x+3],0)}(t_0) \Big)$$

where the supremum is taken over choices of scale i dyadic rational $z \in [0, 2^{-k_0}]$ and dyadic interval $[u_1, u_2] \subset [0, 2^{-k_0}]$ of scale at least i. Indeed, since the number of dyadic rationals of scale i in $[0, 2^{-k_0}]$ is at most $2^{i-k_0} + 1$, while the number of scale j dynamic intervals contained in $[0, 2^{-k_0}]$ equals 2^{j-k_0}, we may sum over $j \geq i$, also using that $K_0 \geq 2^6$, to find that the displayed probability is at most

$$(2^{i-k_0} + 1) \cdot 2^{i-k_0} \cdot \tfrac{2e}{e-1} \cdot 2^{3/2}\pi \cdot 60 \exp\big\{ -2^{-12} K_0^2 \big\} \, .$$

Using $i \geq k_0$, this quantity is found to be bounded above by

$$2^{2(i-k_0)+9/2} \cdot 60\pi \exp\big\{ -2^{-12} K_0^2 \big\} \leq 2^{2(i-k_0)} \cdot 4266 \exp\big\{ -2^{-12} K_0^2 \big\} \, .$$

We have obtained the upper bound claimed in Proposition 3.1 on the probability (16) and, since this upper bound on (15) is similarly obtained, we have completed the proof of this proposition. □

We now state an estimate also needed for the proof of Theorem 1.3. Suppose that n is an even integer that satisfies

$$n \geq 10^{29} \vee 2(c/3)^{-18} \, , \; \big|x-y\big| + 4 \leq 3^{-1} 2^{-2/3} c n^{1/18} \text{ and } t_0 \in \big[33 \, , \, 4n^{1/18}\big] \, . \tag{21}$$

Corollary 1.5 with $\mathbf{n} = n$, $\mathbf{t_1} = 0$, $\mathbf{t_2} = 1$, $\mathbf{x} = x - 2$, $\mathbf{y} = y - 2$, $\mathbf{a} = \mathbf{b} = 5$ and $\mathbf{r} = t_0$ implies that

$$\mathbb{P}\Big(\neg \, \mathsf{PolyWgtReg}^{([y-2,y+3],1)}_{n;([x-2,x+3],0)}(t_0) \Big) \leq 5^2 \cdot 400 \, C \exp\big\{ -c_1 2^{-8} t_0^{3/2} \big\} \, . \tag{22}$$

Note that $\big|x-y\big| + 4 \leq 3^{-1} 2^{-2/3} c n^{1/18}$ is implied by $\big|x-y\big| \leq 2^{-1} 3^{-1} 2^{-2/3} c n^{1/18}$ and $4 \leq 2^{-1} 3^{-1} 2^{-2/3} c n^{1/18}$ and the latter is implied by $n \geq \big(2^{11/3} 3\big)^{18}$.

Proof of Theorem 1.3. Write $I = [x, x+\epsilon]$ and $J = [y, y+\epsilon]$, and let $u \in I$ and $v \in J$ be arbitrary. Recalling that $\epsilon \leq 2^{-4}$ is less than one, we consider the binary expansion $u - x = \sum_{j=1}^{\infty} s_j 2^{-j}$. If the expansion is not unique, we choose its terminating version for definiteness. Let the increasing sequence $u_0, u_1, \cdots$ enumerate the set

$$\left\{ x + \sum_{j=1}^{k} s_j 2^{-j} : k \in \mathbb{N} \right\},$$

so that $u_0 = x$, with u_n equalling the sum of x and the quantity given by the truncation of the binary expansion of $u - x$ that contains n instances of the digit one. Let $n_1 \in \mathbb{N} \cup \{\infty\}$ denote the maximal index of a term in the u-sequence. If $n_1 < \infty$, then $u_{n_1} = u$, and if $n_1 = \infty$, then $u_n \nearrow u$, and we set $u_\infty = u$.

Similarly, we specify an increasing sequence $v_0, v_1, \cdots$ by replacing (x, u) by (y, v) above, and let n_2 denote the maximal index of a term in the v-sequence. If $n_2 = \infty$, set $v_\infty = v$.

Call the planar points $\big\{(u_i, 0) : 0 \leq i \leq n_1\big\}$ lower pegs, and the points $\big\{(v_i, 1) : 0 \leq i \leq n_2\big\}$ upper pegs. Think of a cord, which may be depicted as a planar line segment, that runs in the first instance between $(u_0, 0) = (x, 0)$ and $(v_0, 1) = (y, 1)$. The cord may be pegged at its lower and upper end to any of the pegs, so that the cord begins in its leftmost possible location. The rightmost available location is given by lower peg $(u_\infty, 0) = (u, 0)$ and upper peg $(v_\infty, 1) = (v, 1)$. We now specify a possibly infinite sequence of cord moves by which the cord will achieve, or at least converge towards, this rightmost location. Let $(L_i, U_i) \in \{0, \cdots, n_1\} \times \{0, \cdots, n_2\}$ denote the indices of lower and upper peg locations at step $i \in \mathbb{N}$, where the original location is indexed by $i = 0$, so that $(L_0, U_0) = (0, 0)$. Let $k \in \mathbb{N}$ and consider the value of (L_k, U_k). If this value is (n_1, n_2), then the cord movement is complete and the value of (L_{k+1}, U_{k+1}) is not recorded. In the other case, there are two possible moves for the cord at the next step: a lower peg advance, in which $L_{k+1} = L_k + 1$ and $U_{k+1} = U_k$, or an upper peg advance, in which $U_{k+1} = U_k + 1$ and $L_{k+1} = L_k$. It may be that one of these moves is inadmissible, because $L_k = n_1$, which renders the lower peg advance unavailable, or $U_k = n_2$, which does likewise for the upper peg advance. If this is so, then (L_{k+1}, U_{k+1}) is set equal to the value given by the only available advance. In the case where both advances are possible, note that each move entails displacing a peg to the right by a distance of the form 2^{-i} for some $i \in \mathbb{N}$. The decision of which advance to make is taken so that this distance is the larger for the available two advances, with say the upper advance being made if the distances are equal. In this way, we specify the value of (L_{k+1}, U_{k+1}); we may also record the dyadic scale, $D_{k+1} \in \mathbb{N}$, of the advance associated to this index increase $k \to k+1$: this scale is the value of $i \in \mathbb{N}$ such that the peg displacement made in the peg advance resulting in the new peg locations (L_{k+1}, U_{k+1}) equals 2^{-i}.

Note that the sequence of location pairs $\big(u_{L_k}, v_{U_k}\big)$ either reaches its terminal state (u, v) after finitely many moves, or it converges to this state as k increases. Note also that the dyadic scale sequence $D_1, D_2, \cdots$ is a non-decreasing $\mathbb{N}$-valued sequence that assumes any given value at most twice. This sequence depends on the pair (u, v), and we may indicate this dependence by writing $D_k = D_k(u, v)$.

Moreover, we define a dyadic scale $i_0 \in \mathbb{N}$ by setting $i_0 \geq 0$ to be minimal such that $2^{i_0}\epsilon \geq 1$. Then, since the first peg is displaced by a distance 2^{-D_1} which is at most $\epsilon < 2^{1-i_0}$, we see that $D_1 \geq i_0$. We see then that

$$D_j \geq i_0 - 1 + j/2 \ \text{ for } \ j \geq 1 \,. \tag{23}$$

To any cord location we may associate the weight of the polymer whose endpoints are the pegs to which the cord is pinned. For $k \geq 0$, we may further set

$$W_{k+1} = \mathsf{Wgt}_{n;(L_{k+1},0)}^{(U_{k+1},1)} - \mathsf{Wgt}_{n;(L_k,0)}^{(U_k,1)} \, ,$$

this being the difference in weight of this polymer as a result of the cord move from its index k to $k+1$ location.

For given n, the map $\mathbb{R}^2 \to [0,\infty) : (u,v) \to \mathsf{Wgt}_{n;(u,0)}^{(v,1)}$ is continuous by Lemma 1.6(1), so that $\lim_k \mathsf{Wgt}_{n;(L_k,0)}^{(U_k,1)} = \mathsf{Wgt}_{n;(u,0)}^{(v,1)}$. Thus,

$$\mathsf{Wgt}_{n;(u,0)}^{(v,1)} - \mathsf{Wgt}_{n;(x,0)}^{(y,1)} = \sum_{k=1}^{\infty} W_k \, , \tag{24}$$

where it is understood that the right-hand sum may have only finitely non-zero terms.

For each $k \in \mathbb{N}$ at least one, we set $W^*[k]$ to be the supremum of the values of $|W_k|$ over all choices of (u,v) in our construction. By a second use of polymer weight continuity, we may let $(u^*[k], v^*[k])$ be such that $W^*[k] = W_k(u^*[k], v^*[k])$. We also write $D^*[k] = D_k(u^*[k], v^*[k])$.

By (23), $D_\ell(u,v) \geq i_0 - 1 + k$ provided that $\ell \geq 2k$, whatever that the value of (u,v) among elements of the set $[x, x+\epsilon] \times [y, y+\epsilon]$. Thus, when $\ell \in \{2k, 2k+1\}$, the quantity $W^*[\ell]$ takes the form $\big|\Delta\mathsf{Wgt}_{n;(x+z,0)}^{(y+U,1)}\big|$ or $\big|\Delta\mathsf{Wgt}_{n;(x+U,0)}^{(y+z,1)}\big|$ where $z \in [0, 2^{1-i_0})$ is a dyadic rational of scale $D^*[\ell]$ at least $i_0 - 1 + k$ and $U \subset [0, 2^{1-i_0})$ is a dyadic interval whose scale is at least that of z. For this reason, we may use Proposition 3.1 with $\mathbf{k_0} = i_0 - 1$ and $\mathbf{i} = D^*[\ell]$ to find an upper bound on the probability

$$\mathbb{P}\Big(W^*[\ell] \geq K_0 \, 2^{-D^*[\ell]/2} \, , \, \mathsf{PolyWgtReg}_{n;([x-2,x+3],0)}^{([y-2,y+3],1)}\big(2^{-8} K_0/3\big) \Big) \, ,$$

where recall from the proposition that $K_0 \geq 3 \cdot 2^{19/2}$ is a parameter. Indeed, the proposition implies that, whenever $k \in \mathbb{N}$ and $\ell \in \{2k, 2k+1\}$, this probability is at most

$$2 \cdot 2^{2(D^*[\ell] - i_0 + 1)} \cdot 4266 \exp\big\{ - 2^{-12} K_0^2 \big\} \, .$$

We now make a choice of the parameter K_0 that depends on $\ell \in \mathbb{N}$, setting $K_0 = S \cdot 2^{-i_0/2 - k/4} \cdot 2^{D^*[\ell]/2}$. The new quantity $S > 0$ will be specified later. (Since $k = \lfloor \ell/2 \rfloor$, this specification of K_0 is indeed determined by ℓ.) Using $D^*[\ell] \geq i_0 - 1 + k$, we see that $K_0 \geq 2^{k/4 - 1/2} S$. Since the event $\mathsf{PolyWgtReg}_{n;([x-2,x+3],0)}^{([y-2,y+3],1)}(t_0)$ is increasing in $t_0 > 0$, we find that

$$\begin{aligned} & \mathbb{P}\Big(W^*[\ell] \geq S \cdot 2^{-i_0/2 - k/4} \, , \, \mathsf{PolyWgtReg}_{n;([x-2,x+3],0)}^{([y-2,y+3],1)}\big(2^{-8-1/2} S/3\big) \Big) \\ \leq \; & 2^{2(D^*[\ell] - i_0 + 1)} \cdot 8532 \exp\big\{ - 2^{-12 - i_0 - k/2 + D^*[\ell]} S^2 \big\} \, . \end{aligned}$$

Summing over $k \geq 1$ and the two values of ℓ for each k, we learn that

$$\begin{aligned} & \mathbb{P}\Big(\sum_{\ell=0}^{\infty} W^*[\ell] \geq 2S \sum_{k=0}^{\infty} 2^{-i_0/2 - k/4} \, , \, \mathsf{PolyWgtReg}_{n;([x-2,x+3],0)}^{([y-2,y+3],1)}\big(2^{-17/2} S/3\big) \Big) \\ \leq \; & 2 \sum_{\ell=0}^{\infty} 2^{2(D^*[\ell] - i_0 + 1)} \cdot 8532 \exp\big\{ - 2^{-12 - i_0 - k/2 + D^*[\ell]} S^2 \big\} \\ \leq \; & 15064 \sum_{\ell=0}^{\infty} 2^{2(D^*[\ell] - i_0 + 1)} \exp\big\{ - 2^{-12 - i_0 - \ell/4 + D^*[\ell]} S^2 \big\} \, . \end{aligned}$$

Consider the function $x \to 2^{2x} \exp\big\{ - 2^{a+x} S^2 \big\}$. Taking $x = D^*[\ell] - i_0 + 1$ and $a = -13 - \ell/4$, we recover the summand in the preceding line. Note that the logarithm of this function has derivative which is at most zero for x such that $1 \leq S^2 2^{a+x-1}$; when $S \geq 2^8$, this condition is met when $x \geq -a - 15$. For the choice of (x,a) dictated by the form of the summand, the condition is given by $D^*[\ell] \geq \ell/4 + i_0 - 3$. Since it is known that $D^*[\ell]$ is at least $i_0 - 2 + \ell/2$, and the replacement of $D^*[\ell]$ by the latter value results in the condition being satisfied (due to $\ell \geq 0$), we see that making this replacement only causes the value of the summand to rise. Thus, the last displayed quantity is seen to be at most

$$15064 \sum_{\ell=0}^{\infty} 2^{-2+\ell} \exp\big\{ - 2^{-14+\ell/4} S^2 \big\} \, . \tag{25}$$

The ratio of each summand, indexed by $\ell \geq 1$, to its predecessor is at most

$$2 \exp\big\{ - S^2 \cdot 2^{-14} (2^{1/4} - 1) \big\}$$

which when $S \geq 2^{14/2+2} = 2^9$ is at most $2 \exp\big\{ - 2^{2 \cdot 2} (2^{1/4} - 1) \big\} \leq 3/4$, so that (25) is at most

$$15064 \cdot 4 \cdot 2^{-2} \exp\big\{ - 2^{-14} S^2 \big\} = 15064 \, \exp\big\{ - 2^{-14} S^2 \big\} \, .$$

By (24) and the definition of the sequence $\big\{ W^*[\ell] : \ell \geq \mathbb{N}, \ell \geq 1 \big\}$,

$$\sum_{\ell=1}^{\infty} W^*[\ell] \geq \sup_{(u,v) \in [x,x+\epsilon] \times [y,y+\epsilon]} \big| \mathsf{Wgt}_{n;(u,0)}^{(v,1)} - \mathsf{Wgt}_{n;(x,0)}^{(y,1)} \big| \, .$$

Thus,

$$\mathbb{P} \Bigg(\sup_{(u,v) \in [x,x+\epsilon] \times [y,y+\epsilon]} \big| \mathsf{Wgt}_{n;(u,0)}^{(v,1)} - \mathsf{Wgt}_{n;(x,0)}^{(y,1)} \big| \geq 2^{1 - i_0/2} (1 - 2^{-1/4})^{-1} S \, , \\ \mathsf{PolyWgtReg}_{n;([x-2,x+3],0)}^{([y-2,y+3],1)} \big(2^{-17/2} R/3 \big) \Bigg) \, \leq \, 15064 \, \exp\big\{ - 2^{-14} S^2 \big\} \, .$$

We find that

$$\mathbb{P} \Bigg(\sup \big| \mathsf{Wgt}_{n;(u_1,0)}^{(v_1,1)} - \mathsf{Wgt}_{n;(u_2,0)}^{(v_2,1)} \big| \geq 2 \cdot 2^{1 - i_0/2} (1 - 2^{-1/4})^{-1} S \, , \\ \mathsf{PolyWgtReg}_{n;([x-2,x+3],0)}^{([y-2,y+3],1)} \big(2^{-17/2} S/3 \big) \Bigg) \, \leq \, 15064 \, \exp\big\{ - 2^{-14} S^2 \big\} \, ,$$

where the supremum is over arbitrary $u_1, u_2 \in [x, x+\epsilon]$ and $v_1, v_2 \in [y, y+\epsilon]$.

Applying (22), which is something that requires that n be even, and recalling that $\epsilon \leq 2^{1-i_0}$,

$$\begin{aligned} & \mathbb{P} \Big(\sup \big| \mathsf{Wgt}_{n;(u_1,0)}^{(v_1,1)} - \mathsf{Wgt}_{n;(u_2,0)}^{(v_2,1)} \big| \geq \epsilon^{1/2} 2^{3/2} (1 - 2^{-1/4})^{-1} S \Big) \\ \leq \; & \mathbb{P} \Big(\sup \big| \mathsf{Wgt}_{n;(u_1,0)}^{(v_1,1)} - \mathsf{Wgt}_{n;(u_2,0)}^{(v_2,1)} \big| \geq 2^{2 - i_0/2} (1 - 2^{-1/4})^{-1} S \Big) \\ \leq \; & 15064 \, \exp\big\{ - 2^{-14} S^2 \big\} \, + \, 5^2 \cdot 400 \, C \exp\big\{ - c_1 2^{-8} t_0^{3/2} \big\} \, , \end{aligned}$$

where $t_0 = 2^{-17/2} S/3$. The right-hand side is at most

$$\begin{aligned} & 15064 \, \exp\big\{ - 2^{-14} S^2 \big\} \, + \, 10000 \, C \exp\big\{ - c_1 2^{-8-51/4} 3^{-3/2} S^{3/2} \big\} \\ \leq \; & 25064 \, C \exp\big\{ - c_1 2^{-21} 3^{-3/2} S^{3/2} \big\} \, , \end{aligned}$$

the displayed bound due to $S \geq 1$, $C \geq 1$ and $c_1 \leq 1$. Setting $R = 2^{3/2}(1-2^{-1/4})^{-1}S$ and noting that $2^{-21-9/4}3^{-3/2}(1-2^{-1/4})^{3/2} \geq 2^{-31}$ completes the proof of Theorem 1.3. $\square$

4. Weak limit point regularity: proving Theorem 1.2

We begin this section by developing a preliminary, before giving the proof of Theorem 1.2. It will be useful in that proof to understand that, typically, every f-rewarded polymer that ends in the interval $[-1,1]\times\{1\}$ begins in a compact subset of the x-axis $\mathbb{R}\times\{0\}$. We first introduce a suitable regular fluctuation event $\mathsf{RegFluc}$, and then state and prove a result to this effect, Lemma 4.1.

Recalling Definition 1.1, let $\overline{\Psi} \in (0,\infty)^3$ and $f \in \mathcal{I}_{\overline{\Psi}}$. For $R \geq 0$, define the event

$$\mathsf{RegFluc}^{\big(\{-1,1\},1\big)}_{n;(*:f,0)}(R) = \Big\{ \rho^{(-1,1)}_{n;(*:f,0)}(0) \geq -(R+1)\,,\, \rho^{(1,1)}_{n;(*:f,0)}(0) \leq R+1 \Big\}\,.$$

In fact, the notation used in this definition is slightly abusive. The almost sure uniqueness of f-rewarded line-to-point polymers with given endpoint is proved in [Ham17c, Lemma 4.6(2)]; if we invoke this result, then naturally the polymers $\rho^{(-1,1)}_{n;(*:f,0)}$ and $\rho^{(1,1)}_{n;(*:f,0)}$ are seen to be well defined almost surely. Preferring as do to avoid invoking polymer uniqueness, our interpretation of the above event is instead that: any f-rewarded line-to-point polymer that ends at $(-1,1)$ begins at a location $(x,0)$ where x is at least $-(R+1)$; and any such polymer that ends at $(1,1)$ begins at $(x,0)$, where x is at most $R+1$.

We further remark that our $\mathsf{RegFluc}$ event entails that any f-rewarded line-to-point polymer that ends at a location $(y,1)$ with $y \in [-1,1]$ must begin at a location of the form $(x,0)$, where $|x| \leq R+1$. Indeed, should for example such a polymer $\rho^{(y,1)}_{n;(*:f,0)}$ begin at $(x,0)$, with $x < -(R+1)$, then, in the event $\mathsf{RegFluc}$, it would cross any example of $\rho^{(-1,1)}_{n;(*:f,0)}$, where, in accordance with the usage set out in Subsection 1.3.15, we refer to any such polymer ending at $(-1,1)$. The rewiring of these two polymers also described in that paragraph would then furnish an example of an f-rewarded line-to-point polymer ending at $(-1,1)$ that begins at $(x,0)$, where $x < -(R+1)$, something that is in conflict with the occurrence of $\mathsf{RegFluc}$. Thus, such polymer crossing cannot happen when $\mathsf{RegFluc}$ occurs.

Lemma 4.1. *Let $n \in \mathbb{N}$, $R > 0$ and $\overline{\Psi} \in (0,\infty)^3$ satisfy*

$$n \geq c^{-18} \max\Big\{ (\Psi_2+1)^9\,,\, 10^{23}\Psi_1^9\,,\, 3^9 \Big\}\,,$$

$$R \geq \max\Big\{ 39\Psi_1\,,\, 5\,,\, 3c^{-3}\,,\, 2\big((\Psi_2+1)^2 + \Psi_3\big)^{1/2} \Big\}\,,$$

and $R \leq 6^{-1}cn^{1/9}$. Then, for any $f \in \mathcal{I}_{\overline{\Psi}}$,

$$\mathbb{P}\Big(\neg\, \mathsf{RegFluc}^{\big(\{-1,1\},1\big)}_{n;(*:f,0)}(R) \Big) \leq 38RC \exp\big\{ -2^{-6}cR^3\big(2^{-1/2}-2^{-1}\big)^{3/2} \big\}\,.$$

Proof. The event $\neg\, \mathsf{RegFluc}^{\big(\{-1,1\},1\big)}_{n;(*:f,0)}(R)$ is the union of A_1 and A_2, where A_1 is the event that $y \to \mathcal{L}^{\downarrow;(-1,1)}_{n;0}(1,y) + f(y)$ achieves its maximum for a value of y that is less than $-1-R$, and A_2 is the event that $y \to \mathcal{L}^{\downarrow;(1,1)}_{n;0}(1,y) + f(y)$ achieves its maximum for a value of y that is at greater than $1+R$.

Note the inclusion

$$\Big\{ \sup_{x \in [-\Psi_2, \Psi_2]} \big(\mathcal{L}^{\downarrow;(-1,1)}_{n;0}(1,x) + f(x) \big) > -R^2/2 \, , \, \sup_{x \leq -1-R} \big(\mathcal{L}^{\downarrow;(-1,1)}_{n;0}(1,x) + f(x) \big) \leq -R^2/2 \Big\} \subseteq A_1^c \, . \tag{26}$$

Upper bounds on the failure probability of the left-hand events will now be found. The first event will be shown to be probable because, in view of Definition 1.1, the function f is known to assume a not highly negative value somewhere in a compact interval about the origin. The second event is probable due to the at most linear growth of f far from the origin, combined with decay estimates on the curve $\mathcal{L}^{\downarrow;(-1,1)}_{n;0}(1,x)$ for large x. These estimates take two forms: when x is large, but less than order $n^{1/9}$, the curve hews to a parabola, in accordance with the No Big Max Proposition 1.9(2), applied to the normalized cousin of the ensemble in question; when x becomes even larger, the curve may escape the reaches of this parabola, but it continues to decay rapidly, in accordance with collapse near infinity Proposition 1.9(4).

Since $f \in \mathcal{I}_{\overline{\Psi}}$, there exists $x_0 \in [-\Psi_2, \Psi_2]$ such that $f(x_0) \geq -\Psi_3$. As such, the first left-hand event in (26) fails with a probability that satisfies

$$\begin{aligned}
& \mathbb{P}\Big(\sup_{x \in [-\Psi_2, \Psi_2]} \big(\mathcal{L}^{\downarrow;(-1,1)}_{n;0}(1,x) + f(x) \big) \leq -R^2/2 \Big) \leq \mathbb{P}\Big(\mathcal{L}^{\downarrow;(-1,1)}_{n;0}(1,x_0) \leq -R^2/2 + \Psi_3 \Big) \\
= \; & \mathbb{P}\Big(\mathsf{Nr}\mathcal{L}^{\downarrow;(-1,1)}_{n;0}(1,x_0+1) \leq -R^2/2 + \Psi_3 \Big) \\
\leq \; & \mathbb{P}\Big(\mathsf{Nr}\mathcal{L}^{\downarrow;(-1,1)}_{n;0}(1,x_0+1) + 2^{-1/2}(x_0+1)^2 \leq -R^2/2 + 2^{-1/2}(\Psi_2+1)^2 + \Psi_3 \Big) \\
\leq \; & \mathbb{P}\Big(\mathsf{Nr}\mathcal{L}^{\downarrow;(-1,1)}_{n;0}(1,x_0+1) + 2^{-1/2}(x_0+1)^2 \leq -R^2/4 \Big) \leq C \exp\big\{ -2^{-3} c R^3 \big\} \, ,
\end{aligned}$$

where the penultimate inequality depends on $R^2/4 \geq 2^{-1/2}(\Psi_2+1)^2 + \Psi_3$. The final inequality was obtained by applying the one-point lower tail Reg(2) with parameter choices $\mathbf{z} = x_0$ and $\mathbf{s} = R^2/4$ to the (c,C)-regular ensemble $\mathsf{Nr}\mathcal{L}^{\downarrow;(-1,1)}_{n;0}$. Since the ensemble has $n+1$ curves, this application of Reg(2) may be made provided that $|x_0| + 1 \leq cn^{1/9}$ and $R^2/4 \in [1, n^{1/3}]$. The first of these conditions due to $n \geq c^{-9}(\Psi_2+1)^9$ alongside $|x_0| \leq \Psi_2$; the second we assume.

The failure probability of the second left-hand event in (26) may be gauged as follows: since $f(x)$ is at most $\Psi_1\big(1+|x|\big)$ for any $x \in \mathbb{R}$, we may note that

$$\begin{aligned}
& \mathbb{P}\Big(\sup_{x \leq -1-R} \big(\mathcal{L}^{\downarrow;(-1,1)}_{n;0}(1,x) + f(x) \big) > -R^2/2 \Big) \\
= \; & \mathbb{P}\Big(\sup_{x \leq -R} \big(\mathsf{Nr}\mathcal{L}^{\downarrow;(-1,1)}_{n;0}(1,x) + f(x-1) \big) > -R^2/2 \Big) \\
\leq \; & \mathbb{P}\Big(\sup_{x \leq -R} \big(\mathsf{Nr}\mathcal{L}^{\downarrow;(-1,1)}_{n;0}(1,x) + \Psi_1(2+|x|) \big) > -R^2/2 \Big) \, ;
\end{aligned} \tag{27}$$

the latter term may then be bounded above by

$$\sum \mathbb{P}\Big(\sup_{x \in -R[2^j, 2^{j+1}]} \mathsf{Nr}\mathcal{L}^{\downarrow;(-1,1)}_{n;0}(1,x) > -R^2/2 - \Psi_1\big(2 + 2^{j+1}R\big) \Big) \; + \; E_1 + E_2 \, ,$$

where the first sum is indexed by a parameter j that varies over an initial integer interval $[\![0,k]\!]$ where $k \in \mathbb{N}$ to chosen to be the maximal subject to $2^{k+1}R \leq 3^{-1}cn^{1/9}$. (Such a k exists because we suppose that $2R \leq 3^{-1}cn^{1/9}$.) The term E_1 corresponds to part of a dyadic scale that has been sliced in two by the value $-3^{-1}cn^{1/9}$: this term is specified by the expression in (27) when the supremum in the variable x is chosen to be over the interval $\big[-3^{-1}cn^{1/9}, -2^{k+1}R\big]$. The remaining term E_2 is a long-range error term corresponding to the interval $\big[-z_{\mathcal{L}}, -3^{-1}cn^{1/9}\big]$. Since $R \leq 3^{-1}cn^{1/9}$, this term satisfies

$$E_2 \leq \mathbb{P}\Big(\sup \Big\{ \mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(-1,1)}(1,x) + \Psi_1(2+|x|) : x \in \big[-z_{\mathcal{L}}, -3^{-1}cn^{1/9}\big] \Big\} \geq -2^{-1}(c/3)^2 n^{2/9} \Big) .$$

This right-hand side will be bounded above by applying collapse-near-infinity Proposition 1.9(4) to the the $(n+1)$-curve (c,C)-regular ensemble $\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(-1,1)}$. We apply Proposition 1.9(4) with its parameter $\boldsymbol{\eta}$ chosen so that $\boldsymbol{\eta}(n+1)^{1/9} = -3^{-1}cn^{1/9}$. In order to make the application, we first claim that the affine function $x \to \ell(x)$ in the proposition lies below the function

$$x \to -2^{-1}(c/3)^2 n^{2/9} - \Psi_1\big(2+|x|\big) \tag{28}$$

whenever $x \leq -3^{-1}cn^{1/9}$. To verify this, note that, when $x = -3^{-1}cn^{1/9}$, the assertion takes the form $\Psi_1\big(2+3^{-1}cn^{1/9}\big) \leq \big(2^{-1/2} - 2^{-5/2} - 2^{-1}\big)(c/3)^2 n^{2/9}$, which holds due to the supposed $1 \leq 3^{-1}cn^{1/9}$ and $\Psi_1 \leq \big(2^{-1/2} - 2^{-5/2} - 2^{-1}\big)(c/9)n^{1/9}$. Confirming the claim is then a matter of checking that the gradient of ℓ exceeds that of the function (28), which holds due to $\Psi_1 \leq 5 \cdot 2^{3/2}c/3 \cdot n^{1/9}$.

We may thus apply Proposition 1.9(4) when $n+1 \geq 2^{45/4}c^{-9}$, finding that

$$E_2 \leq 6C \exp\big\{ -2^{-15/4}3^{-3}c^4 n^{1/3} \big\} .$$

Note that

$$\begin{aligned}
& \mathbb{P}\Big(\sup_{x \in -R[2^j,2^{j+1}]} \mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(-1,1)}(1,x) > -R^2/2 - \Psi_1\big(2+2^{j+1}R\big) \Big) \\
\leq \; & \mathbb{P}\Big(\sup_{x \in -R[2^j,2^{j+1}]} \big(\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(-1,1)}(1,x) + 2^{-1/2}x^2 \big) > R^2\big(2^{2j-1/2} - 1/2\big) - \Psi_1\big(2+2^{j+1}R\big) \Big) \\
\leq \; & \mathbb{P}\Big(\sup_{x \in -R[2^j,2^{j+1}]} \big(\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(-1,1)}(1,x) + 2^{-1/2}x^2 \big) > 2^{-1}R^2\big(2^{2j-1/2} - 1/2\big) \Big) \\
\leq \; & 6C(2^{j-1}R+1)\exp\big\{ -2^{-6}cR^3\big(2^{2j-1/2} - 2^{-1}\big)^{3/2} \big\} ,
\end{aligned}$$

where in the second inequality we used $R \geq 1 \vee 39\Psi_1$ in the form

$$2^{-1}R^2\big(2^{2j-1/2} - 1/2\big) \geq \Psi_1\big(2+2^{j+1}R\big)$$

for each $j \geq 0$.

The final inequality arises from an application of Proposition 1.9(2) to the ensemble $\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(-1,1)}$. The parameters of the application are set to be

$$\mathbf{y} = -2^{-1}R\big(2^j + 2^{j+1}\big) , \ \mathbf{r} = 2^{-1}R\big(2^{j+1} - 2^j\big) \text{ and } \mathbf{t} = 2^{-1}R^2\big(2^{2j-1/2} - 2^{-1}\big) .$$

The application's hypotheses are implied by

$$3 \cdot 2^j R \leq cn^{1/9} , \ 2^{j+1}R \leq cn^{1/9} , \ 2^{-1}R^2\big(2^{2j-1/2} - 1/2\big) \in \big[2^{7/2}, 2n^{1/3}\big] \text{ and } n \geq c^{-18} .$$

The first three of these conditions are valid when $j \in [\![0,k]\!]$ in light of the assumed bound $2^{k+1}R \leq 3^{-1}cn^{1/9}$ (and $c \leq 1$); indeed, they are also valid when $j = k+1$, a fact that we will use momentarily.

The term E_1 is bounded above

$$\mathbb{P}\Big(\sup_{x \in -R[2^j,2^{j+1}]} \mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(-1,1)}(1,x) > -R^2/2 - \Psi_1\big(2+2^{j+1}R\big) \Big)$$

$$\mathbb{P}\Big(\sup_{x \in -(R+1)[2^j,2^{j+1}]} \mathcal{L}^{\downarrow;(-1,1)}_{n;0}(1,x) > -R^2/2 - \Psi_1\big(1 + (R+1)2^{j+1}\big)\Big)$$

with $j = k+1$, so that the preceding argument shows that

$$E_1 \leq 6C(2^k R + 1)\exp\big\{ -2^{-6}cR^3\big(2^{2k+3/2} - 2^{-1}\big)^{3/2}\big\}\,.$$

Thus,

$$\begin{aligned}
& \mathbb{P}\Big(\sup_{x \leq -1-R} \big(\mathcal{L}^{\downarrow;(-1,1)}_{n;0}(1,x) + f(x)\big) > -R^2/2\Big) \\
\leq \;\; & 6C\sum_{j=0}^{k+1}(2^j R + 1)\exp\big\{ -2^{-6}cR^3\big(2^{2j-1/2} - 2^{-1}\big)^{3/2}\big\} \, + \, 6C\exp\big\{ -2^{-15/4}3^{-3}c^4 n^{1/3}\big\} \\
\leq \;\; & 12RC\exp\big\{ -2^{-6}cR^3\big(2^{-1/2} - 2^{-1}\big)^{3/2}\big\} \, + \, 6C\exp\big\{ -2^{-15/4}3^{-3}c^4 n^{1/3}\big\} \\
\leq \;\; & 18RC\exp\big\{ -2^{-6}cR^3\big(2^{-1/2} - 2^{-1}\big)^{3/2}\big\}\,,
\end{aligned}$$

where the second inequality used $R \geq (\log 4)^{1/3}2^2c^{-3}\big((2^{3/2}-2^{-1})^{3/2} - (2^{-1/2}-2^{-1})^{3/2}\big)^{-1/3}$ in order to ensure that each ratio of consecutive summands in the sum is at most one-half; the third makes use of $1 \leq R \leq (2^{-1/2}-2^{-1})^{-1/2}2^{3/4}3^{-1}cn^{1/9}$.

Thus,

$$\begin{aligned}
\mathbb{P}\big(A_1\big) \;\; & \leq \;\; C\exp\big\{ -2^{-3}cR^3\big\} \, + \, 18RC\exp\big\{ -2^{-6}cR^3\big(2^{-1/2} - 2^{-1}\big)^{3/2}\big\} \\
& \leq \;\; 19RC\exp\big\{ -2^{-6}cR^3\big(2^{-1/2} - 2^{-1}\big)^{3/2}\big\}\,,
\end{aligned}$$

the latter inequality due to $R \geq 1$. The same argument yields that $\mathbb{P}\big(A_2\big)$ satisfies the same upper bound. Combining the estimates completes the proof of Lemma 4.1. $\square$

Proof of Theorem 1.2. First we mention that Theorem 1.2(1) has already been proved: the statement is Lemma 1.6(3).

Before beginning the substance of the derivation, we dispatch a peculiar little difficulty. During the upcoming proof, we will be applying the local weight regularity Theorem 1.3, and this will entail making the assumption that the index $n \in \mathbb{N}$ is even. In fact, the proof we are about to give will yield what we may call *even* Theorem 1.2, namely the variant of the theorem in which all instances of $\mathbb{N}$ in the statement are replaced by $2\mathbb{N}$ (so that only even choices of n are considered). Before beginning, then, we want to undo the need for this slightly unwelcome hypothesis.

Our method of doing so begins by noting that, whenever $n \in \mathbb{N}$, and $x, y \in \mathbb{R}$, the weight $\mathsf{Wgt}^{(y,1)}_{n+1;(x,0)}$ is at least $\mathsf{Wgt}^{\big(y,1-(n+1)^{-1}\big)}_{n+1;(x,0)}$. After all, any $(n+1)$-zigzag from $(x,0)$ to $\big(y, 1-(n+1)^{-1}\big)$ can be extended by a microscopic unit upwards to form an $(n+1)$-zigzag from $(x,0)$ to $(y,1)$. Now, by the scaling principle, the latter weight is seen to be distributionally equal to $\big(1+n^{-1}\big)^{1/3}\mathsf{Wgt}^{\big(y(1+n^{-1})^{2/3},1\big)}_{n;\big(x(1+n^{-1})^{2/3},0\big)}$. By translation invariance and Taylor expansion, there is also distributional equality with the quantity $\big(1+O(n^{-1})\big)\mathsf{Wgt}^{(\tilde{y},1)}_{n;(x,0)}$ where $\tilde{y} = y + O\big(|x|+|y|\big)n^{-1}$. Indeed, for given x, these equalities in law hold even as y varies, again by the scaling principle.

As we pass to weak limit points along subsequences, as we do in Theorem 1.2(3), we see that any weak limit point obtained along an odd-indexed subsequence stochastically dominates its counterpart along the even-indexed subsequence given by subtracting one from each index. However, the argument may be played in reverse, so that the weak limit point is stochastically bounded above by the counterpart where one is added to each index. We see then that any limit point obtained along an odd-indexed subsequence is stochastically sandwiched, above and below, by a given limit point obtained on an even indexed subsequence. So the weak limit point set has been shown to be unchanged if we impose the condition that subsequential limits be taken along even-indexed sequences.

It remains to address the reduction of Theorem 1.2(2) to its even counterpart. This will do so during the upcoming proof at a moment when the reduction has become straightforward.

Keeping this in mind, it is enough to prove the even theorem, and this we now do. The result has three parts, of which the first is already treated. The uniform tightness claim in the second part will follow once we demonstrate that uniform boundedness and equicontinuity properties obtain uniformly for our prelimiting measures. Indeed, we will begin by specifying these properties and showing that they are typical, thereby proving even Theorem 1.2(2). In fact, the equicontinuity assertion is stronger, establishing that our general initial condition weight profiles in essence enjoy a shared modulus of continuity of the form $x^{1/2}\big(\log x^{-1}\big)^{2/3}$. Thus, we will be able to establish the third part of the even theorem using the equicontinuity claim.

Let $f \in \mathcal{I}_{\overline{\Psi}}$. For $K > 0$, define the uniform boundedness event

$$\mathsf{UnifBd}^{[-1,1]}_{n;(*:f,0)}(K) \, = \, \Big\{ \sup_{y \in [-1,1]} \big|\mathsf{Wgt}^{(y,1)}_{n;(*:f,0)}\big| \leq K \Big\} \, .$$

For $\epsilon \in (0,2)$ and $\rho > 0$, define the equicontinuity event

$$\mathsf{EquiCty}^{[-1,1]}_{n;(*:f,0)}\big(\rho,\epsilon\big) = \Big\{ \, \omega_{[-1,1],\epsilon}\big(y \to \mathsf{Wgt}^{(y,1)}_{n;(*:f,0)}\big) < \rho \, \Big\} \, ,$$

where

$$\omega_{[-1,1],\epsilon}(h) \, = \, \sup \Big\{ \, \big|h(x) - h(y)\big| : x,y \in [-1,1] \, , \, |x-y| \leq \epsilon \, \Big\}$$

denotes the modulus of continuity of a function $h : [-1,1] \to \mathbb{R}$.

Let $f \in \mathcal{I}_{\overline{\Psi}}$ be given. From [Bil99, Theorem 8.2], the sequence of probability measures $\big\{\nu^{([-1,1],1)}_{n;(*:f,0)} : n \in \mathbb{N}\big\}$ is tight if, first, the one-point distribution is tight, in the sense that for all $\eta > 0$, there exists $K > 0$ such that

$$\mathbb{P}\Big(\, \big|\mathsf{Wgt}^{(0,1)}_{n;(*:f,0)}\big| \leq K \, \Big) \geq 1 - \eta \tag{29}$$

for all $n \in \mathbb{N}$; and, second, if, for each $\epsilon > 0$ and $\eta > 0$, there exist a $\rho > 0$ and $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$,

$$\mathbb{P}\Big(\mathsf{EquiCty}^{[-1,1]}_{n;(*:f,0)}\big(\rho,\epsilon\big)\Big) \geq 1 - \eta \, . \tag{30}$$

Moreover, if a choice of $n_0 = n_0(\epsilon,\eta)$ such that (29) and (30) hold whenever $n \geq n_0$ may be made independently of $f \in \mathcal{I}_{\overline{\Psi}}$, then the collection of measures $\big\{\nu^{([-1,1],1)}_{n;(*:f,0)} : n \in \mathbb{N}\big\}$ is uniformly tight in the indexing variable $f \in \mathcal{I}_{\overline{\Psi}}$. (A little work is needed to use the proof of [Bil99, Theorem 8.2] to establish this last assertion. We need to understand that, if the two bounds (29) and (30) hold whenever $n \geq n_0(\epsilon,\eta)$, we are able to assert that the same bounds also hold whenever $n \geq n_0$ where the new selection of n_0 is made merely as a function of $\overline{\Psi}$. For this, what is needed is that, for a *given* value of n that exceeds an $\overline{\Psi}$-determined level, these two bounds may be asserted with the parameters K and η being selected independently of $f \in \mathcal{I}_{\overline{\Psi}}$. We omit this fact's proof, but mention that the essence of the derivation lies in the argument for Lemma 1.6(1) and (3).)

In order to prove even Theorem 1.2(2), it thus suffices to prove the next two claims.

Equicontinuity claim. Define the function $g(\epsilon) = 2C_- \, \epsilon^{1/2} \big(\log \epsilon^{-1} \big)^{2/3}$. There exists a choice of the positive parameter C_-, and a small value $\epsilon_0 > 0$, such that, for all $\epsilon \in (0, \epsilon_0)$ and $f \in \mathcal{I}_{\overline{\Psi}}$,

$$\mathbb{P}\Big(\neg \, \mathsf{EquiCty}^{[-1,1]}_{n;(*:f,0)} \big(g(\epsilon), \epsilon \big) \Big) \leq \epsilon \, , \tag{31}$$

provided that n, even, exceeds a value that is determined by ϵ.

Uniform boundedness claim. For any $\epsilon > 0$, there exists $K = K(\epsilon, \overline{\Psi}) > 0$ such that, for all $f \in \mathcal{I}_{\overline{\Psi}}$,

$$\mathbb{P}\Big(\neg \, \mathsf{UnifBd}^{[-1,1]}_{n;(*:f,0)}(K) \Big) \leq \epsilon$$

whenever n exceeds a value that is determined by ϵ.

(Now is the moment when we can easily reduce the general version of Theorem 1.2(2) to its even counterpart. Indeed, the reduction follows from the distributional equality of the process $y \to \mathsf{Wgt}^{(y,1)}_{n+1;(x,0)}$ with a similar process indexed by n, which we noted at the outset of the proof, and the characterization of even Theorem 1.2(2) just made.)

We prove the two claims in order. To prove the equicontinuity claim, we will first argue that, whenever $R \geq 1$,

$$\begin{aligned} & \mathsf{RegFluc}^{\big(\{-1,1\},1\big)}_{n;(*:f,0)}(R-1) \cap \bigcap_{u \in \epsilon\mathbb{Z} \cap [-R,R], v \in \epsilon\mathbb{Z} \cap [-1,1]} \mathsf{LocWgtReg}^{([v,v+\epsilon],1)}_{n;([u,u+\epsilon],0)} \Big(\epsilon, C_- \big(\log \epsilon^{-1} \big)^{2/3} \Big) \\ \subseteq \quad & \mathsf{EquiCty}^{[-1,1]}_{n;(*:f,0)} \big(g(\epsilon), \epsilon \big) \, . \end{aligned} \tag{32}$$

To verify this inclusion, suppose that $\mathsf{RegFluc}^{\big(\{-1,1\},1\big)}_{n;(*:f,0)}(R-1)$ occurs, and consider $y \in [-1,1]$. After the $\mathsf{RegFluc}$ event was defined, we remarked that its occurrence forces a sandwiching of all f-rewarded line-to-point polymers that abut at time one on $[-1,1]$: they must all begin at time zero somewhere on $[-R,R]$, (where here of course the present parameter value $R-1$ is involved). For this reason, the quantity $\mathsf{Wgt}^{(y,1)}_{n;(*:f,0)}$ is seen to equal $\mathsf{Wgt}^{(y,1)}_{n;(x,0)} + f(x)$ for some $x \in [-R,R]$. Note that the event $\mathsf{LocWgtReg}^{([y,y+\epsilon],1)}_{n;(x,0)} \Big(\epsilon, 2C_- \big(\log \epsilon^{-1} \big)^{2/3} \Big)$ occurs when the intersection of the $\mathsf{LocWgtReg}$ events displayed above occurs; in this circumstance, we thus see that, for any $\eta \in (0,\epsilon)$, $\mathsf{Wgt}^{(y+\eta,1)}_{n;(x,0)} \geq \mathsf{Wgt}^{(y,1)}_{n;(x,0)} + f(x) - \epsilon^{1/2} \cdot 2C_- \big(\log \epsilon^{-1} \big)^{2/3}$ and thus $\mathsf{Wgt}^{(y+\eta,1)}_{n;(*:f,0)} \geq \mathsf{Wgt}^{(y,1)}_{n;(*:f,0)} - \epsilon^{1/2} \cdot 2C_- \big(\log \epsilon^{-1} \big)^{2/3}$. Provided that we further suppose that $y+\eta \leq 1$, the inequality with the roles of y and $y+\eta$ reversed is similarly obtained, so that

$$\Big| \mathsf{Wgt}^{(y+\eta,1)}_{n;(*:f,0)} - \mathsf{Wgt}^{(y,1)}_{n;(*:f,0)} \Big| \, \leq \, \epsilon^{1/2} \cdot 2C_- \big(\log \epsilon^{-1} \big)^{2/3} \, .$$

Thus, (32) is obtained. Verifying the equicontinuity claim is now a matter of arguing that the $\mathsf{RegFluc}$ and the intersection $\mathsf{LocWgtReg}$ events on the left-hand side of (32) both have probability at least $1 - \epsilon/2$.

Treating the intersection $\mathsf{LocWgtReg}$ event first, we now set the value of R equal to $1 + C_+ \big(\log \epsilon^{-1} \big)^{1/3}$ where C_+ is a further positive parameter on which we will impose certain lower bounds.

Let $u, v \in \mathbb{R}$ satisfying $|u| \leq R$ and $|v| \leq 1$ be given. We apply Theorem 1.3 with $\mathbf{x} = u$, $\mathbf{y} = v$, $\boldsymbol{\epsilon} = \epsilon$ and $\mathbf{R} = C_- \big(\log \epsilon^{-1} \big)^{2/3}$ to find that

$$\mathbb{P}\Big(\neg \, \mathsf{LocWgtReg}^{([v,v+\epsilon],1)}_{n;([u,u+\epsilon],0)} \Big(\epsilon, C_- \big(\log \epsilon^{-1} \big)^{2/3} \Big) \leq 25064 \, C \epsilon^{c_1 2^{-31} C_-^{3/2}} \, .$$

Since $|u| \le R = 1 + C_+\big(\log \epsilon^{-1}\big)^{1/3}$ and $|v| \le 1$, this application may be made provided that $\epsilon \in (0, 2^{-4}]$, $n \ge 10^{29} c^{-18}$, $C_+\big(\log \epsilon^{-1}\big)^{1/3} + 2 \le 2^{-5/3} 3^{-1} c n^{1/18}$, and $C_-\big(\log \epsilon^{-1}\big)^{2/3} \in \big[10^6 \, , \, 10^4 n^{1/18}\big]$. Thus, it may be made for $\epsilon > 0$ sufficiently small, and with n exceeding an ϵ-determined level whose order is $\big(\log \epsilon^{-1}\big)^{12}$. We also mention that this application makes use of n being even.

Allowing u and v to vary over $\epsilon\mathbb{Z} \cap [-R, R]$ and $\epsilon\mathbb{Z} \cap [-1, 1]$, the probability that any of the $\mathsf{LocWgtReg}$ events so indexed fails is seen to be at most

$$\big(2R\epsilon^{-1} + 1\big)\big(2\epsilon^{-1} + 1\big) \cdot 25064 \, C \epsilon^{c_1 2^{-31} C_-^{3/2}}$$

and thus at most $\epsilon/2$ provided that $C_- > 0$ is chosen to satisfy $c_1 2^{-31} C_-^{3/2} - 2 > 1$, and $\epsilon > 0$ is small enough.

Lemma 4.1 shows that the failure probability of the $\mathsf{RegFluc}$ event is governed by a similar bound. Indeed, setting $\mathbf{R} = R - 1$ in the lemma, we see that

$$\mathbb{P}\Big(\neg \, \mathsf{RegFluc}_{n;(*:f,0)}^{\big(\{-1,1\},1\big)}(R-1)\Big) \le 38(R-1)C \exp\big\{ -2^{-6} c (R-1)^3 \big(2^{-1/2} - 2^{-1}\big)^{3/2} \big\} \, ,$$

provided that $n \ge c^{-18} \max\big\{(\Psi_2 + 1)^9 \, , \, 10^{23} \Psi_1^9 \, , \, 3^9\big\}$,

$$R \ge 1 + \max\Big\{ 39\Psi_1 \, , \, 5 \, , \, 3c^{-3} \, , \, 2\big((\Psi_2+1)^2 + \Psi_3\big)^{1/2} \Big\} \, ,$$

and $R - 1 \le 6^{-1} c n^{1/9}$.

Recalling that $R = 1 + C_+\big(\log \epsilon^{-1}\big)^{1/3}$, we see that, by choosing $C_+ > 0$ high enough that $2^{-6} c C_+^3 \big(2^{-1/2} - 2^{-1}\big)^{3/2}$ exceeds one, we ensure that

$$\mathbb{P}\Big(\neg \, \mathsf{RegFluc}_{n;(*:f,0)}^{(\{-1,1\},1)}(R)\Big) \, \le \, \epsilon/2 \, , \tag{33}$$

for $\epsilon > 0$ small enough.

We infer then from (32) that the equicontinuity claim holds.

We now demonstrate the uniform boundedness claim. We begin by arguing that, for $R \ge \Psi_2$,

$$\begin{aligned} & \mathsf{RegFluc}_{n;(*:f,0)}^{\big(\{-1,1\},1\big)}(R-1) \cap \mathsf{PolyWgtReg}_{n;([-R,R],0)}^{([-1,1],1)}(R^2) \\ \subseteq \; & \mathsf{UnifBd}_{n;(*:f,0)}^{[-1,1]}\Big(R^2 + 2^{-1/2}(R+1)^2 + \max\{\Psi_3, \Psi_1(1+R)\}\Big) \, . \end{aligned} \tag{34}$$

Indeed, we have noted already that $\mathsf{Wgt}_{n;(*:f,0)}^{(y,1)}$ equals the supremum of $\mathsf{Wgt}_{n;(x,0)}^{(y,1)} + f(x)$ over $x \in [-R, R]$ when the event $\mathsf{RegFluc}_{n;(*:f,0)}^{\big(\{-1,1\},1\big)}(R-1)$ occurs; since $R \ge \Psi_2$, $-\Psi_3 \le \sup_{|x| \le R} f(x) \le \Psi_1(1+R)$. On the event $\mathsf{PolyWgtReg}_{n;([-R,R],0)}^{([-1,1],1)}(R^2)$, $\big|\mathsf{Wgt}_{n;(x,0)}^{(y,1)}\big| \le R^2 + 2^{-1/2}(R+1)^2$ whenever $|x| \le R$ and $|y| \le 1$; this proves (34).

Recall that we have set R equal $1 + C_+\big(\log \epsilon^{-1}\big)^{1/3}$. We will shortly exercise the right to increase the value of the positive constant C_+.

We now apply Corollary 1.5 with parameter settings $\mathbf{t_1} = 0$, $\mathbf{t_2} = 1$, $\mathbf{x} = -R$, $\mathbf{y} = -1$, $\mathbf{a} = \lceil 2R \rceil$, $\mathbf{b} = 2$ and $\mathbf{r} = R^2$. In so doing, we find that

$$\mathbb{P}\Big(\neg \, \mathsf{PolyWgtReg}_{n;([-R,R],0)}^{([-1,1],1)}(R^2)\Big) \le (2R+1) \cdot 400 C \exp\big\{ -c_1 2^{-8} R^3 \big\}$$

provided that n exceeds an ϵ-determined level (which is of the order $\big(\log \epsilon^{-1}\big)^{12}$, in order that the hypothesis $\mathbf{r} \leq 4n^{1/18}$ be satisfied). This upper bound is at most $\epsilon/2$ for small enough $\epsilon > 0$, provided that we insist that the constant $C_+ > 0$ satisfies $c_1 2^{-8} C_+^3 > 1$.

Set $K = R^2 + 2^{-1/2}(R+1)^2 + \max\{\Psi_3, \Psi_1(1+R)\}$. From (34), we combine the last inference with (33) to confirm that the uniform boundedness claim holds, and so complete the proof of even Theorem 1.2(2).

To prove even Theorem 1.2(3), let $\nu \in \mathrm{WLP}_{\overline{\Psi}}$. Let X be ν-distributed. By the definition of ν and properties of weak convergence of measures, we may couple X to a sequence $\big\{X_n : n \in \mathbb{N}\big\}$ of processes so that X_n converges uniformly to X almost surely, where, for each $n \in \mathbb{N}$, X_n has the law of $[-1,1] \to \mathbb{R} : y \to \mathsf{Wgt}^{(y,1)}_{n;(*:f_n,0)}$ with f_n being an element of $\mathcal{I}_{\overline{\Psi}}$.

The bound (31) provides a value of $\epsilon_0 > 0$ that has no dependence on $\nu \in \mathrm{WLP}_{\overline{\Psi}}$ or on the sequence $f_n \in \mathcal{I}_{\overline{\Psi}}$, $n \in \mathbb{N}$, such that, for $\epsilon \in (0, \epsilon_0)$, the probability that there are infinitely many values of $n \in \mathbb{N}$ for which

$$\omega_{[-1,1],\epsilon}\big(y \to X_n(y)\big) < g(\epsilon)$$

is at least $1-\epsilon$. If $j \in \mathbb{N}$ satisfies $2^{-j} \leq \epsilon_0$, we see that, on an event whose probability is at least $1 - 2^{1-j}$, there are infinitely many such $n \in \mathbb{N}$ associated to the value $\epsilon = 2^{-k}$ for every k that is at least j. Since X_n converges to X uniformly except on a null set, we see that, on this same event, $\omega_{[-1,1],\epsilon}\big(y \to X(y)\big) \leq g(\epsilon)$ for these values of ϵ. Since $g:(0,1) \to (0,\infty)$ is decreasing on the interval $\big(0, e^{-4/3}\big)$ (and naturally ϵ_0 may be supposed to be at most $e^{-4/3}$), we find that

$$\mathbb{P}\Big(\omega_{[-1,1],\rho}\big(y \to X(y)\big) \leq g(2\rho) \;\; \forall\, \rho \in \big(0, 2^{-j}\big)\Big) \geq 1 - 2^{1-j}$$

whenever $j \in \mathbb{N}$ satisfies $2^{-j} \leq \epsilon_0$.

Define the random variable $\zeta \in \big[0, \epsilon_0\big]$ to be maximal value on this interval such that, for all $\rho \in \big(0, \zeta\big)$, $\omega_{[-1,1],\rho}\big(y \to X(y)\big) \leq g(2\rho)$. We see that ζ is almost surely positive, and indeed that $\mathbb{P}\big(\zeta \leq x\big) \leq 4x$ for $x \in (0, \epsilon_0]$.

We now fix $x, y \in [-1,1]$ with $x < y$, and suppose in the first instance that $y \leq x + e^{-1}$. Define $K \in \mathbb{N}$ to be the random integer $\lceil (y-x)\zeta^{-1} \rceil$. We may then set $h = (y-x)K^{-1}$, note that $h \in (0, \zeta]$, and find that

$$\begin{aligned} \big|X(y) - X(x)\big| &\leq \sum_{k=0}^{K-1} \big|X\big(x + (k+1)h\big) - X\big(x+kh\big)\big| \leq K g(2h) \\ &\leq 2^{3/2} K C_- h^{1/2} \big(\log h^{-1}\big)^{2/3} . \end{aligned} \tag{35}$$

In the case that $K = 1$, we may now apply $h \leq \zeta \leq e^{-4/3}$ in order to learn that $\big|X(y) - X(x)\big|$ is at most $2^{3/2} K C_- \zeta^{1/2}\big(\log \zeta^{-1}\big)^{2/3}$.

We will now treat the case that $K \geq 2$. Distinctive to this case is the bound $h \geq \zeta/2$, which follows from $(y-x)\zeta^{-1} > 1$. Before we make use of this bound, we first note that, since $Kh = (y-x)$, the quantity (35) equals

$$2^{3/2} C_- (y-x)^{1/2} h^{-1/2} \bigg(\frac{\log h^{-1}}{\log (y-x)^{-1}}\bigg)^{2/3} (y-x)^{1/2} \big(\log (y-x)^{-1}\big)^{2/3}$$

and thus, in view of $y-x \leq e^{-1}$ and $\zeta/2 \leq h \leq e^{-4/3}$, may be bounded above by

$$\begin{aligned} & 2^{3/2} C_- h^{-1/2} \big(\log h^{-1} \big)^{2/3} (y-x)^{1/2} \big(\log (y-x)^{-1} \big)^{2/3} \\ \leq \; & 2^{3/2} C_- (\zeta/2)^{-1/2} \big(\log (\zeta/2)^{-1} \big)^{2/3} (y-x)^{1/2} \big(\log (y-x)^{-1} \big)^{2/3} \, . \end{aligned}$$

Further using $\zeta \leq 1/2$, we find that $2^{3/2+1/2+2/3} C_- \zeta^{-1/2} \big(\log \zeta^{-1} \big)^{2/3} (y-x)^{1/2} \big(\log (y-x)^{-1} \big)^{2/3}$ serves as an upper bound on $\big\vert X(y) - X(x) \big\vert$ in the case that $K \geq 2$. Whether this case applies, or rather $K = 1$, we see that the random variable

$$S := \sup \bigg\{ \big\vert X(y) - X(x) \big\vert (y-x)^{-1/2} \big(\log (y-x)^{-1} \big)^{-2/3} : (x,y) \in [-1,1]^2_< : y \leq x + e^{-1} \bigg\}$$

is bounded above by $2^{8/3} C_- \zeta^{-1/2} \big(\log \zeta^{-1} \big)^{2/3}$. Recalling that $\mathbb{P} \big(\zeta \leq x \big) \leq 4x$ for $x \in (0, \epsilon_0]$, we see that, for such x,

$$\mathbb{P} \Big(S \geq 2^{3/2} C_- x^{-1/2} \big(\log x^{-1} \big)^{2/3} \Big) \leq 4x \, .$$

Setting $y = 2^{8/3} C_- x^{-1/2} \big(\log x^{-1} \big)^{2/3}$, we use $C_- \geq 1$ in finding that $x \leq 2^{20/3} C_-^2 y^{-2} \big(\log y \big)^{4/3}$. That is,

$$\mathbb{P} \big(S \geq y \big) \leq 2^{26/3} C_-^2 y^{-2} \big(\log y \big)^{4/3} \, . \tag{36}$$

Note here that the constant C_- does not depend on our choice of the element ν of $\mathrm{WLP}_{\overline{\Psi}}$ that determines the distribution of X, nor on the sequence $f_n \in \mathcal{I}_{\overline{\Psi}}$, $n \in \mathbb{N}$, that subsequentially indexes convergence to ν. Indeed, because $\epsilon_0 > 0$ also enjoys this lack of dependence, the last display may be asserted for any $y > 1$, by an increase if necessary in the value of C_- that nonetheless does not jeopardise this constant's independence. To complete the proof of even Theorem 1.2(3), we merely need to reach a similar conclusion about the version of the random variable S in whose definition the restriction $y \leq x + e^{-1}$ is absent. If we multiply the right-hand side of (36) by six, the resulting bound may be directly obtained.

The even theorem is proved and thus, as we have noted, so is the theorem itself. □

Appendix A. Calculational derivations

We mentioned in Subsection 1.6.1 that, during the proofs of our results, we have taken care to record the hypotheses needed in order to invoke results needed along the way. Clearly, in each proof, it is necessary that the hypotheses of the result being proved apply all the conditions so invoked. What we have not done during the course of the proofs is to justify that this is the case. In this appendix, we provide these justifications, which we call *calculational derivations.*

There are four results where this work needs to be done. We present our working for the results in the order in which the proofs have appeared. Thus, we treat in the consecutive sections of this appendix Proposition 1.4, Proposition 3.1, Theorem 1.3 and Lemma 4.1.

In the typical calculational derivation, we begin by recording all of the conditions invoked during the proof of the result in question. Some analysis then follows, sometimes involving introducing further conditions, whose role is to imply several of the needed hyptoheses, with a view to producing a simplified list of conditions that imply the complete list of needed hypotheses. Sometimes after a little further simplification, we produce a final list of conditions, which coincide with the hypotheses set of the result in question.

There are a few notational devices that we will use in the derivations. Conditions are given square bracketed names such as [1] or [$n4$], shown on the left. These names may be recycled from one calculational derivation to the next. The notation $[1, n4]$ means 'conditions [1] and [$n4$]'. Implication is denoted by a right arrow, so that $[1, n4] \to [r3]$ means 'conditions [1] and [$n4$] imply condition [$r3$]'.

The notation $[n2, 4, 5]$ is used as a shorthand for $[n2, n4, n5]$. This meaning would be ambiguous, were there named conditions [4] or [5], but we employ this condition only when the meaning is unambiguous.

A.1. **Proposition 1.4: derivation.** The proof of this result consists of the derivation of (3) and that of (2). The calculational derivation of the proposition treats the two pieces separately. In the first case, we follow the plan described above: recording of all needed hypotheses is followed by an analysis that reveals that all of these conditions are implied by the hypotheses of Proposition 1.4. In the second, we split the derivation again into two pieces, and analyse them separately.

In the derivation of (3), the following assertions are used:

$$[1] \; n/2 \geq (c/3)^{-1/18} \vee 6^{36}$$

$$[2] \; 2^{-1/3} \leq (n/2)^{1/18}$$

$$[3] \; 2^{-5/3} t = s \in [2^{3/2}, 2(n/2)^{1/18}]$$

and

$$[4] \; 2^{-1/3}(|x-y|+1) \leq 2^{-1} c (n/2)^{1/18}$$

Thus, $[1,2,3,4]$ is our list of needed conditions. Turning to analyse them, we first note that $[1] \to [2]$. Introducing a further condition

$$[5] \; n \geq 10^{29} \vee 2(c/3)^{-18} \, ,$$

we have $[5] \to [1]$.

[3] is equivalent to

$$t \in [2^{19/6}, 2^{47/18} n^{1/18}]$$

and thus is implied by

$$[6] \; t \in [33, 2^{47/18} n^{1/18}]$$

since $33 \geq 2^{19/6}$.

[4] is equivalent to

$$|x-y| + 1 \leq 2^{-13/18} c n^{1/18}$$

Since [5] entails that $1 \leq 3^{-1} 2^{-1/18} c n^{1/18}$, [4] is implied by [5] and

$$|x-y| \leq (2^{-13/18} - 3^{-1} 2^{-1/18}) c n^{1/18}$$

Since $2^{-13/18} - 3^{-1} 2^{-1/18} = 0.28542 \cdots \geq 0.20998 \cdots = 3^{-1} 2^{-2/3}$, we see that, if we write

$$[7] \; |x-y| \leq 3^{-1} 2^{-2/3} c n^{1/18}$$

then $[5,7] \to [4]$.

In summary, $[5,6,7] \to [1,2,3,4]$. Since $[5,6,7]$ are hypothesised in Proposition 1.4, we see that the hypotheses are adequate for deriving (3).

Derivation of (2).

This derivation involves a use of Proposition 1.9(1) and a use of Proposition 1.9(2). We structure the calculational derivational of (2) by analysing each of these applications in turn.

In the application of Proposition 1.9(1), the bounds needed are

$$n \geq 1 \vee (c/3)^{-18} \vee 6^{36}$$

$$1/2 \leq n^{1/18}$$

$$t/2 \in \big[2^{3/2}\,,\, 2n^{1/18}\big]$$

$$\big|x - z + 1/2\big| = \big|y - x + 1/2\big| \leq c/2 \cdot n^{1/18}$$

The first two follow directly from the hypotheses. The third follows from the hypothesised $t \in [33, 4n^{1/18}]$. For the fourth,

$$[a] \; |x - y| + 1/2 \leq 2^{-1} c n^{1/18}$$

suffices. By hypothesis, we know $n/2 \geq (c/3)^{-18}$ or equivalently $c/6 \cdot 2^{-1/18} n^{1/18} \geq 1/2$. To show $[a]$, it thus suffices to show that

$$|x - y| \leq c\big(2^{-1} - 6^{-1} 2^{-1/18}\big) n^{1/18}$$

Since we suppose that the left-hand side is at most $c \cdot 3^{-1} 2^{-2/3} n^{1/18}$, this inequality holds in view of $3^{-1} 2^{-2/3} \leq 2^{-1} - 6^{-1} 2^{-1/18}$.

In the application of Proposition 1.9(2), we need

$$[b] \; \big|2^{-2/3}(y - z) + 2^{-5/3}\big| = \big|2^{1/3}(y - x) + 2^{-5/3}\big| \leq c/2 \cdot (2n)^{1/9}$$

$$[c] \; 2^{-5/3} \leq c/4 \cdot (2n)^{1/9}$$

$$[d] \; 2^{-4/3} t - 5 \cdot 2^{-11/6} \in \big[2^{7/2}, 2(2n)^{1/3}\big]$$

$$[e] \; 2n \geq c^{-18}$$

Note that $[b]$ is equivalent to

$$[f] \; |x - y| + 4^{-1} \leq 2^{-11/9} c n^{1/9}$$

When $|y - x| \geq 1$, $[f]$ is implied by squaring $[a]$ and using $c \leq 1$. When $|y - x| \leq 1$, $[f]$ is implied by $5^9 2^{-7} c^{-9} \leq n$, a condition which follows from the hypothesised $n \geq 2(c/3)^{-18}$ and $c \leq 1$.

$[c]$ is equivalent to $n \geq 4c^{-9}$ which follows by the hypothesised lower bound on n as well as $c \leq 1$.

$[d]$ follows from the hypothesised $t \in [33, 4n^{1/18}]$ via

$$2^{-4/3} \cdot 33 - 5 \cdot 2^{-11/6} = 11.69 \cdots \geq 11.31 \cdots = 2^{7/2}$$

$[e]$ follows from the hypothesised lower bound on n.

A.2. **Proposition 3.1: derivation.** In this proof, we make use of $|y + u_1 - x - z| \leq 2^{-1} c n^{1/9}$ and $|x + u_1 - y - z| \leq 2^{-1} c n^{1/9}$. Since $u_1, z \in [0, 1]$, both of these are implied by

$$[1] \; |y - x| + 1 \leq 2^{-1} c n^{1/9}\,.$$

Note that $[1]$ is implied by the hypothesised inequalities that $|y - x| \leq 4^{-1} c n^{1/9}$ and $1 \leq 4^{-1} c n^{1/9}$.

We also use $n \geq 2$, which follows from $n \geq (4/c)^9$ and $c \leq 1$.

A.3. **Theorem 1.3: derivation.** For this calculational derivation, in defiance of the guideline offered in the paragraphs that begin this appendix, we begin by labelling the actual hypotheses of Theorem 1.3.

These hypotheses concern parameters $n \in \mathbb{N}$, $\epsilon > 0$, $x, y \in \mathbb{R}$ and $R \geq 0$. The conditions in question are:

$$[h1] \ \epsilon \in (0, 2^{-4}]$$
$$[h2] \ n \geq 10^{29} c^{-18}$$

Let $x, y \in \mathbb{R}$ satisfy

$$[h3] \ \big|x - y\big| \leq 2^{-5/3} 3^{-1} c n^{1/18}$$
$$[h4] \ R \geq 10^6$$
$$[h5] \ R \leq 10^4 n^{1/18}$$

Naturally we must now argue that these conditions imply all conditions invoked during the proof of Theorem 1.3. With a few exceptions, discussed at the end of the derivation, these needed conditions are invoked in two contexts: an application of Corollary 1.5 and an application of Proposition 3.1. We discuss the two contexts in turn.

The first context is the use of Corollary 1.5 via (22). Here, the following conditions are needed:

$$[c1] \ n \geq 10^{29} \vee 2(c/3)^{-18} \, ,$$
$$[c2] \ \big|x - y\big| + 4 \leq 3^{-1} 2^{-2/3} c n^{1/18}$$
$$[c3] \ t_0 \geq 33$$
$$[c4] \ t_0 \leq 4 n^{1/18}$$

where

$$t_0 = 2^{-17/2} S/3$$

and

$$R = 2^{3/2} (1 - 2^{-1/4})^{-1} S$$

Note that $[h2, h3] \to [c1]$.

$[c2]$ is implied by

$$\big|x - y\big| \leq 2^{-1} 3^{-1} 2^{-2/3} c n^{1/18} \, \text{and} \, n \geq \big(2^{11/3} 3 c^{-1}\big)^{18} = 2^{66} 3^{18} c^{-18}$$

the first of which is $[h3]$ and the second of which is implied by $[h2]$ since $2^{66} 3^{18} \in [10^{28}, 10^{29}]$.

$[c3]$ expressed in terms of R is

$$[c3] \ R \geq 33 \cdot 3 \cdot 2^{17/2} \cdot 2^{3/2} (1 - 2^{-1/4})^{-1}$$

$[c4]$ expressed in terms of R is

$$[c4] \ R \leq 4 n^{1/18} \cdot 3 \cdot 2^{17/2} \cdot 2^{3/2} (1 - 2^{-1/4})^{-1}$$

The second of the two contexts is the application of Proposition 3.1. For this purpose, we must have:

$$[p1] \ n \geq (4/c)^9$$
$$[p2] \ \big|x - y\big| \leq c/4 \cdot n^{1/9}$$
$$[p3] \ K_0 \geq 3 \cdot 2^{19/2}$$

Note that $[h2] \to [p1]$ since $c \leq 1$. Also $[h3] \to [p2]$.

In $[p3]$, the value of K_0 ranges over infinitely many values as Proposition 3.1 is repeatedly applied, but in each case, this value satisfies

$$K_0 \geq 2^{k/4-1/2} S$$

where $k \in \mathbb{N}$. Thus, it is enough to ensure that $[p3]$ is satisfied to impose that

$$S \geq 3 \cdot 2^{10}$$

or equivalently

$$R \geq 3 \cdot 2^{10} \cdot 2^{3/2} (1 - 2^{-1/4})^{-1}$$

which is implied by $[c3]$.

Returning to $[c3, c4]$, note that $(1 - 2^{-1/4})^{-1} = 6.285 \cdots$, so that $[c3]$ is implied by

$$R \geq 10^6$$

which is $[h4]$, while $[c4]$ is implied by

$$R \leq 10^4 n^{1/18}$$

which is $[h5]$.

During the proof of the theorem, we also use that S is bounded below by 2^8, $2^{14/2+2} = 2^9$ and 1. All of these bounds follow from $S \geq 3 \cdot 2^{10}$ which we have seen to be implied by $[c3]$.

A.4. **Lemma 4.1: derivation.** We begin this longer derivation by gathering together the collection of conditions that were used during the proof of this lemma.

First recall that the parameters on which conditions are imposed are $n \in \mathbb{N}$, $R > 0$, and the three positive components of $\overline{\Psi}$.

The penultimate inequality of the first displayed set of the equations in the proof makes use of

$$R^2/4 \geq 2^{-1/2} (\Psi_2 + 1)^2 + \Psi_3 \, .$$

When Reg(2) with parameter choices $\mathbf{z} = x_0$ and $\mathbf{s} = R^2/4$ is applied to the ensemble $\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(-1,1)}$, the hypotheses

$$n \geq c^{-9} (\Psi_2 + 1)^9$$

and

$$R^2/4 \in [1, n^{1/3}]$$

are used.

Next, it is supposed that

$$2R \leq 3^{-1} c n^{1/9} \, .$$

When Proposition 1.9(4) is applied to $\mathsf{Nr}\mathcal{L}_{n;0}^{\downarrow;(-1,1)}$, use is made of the conditions:

$$1 \leq 3^{-1} c n^{1/9}$$
$$\Psi_1 \leq \big(2^{-1/2} - 2^{-5/2} - 2^{-1} \big) (c/9) n^{1/9}$$
$$\Psi_1 \leq 5 \cdot 2^{3/2} c/3 \cdot n^{1/9}$$
$$n + 1 \geq 2^{45/4} c^{-9}$$

Next, the condition

$$R \geq 1 \vee 39 \Psi_1$$

is used to assert that

$$2^{-1}R^2\big(2^{2j-1/2}-1/2\big) \geq \Psi_1\big(2+2^{j+1}R\big)$$

for each $j \geq 0$.

When Proposition 1.9(2) is applied to the ensemble $\mathsf{Nr}\mathcal{L}^{\downarrow;(-1,1)}_{n;0}$, the conditions

$$3\cdot 2^j R \leq cn^{1/9}\,,$$
$$2^{j+1}R \leq cn^{1/9}\,,$$
$$2^{-1}R^2\big(2^{2j-1/2}-1/2\big) \leq 2n^{1/3}$$
$$2^{-1}R^2\big(2^{2j-1/2}-1/2\big) \geq 2^{7/2}$$

and

$$n \geq c^{-18}$$

are used. The first three of these conditions are valid when $j \in [\![0,k]\!]$ in light of the condition $2^{k+1}R \leq 3^{-1}cn^{1/9}$ that defines k (and $c \leq 1$ is used as well); indeed, they are also valid when $j = k+1$, a fact which is used immediately after in this proof. For this reason, none of these three conditions impose any additional requirements, and they will be omitted henceforth in our analysis.

Finally, the following bounds are needed:

$$R \geq (\log 4)^{1/3}2^2c^{-3}\big((2^{3/2}-2^{-1})^{3/2}-(2^{-1/2}-2^{-1})^{3/2}\big)^{-1/3}$$
$$R \leq (2^{-1/2}-2^{-1})^{-1/2}2^{3/4}3^{-1}cn^{1/9}$$

and

$$R \geq 1\,.$$

Our analysis begins by confirming that the condition $R \geq 1 \vee 39\Psi_1$ indeed suffices for its role above.

Claim. The condition

$$R \geq 1 \vee 39\Psi_1$$

implies that

$$2^{-1}R^2\big(2^{2j-1/2}-1/2\big) \geq \Psi_1\big(2+2^{j+1}R\big)$$

for each $j \geq 0$.

Proof. We claim that $R \geq 1 \vee 39\Psi_1$ implies that

$$2^{-1}R^2\big(2^{2j-1/2}-1/2\big) \geq \Psi_1\big(2+2^{j+1}R\big) \qquad \text{for each } j \geq 0\,. \tag{37}$$

The displayed inequality is equivalent to

$$R^2\big(2^{2j+1/2}-1\big) \geq 2^3\Psi_1\big(1+2^jR\big)$$

for $j \geq 0$. Since $2^{2j+1/2}-1 \geq 2^{2j}\big(2^{1/2}-1\big)$ for $j \geq 0$ and $1+2^jR \leq 2^{j+1}R$ (this due to $R \geq 1$), the last displayed bound is implied by

$$R^2 2^j\big(2^{1/2}-1\big) \geq 2^4\Psi_1 R\,.$$

This holds for all $j \geq 0$ provided that $R \geq 2^4\big(2^{1/2}-1\big)^{-1}\Psi_1$. Since $2^4\big(2^{1/2}-1\big)^{-1} = 38.62\cdots$, it is also implied by $R \geq 39\Psi_1$.

Thus, $R \geq 1 \vee 39\Psi_1$ implies (37), as we claimed. □

Next we write the set of needed conditions in a list:

$$R^2/4 \geq 2^{-1/2}(\Psi_2+1)^2 + \Psi_3\,.$$
$$n \geq c^{-9}(\Psi_2+1)^9$$
$$R^2 \geq 4$$
$$R^2 \leq 4n^{1/3}$$
$$2R \leq 3^{-1}cn^{1/9}\,.$$
$$1 \leq 3^{-1}cn^{1/9}$$
$$\Psi_1 \leq \big(2^{-1/2} - 2^{-5/2} - 2^{-1}\big)(c/9)n^{1/9}$$
$$\Psi_1 \leq 5 \cdot 2^{3/2}c/3 \cdot n^{1/9}$$
$$n+1 \geq 2^{45/4}c^{-9}$$
$$R \geq 1 \vee 39\Psi_1$$
$$2^{-1}R^2\big(2^{1/2} - 2^{-1}\big) \geq 2^{7/2}$$
$$n \geq c^{-18}\,.$$
$$R \geq (\log 4)^{1/3}2^2c^{-3}\big((2^{3/2}-2^{-1})^{3/2} - (2^{-1/2}-2^{-1})^{3/2}\big)^{-1/3}$$
$$R \leq (2^{-1/2}-2^{-1})^{-1/2}2^{3/4}3^{-1}cn^{1/9}$$
$$R \geq 1\,.$$

We now split this list into categories:

- Lower bounds on R.
- Upper bounds on R.
- Lower bounds on n without dependence on R.

Lower bounds on R.

$$[L1]\ R^2/4 \geq 2^{-1/2}(\Psi_2+1)^2 + \Psi_3\,.$$
$$[L2]\ R^2 \geq 4$$
$$[L3]\ R \geq 39\Psi_1$$
$$[L4]\ 2^{-1}R^2\big(2^{1/2} - 2^{-1}\big) \geq 2^{7/2}$$
$$[L5]\ R \geq (\log 4)^{1/3}2^2c^{-3}\big((2^{3/2}-2^{-1})^{3/2} - (2^{-1/2}-2^{-1})^{3/2}\big)^{-1/3}$$
$$[L6]\ R \geq 1\,.$$

Upper bounds on R.

$$[U1]\ R^2 \leq 4n^{1/3}$$
$$[U2]\ 2R \leq 3^{-1}cn^{1/9}\,.$$
$$[U3]\ R \leq (2^{-1/2}-2^{-1})^{-1/2}2^{3/4}3^{-1}cn^{1/9}$$

Lower bounds on n.

$$[n1]\ n \geq c^{-9}(\Psi_2+1)^9$$
$$[n2]\ 1 \leq 3^{-1}cn^{1/9}$$
$$[n3]\ \Psi_1 \leq \big(2^{-1/2} - 2^{-5/2} - 2^{-1}\big)9^{-1} \cdot cn^{1/9}$$
$$[n4]\ \Psi_1 \leq 5 \cdot 2^{3/2}3^{-1} \cdot cn^{1/9}$$

$$[n5]\ n+1 \geq 2^{45/4} c^{-9}$$
$$[n6]\ n \geq c^{-18}\,.$$

Simplifying the lists.

For each of the three lists, we now find a simpler set of conditions that imply the conditions listed.

Simplifying the list of lower bounds on R. We consider three further conditions:

$$[L7]\ R \geq 5$$
$$[L8]\ R \geq 3c^{-3}$$

and

$$[L9]\ R \geq 2\big((\Psi_2+1)^2+\Psi_3\big)^{1/2}\,.$$

$[L4]$ takes the form $R \geq \big(2^{1/2}-2^{-1}\big)^{-1/2} 2^{9/4}$. Since $\big(2^{1/2}-2^{-1}\big)^{-1/2} 2^{9/4} = 4.975\cdots$, $[L7] \to [L4]$.

Note also that $[L7] \to [L2,6]$.

Since

$$\begin{aligned} & (\log 4)^{1/3} 2^2 \big((2^{3/2}-2^{-1})^{3/2} - (2^{-1/2}-2^{-1})^{3/2}\big)^{-1/3} \\ = \quad & 3.37758\cdots \times \big(3.55298\cdots - 0.09425\cdots\big)^{-1/3} = 3.37758\cdots \times 0.66124\cdots = 2.23\cdots\,, \end{aligned}$$

we see that $[L8] \to [L5]$.

Note that $[L9] \to [L1]$.

Thus $[L7,8,9] \to [L1,2,4,5,6]$.

We see then that $[L3,7,8,9]$ is a simplified list, implying the entire set of conditions $[L1,2,3,4,5,6]$.

Simplifying the list of upper bounds on R.

Note that the conditions $[U1]$ and $[U2]$ may be rewritten:

$$[U1]\ R \leq 2n^{1/9}$$

and

$$[U2]\ R \leq 6^{-1} c n^{1/9}\,.$$

Since $(2^{-1/2}-2^{-1})^{-1/2} 2^{3/4} 3^{-1} = 1.231\cdots$, $[U2] \to [U3]$. That $[U2] \to [U1]$ follows from $c \leq 1$. We see that the simplified list for the upper bounds on R may be chosen to consist of the condition $[U2]$.

Simplifying the list of lower bounds on n. Note that $[n2,3,4]$ may be rewritten:

$$[n2]\ n \geq 3^9 c^{-9}$$
$$[n3]\ n \geq \Psi_1^9 \big(2^{-1/2}-2^{-5/2}-2^{-1}\big)^{-9} 9^9 c^{-9}$$

and

$$[n4]\ n \geq \Psi_1^9\, 5^{-9} 2^{-27/2} 3^9 c^{-9}$$

Noting that

$$\big(2^{-1/2}-2^{-5/2}-2^{-1}\big)^{-9} 9^9 = (0.03033\cdots)^{-9} 9^9 = 1.783\cdots \times 10^{22}\,,$$

we see that, if we write

$$[n7]\ n \geq \Psi_1^9 10^{23} c^{-9}\,,$$

then $[n7] \to [n3,4]$.

We also introduce

$$[n8] \ n \geq 3^9 c^{-18}$$

Since $c \leq 1$, $[n8] \to [n2, 5]$. Also $[n8] \to [n6]$.

Thus, $[n7, 8] \to [n2, 3, 4, 5, 6]$. We see that $[n1, 7, 8] \to [n1, 2, 3, 4, 5, 6]$. The list $[n1, 7, 8]$ may be written:

$$n \geq \max \Big\{ c^{-9} (\Psi_2 + 1)^9 \, , \, \Psi_1^9 10^{23} c^{-9} \, , \, 3^9 c^{-18} \Big\}$$

which since $c \leq 1$ is implied by

$$[n9] \ n \geq c^{-18} \max \Big\{ (\Psi_2 + 1)^9 \, , \, 10^{23} \Psi_1^9 \, , \, 3^9 \Big\} \, .$$

The condition $[n9]$ may thus be chosen as the simplified list of lower bounds on n.

Summary. Drawing together the three simplified lists, we see that the set of conditions

$$[L3, L7, L8, L9, U2, n9]$$

are sufficient for all the conditions that are applied during the proof of Lemma 4.1. These conditions are:

$$R \geq \max \Big\{ \, 39\Psi_1 \, , \, 5 \, , \, 3c^{-3} \, , \, 2 \big((\Psi_2 + 1)^2 + \Psi_3 \big)^{1/2} \, \Big\} \, ,$$

$$R \leq 6^{-1} c n^{1/9} \, ,$$

and

$$n \geq c^{-18} \max \Big\{ (\Psi_2 + 1)^9 \, , \, 10^{23} \Psi_1^9 \, , \, 3^9 \Big\} \, .$$

Since these are the set of hypotheses of Lemma 4.1, the calculational derivation of this result is complete. □

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