Local time on the exceptional set of dynamical percolation, and the Incipient Infinite Cluster

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Abstract

In dynamical critical site percolation on the triangular lattice or bond percolation on $\mathbb{Z}^2$, we define and study a local time measure on the exceptional times at which the origin is in an infinite cluster. We show that at a typical time with respect to this measure, the percolation configuration has the law of Kesten's Incipient Infinite Cluster. In the most technical result of this paper, we show that, on the other hand, at the first exceptional time, the law of the configuration is different. We also study the collapse of the infinite cluster near typical exceptional times, and establish a relation between static and dynamic exponents, analogous to Kesten's near-critical relation.

Contents

1 Introduction

Critical planar percolation is a central object of probability theory and statistical mechanics; see [Gri99, Wer09] for background. The best understood example is $\mathsf{Bernoulli}(1/2)$ site percolation on the triangular lattice $\mathbb{T}$, where conformal invariance and hence convergence of interfaces to SLE_6 is known [Smi01, Sch00, Smi06, CN07]. Nevertheless, many results are known for critical bond percolation on $\mathbb{Z}^2$ and other nice lattices, as well. In particular, almost everything in the present paper will apply equally to site percolation on $\mathbb{T}$ and bond percolation on $\mathbb{Z}^2$.

In **dynamical percolation**, a model introduced independently by [HägPS97] and Itai Benjamini, the status of each bit (site or bond) is continuously and independently resampled from the $\mathsf{Bernoulli}(p)$ measure, at times given by independent Poisson clocks of rate one. We will always consider site percolation on $\mathbb{T}$ and bond percolation on $\mathbb{Z}^2$, at the critical value $p = p_c = 1/2$. One of the principal reasons that dynamical percolation is interesting is that it provides a natural coupling of an uncountable number of copies of the underlying percolation process, and there may exist some exceptional instances of these copies that satisfy certain events that have zero probability in static percolation. The existence (or non-existence) of such **exceptional times** is called dynamical sensitivity (or stability) of the event, and the key event in question is of course the existence of an infinite cluster. See [Ste09] for a survey, but here is a brief summary of the subject. It was proved in [HägPS97] that for $p \neq p_c$ on any graph, both the existence and non-existence of infinite clusters are dynamically stable; then, dynamical stability of non-existence also holds at $p = p_c$ on $\mathbb{Z}^d$ with $d \geq 19$ and on regular trees; and finally, there exist non-regular but spherically symmetric trees with no infinite clusters at p_c in static percolation, but with exceptional times in dynamical percolation. See [Kho08, PSS09] for more recent results on trees. The first example of dynamical sensitivity at p_c in a transitive graph was given by [SchSt10], proving it for the triangular lattice $\mathbb{T}$. This paper used discrete Fourier analysis, a tool that was introduced by [BKS99] for the closely related problem of noise sensitivity of percolation. This technique was further developed in [GPS10], proving that the set of exceptional times almost surely has Hausdorff dimension 31/36, and showing dynamical sensitivity of critical percolation also for bond percolation on $\mathbb{Z}^2$. Further studies of dynamical sensitivity and stability include [BS98, BrGS12, Ahl11] for percolation type processes, [BrS06] and [DCGP11, Section 5] for Ising and random cluster Glauber dynamics, and [BHPS03, Hof06, FNRS09] for some other processes.

The rare appearances of infinite structure at the exceptional times are reminiscent of the **Incipient Infinite Cluster**: a term used by physicists to refer to the large-scale connected structure present in critical percolation, and defined mathematically by Kesten as follows.

Definition 1.1. *The incipient infinite cluster, denoted by* IIC*, is the weak limit of the probability measures* $\mathbb{P}_{p_c}(\,\cdot\,|0 \leftrightarrow n)$ *as* $n \to \infty$*, provided that the limit exists.*

Here, $\{0 \leftrightarrow n\}$ denotes the event that the open cluster of the origin reaches to distance n. (We will formulate a precise definition shortly.) The existence of the IIC for numerous lattices in two dimensions was proved by Kesten [Kes86]. In high dimensions, properties of IIC and its scaling limits have been investigated in detail using the lace expansion [HarS00a, HarS00b]. In two dimensions, several other natural means of locating large structures at criticality — such as using the above definition with the condition $0 \leftrightarrow n$ replaced by the requirement that the open cluster of the origin have size at least n, or the weak limit as $n \to \infty$ of the largest cluster in $[-n,n]^2$ viewed from a uniformly chosen vertex in the cluster — have been shown to also be equal to IIC [Jar03]. These results support the view that, at least in dimension two, any natural means of selecting a limit of large scale critical structure is the IIC. One may ask then how the IIC may be found in dynamical percolation — and this question is central to the present paper.

1.1 The first exceptional time

There is one very natural means of selecting an exceptional time at which the cluster of the origin in dynamical percolation is infinite:

Definition 1.2. *Consider dynamical percolation $\{\omega_t : t \in \mathbb{R}\}$ at criticality. Let $\mathcal{E}$ denote the random set of times at which the cluster of the origin is infinite. We define the first exceptional time FET to be $\inf\{\mathcal{E} \cap (0,\infty)\}$. That $\mathsf{FET} < \infty$ almost surely follows from the principal result of [SchSt10] for $\mathbb{T}$ and from [GPS10] for $\mathbb{Z}^2$. Note that FET is positive almost surely, since some positive time passes before there is a change in any bit (be it site or bond) in the boundary of the finite cluster of the origin in the time zero configuration ω_0. The law of ω_{FET} will be denoted by FETIC, the first exceptional time infinite cluster.*

Although it may be a natural candidate for the appearance of the incipient infinite cluster in dynamical percolation, FETIC is not the right choice:

Theorem 1.3. *The laws FETIC and IIC are not equal.*

Proving Theorem 1.3 is this paper's most complex task. Roughly speaking, we show that the cluster of the origin under FETIC is somewhat thinner than under IIC. Indeed, as we will state more precisely in the next subsection, while the configuration at a "typical" exceptional time turns out to have the law of IIC, with many other exceptional times nearby, FET appears at the endpoint of a unit-order interval in which exceptional times are absent; in fact, finite approximations to FETIC may be constructed by size-biasing dynamical percolation according to the length of the interval lacking connection from 0 to a high distance R leading up to a moment of such a connection. As such, FETIC assigns more mass to configurations which are liable to break apart easily under the perturbation provided by dynamical percolation. What makes the proof difficult is to detect this imbalance also in the limit $R \to \infty$. We will explain these vague ideas in more detail when we start proving Theorem 1.3 in Section 4.

We believe that the two measures differ to a greater degree:

Conjecture 1.4. *The measures* FETIC *and* IIC *are singular with respect to each other.*

The above intuitive explanation about how biasing by the length of the waiting time makes FETIC thinner than IIC might suggest that IIC stochastically dominates FETIC. However, IIC does not satisfy the FKG inequality (which we shortly review), and so it may be that such a general conclusion does not follow from the negative conditioning represented by longer waiting times.

Question 1.5. *Does* IIC *stochastically dominate* FETIC*?*

The invasion percolation cluster IPC is an infinite cluster associated to the critical point which is built by self-organized criticality. It was shown in [DSV09] that IIC and IPC are singular with respect to each other on $\mathbb{Z}^2$. On the other hand, although IIC dominates IPC on regular trees [AGdHS08], this is not so on $\mathbb{Z}^2$ [Sap11].

It was pointed out to us by Alain-Sol Sznitman that, instead of considering the distribution at the first entry to a given subset of the state space in a Markov process, which is FETIC in our case, it is often more convenient to study the so-called equilibrium measure on the subset. For dynamical percolation on the ball B_R and the subset $\mathcal{A} := \left\{\zeta \in \{0,1\}^{B_R} : 0 \leftrightarrow R \text{ in } \zeta\right\}$, this measure is proportional at $\zeta \in \mathcal{A}$ to the probability that dynamical percolation started at ζ and stopped at an independent exponential time T leaves the set $\mathcal{A}$ at the first update and does not return to it before T. The virtue of considering this measure could be that is has closer connections to the potential theory of the Markov process (Green's functions, Dirichlet forms, etc.; see [Szn11, Section 1.3]) than the first entry time, hence it might be easier to address the analogues of Theorem 1.3, Conjecture 1.4 and Question 1.5 for this measure.

1.2 The local time measure and the IIC

Our first effort to seek the IIC in dynamical percolation was hampered by biasing created by the procedure for selection. In light of this, it is natural to try again by considering the law of the configuration obtained by selecting an exceptional time at a "uniform" moment. However, this notion of uniformity requires more structure on the exceptional time set in order to make sense. For this reason, and because of its intrinsic interest, we construct a local time measure μ on the exceptional time set $\mathcal{E}$ as a weak limit of certain measures μ_r on the set of connection times to a large distance $r \in \mathbb{N}$.

The simplest construction would be to define an approximative local time $\overline{\mu}_r$ for distance $r \in \mathbb{N}$ by setting

$$\overline{M}_r(\omega) := \frac{\mathbb{1}\{0 \leftrightarrow r\}}{\mathbb{P}(0 \leftrightarrow r)}, \qquad \text{and} \qquad \overline{\mu}_r[a,b] := \int_a^b \overline{M}_r(\omega_s)\, ds\,, \tag{1.1}$$

and then hope that these measures have a limit $\overline{\mu}[a,b]$ in some sense, as $r \to \infty$. However, we have encountered some technical difficulties in trying to prove this convergence, hence will rely on the following slightly more complicated, but still very natural definition, which turns out to be easier to handle.

A local time is supposed to measure how much time the dynamical percolation process ω_s spends near $\mathcal{E}$. For this, we need some notion of how close a percolation configuration ω is to satisfying $0 \leftrightarrow \infty$. The simplest such notion was proposed in (1.1): the existence of a connection to a large distance r. But we seem to get a more canonical notion by looking at how much a finite piece of the percolation configuration actually helps in realizing a connection to infinity. Namely, for any finite set H of bits, we let ω^H denote the restriction of ω to H, and define the random variable

$$M_H(\omega) := \lim_{R\to\infty} \frac{\mathbb{P}(0 \leftrightarrow R \,|\, \omega^H)}{\mathbb{P}(0 \leftrightarrow R)}. \tag{1.2}$$

Of course, it is not at all obvious that the limit over R exists. However,

$$\frac{\mathbb{P}(0 \leftrightarrow R \,|\, \omega^H)}{\mathbb{P}(0 \leftrightarrow R)} = \frac{\mathbb{P}(0 \leftrightarrow R,\ \omega^H)}{\mathbb{P}(0 \leftrightarrow R)\mathbb{P}(\omega^H)} = \frac{\mathbb{P}(\omega^H \,|\, 0 \leftrightarrow R)}{\mathbb{P}(\omega^H)}, \tag{1.3}$$

whose right-hand side indeed has a weak limit in high R — this is nothing other than the IIC, whose construction was carried out in dimension two by Kesten [Kes86]. Thus, the limit in (1.2) indeed exists, so that we may define

$$M_r(\omega_s) := M_{B_r}(\omega_s), \qquad \mu_r[a,b] := \int_a^b M_r(\omega_s)\, ds\,. \tag{1.4}$$

Note that $\mathbb{E}M_H(\omega_s) = 1$ for any H, hence $\mathbb{E}\mu_r[a,b] = b-a$, independently of r, and we may hope to get a non-degenerate random measure in the limit $r \to \infty$. Moreover, and this is the main advantage of M_r over $\overline{M}_r$, the sequence $\{\mu_r[a,b]\}_{r\in\mathbb{N}}$ is a martingale with respect to the full filtration $\mathscr{F}_r[a,b]$ generated by $\{\omega_s^{B_r} : s \in [a,b]\}$ (see (2.1) in Section 2 for the proof). Thus, martingale convergence results can be used to prove the following:

Theorem 1.6. *The limit $\mu[a,b] = \lim_{r\to\infty} \mu_r[a,b]$ of (1.4) exists almost surely, simultaneously for all $a, b \in \mathbb{R}$; moreover, the convergence holds in L^2 for any interval $[a,b]$.*

Assuming that the limit $\overline{\mu}[a,b] = \lim_{r\to\infty} \overline{\mu}_r[a,b]$ of (1.1) exists in L^2 for all $a,b \in \mathbb{R}$, the two local time measures obtained this way almost surely coincide: $\overline{\mu}[a,b] = \mu[a,b]$ for all $a,b\in\mathbb{R}$.

So, we now have a measure from which we wish to sample uniformly to obtain a candidate for a law coinciding with IIC. However, μ is a σ-finite measure on $\mathbb{R}$ so that further work is needed to make valid the notion of sampling a uniform point with respect to the measure. The next two theorems give constructions of such a point and show that indeed the law of the configuration at the selected time is IIC.

Theorem 1.7 (Quenched sampling). *For almost every realization of the dynamical percolation process $\{\omega_s : s \in [0,\infty)\}$, and the corresponding local time measure μ, there exists some $T_0 < \infty$ such that for all $T > T_0$ we have $\mu[0,T] > 0$. For such T, let χ_T be a random point from $[0,T]$ with law $\mu/\mu[0,T]$. Then, for almost all $\{\omega_s : s \in [0,\infty)\}$, the configuration $\omega(\chi_T)$ converges in law to IIC, as $T \to \infty$.*

Theorem 1.8 (Annealed sampling).

- **(a)** *For any fixed $T > 0$, let $\{\omega^*_s : s \in [0,T]\}$ be dynamical percolation reweighted (size-biased) by $\mu[0,T]$. Let χ^*_T be a random time from $[0,T]$ with law $\mu/\mu[0,T]$ for $\mu = \mu(\omega^*)$. Then, the configuration $\omega^*(\chi^*_T)$ has the distribution of the IIC.*
- **(b)** *Given a sample of $\mu = \mu(\omega)$ on $\mathbb{R}$, let Π_μ be the Poisson point process with intensity μ. One can make sense of conditioning $(\omega, \Pi_{\mu(\omega)})$ on $0 \in \Pi_{\mu(\omega)}$; this is called (ω^*, Π^*_μ), the Palm version of (ω, Π_μ). Then ω^*_0 has the law of the IIC.*

A concrete means of realizing the Palm version of (ω, Π_μ) from dynamical percolation ω is Liggett's extra head construction, which we will describe in Section 3; see Figure 3.1.

Another application of the local time μ could be to run the dynamical percolation process ω according to $\mu(\omega)$. It should be possible to consider this time-changed dynamical percolation as a Markov process on configurations satisfying $0 \leftrightarrow \infty$, with stationary measure IIC; however, even the definition of the right state-space is unclear, especially if one wants IIC to be the unique stationary measure. We will not study these questions here.

1.3 Structure of the paper

In the rest of this Introduction, we summarize the necessary background in static and dynamical critical percolation. In Section 2, we prove Theorem 1.6, and collect some properties of the finite and the limiting local time measures $\overline{\mu}_r$, μ_r, μ. We then locate the IIC using the local time, proving Theorems 1.7 and 1.8 in Section 3. The more substantial Section 4 is devoted to telling apart FETIC and IIC, with a thinning procedure on bounded configurations being introduced and analysed in order to prove Theorem 1.3. The proof of Theorem 1.3 in fact exploits our identification of the IIC in dynamical percolation, because the proof considers a uniform right-hand endpoint of a period of connection $0 \leftrightarrow R$ and examines how long it takes for this connection to be reestablished as time advances; in finding an answer, we will exploit the fact that the law of the configuration in B_R at this endpoint time is a close relative of critical percolation given $0 \leftrightarrow R$ (and thus also of IIC). Section 5 contains Theorem 5.1, a result addressing the question of how instances of the IIC embedded within dynamical percolation typically collapse as the time parameter is tuned at short distances to the moment at which the IIC appears.

As mentioned above, all our results apply equally to critical site percolation on the triangular lattice $\mathbb{T}$ and critical bond percolation on $\mathbb{Z}^2$, except for the existence and values of some critical exponents, of course, but we will formulate our results without using these exponents. For the sake of definiteness, we will work with critical site percolation on $\mathbb{T}$, or rather, with critical percolation on the faces of the dual hexagonal lattice.

1.4 Notation and percolation background

Let e_1 and e_2 denote the Euclidean unit vectors. The lattice in $\mathbb{R}^2$ with generators e_1 and $\frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2$ induces a Voronoi tiling of the plane whose faces are hexagons. We refer to the set of these hexagons, with the adjacency relation given by two hexagons sharing a common edge, as the **hexagonal lattice** $\mathcal{H}$. The hexagon centred at the origin will be denoted by 0. Note that the set of hexagons intersecting the x-axis forms a bi-infinite simple path. Define $d : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{N}$ to be graphical distance and set $B_R = \{h \in \mathcal{H} : d(0, h) \leq R\}$ for $R \in \mathbb{N}$. For $R_1, R_2 \in \mathbb{N}$ such that $R_1 < R_2$, write $A_{R_1,R_2} = B_{R_2} \setminus B_{R_1}$ for the annulus with inner and outer radii R_1 and R_2. The (outer) boundary of a set $A \subset \mathcal{H}$ is $\partial A := \{h \in \mathcal{H} \setminus A : d(h, A) = 1\}$.

In critical percolation on $\mathcal{H}$, each $h \in \mathcal{H}$ is independently open or closed with probability one-half. The set $\{0,1\}^{\mathcal{H}}$ of percolation configurations is equipped with the usual product topology, and the events are the subsets $\mathcal{A} \subseteq \{0,1\}^{\mathcal{H}}$ that are measurable with respect to the corresponding Borel sigma-algebra. For $a, b \in \mathcal{H}$, we write $a \leftrightarrow b$ for the event that an open path of hexagons connects a and b. For $A, B \subseteq \mathcal{H}$, we write $A \leftrightarrow B$ if there exist $a \in A$ and $b \in B$ such that $a \leftrightarrow b$. For $R_1, R_2 \in \mathbb{N}$ such that $1 \leq R_1 < R_2$, we write $R_1 \leftrightarrow R_2$ to indicate that $\partial B_{R_1} \leftrightarrow \partial B^c_{R_2}$. For $R \in \mathbb{N}$, we also write $0 \leftrightarrow R$ for $0 \leftrightarrow \partial B^c_R$.

The open cluster of 0, $\{h \in \mathcal{H} : 0 \leftrightarrow h\}$, will be denoted by $\mathscr{C}_0$.

We will use the notation $\alpha_1(R_1, R_2) := \mathbb{P}(R_1 \leftrightarrow R_2)$ and $\alpha_1(R) := \alpha_1(1, R)$, this being the **one-arm** probability. Furthermore, $\alpha_4(R_1, R_2)$ denotes the **alternating four-arm** probability: the probability that there are two open and two closed paths connecting ∂B_{R_1} and $\partial B^c_{R_2}$, in an alternating order: open-closed-open-closed. Again, $\alpha_4(R) := \alpha_4(1, R)$.

Given a percolation configuration $\omega \in \{0,1\}^{\mathcal{H}}$ and an event $\mathcal{A} \subseteq \{0,1\}^{\mathcal{H}}$, we call a hexagon h **pivotal** for $\mathcal{A}$ in ω if changing the status of h changes the outcome of the event. The set of pivotal hexagons will be denoted by $\mathsf{Piv}_{\mathcal{A}}(\omega)$. For instance, note that h is pivotal for the left-right crossing event in a rectangular region of $\mathcal{H}$ if and only if there are four alternating arms connecting h to the corresponding sides of the rectangle.

Let us now recall some standard tools in percolation theory [Wer09].

The Harris-FKG inequality. The set $\{0,1\}^{\mathcal{H}}$ of percolation configurations on the hexagonal lattice has a natural partial order $\leq$. A percolation event $\mathcal{A} \subseteq \{0,1\}^{\mathcal{H}}$ is called increasing if $\omega \in \mathcal{A}$ and $\omega \leq \omega'$ implies that $\omega' \in \mathcal{A}$. The inequality of Harris and Fortuin-Kesteleyn-Ginibre states that if $\mathcal{A}$ and $\mathcal{B}$ are increasing events, then $\mathbb{P}(\mathcal{A} \cap \mathcal{B}) \geq \mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{B})$. In other words, the percolation measure has positive associations.

RSW estimates. For any $L > 0$, there exists a constant $c_L > 0$ such that the probability of an open path in critical percolation between the left and right sides of the region $\mathcal{H} \cap [0, Ln] \times [0, n]$ is at least c_L, independently of n.

Quasi-muliplicativity of arm probabilities. For $\ell \in \{1, 4\}$, there exists a constant $0 < c_\ell$ such that, for any radii $R_1 < R_2 < R_3$, we have

$$c_\ell < \frac{\alpha_\ell(R_1, R_3)}{\alpha_\ell(R_1, R_2)\,\alpha_\ell(R_2, R_3)} \leq 1\,. \tag{1.5}$$

The right-hand inequality is trivial; for $\ell = 1$, the left-hand one is a simple consequence of FKG and RSW; for $\ell = 4$, more work is needed, done in [Kes87a]; see also [Nol08, SchSt10]. Similarly to quasi-multiplicativity, one can show that we lose only a constant factor in probability if we require our four alternating arms to have their endpoints on nice prescribed arcs of the boundary. This implies the following bounds on the number of pivotals: if $\mathcal{A}(R)$ is the left-right crossing event in the square $[0, R]^2$, then $\mathbb{E}|\mathsf{Piv}_{\mathcal{A}(R)}| \asymp \alpha_4(R)\, R^2$, and if $\mathcal{A}(R_1, R_2)$ is the annulus crossing event $R_1 \leftrightarrow R_2$ with $R_1 < R_2/2$, then $\mathbb{E}|\mathsf{Piv}_{\mathcal{A}(R_1,R_2)}| \asymp \alpha_1(R_1, R_2)\, \alpha_4(R_2)\, R_2^2$, with absolute constant factors. Both upper bounds use the fact there are not many pivotals close to a smooth boundary, which follows from some simple results on arm probabilities in the half-plane; see, e.g., the beginning of [GPS10, Section 7.2].

For critical percolation on $\mathcal{H}$, we also know the existence and values of critical exponents: $\alpha_1(R_1, R_2) = (R_1/R_2)^{5/48+o(1)}$ by [LSW02], and $\alpha_4(R_1, R_2) = (R_1/R_2)^{5/4+o(1)}$ by [SmW01], as $R_2/R_1 \to \infty$. In particular, $\mathbb{E}|\mathsf{Piv}_{\mathcal{A}(R)}| = R^{3/4+o(1)}$ as $R \to \infty$. On $\mathbb{Z}^2$, we have the bounds

$$C^{-1}\,(r/R)^{2-\eta} \leq \alpha_4(r, R) \leq C\,(r/R)^{1+\eta} \tag{1.6}$$

for some fixed constants $C > 0$, $\eta \in (0, 1)$ and every $1 \leq r \leq R$. See [SSmG11, Appendix B] and the references at [GPS10, Eq. (2.6)]. Consequently, with some different value of the constant C,

$$C^{-1}\,R^{\eta} \leq \mathbb{E}|\mathsf{Piv}_{\mathcal{A}(R)}| \leq C\,R^{1-\eta}\,. \tag{1.7}$$

The near-critical window. One can consider monotone versions of dynamical percolation, in which dynamical updates lead always either to the closure or to the opening of hexagons. These give couplings between dynamical and off-critical percolation (and also a coupling of percolation measures at different densities), and therefore information on off-critical percolation can yield bounds on dynamical percolation questions. We will use these relations (which turn out to be sharp) several times.

Kesten found the near-critical window of percolation precisely [Kes87a] (see [Nol08, Wer09] for more modern accounts): for a system of linear size R, the window is given by the reciprocal of the expected number of pivotals for the left-right crossing event $\mathcal{A}(R)$ at criticality. More precisely, for the annulus crossing event $\mathcal{A}(R, 2R)$, as $R \to \infty$, we have

$$\frac{\mathbb{P}_{p_c \pm \epsilon}(\mathcal{A}(R, 2R))}{\mathbb{P}_{p_c}(\mathcal{A}(R, 2R))} \to 1 \qquad \text{if} \quad \epsilon \ll \frac{1}{|\mathsf{Piv}_{\mathcal{A}(R)}|}\,, \tag{1.8}$$

while

$$\delta < \mathbb{P}_{p_c \pm \epsilon}(\mathcal{A}(R,2R)) < 1-\delta \qquad \text{if} \quad \epsilon \asymp \frac{1}{|\mathsf{Piv}_{\mathcal{A}(R)}|}\,, \tag{1.9}$$

with $\delta \in (0,1)$ depending only on the constant factors giving the size of ϵ, and finally,

$$\mathbb{P}_{p_c+\epsilon}(\mathcal{A}(R,2R)) \to \begin{cases} 1 & \text{if} \quad \epsilon \gg \frac{1}{|\mathsf{Piv}_{\mathcal{A}(R)}|}\,, \\ 0 & \text{if} \quad -\epsilon \gg \frac{1}{|\mathsf{Piv}_{\mathcal{A}(R)}|}\,. \end{cases} \tag{1.10}$$

Kesten also proved the stability of one- and alternating four-arm probabilities inside the window:

$$\begin{aligned} \frac{\mathbb{P}_{p_c\pm\epsilon}(\mathcal{A}_\ell(1,R))}{\mathbb{P}_{p_c}(\mathcal{A}_\ell(1,R))} &\to 1 \qquad \text{if} \quad \epsilon \ll \frac{1}{|\mathsf{Piv}_{\mathcal{A}(R)}|}\,, \\ &\asymp 1 \qquad \text{if} \quad \epsilon \asymp \frac{1}{|\mathsf{Piv}_{\mathcal{A}(R)}|}\,, \end{aligned} \tag{1.11}$$

for $\ell \in \{1,4\}$. The $\epsilon \ll 1/\mathbb{E}|\mathsf{Piv}_{\mathcal{A}(R)}|$ case of (1.11) and (1.8) are not stated explicitly in [Kes87a], but they clearly follow from his proof using differential inequalities.

Using the stability of the 1-arm and 4-arm probabilities in the near-critical window, he also found the off-critical exponent, a relation usually called Kesten's scaling relation [Kes87a, Corollary 1]:

$$\mathbb{P}_{p_c+\epsilon}(0 \longleftrightarrow \infty) \asymp \alpha_1(\rho(1/\epsilon))\,, \tag{1.12}$$

where $\rho(r) := \inf\{s \in \mathbb{N}_+ : s^2\,\alpha_4(s) \geq r\}$ for $r \geq 1$, the inverse function of $R \mapsto \mathbb{E}|\mathsf{Piv}_{\mathcal{A}(R)}|$. We have $\rho(r) = r^{4/3+o(1)}$ on $\mathcal{H}$, and $C^{-1}r^\eta \leq \rho(r) \leq C r^{1/\eta}$ for some $0 < \eta, C < \infty$ on $\mathbb{Z}^2$, by (1.7). Note here that Kesten formulated his result in terms of critical exponents, which would not be enough for us later because of the unspecified $o(1)$ terms in the exponent, but the proof clearly gives the stronger result we stated; see [Wer09, Chapter 6].

Dynamical percolation and a dynamical FKG inequality. As mentioned above, we will consider dynamical critical percolation with updates from the stationary distribution (resampling the bits) at times given by Poisson clocks of rate one, with time indexed by $\mathbb{R}$, and, just for the sake of definiteness, with càdlàg trajectories.

We will need the following extension of the FKG inequality to increasing events of dynamical percolation, an immediate consequence of [Lig05, Corollary II.2.21]. A weaker form (with a very different proof) was given in [HamMP12, Lemma 4.2].

Lemma 1.9 (Dynamical FKG inequality)**.** *Let $\omega, \omega' : \mathcal{H} \times \mathbb{R} \longrightarrow \{0,1\}$ denote two realizations of dynamical percolation on the hexagonal lattice $\mathcal{H}$. We say that $\omega \leq \omega'$ if $\omega_t(x) \leq \omega'_t(x)$ for all $(x,t) \in \mathcal{H} \times \mathbb{R}$. Let $\mathcal{A}, \mathcal{B} \subseteq \{0,1\}^{\mathcal{H}\times\mathbb{R}}$ be two increasing events (i.e., if $\omega \in \mathcal{A}$ and $\omega \leq \omega'$, then $\omega' \in \mathcal{A}$). Then $\mathbb{P}(\mathcal{A} \cap \mathcal{B}) \geq \mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{B})$.*

The same holds if the dynamics is not stationary, but started at time 0 from an arbitrary distribution on $\{0,1\}^{\mathcal{H}}$ that satisfies the static FKG inequality (i.e., has positive associations).

Proof. Corollary II.2.21 of [Lig05] states this for increasing events that depend on the configuration at finitely many time instances $t_1 < \cdots < t_n$, proved using induction on n and the infinitesimal generator of the process. Since all measurable dynamical events can be approximated by events depending on finitely many time instances, our statement follows. □

1.5 The Fourier spectrum of critical percolation

A key tool for the analysis of dynamical percolation is discrete Fourier analysis. Here we provide the definition of the Fourier spectrum of a percolation event, explain the basic relation between the spectrum and decorrelation for the event under dynamical percolation, and collect the results from the literature that we will use. A far more thorough overview of this theory is provided by the survey article [GS12].

Let $\mathcal{A}$ denote a percolation event in B_R, so that $\mathcal{A}$ is a subset of percolation configurations in B_R. Define the usual inner product on the L^2-space on percolation configurations on B_R by $\langle f, g \rangle = \mathbb{E}(fg) = 2^{-|B_R|} \sum_{\omega \in \{-1,1\}^{B_R}} f(\omega) g(\omega)$, and note that the collection $\big\{ \chi_S := \prod_{i \in S} \omega(i) : S \subseteq B_R \big\}$ is an orthonormal basis for this L^2-space. As such, the $\{-1,1\}$-indicator function $f_{\mathcal{A}}$ of $\mathcal{A}$ has a Fourier decomposition $f_{\mathcal{A}} = \sum_{S \subseteq B_R} \widehat{f}_{\mathcal{A}}(S) \chi_S$. Parseval's identity $\sum_{S \subseteq B_R} \widehat{f_{\mathcal{A}}}^2(S) = 1$ allows us to define a random variable $\mathsf{Spec}_{\mathcal{A}}$, the spectral sample of $\mathcal{A}$, on subsets of B_R according to $\mathbb{P}(\mathsf{Spec}_{\mathcal{A}} = C) = \widehat{f_{\mathcal{A}}}^2(C)$ for $C \subseteq B_R$.

Recall that the dynamical percolation process $\{\omega_t\}_{t \in \mathbb{R}}$ is defined using i.i.d. rate one Poissonian updates for each bit. Now, the basic relation between the spectral sample and decorrelation under this dynamics is that, for percolation events $\mathcal{A}$ and $\mathcal{B}$ in B_R,

$$\mathbb{E}(\omega_0 \in \mathcal{A}, \omega_t \in \mathcal{B}) = \sum_{S \subseteq B_R} \widehat{f}_{\mathcal{A}}(S) \widehat{f}_{\mathcal{B}}(S) e^{-t|S|} \,. \tag{1.13}$$

This shows that if most of the measure for at least one of the spectral samples $\mathsf{Spec}_{\mathcal{A}}$, $\mathsf{Spec}_{\mathcal{B}}$ is supported on large sets S, then fast decorrelation occurs.

The spectral sample $\mathsf{Spec}_{\mathcal{A}}$ is a random subset of B_R with some similarities to, and marked differences from, the random set $\mathsf{Piv}_{\mathcal{A}}$ of hexagons in B_R that are pivotal for the occurrence of $\mathcal{A}$ under critical percolation. As first observed by Gil Kalai, the two random variables share their first and second moments (see [GPS10, Section 2.3]),

$$\mathbb{E}|\mathsf{Piv}_{\mathcal{A}}| = \mathbb{E}|\mathsf{Spec}_{\mathcal{A}}| \,, \qquad \mathbb{E}|\mathsf{Piv}_{\mathcal{A}}|^2 = \mathbb{E}|\mathsf{Spec}_{\mathcal{A}}|^2 \,, \tag{1.14}$$

but not the higher ones, and their large deviations usually differ (see [GPS10, Remark 4.6]).

Of particular import to us is the case where $\mathcal{A}$ is a crossing event from one boundary arc to another in some planar domain. Let us first consider $\mathcal{A}(R, 2R) = \big\{ R \leftrightarrow 2R \big\}$. A standard second moment argument yields the conclusion that there exists $C > 0$ such that, for all R, $\mathbb{E}|\mathsf{Piv}_{\mathcal{A}(R,2R)}|^2 \leq C \left(\mathbb{E}|\mathsf{Piv}_{\mathcal{A}(R,2R)}| \right)^2$. In light of (1.14) and the second moment method, we see that there exists $c > 0$ such that, for all $R \in \mathbb{N}$,

$$\mathbb{P}\big(|\mathsf{Spec}_{\mathcal{A}(R,2R)}| \geq c\,\mathbb{E}|\mathsf{Spec}_{\mathcal{A}(R,2R)}|\big) \geq c\,.$$

Thus, (1.13) and (1.14) show that, for each $s > 0$, there exists $c(s) < 1$ (with the supremum of $c(s)$ strictly less than one over any interval of the form (ϵ, ∞)) such that, for all $R \in \mathbb{N}$,

$$\mathbb{P}\big(\omega_0 \in \mathcal{A}(R,2R), \omega_t \in \mathcal{A}(R,2R)\big) \leq c(s)\,\mathbb{P}\big(\omega_0 \in \mathcal{A}(R,2R)\big) \tag{1.15}$$

where $t = s\left(\mathbb{E}|\mathsf{Piv}_{\mathcal{A}(R,2R)}|\right)^{-1}$; thus, the characteristic time-scale for at least partial decorrelation of the crossing event is determined by the mean number of pivotals. We will also need the much stronger assertion, proved in [GPS10], that, as $s \to \infty$,

$$\mathbb{P}\big(\omega_0 \in \mathcal{A}(R,2R), \omega_t \in \mathcal{A}(R,2R)\big) - \mathbb{P}\big(\omega_0 \in \mathcal{A}(R,2R)\big)^2 \to 0\,, \tag{1.16}$$

where $t = s\left(\mathbb{E}|\mathsf{Piv}_{\mathcal{A}(R,2R)}|\right)^{-1}$, uniformly in $R \in \mathbb{N}$; on $\mathcal{H}$, we have the sharp upper bound $s^{-2/3+o(1)}$. That is, the crossing event in fact decorrelates fully at large multiples of the scale determined by the mean pivotal number. The bound (1.16) arises from a detailed examination of the lower-tail of the size $|\mathsf{Spec}_{\mathcal{A}(R,2R)}|$ of the spectral sample.

Similar sharp results are proved in [GPS10] for the decorrelation of the crossing events $\mathcal{A}(0,R) = \{0 \leftrightarrow R\}$, which are the key for the applications to exceptional times. Namely, [GPS10, Equation (9.2)] says that, for all $s,t \in \mathbb{R}$ with $|s-t| = O(1)$,

$$\frac{\mathbb{P}\big(0 \overset{\omega_s}{\longleftrightarrow} R,\ 0 \overset{\omega_t}{\longleftrightarrow} R\big)}{\mathbb{P}(0 \longleftrightarrow R)^2} \leq O(1)\frac{1}{\alpha_1(\rho(1/|t-s|))} \tag{1.17}$$

$$\leq O(1)\,|s-t|^{-1+\delta+o(1)} \tag{1.18}$$

for some $\delta > 0$, uniformly in $R \in \mathbb{N}_+$, the $o(1)$ term being understood as $|s-t| \to 0$. On $\mathcal{H}$, also the sharp result $\delta = 31/36$ is known. For exceptional times, the importance of these decorrelation bounds lies in the fact that the exponent δ of (1.18) is a lower bound on the Hausdorff dimension of the set $\mathcal{E}$, using the so-called Mass Distribution Principle.

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2 Construction and basic properties of the local time

In this section, we present the proof of Theorem 1.6, and collect some basic and less basic properties of the finite and the limiting local time measures. We begin by examining the martingale property for the approximating local time measures $\overline{\mu}_r[a,b]$ and $\mu_r[a,b]$, defined in (1.1) and (1.4).

Note that $\overline{M}_R(\omega)$ is a martingale with respect to the filtration $\overline{\mathscr{F}}_R$ of the percolation space generated by the variables $\big\{\mathbb{1}\{0 \leftrightarrow r\} : r \le R\big\}$; indeed, for any $r' > r$,

$$\mathbb{E}\left(\frac{\mathbb{1}\{0 \leftrightarrow r'\}}{\mathbb{P}(0 \leftrightarrow r')} \,\middle|\, \overline{\mathscr{F}}_r \right) = \frac{\mathbb{P}(0 \leftrightarrow r' \,|\, 0 \leftrightarrow r)}{\mathbb{P}(0 \leftrightarrow r')} \mathbb{1}\{0 \leftrightarrow r\} = \frac{\mathbb{1}\{0 \leftrightarrow r\}}{\mathbb{P}(0 \leftrightarrow r)}.$$

Similarly, it is clear from (1.2) that $M_r(\omega)$ is a martingale with respect to the full filtration $\mathscr{F}_r$ generated by ω^{B_r}. Being a martingale w.r.t. this larger sigma-algebra is more useful:

$$\begin{aligned} \mathbb{E}\Big(\mu_R[a,b] \,\Big|\, \mathscr{F}_r[a,b]\Big) &= \int_a^b \mathbb{E}\Big(M_R(\omega_s) \,\Big|\, \mathscr{F}_r[a,b]\Big)\, ds \\ &= \int_a^b \mathbb{E}\Big(M_R(\omega_s) \,\Big|\, \mathscr{F}_r(\omega_s)\Big)\, ds = \int_a^b M_r(\omega_s)\, ds = \mu_r[a,b]\,; \end{aligned} \tag{2.1}$$

that is, $\mu_r[a,b]$ is a martingale w.r.t. $\mathscr{F}_r[a,b]$. On the other hand, $\overline{\mu}_r[a,b]$ does not seem to be a martingale w.r.t. $\overline{\mathscr{F}}_r[a,b]$, since

$$\mathbb{E}\Big(\overline{M}_R(\omega_s) \,\Big|\, \overline{\mathscr{F}}_r[a,b]\Big) \neq \mathbb{E}\Big(\overline{M}_R(\omega_s) \,\Big|\, \overline{\mathscr{F}}_r(s)\Big)$$

in general, because of the extra information provided by $\overline{\mathscr{F}}_r(t)$, $t \in [a,b] \setminus \{s\}$.

Consequently, it is much simpler to prove the convergence of μ_r to some limit μ than the convergence of $\overline{\mu}_r$, though we expect that the latter also holds: as we will see in the forthcoming proof, the local time densities $\overline{M}_r$ and M_r are closely related to each other.

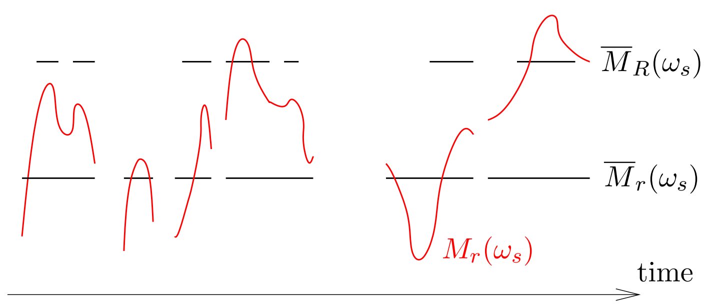

Figure 2.1: Schematic pictures of the approximate local time densities for $\overline{\mu}_r$ and μ_r.

Proof of Theorem 1.6. We begin by proving the statements for any fixed interval $[a,b]$.

First recall the quasi-multiplicativity relation (1.5), which implies, for $R > r > 0$,

$$\frac{\mathbb{P}(0 \leftrightarrow R \,|\, \omega^{B_r})}{\mathbb{P}(0 \leftrightarrow R)} \asymp \frac{\mathbb{P}(0 \leftrightarrow R \,|\, \omega^{B_r})}{\mathbb{P}(0 \leftrightarrow r)\mathbb{P}(r \leftrightarrow R)} \le \frac{\mathbb{P}(r \leftrightarrow R \,|\, \omega^{B_r})\mathbb{1}\{0 \leftrightarrow r\}}{\mathbb{P}(0 \leftrightarrow r)\mathbb{P}(r \leftrightarrow R)} = \frac{\mathbb{1}\{0 \leftrightarrow r\}}{\mathbb{P}(0 \leftrightarrow r)}.$$

Therefore, with an absolute constant $C_1 < \infty$,

$$M_r(\omega) \le C_1 \overline{M}_r(\omega) \quad \text{and} \quad \mu_r[a,b] \le C_1 \overline{\mu}_r[a,b]. \tag{2.2}$$

Second, recall from (1.18) the bound $O(1)|s-t|^{-1+\delta+o(1)}$, with $\delta > 0$. Integrating over s and t, this gives the second moment estimate

$$\mathbb{E}\big(\overline{\mu}_r[a,b]^2\big) \le \begin{cases} |b-a|^{1+\delta+o(1)} & \text{as } |b-a| \to 0\,, \\ C_2\,|b-a| & \text{for all } a<b \text{ with } b-a = O(1)\,, \end{cases} \tag{2.3}$$

uniformly in r, with an absolute constant $C_2 < \infty$. Therefore, by (2.2), the sequence $\mu_r[a,b]$ is an L^2-bounded martingale w.r.t. $\mathscr{F}_r[a,b]$, and the L^2 martingale convergence theorem implies the existence of the limit

$$\mu_r[a,b] \xrightarrow[L^2]{\text{a.s.}} \mu[a,b]\,. \tag{2.4}$$

We also have this convergence almost surely simultaneously over $a, b \in \mathbb{Q}$. To extend it to all $a, b \in \mathbb{R}$ and to prove that the resulting random variables $\mu[a,b]$ together form a measure on $\mathbb{R}$ (finite additivity is clear, but σ-additivity is not), we will use the following lemma:

Lemma 2.1 (No atoms)**.** *There are almost surely no atoms in any of the measures $\overline{\mu}_r$, μ_r or μ.*

To be more precise, we do not yet know at this point that μ is actually a measure on $\mathbb{R}$, hence what we mean is that the non-decreasing map $q \mapsto \mu[0,q]$ for $q \in \mathbb{Q}$ is a.s. continuous (using the convention that $\mu[0,q] = -\mu[q,0]$ for $q < 0$).

Proof. Fix any large $n \in \mathbb{N}$, and cover the interval $[0,1]$ by the intervals $I_i^n := [\frac{i}{2n}, \frac{i}{2n} + \frac{1}{n}]$, $i = 0, 1, \dots, 2n-2$. By (2.3) and Chebyshev's inequality, for any $c > 0$ and any index i, we have $\mathbb{P}(\overline{\mu}_r(I_i^n) > c) \le c^{-2} n^{-1-\delta+o(1)}$ as $n \to \infty$, uniformly in r. By a union bound, $\mathbb{P}(\exists i : \overline{\mu}_r(I_i^n) > c) \le c^{-2} n^{-\delta+o(1)} \to 0$, which implies the claim for $\overline{\mu}_r$, and then (2.2) implies it for μ_r. Now note that Fatou's lemma gives that

$$\mathbb{P}\big(\text{for infinitely many } r,\, \forall i : \mu_r(I_i^n) \le c\big) > 1 - c^{-2} n^{-\delta+o(1)},$$

and then, since all the intervals I_i^n have rational endpoints, the simultaneous almost sure convergence of (2.4) shows that $\mathbb{P}(\exists i : \mu(I_i^n) > c) \le c^{-2} n^{-\delta+o(1)}$. This implies the continuity claim for μ. □

The continuity of $q \mapsto \mu[0,q]$ for $q \in \mathbb{Q}$ gives us a unique way to extend $\mu[0,x]$ continuously to all $x \in \mathbb{R}$. The simultaneous almost sure convergence $\mu_r[0,q] \to \mu[0,q]$ for all $q \in \mathbb{Q}$ and the obvious monotonicity $\mu_r[0,q] \le \mu_r[0,x] \le \mu_r[0,q']$ for $q < x < q'$ clearly implies simultaneous convergence for all $\mu_r[0,x]$, and the finite additivity of μ_r implies the simultaneous a.s. convergence $\mu_r[a,b] \to \mu[a,b]$ for all $a,b \in \mathbb{R}$.

Now, we turn to the sequence $\overline{\mu}_r[a,b]$. If we fix $r>0$, and take $R \to \infty$, then

$$\mathbb{E}\big(\overline{M}_R \mid \mathscr{F}_r\big) = \frac{\mathbb{P}(0 \leftrightarrow R \,|\, \mathscr{F}_r)}{\mathbb{P}(0 \leftrightarrow R)} \xrightarrow[L^\infty]{\text{a.s.}} M_r\,,$$

by the very definition of M_r, a random variable on the finite space B_r. Thus, for fixed r, the random variables $\mathbb{E}\big(\overline{M}_R \mid \mathscr{F}_r\big)$ are uniformly bounded in R, and

$$\int_a^b \mathbb{E}\Big(\overline{M}_R(\omega_s) \,\Big|\, \mathscr{F}_r(\omega_s)\Big)\, ds \xrightarrow[L^\infty]{\text{a.s.}} \int_a^b M_r(\omega_s)\, ds = \mu_r[a,b].$$

On the other hand, for random variables, convergence in L^2 is stronger than in L^1, hence the hypothetical L^2-convergence of the unconditional $\overline{\mu}_R[a,b]$ implies

$$\begin{aligned}\int_a^b \mathbb{E}\Big(\overline{M}_R(\omega_s) \,\Big|\, \mathscr{F}_r(\omega_s)\Big)\, ds &= \int_a^b \mathbb{E}\Big(\overline{M}_R(\omega_s) \,\Big|\, \mathscr{F}_r[a,b]\Big)\, ds \\ &= \mathbb{E}\Big(\overline{\mu}_R[a,b] \,\Big|\, \mathscr{F}_r[a,b]\Big) \xrightarrow[L^2]{} \mathbb{E}\Big(\overline{\mu}[a,b] \,\Big|\, \mathscr{F}_r[a,b]\Big).\end{aligned}$$

One sequence can have only one L^2-limit, and convergence in L^∞ is stronger than in L^2, thus

$$\mathbb{E}\Big(\overline{\mu}[a,b] \,\Big|\, \mathscr{F}_r[a,b]\Big) = \mu_r[a,b] \quad \text{ in } L^2, \text{ hence almost surely.} \tag{2.5}$$

As $r \to \infty$, $\mathscr{F}_r[a,b]$ converges to the full sigma-algebra, hence the left-hand side of (2.5) converges a.s. to $\overline{\mu}[a,b]$ by Lévy's zero-one law, while the right-hand side converges to $\mu[a,b]$, by (2.4). The two limits coincide a.s., simultaneously for all $a,b \in \mathbb{Q}$. As before, applying Lemma 2.1 gives us a measure $\overline{\mu}[a,b]$ which agrees with $\mu[a,b]$ simultaneously for all $a,b \in \mathbb{R}$, and the proof of Theorem 1.6 is complete. □

Conjecture 2.2. *The L^2-limit $\overline{\mu}[a,b] = \lim_{r\to\infty} \overline{\mu}_r[a,b]$ exists, and then, by Theorem 1.6, $\overline{\mu} = \mu$ almost surely.*

We collect now some basic properties of the dynamical percolation process, the exceptional set, and the associated local time.

Lemma 2.3 (Ergodicity)**.** *The dynamical percolation process ω on the infinite lattice (in particular, the local time $\mu = \mu(\omega)$) is ergodic with respect to time shifts.*

Proof. This argument is certainly classical, but, having been unable to find an exact reference, we include it here for completeness.

For any dynamic event $\mathcal{A}$ and any $\epsilon > 0$, there exists a radius $r \in \mathbb{N}$, a time $T>0$, and an event $\mathcal{A}_{r,T}$ measurable with respect to $\omega^{B_r}(-T,T)$ such that $\mathbb{P}\big(\mathcal{A} \triangle \mathcal{A}_{r,T}\big) < \epsilon$. Now, by the ergodicity of dynamical percolation in B_r (a Markov chain on a finite state space), there exists $t = t(r,T)$ such that $\left|\mathbb{P}(\mathcal{A}_{r,T} \cap (\mathcal{A}_{r,T}+t)) - \mathbb{P}(\mathcal{A}_{r,T})^2\right| < \epsilon$, where $\mathcal{A}_{r,T}+t$ represents the event $\mathcal{A}_{r,T}$ evaluated for the dynamical configuration shifted back by time t. Now, if $\mathcal{A}$ is invariant under time shifts, then $\left|\mathbb{P}(\mathcal{A} \cap (\mathcal{A}+t)) - \mathbb{P}(\mathcal{A}_{r,T} \cap (\mathcal{A}_{r,T}+t))\right| \leq \mathbb{P}\big(\mathcal{A} \triangle (\mathcal{A}_{r,T} \cap (\mathcal{A}_{r,T}+t))\big) < 2\epsilon$. Altogether, $\left|\mathbb{P}(\mathcal{A}) - \mathbb{P}(\mathcal{A})^2\right| < 2\epsilon + \epsilon + \epsilon^2$. This holds for any $\epsilon > 0$, hence $\mathbb{P}(\mathcal{A}) \in \{0,1\}$. □

Lemma 2.4 (Perfectness)**.** *Almost surely, the set $\mathcal{E}$ of exceptional times*
(i) *is disjoint from the set of times at which the status of a hexagon is updated;*
(ii) *is topologically closed;*
(iii) *has no isolated points.*

Proof. Parts (i) and (ii) are proved in [HägPS97, Lemma 3.2]. Part (iii) is proved in [HägPS97, Lemma 3.4] and the remark following it. □

It is clear that μ is supported inside $\mathcal{E}$. The following statement is very natural, but it seems hard to prove:

Conjecture 2.5. *The support of the local time measure μ is almost surely the entire exceptional time set $\mathcal{E}$.*

This conjecture cannot fail by much: the Hausdorff dimension of $\operatorname{supp}\mu$ is the same as the dimension of $\mathcal{E}$, namely $31/36$. The reason is that the proof of the lower bound in [GPS10] (just like in [SchSt10]) uses the approximate local time measures $\overline{\mu}_r$ and a version of the Mass Distribution Principle, and, via (2.2), it could also have used the measures μ_r, hence it actually yields a lower bound on $\dim_H(\operatorname{supp}\mu)$. The next lemma, which will be of use later, provides a little further evidence for the conjecture.

Lemma 2.6. *For any $\epsilon > 0$, let μ_ϵ denote $\overline{\mu}_r[0,\epsilon]$ or $\mu_r[0,\epsilon]$ or the limit $\mu[0,\epsilon]$. Then there is an absolute constant $C < \infty$ such that*

$$\mathbb{E}\big(\mu_\epsilon^2 \mid \mu_\epsilon > 0\big) \leq C\,\mathbb{E}\big(\mu_\epsilon \mid \mu_\epsilon > 0\big)^2, \tag{2.6}$$

and another such constant $c > 0$ such that

$$\mathbb{P}\Big(\mu_\epsilon > c\,\mathbb{E}(\mu_\epsilon \mid \mu_\epsilon > 0) \;\Big|\; \mathcal{E} \cap [0,\epsilon] \neq \emptyset\Big) > c. \tag{2.7}$$

Proof. The left-hand side of (2.6) equals $\mathbb{E}(\mu_\epsilon^2)/\mathbb{P}(\mu_\epsilon > 0)$, while the right-hand side equals $\mathbb{E}(\mu_\epsilon)^2/\mathbb{P}(\mu_\epsilon > 0)^2 = \epsilon^2/\mathbb{P}(\mu_\epsilon > 0)^2$. Hence, we need to show that

$$\mathbb{E}\big(\mu_\epsilon^2\big) \leq C\,\frac{\epsilon^2}{\mathbb{P}(\mu_\epsilon > 0)}.$$

By a usual coupling between dynamical and near-critical percolation, in which dynamical updates lead always to the opening of hexagons in the latter case, we have

$$\begin{aligned}\mathbb{P}(\mu_\epsilon > 0) \leq \mathbb{P}(\mathcal{E} \cap [0,\epsilon] \neq \emptyset) &\leq \mathbb{P}_{p_c+O(\epsilon)}(0 \longleftrightarrow \infty)\\ &= O(1)\,\alpha_1(\rho(1/\epsilon)),\end{aligned} \tag{2.8}$$

by Kesten's scaling relation (1.12). On the other hand, taking the double integral of (1.17) over $s,t\in[0,\epsilon]$, we claim that

$$\mathbb{E}\big(\mu_\epsilon^2\big) \le O(1)\,\frac{\epsilon^2}{\alpha_1(\rho(1/\epsilon))}\,, \tag{2.9}$$

which will finish the proof of (2.6).

By (2.2) and (2.4), it is enough to verify (2.9) for $\mu_\epsilon = \overline{\mu}_r[0,\epsilon]$. Set $R=\rho(1/\epsilon)$ and $A_i=[C^iR, C^{i+1}R]$, $i\in\mathbb{N}$, where $C>0$ is a large constant to be specified shortly. For $i\in\mathbb{N}$, write

$$B_i = \Big\{(s,t)\in[0,\epsilon]^2 : \rho\big(|s-t|^{-1}\big)\in A_i\Big\}\,,$$

so that

$$\mathbb{E}\big(\mu_\epsilon^2\big) = \int_{[0,\epsilon]^2} \frac{\mathbb{P}\Big(\mathbb{1}\{0 \overset{\omega_s}{\longleftrightarrow} r\}\mathbb{1}\{0 \overset{\omega_t}{\longleftrightarrow} r\}\Big)}{\mathbb{P}(0\longleftrightarrow r)^2}\,\mathrm{d}s\,\mathrm{d}t \le O(1)\sum_{i\ge 0}\phi_i\,,$$

with $\phi_i = \int_{B_i}\frac{1}{\alpha_1(\rho(1/|t-s|))}\,\mathrm{d}s\,\mathrm{d}t$; the latter inequality is due to (1.17).

Note that $\rho(\cdot)$ is a non-strictly increasing function. By (1.6), there exists an absolute constant $K>0$ such that $s^2\alpha_4(s) < (Ks)^2\alpha_4(Ks)$ for all $s\in\mathbb{Z}^+$, hence $\rho(s^2\alpha_4(s))\in(s/K,s]$ for all $s\in\mathbb{Z}^+$. This implies that

$$\rho^{-1}(A_i) \subseteq \big[(C^iR)^2\alpha_4(C^iR),\,(C^{i+1}KR)^2\alpha_4(C^{i+1}KR)\big]\,. \tag{2.10}$$

If C is large enough, so that $(CR)^2\alpha_4(CR) > 2(KR)^2\alpha_4(KR)$, then $(1/\epsilon, 2/\epsilon)\subseteq\rho^{-1}(A_0)$, hence the Lebesgue measure of B_0 is at least $\epsilon^2/2$. Therefore,

$$\phi_0 = \int_{B_0}\frac{1}{\alpha_1(\rho(1/|t-s|))}\,\mathrm{d}s\,\mathrm{d}t \ge \frac{\epsilon^2}{2}\,\alpha_1\big(\rho(1/\epsilon)\big)^{-1}\,.$$

On the other hand, for $i\ge 1$, using (2.10),

$$\phi_i = \int_{B_i}\frac{1}{\alpha_1(\rho(1/|t-s|))}\,\mathrm{d}s\,\mathrm{d}t \le 2\epsilon C^{-2i}R^{-2}\alpha_4\big(C^iR\big)^{-1}\alpha_1\big(C^{i+1}R\big)^{-1}\,.$$

Thus,

$$\begin{aligned}\frac{\phi_i}{\phi_0} &\le 4\epsilon^{-1}C^{-2i}R^{-2}\frac{\alpha_1(R)}{\alpha_4(C^iR)\alpha_1(C^{i+1}R)}\\ &\le \frac{4C^{-2i}}{\alpha_4(R,C^iR)\alpha_1(R,C^{i+1}R)}\,,\end{aligned}$$

where in the second inequality we used that $\epsilon^{-1}\le R^2\alpha_4(R)$. Now, [GPS10, Appendix] says that the sum of the 1-arm and 4-arm exponents is strictly less than 2 — properly interpreted in the case of $\mathbb{Z}^2$ where these exponents are not known to exist. That is, there exists some $c\in(0,1)$ such that $\phi_i/\phi_0\le O(1)\,c^i$ for all $i\ge 1$. Thus,

$$\int_{[0,\epsilon]^2}\frac{1}{\alpha_1(\rho(1/|t-s|))}\,\mathrm{d}s\,\mathrm{d}t \le O(1)\int_{A_0}\frac{1}{\alpha_1(\rho(1/|t-s|))}\,\mathrm{d}s\,\mathrm{d}t \le O(1)\,\epsilon^2\alpha_1\big(\rho(1/\epsilon)\big)^{-1}\,,$$

and we have confirmed (2.9).

By the Paley-Zygmund second moment inequality (a simple consequence of Cauchy-Schwarz; see, e.g., [LyP11, Section 5.5]), the above computations show that

$$\mathbb{P}(\mu_\epsilon > 0) \geq \frac{(\mathbb{E}\mu_\epsilon)^2}{\mathbb{E}(\mu_\epsilon^2)} \geq c_1\, \alpha_1(\rho(1/\epsilon))\,,$$

matching the upper bound (2.8) up to a constant factor. Therefore,

$$\mathbb{P}\big(\mu_\epsilon > 0 \,\big|\, \mathcal{E} \cap [0,\epsilon] \neq \emptyset\big) > c_2 > 0\,.$$

On the other hand, again by the Paley-Zygmund inequality, (2.6) implies that

$$\mathbb{P}\Big(\mu_\epsilon > c_3\, \mathbb{E}(\mu_\epsilon \mid \mu_\epsilon > 0) \,\Big|\, \mu_\epsilon > 0\Big) > c_3 > 0\,,$$

for some $c_3 > 0$. Combining the last two displayed inequalities proves (2.7). □

We conclude this section with a natural question:

Question 2.7. *Is the local time μ the 31/36-dimensional Minkowski content of the set $\mathcal{E}$? Is μ the Hausdorff measure of $\mathcal{E}$ for some Hausdorff gauge function?*

3 Finding the Incipient Infinite Cluster

Given the description of the local time measure using (1.3), it is natural to guess that the infinite cluster at a "typical" exceptional time (typical with respect to μ) has the law of IIC. The first exceptional time having been discredited as a candidate for the IIC by Theorem 1.3, we now prove Theorems 1.7 and 1.8, thereby verifying what may be the simplest relationship between exceptional times and the IIC.

Unsurprisingly, the proofs go through the finite approximations, about which we provide a further definition.

Definition 3.1. *Let IIC_r denote the law on percolation configurations in B_r given by $\mathbb{P}_{p_c}\big(\cdot \,\big|\, 0 \leftrightarrow r\big)$.*

Note that $\overline{M}_r(\omega)$ is the Radon-Nikodym derivative $d\mathsf{IIC}_r/d\mathbb{P}$, while $M_r(\omega)$ is the Radon-Nikodym derivative $d\mathsf{IIC}^{B_r}/d\mathbb{P}$, where $\mathbb{P} = \mathbb{P}_{p_c}$ is critical percolation. Since both IIC_r and IIC^{B_r} converge to IIC as $r \to \infty$, both $\overline{\mu}_r$ and μ_r can be useful in studying the relationship between dynamical percolation and the IIC. Indeed, in the forthcoming lemmata, the versions about μ_r will be used in finding the IIC in dynamical percolation, while the versions for $\overline{\mu}_r$ will be used in Section 4 to prove that $\mathsf{FETIC} \neq \mathsf{IIC}$. The finite versions of our results will be slightly stronger than the infinite ones, in that they identify not only a moment where we get IIC^{B_r} or IIC_r, but also an equality of entire processes. We will use the stronger, dynamic version for $\overline{\mu}_r$ in Section 4.

Lemma 3.2 (Finite r quenched sampling)**.** *Let $\{\omega(s) : s \in [0,\infty)\}$ be dynamical percolation in B_r. Let $\overline{\chi}_{r,T} \in \mathbb{R}$ be a random time sampled from $\overline{\mu}_r/\overline{\mu}_r[0,T]$, defined only when $\overline{\mu}_r[0,T] > 0$. Then, the finite dimensional distributions of $\{\omega(\overline{\chi}_{r,T} + s) : s \in [0,\infty)\}$ converge for almost all ω as $T \to \infty$ to those of standard dynamical percolation started from IIC_r at time zero. Moreover, the law of the entire process in the Skorokhod topology converges in probability to the same limit process.*

Similarly, if $\chi_{r,T} \in \mathbb{R}$ is a random time sampled from $\mu_r/\mu_r[0,T]$, then the same results hold for the process $\{\omega(\chi_{r,T} + s) : s \in [0,\infty)\}$.

Lemma 3.3 (Finite r annealed sampling)**.**

(a) *Let $\{\overline{\omega}^*(s) : s \in [0,\infty)\}$ be dynamical percolation in B_r size-biased by $\overline{\mu}_r[0,T]$, and $\overline{\chi}^*_{r,T} \in \mathbb{R}$ be a random time with law $\overline{\mu}_r/\overline{\mu}_r[0,T]$ for $\overline{\mu}_r = \overline{\mu}_r(\overline{\omega}^*)$. Then the process $\{\overline{\omega}^*(\overline{\chi}^*_{r,T} + s) : s \in [0,\infty)\}$ is equal in law to standard dynamical percolation started from IIC_r at time zero.*

Similarly, if $\{\omega^*(s) : s \in [0,\infty)\}$ is dynamical percolation in B_r size-biased by $\mu_r[0,T]$, and $\chi^*_{r,T} \in \mathbb{R}$ is a random time with law $\mu_r/\mu_r[0,T]$ for $\mu_r = \mu_r(\omega^*)$, then the process $\{\omega^*(\chi^*_{r,T} + s) : s \in [0,\infty)\}$ is equal in law to standard dynamical percolation started from IIC^{B_r} at time zero.

(b) *The Palm version $(\overline{\omega}^*, \overline{\Pi}^*_r)$ of the process $(\omega, \Pi_{\overline{\mu}_r(\omega)})$ in B_r is standard dynamical percolation started from IIC_r at time zero. A somewhat concrete way to realize the Palm version is* **Liggett's extra head construction** *[Lig02], see Figure 3.1:*

Let $\{p_i \in [0,\infty) : i \in \mathbb{N}\}$ enumerate a Poisson point process Θ with intensity measure Lebesgue on $[0,\infty)$, and set $\bar{q}_{r,i} = \inf\{t > 0 : \overline{\mu}_r[0,t] > p_i\}$. Clearly, $\Pi_{\overline{\mu}_r} := \{\bar{q}_{r,i} : i \in \mathbb{N}\}$ is a Poisson point process with intensity $\overline{\mu}_r$. Set $m_r := \mathbb{E}\overline{\mu}_r[0,1]$. Now let $J \in \mathbb{N}$ be the first integer with $\left|\Pi_{\overline{\mu}_r} \cap [0, m_r^{-1}J]\right| > J$. Then shifting back time by $\bar{q}_{r,J}$ gives the Palm version of $(\omega, \Pi_{\overline{\mu}_r})$.

Similarly, the Palm version $(\omega^*, \Pi^*_{\mu_r})$ of the process $(\omega, \Pi_{\mu_r(\omega)})$ in B_r is standard dynamical percolation started from IIC^{B_r} at time zero. As above, the Palm versions $(\omega^*, \Pi^*_{\mu_r})$ and (ω^*, Π^*_{μ}) can be constructed using time shifts by $q_{r,J}$ and q_J.

It should be intuitively quite clear why the ergodic quenched limits in Lemma 3.2 lead to the size-biased finite averages in Lemma 3.3: each dynamic configuration of a finite time interval appears in the ergodic quenched limit with a frequency proportional to its probability.

Proof of Lemma 3.2. Note that for any percolation configuration ζ on B_r satisfying $0 \longleftrightarrow r$, by definition, $\mathbb{P}\big(\omega(\overline{\chi}_{r,T}) = \zeta\big) = \mathbb{E}\big(\int_0^T \mathbb{1}\{\omega_t = \zeta\}\, \mathrm{d}t/\overline{\mu}_r[0,T]\big)$, where the event on the left-hand side is taken to be unsatisfied and the ratio on the right-hand side is taken to

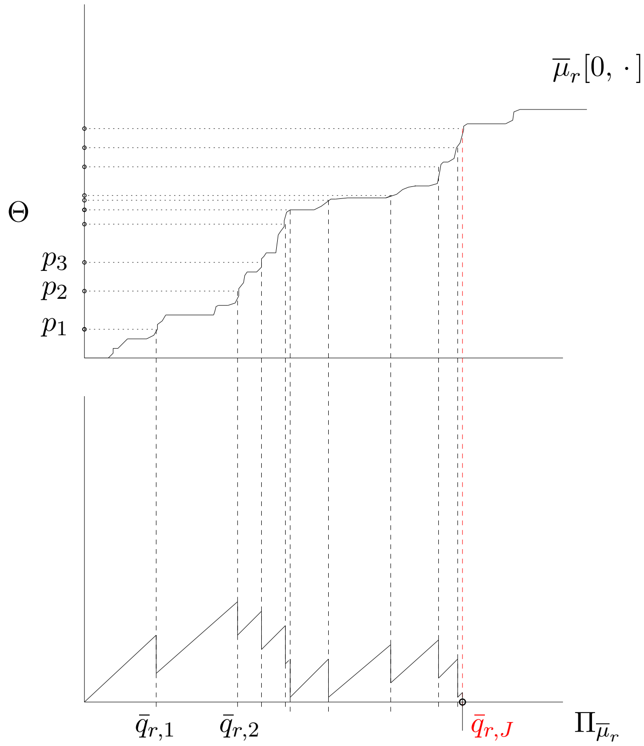

Figure 3.1: Depicting Liggett's extra head construction.

be zero on the event that $\overline{\mu}_r[0,T]=0$. Similarly and more generally, for any time instances $0=s_0\le s_1\le\cdots\le s_k$ and configurations $\zeta_0,\ldots,\zeta_k$,

$$\mathbb{P}\Big(\omega(\chi_{r,T}+s_i)=\zeta_i,\ i=0,\ldots,k\Big)=\mathbb{E}\Big(\frac{1}{\mu_r[0,T]}\int_0^T M_r(\zeta_0)\prod_{i=0}^{k}\mathbb{1}\{\omega_{t+s_i}=\zeta_i\}\,\mathrm{d}t\Big)\,, \quad (3.1)$$

where the random variables on both sides are again interpreted appropriately if $0\longleftrightarrow r$ at no time in $[0,T]$. There is a very similar multi-point formula in the case of $\overline{\chi}_{r,T}$; in fact, the entire argument for the first part of the lemma runs in parallel to that for the second, and we omit it.

Dynamical percolation $\{\omega_t : t\in\mathbb{R}\}$ in B_r is a tail trivial process, hence not only is this process ergodic, but so is the process $\big\{\omega_{[t,t+s]} : t\in\mathbb{R}\big\}$ for any fixed $s\ge 0$. Thus, by the ergodic theorem and the Markov property, the integral in (3.1), divided by T, converges almost surely as $T\to\infty$ to

$$\mathsf{IIC}^{B_r}(\zeta_0)\prod_{i=0}^{k-1}\mathbb{P}\big(\omega_{s_{i+1}}=\zeta_{i+1}\;\big|\;\omega_{s_i}=\zeta_i\big)\,, \quad (3.2)$$

while $\mu_r[0,T]/T\to\mathbb{E}\mu_r[0,1]=1$, almost surely. Therefore, in (3.1), we are taking the expectation of a random variable that converges almost surely to the formula in (3.2). This random variable is bounded, and hence convergence in expectation also follows. We have thus shown that, for almost all ω, the finite dimensional distributions of $\{\omega(\chi_{r,T}+s) : s \in [0,\infty)\}$ converge as $T \to \infty$ to those of standard dynamical percolation started from IIC^{B_r}.

To ameliorate this conclusion to hold for the Skorokhod topology (but only in probability, not almost surely), note that, alongside finite-dimensional distributional convergence and the càdlàg nature of all the sample paths concerned, it is enough to argue that, for any given $K > 0$ and $\epsilon > 0$, the probability that the process $[0,K] \to \mathbb{R} : t \to \omega(\chi_{r,T}+t)$ has two hexagon switches at times differing by less than ϵ vanishes in the high T then low ϵ limit. To see this, note that the Lebesgue measure of the set $\mathcal{A}_T$ of times $t \in [0,T]$ such that $[t,t+K]$ contains two such switch times behaves like $a_\epsilon T(1+o(1))$ as $T \to \infty$, where $\lim_{\epsilon\to 0} a_\epsilon = 0$; on the other hand, the Lebesgue measure of the set $\mathcal{B}_T$ of times $t \in [0,T]$ such that $\omega\big|_{B_r}$ is the completely open configuration behaves almost surely like $bT(1+o(1))$ as $T \to \infty$, where $b > 0$. Since the Radon-Nikodym derivative of $\chi_{r,T}$ is maximized by each point in $\mathcal{B}_T$, we see that $\mathbb{P}\Big(\chi_{r,T} \in \mathcal{A}_T\Big) \leq |\mathcal{A}_T|/|\mathcal{B}_T| \leq 2a_\epsilon/b$ almost surely for T sufficiently high, where $|\cdot|$ denotes Lebesgue measure. Since $a_\epsilon \searrow 0$ as $\epsilon \searrow 0$, we verify the claim needed for convergence in the Skorokhod topology, and complete the proof. □

Proof of Lemma 3.3. The Palm version of a stationary process (ω,ξ) on $\mathbb{R}$, where ξ is a random measure, is defined in [Kal02, Chapter 11] as follows. For any Borel set $B \subset \mathbb{R}$ of positive Lebesgue measure, and any nonnegative measurable function f on configurations (ω,ξ), consider $\xi_f(B) := \int_B f(\theta_s(\omega,\xi))\,\xi(ds)$, where θ_s is the shift by $-s$. Then the Palm version is the law defined by $Q_{\omega,\xi}[f] := \mathbb{E}\xi_f(B)/\mathbb{E}\xi(B)$. It is not hard to show that this does not depend on B.

If we take $\xi = \overline{\mu}_r$ or μ_r and $B = [0,T]$, then this construction specializes to the processes defined in part (a). Since we know from Lemma 2.3 that $(\omega,\overline{\mu}_r,\mu_r)$ is ergodic, we can apply [Kal02, Theorem 11.6], saying that these Palm versions equal the limit processes defined in Lemma 3.2, hence the claim of part (a) follows from that lemma.

For part (b), there will be no difference between the proofs for $\overline{\mu}_r$ and μ_r, so let us just work with μ_r. Take $\xi = \Pi_{\mu_r}$, and the Borel sets $B_\epsilon := (-\epsilon,\epsilon)$. [Kal02, Theorem 11.5] says that the Palm version of (ω,Π_{μ_r}) is the same as conditioning on $|B_\epsilon \cap \Pi_{\mu_r}| \geq 1$ or on $|B_\epsilon \cap \Pi_{\mu_r}| = 1$, then taking the limit $\epsilon \to 0$. This is the most common form of taking the "Palm version of a point process". Note that for the equivalence of definitions here, we need that μ_r does not have atoms (by Lemma 2.1), hence Π_{μ_r} is a simple point process.

(Let us give a two-sentence intuitive explanation of why the quoted theorem on the equality between the Palm process and the ϵ-conditioning holds, at least for the time-zero configuration. Since μ_r has a density, M_r, for any static percolation configuration ζ in B_r, we have

$$\mathbb{P}\big(B_\epsilon \cap \Pi_{\mu_r} \neq \emptyset \mid \omega(0)^{B_r} = \zeta,\, \mu_r\big) = 1 - \exp\Big(-\int_{-\epsilon}^{\epsilon} M_r(\omega_t)\,\mathrm{d}t\Big)$$
$$\sim 2\epsilon\, M_r(\zeta) \qquad \text{a.s. as } \epsilon \to 0\,,$$

by the Lebesgue integration theorem and Fubini. Therefore, $M_r(\zeta)$ being the Radon-Nikodym derivative $\mathrm{d}\mathsf{IIC}^{B_r}/\mathrm{d}\mathbb{P}$, the ϵ-conditioning gives

$$\lim_{\epsilon\to 0} \mathbb{P}\big(\omega(0)^{B_r} = \zeta \;\big|\; B_\epsilon \cap \Pi_{\mu_r} \neq \emptyset\big) = \mathsf{IIC}^{B_r}(\zeta)\,,$$

as desired.)

Since Π_{μ_r} is obtained from μ_r using independent stationary randomness (the Lebesgue Poisson point process Θ), the ω^* marginal in the Palm version of (ω, Π_{μ_r}) is the same as in the Palm version of (ω, μ_r), which we already described in part (a).

Finally, regarding Liggett's extra head construction, [Lig02, Corollary 4.18] says that shifting back by $q_{r,J}$ as defined in the statement of part (b) produces the Palm version of Π_{μ_r}. Now we need to extend this result from the marginal Π_{μ_r} to (ω, Π_{μ_r}); we will certainly need to use that Liggett's shift coupling acts nicely also on the level of ω and Θ, since the result clearly would not hold for an arbitrary measurable map $(\omega,\Theta) \mapsto f(\omega,\Theta)$ with the property that $\Pi_{\mu_r(f(\omega,\Theta))} \overset{d}{=} \Pi^*_{\mu_r(\omega)}$. The niceness of Liggett's construction lies in the fact that it gives a random time shift $T_{J_{q,r}}$ that is measurable with respect to Π_{μ_r}, where each time shift T_x is a measure-preserving transformation on the space of configurations (ω,Θ). Therefore, if $\mathcal{A}$ and $\mathcal{B}$ are arbitrary events for the Palm version $\Pi^*_{\mu_r}$, and $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ are the events for $(\omega^*, \Pi^*_{\mu_r})$ that project to $\mathcal{A}$ and $\mathcal{B}$ in the second coordinate, then

$$\frac{\mathbb{P}^*(\mathcal{A})}{\mathbb{P}^*(\mathcal{B})} = \frac{\mathbb{P}\big(T^{-1}(\mathcal{A})\big)}{\mathbb{P}\big(T^{-1}(\mathcal{B})\big)} = \frac{\mathbb{P}\big(T^{-1}(\tilde{\mathcal{A}})\big)}{\mathbb{P}\big(T^{-1}(\tilde{\mathcal{B}})\big)}\,,$$

whenever the denominator on either side of this equation is positive. See Figure 3.2. Since $\frac{\mathbb{P}^*(\mathcal{A})}{\mathbb{P}^*(\mathcal{B})} = \frac{\mathbb{P}^*(\tilde{\mathcal{A}})}{\mathbb{P}^*(\tilde{\mathcal{B}})}$ by definition, we get that the effect of T is the same as conditioning on $\{0 \in \Pi_{\mu_r}\}$ not only on Π_{μ_r} but also on (ω, Π_{μ_r}), and we are done. □

We can now turn to sampling from the limit measure $\mu[0,T]$.

Proof of Theorem 1.7. We must argue that $\mu[0,T] > 0$ for all T sufficiently high, and also that, for each $r \in \mathbb{N}$, $\omega(\chi_T)^{B_r}$ converges weakly, as $T \to \infty$, to IIC^{B_r}.

For $n \in \mathbb{N}$, set $I_i^n = [i/n, (i+1/n))$. For $R \in \mathbb{N}$, define $f_R^n : [0,\infty) \to [0,\infty)$ according to $f_R^n(x) = n\mu_R(I_i^n)$ if $x \in I_i^n$ for $i \in \mathbb{N}$. Similarly define $f_\infty^n : [0,\infty) \to [0,\infty)$ according to $f_\infty^n(x) = n\mu(I_i^n)$ if $x \in I_i^n$ for $i \in \mathbb{N}$. We now argue that, for each $\epsilon > 0$ and $n \in \mathbb{N}$, there exists $R \in \mathbb{N}$ such that

$$\limsup_T \int_0^T \Big| \tfrac{f_R^n(t)}{\int_0^T f_R^n(s)ds} - \tfrac{f_\infty^n(t)}{\int_0^T f_\infty^n(s)ds} \Big| \mathrm{d}t \leq \epsilon\,. \tag{3.3}$$

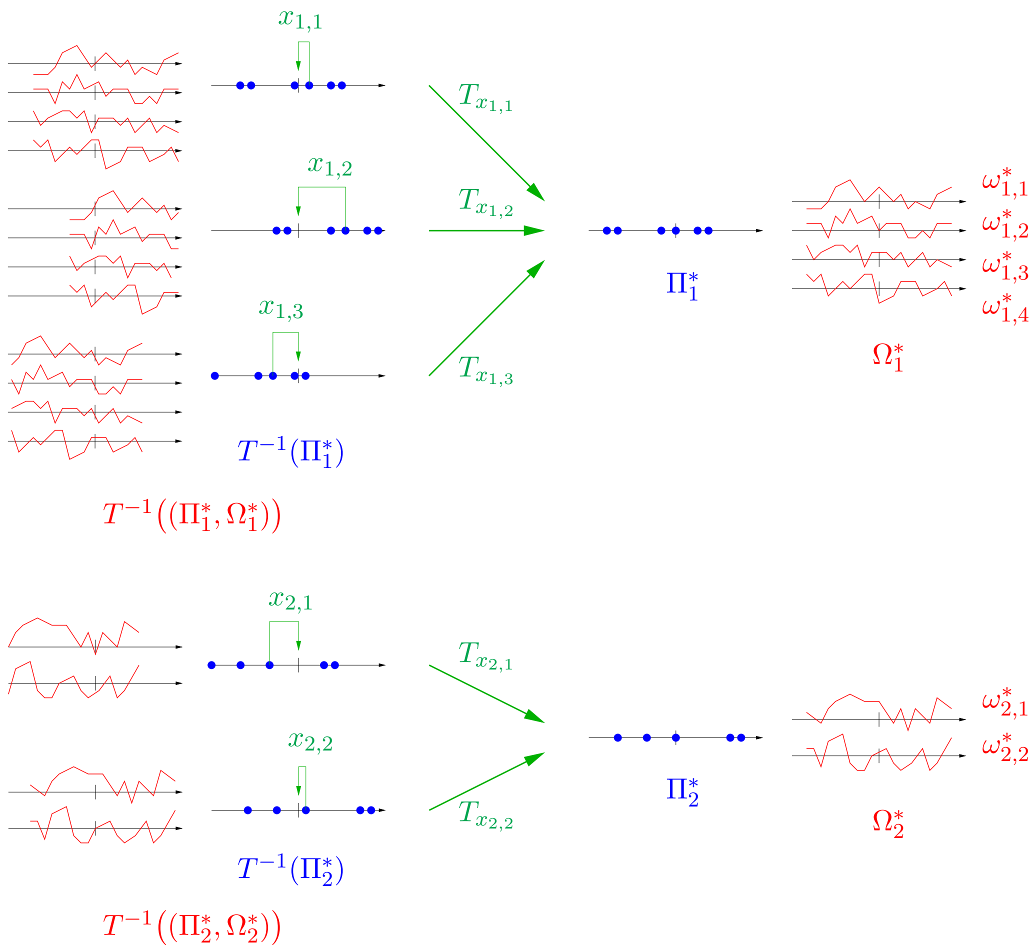

Figure 3.2: A schematic picture of the effect of Liggett's extra head time shift T on (Π_{μ_r}, ω). (For simplicity, the figure pretends that Π is a measurable function of ω, instead of (ω, Θ).) Different Palm point process realizations Π_1^* and Π_2^* may arise from a different "amount" of Palm dynamical percolation realizations $\Omega_1^* = \{\omega_{1,i}^* : i \in I_1\}$ and $\Omega_2^* = \{\omega_{2,i}^* : i \in I_2\}$ (a ratio 4:2 on the right side of the picture), and the preimages $T^{-1}(\Pi_1^*)$ and $T^{-1}(\Pi_2^*)$ might also have different sizes (which gives the reweighting of the Palm measure compared to the ordinary measure, a ratio 3:2 in the middle of the picture). The product of these ratios is the same as the ratio for the preimages $T^{-1}\big((\Pi_1^*, \Omega_1^*)\big)$ and $T^{-1}\big((\Pi_2^*, \Omega_2^*)\big)$.

Note that this assertion allows us to construct couplings $\mathbf{Q}$ of $\chi_{R,T}$ and χ_T with the following property: for each $\epsilon > 0$ and $n \in \mathbb{N}$, there exists $R \in \mathbb{N}$ such that, for all large enough T simultaneously, $\chi_{R,T}$ and χ_T are coupled under $\mathbf{Q} = \mathbf{Q}_R$ so that

$$\limsup_T \mathbf{Q}\big(|\chi_{R,T} - \chi_T| \geq 1/n\big) \leq \epsilon\,. \tag{3.4}$$

Note that (3.3) is a consequence of the next three assertions. First, for each $\epsilon > 0$, there exists $R \in \mathbb{N}$ such that

$$\limsup_T T^{-1} \int_0^T \big|f_R^n(t) - f_\infty^n(t)\big| \mathrm{d}t \leq \epsilon\,, \ \ \mathbb{P}\text{-almost surely}\,. \tag{3.5}$$

Second, for each $\epsilon > 0$, and for this same value of $R \in \mathbb{N}$,

$$\limsup_T T^{-1} \Big| \int_0^T f_R^n(s)\mathrm{d}s - \int_0^T f_\infty^n(s)\mathrm{d}s \Big| \leq \epsilon\,, \ \ \mathbb{P}\text{-almost surely}\,. \tag{3.6}$$

Third,

$$\lim_T T^{-1} \int_0^T f_\infty^n(s)\mathrm{d}s = 1\,, \ \ \mathbb{P}\text{-almost surely}\,. \tag{3.7}$$

We now justify (3.5), (3.6) and (3.7).

To confirm (3.5), note that $\int_0^{1/n} |f_R^n(t) - f_\infty^n(t)|\mathrm{d}t = \int |\mu(0,1/n) - \mu_R(0,1/n)| \mathrm{d}\mathbb{P}$. We fix $R \in \mathbb{N}$ by Theorem 1.6 so that $\mathbb{E} \int |\mu(0,1/n) - \mu_R(0,1/n)|\mathrm{d}\mathbb{P} \leq \epsilon/n$. Lemma 2.3 then provides (3.5) along T values that are multiples of $1/n$. To extend this to all T, we can sandwich the integral up to T between the integrals up to the closest multiples of $1/n$, and use that $\lim_{T\to\infty} T/(T \pm 1/n) = 1$.

Note that (3.6) is a trivial consequence of (3.5).

To show (3.7), note that, by definition, $\mathbb{E}(\mu_r[0,1]) = 1$ for each $r \in \mathbb{N}$. Thus Theorem 1.6 implies that $\mathbb{E}(\mu[0,1]) = 1$. Lemma 2.3 then implies that $\lim_T T^{-1}\mu(0,T) = 1$, $\mathbb{P}$-almost surely. This limit coincides with that in (3.7), which establishes this claim. Note that in this derivation we have confirmed that indeed $\mu(0,T) > 0$ for T sufficiently high, $\mathbb{P}$-almost surely.

We conclude the proof by arguing that, for each $\epsilon > 0$ and $r \in \mathbb{N}$, there exists $R \in \mathbb{N}$ such that

$$\liminf_T \mathbf{Q}\Big(\omega(\chi_{R,T})^{B_r} = \omega(\chi_T)^{B_r}\Big) \geq 1 - \epsilon\,. \tag{3.8}$$

This indeed suffices for Theorem 1.7, by the following argument. Recall that we must argue that, for each $r \in \mathbb{N}$, $\omega(\chi_T)^{B_r}$ converges weakly as $T \to \infty$ to IIC^{B_r}. We know by Lemma 3.2 that the weak limit as $T \to \infty$ of $\omega(\chi_{R,T})$ equals IIC^{B_R}. Thus, fixing any $\epsilon > 0$ and any $R \geq r$, for large enough T, the total variation distance between $\omega(\chi_{R,T})^{B_r}$ and IIC^{B_r} is at most ϵ. (Note here that on the discrete topological space $\{0,1\}^{B_r}$, convergence in law is the same as in total variation distance.) On the other hand, by (3.8), $\omega(\chi_T)$ coincides with $\omega(\chi_{R,T})$ on B_r with $\mathbf{Q}$-probability at least $1 - \epsilon$ for all high enough T.

Thus the total variation distance between $\omega(\chi_T)^{B_r}$ and IIC^{B_r} becomes less than 2ϵ, and we are done.

It remains only to verify (3.8). In light of (3.4), it is enough to argue that, for given $\epsilon > 0$ and $r \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that, for all $R \geq r$ and all T sufficiently high, the $\mathbf{Q}$-probability that a hexagon in B_r flips during $\big[\chi_{R,T} - 1/n, \chi_{R,T} + 1/n\big]$ is at most ϵ. However, by Lemma 3.2, the times of hexagon flips in B_r during $\big[\chi_{R,T} - 1/n, \chi_{R,T} + 1/n\big]$, shiftward backwards in time by $\chi_{R,T}$, converges weakly as $T \to \infty$ to a Poisson process of rate $|B_r|/2$ on $[-1/n, 1/n]$. Choosing $n \geq C_\epsilon r^2$ thus gives the desired statement. □

Proof of Theorem 1.8. Part (a) follows from Theorem 1.7 — by Lemma 2.3 and [Kal02, Theorem 11.6] — just as Lemma 3.3 followed from Lemma 3.2.

Part (b) follows from Lemma 3.3(b) and the next two lemmas. □

Lemma 3.4. *If $\tau \in \mathcal{E}$ is an exceptional time, and $\tau_n \to \tau$, then, for any $r > 0$, we have $\omega(\tau_n)^{B_r} = \omega(\tau)^{B_r}$ for all sufficiently high n.*

Proof. By Lemma 2.4(i), there is an open interval I which contains the exceptional time τ such that, for $t \in I$, $\omega(t)^{B_r} = \omega(\tau)^{B_r}$; hence $\tau_n \to \tau$ implies the lemma. □

Lemma 3.5. *For the times $q_{r,J}$ and q_J defined in Lemma 3.3 (b), the limit $q_J = \lim_{r\to\infty} q_{r,J}$ holds almost surely, and q_J is an exceptional time.*

Proof. If $f_n : [0,\infty) \to [0,\infty)$ is a sequence of non-decreasing functions converging pointwise on $[0,\infty)$ to a function $f : [0,\infty) \to [0,\infty)$, and we write

$$f_n^{-1}(x) = \inf\Big\{ t > 0 : f_n(t) > x \Big\},$$

then, whenever $x \in [0,\infty)$ is a point of increase of $f : [0,\infty) \to [0,\infty)$, we have that $\lim_{n\to\infty} f_n^{-1}(x) = f^{-1}(x)$. The following thus suffices for Lemma 3.5:

Lemma 3.6. *For any $\rho \in \Theta$, $t_\rho := \inf\big\{ s > 0 : \mu(0,s) > \rho \big\}$ is almost surely a point of increase of $\mu(0,\cdot)$; in particular, it is contained in the support of μ.*

Proof. Note that the set of $\rho \in (0,\infty)$ for which t_ρ is a point of increase of $\mu(0,\cdot)$ is given by $\mathbb{R} \setminus \mu(0,Q)$, with $\mu(0,Q) = \big\{ \mu(0,q) : q \in Q \big\}$, where Q is the collection of left-hand endpoints of intervals comprising $\mathrm{supp}(\mu)^c$. Note that Q is countable, and, thus, is so $\mu(0,Q)$. Thus, $\Theta \cap \mu(0,Q) = \emptyset$ a.s., because Θ is independent of $\mu(0,Q)$. □

4 FETIC is not IIC

In this section, we prove Theorem 1.3.

4.1 The skeleton of the argument

Definition 4.1. *Let ω be a sample of dynamical percolation in the R-ball B_R. Write $\mathcal{E}_R$ for the set of times such that $0 \leftrightarrow R$. Let $\mathsf{FET}_R = \inf\left\{t \geq 0 : 0 \xleftrightarrow{\omega_t} R\right\}$, and let FETIC_R be the law of ω_{FET_R} conditioned on $\mathsf{FET}_R > 0$. (Since $\mathscr{C}_0(\omega_0)$ is almost surely finite, it takes positive time for the first bit on its boundary to change, and hence the event $\mathsf{FET}_R > 0$ is the same as $0 \not\leftrightarrow R$ in ω_0, which is almost surely satisfied for large enough R.)*

These finite approximations will be very useful. On the one hand, IIC_R is the law of the configuration at a typical point of $\mathcal{E}_R$, as we saw in Lemmas 3.2 and 3.3. On the other hand, by [HamMP12, Lemma 4.5], we have that $\mathsf{FET}_R \to \mathsf{FET}$ almost surely as $R \to \infty$; hence FETIC_R converges to FETIC in law (by Lemma 3.4).

There is a natural line of attack if we want to distinguish IIC_R from FETIC_R. Let us call the left-isolated points of $\mathcal{E}_R$ **arrivals**, and the right-isolated points **departures**. As we will see, the law of a *typical* arrival configuration can be easily obtained from IIC_R (and will be denoted by IIC'_R): we get it by size-biasing with respect to the number of pivotal hexagons for the event $\{0 \leftrightarrow R\}$. This is different from IIC_R, but not by much: it can be shown (though we will not do so) that its weak limit as $R \to \infty$ coincides with IIC. However, FETIC_R is not given by a typical arrival: as usual when waiting for the first arrival of a stationary point process, the time between FET_R and the last departure before it (somewhere in the negative half-line) is a size-biased sample of the typical reconnection time between departures and arrivals, and if an arrival configuration typically occurs at the end of longer disconnection intervals, then it is more likely to appear in FETIC_R. Since it is harder to think about dynamical percolation ending at a certain configuration than about starting it at such a configuration, our strategy to understand FETIC_R will be to reverse time, start dynamical percolation from certain typical IIC'_R configurations, condition on immediate termination of $\{0 \leftrightarrow R\}$, and then estimate the expected time of reconnection. If we can exhibit two events at time zero that have the same positive probability under the limit measure IIC, but for which the expected reconnection times differ, then these events will turn out to have different probabilities under FETIC, and we will be done.

Roughly, of these two events under IIC'_R, the first will be that the configuration looks "normal" in a bounded neighbourhood of 0, while the second will be that the configuration is "thinner" in the same neighbourhood. (We will in fact define a thinning procedure on normal static configurations satisfying $0 \leftrightarrow R$, changing the configuration in a bounded neighbourhood of 0.) A thinner configuration falls apart more easily, and hence reconnects to distance R with more difficulty; and so, one may expect that such a configuration is more probable under FETIC_R than is a normal configuration, which is to say, FETIC_R is thinner than IIC_R. This is certainly the case if the thin configuration is, say, given by a single straight line segment of open hexagons from 0 to ∂B_R, with all other hexagons in B_R being closed. However, this R-dependent configuration has a vanishing probability in the limit measure

IIC; therefore, while the imbalance in probability of this configuration distinguishes FETIC_R from IIC_R, a distinction between FETIC and IIC cannot be deduced. This is why we want to require the configuration to be thin only in a bounded neighbourhood of 0. However, the main difficulty now is that normal reconnection times are very short if R is large, and that, with high probability, the configuration is entirely static in a bounded neighbourhood of the origin; hence it is not clear that our thinning will have a noticeable effect on the reconnection time. The solution will be that the expected reconnection time, though tiny, turns out to be dominated by times that are macroscopically large (independently of R): large enough that if the configuration close to 0 is thin then it does indeed start falling apart, making expected reconnection time noticeably larger when the thinning procedure has been applied. To argue this, we will need the result from [HamMP12] that FET has finite expectation (in fact, an exponential tail): this will tell us that the normal reconnection time is well behaved, making it possible to prove that, in expectation, it is strictly dominated by the reconnection time of thinned configurations.

In this introductory subsection, we first explain the time-reversal and the size-biasing effects determining the relationship between IIC_R and FETIC_R, then define the thinning procedure, and will finally show that a noticeable difference between expected reconnection times indeed implies that FETIC and IIC are different. In the subsequent subsections, we will prove that there is such a difference.

Recall from the above discussion that, in standard càdlàg dynamical percolation, a time $t \in \mathcal{E}_R$ for which there exists $\epsilon > 0$ such that $[t-\epsilon, t) \cap \mathcal{E}_R = \emptyset$ is called an arrival. Write $\mathcal{A}_R$ for the set of arrivals. Furthermore, for a static percolation configuration ζ in B_R that satisfies $0 \leftrightarrow R$, denote by $\mathsf{Piv} = \mathsf{Piv}_{0\leftrightarrow R}(\zeta)$ the set of hexagons in B_R that are pivotal in the configuration ζ for $0 \leftrightarrow R$, and recall that IIC'_R denotes the law on configurations in B_R whose Radon-Nikodym derivative with respect to IIC_R is given by $|\mathsf{Piv}|$ up to normalization.

Lemma 4.2. *The following three definitions for the process $\mathbb{P}(\,\cdot \mid 0 \in \mathcal{A}_R)$ are equivalent:*

- **(i)** *consider dynamical percolation in B_R conditionally on the event $\{0 \leftrightarrow R\}$ occurring at time 0 but not at time $-\epsilon$, and take the weak limit as $\epsilon \downarrow 0$;*
- **(ii)** *for large $T > 0$, pick uniformly an element $\tau \in \mathcal{A}_R \cap [0, T]$, consider the shifted dynamical percolation configuration $\{\omega_{t-\tau} : t \in \mathbb{R}\}$, and take the weak limit as $T \to \infty$ (conditionally on ω, or averaged);*
- **(iii)** *let ω_0 be distributed according to IIC'_R, choose uniformly an element $S \in \mathsf{Piv}(\omega_0)$, obtain the configuration ω_{0^-} by closing the hexagon S, and let the rest of the evolution $\{\omega_t : t \in \mathbb{R}\}$ be given by càdlàg dynamical percolation updates independently of the values of ω_0 and S.*

Proof. While the weak limits in (i) and (ii) might not exist a priori, the definition of (iii) is clearly well formulated. We first prove the equivalence of (i) and (iii), implying the existence of the weak limit in (i), in particular. It is enough to show that, for all configurations ζ in B_R such that $0 \leftrightarrow R$,

$$\lim_{\epsilon \to 0} \frac{\mathrm{d}\mathbb{P}\big(\cdot \,\big|\, 0 \in \mathcal{E}_R,\, -\epsilon \notin \mathcal{E}_R \big)}{\mathrm{d}\mathsf{IIC}_R}(\zeta) = Z_1^{-1} |\mathsf{Piv}_{0 \leftrightarrow R}(\zeta)| \,, \tag{4.1}$$

where $Z_1 \in (0, \infty)$ is a normalization.

Given a configuration ζ such that $0 \leftrightarrow R$, let $p_\epsilon(\zeta)$ be the probability that dynamical percolation given $\omega_0 = \zeta$ satisfies $0 \not\leftrightarrow R$ at time $-\epsilon$. If ϵ is tiny (depending on R), then the probability of having at least two hexagons flipping in the time interval $(-\epsilon, 0)$ is much less than the probability of any specific hexagon flip. Therefore, $\lim_{\epsilon \to 0} p_\epsilon(\zeta)/\epsilon = |\mathsf{Piv}_{0 \leftrightarrow R}(\zeta)|$, which implies (4.1).

To prove the equivalence of (ii) and (iii), let us reformulate the T-dependent law defined in (ii) as taking uniformly one from all pairs of configurations $(\omega_{t-}, \omega_t) \in \mathcal{E}_R^c \times \mathcal{E}_R$, with $t \in [0, T]$, and then running dynamical percolation in the two directions from here. By the ergodicity of $\{\omega_t : t \in \mathbb{R}\}$ (Lemma 2.3), the weak limit of this law is the same as taking a pair of static configurations (ζ_1, ζ_2) that differ only in one hexagon such that $0 \leftrightarrow R$ in ζ_2 but not in ζ_1 to start the dynamics. This is clearly the same as the law defined in (iii).

The equivalence of (i) and (ii) follows from the above two equivalences; or, just like in Lemma 3.3, we can also quote [Kal02, Theorem 11.6] on the equivalent definitions of the Palm version of the process $(\omega, \mathcal{A}_R)$. □

Now, as we promised, in order to understand the effect of waiting for the first exceptional time on the distribution of the configuration at that time, we time-reverse the dynamics, started from typical arrival times:

Definition 4.3. *Let $\mathbb{P}_{\mathrm{norm}}$ denote the time-reversal of $\mathbb{P}(\cdot \mid 0 \in \mathcal{A}_R)$ (i.e., $t \mapsto -t$ for all $t \in \mathbb{R}$). More explicitly, it is the càg**làd** (left-continuous with right limits) Markov process given as follows. Under $\mathbb{P}_{\mathrm{norm}}$, the distribution of ω_0 is IIC'_R. Given ω_0, a uniform element $S \in \mathsf{Piv}$ is selected, with the configuration ω_{0+} being set equal to ω_0 modified by closing the hexagon S. The rest of the evolution of $\{\omega_t : t \in \mathbb{R}\}$ is given by càglàd dynamical percolation updates independently of the values of ω_0 and S.*

Lemma 4.4. *Under the law $\mathbb{P}_{\mathrm{norm}}$, recall that $0 \leftrightarrow R$ is satisfied by ω_0 but not by ω_{0+}; let the reconnection time $N \in (0, \infty)$ be given by $N = \inf\big\{ t > 0 : 0 \overset{\omega_t}{\longleftrightarrow} R \big\}$. For each static B_R configuration ζ, we have that*

$$\frac{\mathrm{d}\mathsf{FETIC}_R}{\mathrm{d}\mathsf{IIC}_R}(\zeta) = Z^{-1} \mathbb{E}_{\mathrm{norm}}(N \mid \omega_0 = \zeta)\, |\mathsf{Piv}_{0 \leftrightarrow R}| \,,$$

where $Z \in (0, \infty)$ is a normalization.

Proof. We claim that

$$\frac{\mathrm{d}\mathsf{FETIC}_R}{\mathrm{d}\mathbb{P}\big(\cdot \,\big|\, 0 \in \mathcal{A}_R \big)}(\zeta) = Z_2^{-1} \mathbb{E}_{\mathrm{norm}}(N \mid \omega_0 = \zeta) \,, \tag{4.2}$$

where $Z_2 \in (0,\infty)$ is another normalization. From (4.1) and (4.2) follows the statement of the lemma.

To prove (4.2), let $\phi : \mathcal{E}_R^c \longrightarrow \mathcal{A}_R$ associate to each moment of disconnection $0 \not\leftrightarrow R$ in càdlàg dynamical percolation the first connection time to its right (which is necessarily an arrival). Condition the process on $\mathcal{E}_R^c \cap [-n,0] \neq \emptyset$ and pick a random time χ whose conditional law is given by normalized Lebesgue measure on $\mathcal{E}_R^c \cap [-n,0]$; note that FETIC_R is the weak limit as $n \to \infty$ of $\omega_{\phi(\chi)}$. Note that, in this weak limit, the probability that $\omega_{\phi(\chi)}$ is a given static configuration ζ (for which $0 \leftrightarrow R$) is proportional to the mean length of an interval in $\mathcal{E}_R^c$ at whose right-hand endpoint the configuration is ζ. Thus we obtain (4.2). □

Here is a straightforward variant of (4.2). For any non-negative random variable X of finite mean, $\widehat{X}$ will denote the size-biased version; i.e., $\mathbb{P}\big(\widehat{X} \geq t\big) = \mathbb{E}\big(X\big)^{-1}\mathbb{E}\big(X \mathbb{1}_{X \geq t}\big)$.

Lemma 4.5. *Let $\widehat{N}$ be the size-biased version of the reconnection time N under the law $\mathbb{P}_{\mathrm{norm}}$, and let U be an independent $\mathsf{Unif}[0,1]$ random variable. Then $\widehat{N}U$ has the distribution of FET_R.*

The following useful fact was proved in [HamMP12].

Lemma 4.6. *In dynamical percolation we have*

$$\mathbb{P}\big(\mathsf{FET}_R > t\big) \leq \exp\big\{ - ct\big\}$$

for all $t > 0$, where $c > 0$ may be chosen uniformly in $R \in \mathbb{N}$.

Note that the preceding two lemmas imply that $\mathbb{P}(\widehat{N} > t) \leq \exp\{-ct\}$, uniformly in R. In particular, this random variable has finite moments: for each $k \in \mathbb{N}$, $\mathbb{E}(\widehat{N}^k) = \mathbb{E}(N^{k+1})/\mathbb{E}N < \infty$, again uniformly in R.

We now introduce the thinning procedure which is central to our technique for showing that FETIC differs from IIC.

Definition 4.7. *A circuit Γ is a finite self-avoiding path of hexagons such that for no vertex in the hexagonal lattice are all three of the neighbouring hexagons visited by Γ and such that $\mathcal{H} \setminus \Gamma$ has exactly two connected components: a finite one, denoted by $\mathsf{Int}(\Gamma)$, and an infinite one. Note that a partial order on circuits Γ is provided by containment of the enclosed regions $\mathsf{Int}(\Gamma)$.*

Let ζ be a percolation configuration in B_R such that $0 \leftrightarrow R$. Note that if some ζ-open circuit Γ satisfies $B_r \subseteq \mathsf{Int}(\Gamma)$, then there is a unique ζ-open circuit which encloses B_r and is minimal in the partial order among such circuits. If ζ is such that this circuit exists, we label the circuit by Γ_r.

Definition 4.8. *Recall the exponent* $\eta \in (0,1)$ *from (1.6), and fix* $\epsilon > 0$ *small enough that* $(1+2\epsilon)(1-\eta) < 1$. *Now assume that* r *satisfies* $r^{2(1+2\epsilon)}\alpha_4(r^{1+2\epsilon}) < r/2$, *which holds for all large enough* r, *by (1.7). Let* $R \in \mathbb{N}$ *satisfy* $R \geq r^{1+2\epsilon}$. *A configuration* ζ *in* B_R *is said to satisfy* $\zeta \in \mathsf{Fine}$ *if the following conditions hold:*

- $0 \longleftrightarrow R$;
- *the circuit* Γ_r *exists and satisfies* $\Gamma_r \subseteq B_{r^{1+\epsilon}}$;
- *the pivotal set* $\mathsf{Piv}_{0\leftrightarrow\Gamma_r} = \mathsf{Piv}_{0\leftrightarrow R} \cap \mathsf{Int}(\Gamma_r)$ *satisfies* $|\mathsf{Piv}_{0\leftrightarrow\Gamma_r}(\zeta)| \leq r^{2(1+2\epsilon)}\alpha_4(r^{1+2\epsilon})$.

Finally, a dynamical configuration $\{\omega_t : t \in \mathbb{R}\}$ *is said to satisfy* Fine *if* $\omega_0 \in \mathsf{Fine}$.

Definition 4.9. *Let* $r \in \mathbb{N}$ *be even. Let* Γ *denote a circuit such that* $B_r \subseteq \mathsf{Int}(\Gamma)$. *Let* $a \in \{0,\ldots,r/2\}$. *The* (r,Γ,a)*-slim configuration* $\chi_{r,\Gamma,a}$ *is a particular percolation configuration in* $\mathsf{Int}(\Gamma)$, *as shown in Figure 4.1, whose set of open hexagons in* $\mathsf{Int}(\Gamma) \cap B_{r/2}$ *consists of the hexagons in* $B_{r/2}$ *that intersect the* x*-axis, and for which* $|\mathsf{Piv}_{0\leftrightarrow\Gamma}| = a$.

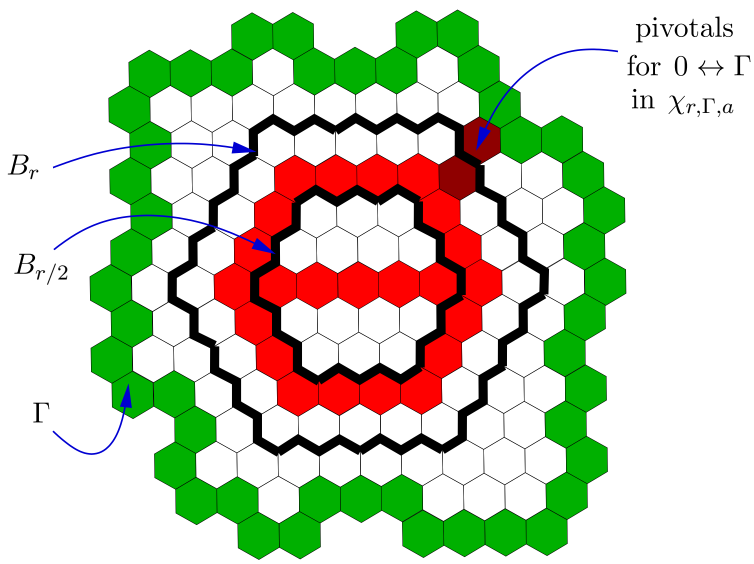

Figure 4.1: The boundary paths delimiting B_2 and B_4 are black, and the circuit Γ is green. The red and dark red hexagons are the open hexagons of $\chi_{4,\Gamma,2}$: the red hexagonal circuit is set in such a way that its distance from Γ is $a = 2$, and the dark red path is chosen in some arbitrary but fixed way so that it realizes this distance a. Note that this dark red path is the set of pivotals for $0 \leftrightarrow \Gamma$.

Definition 4.10. *The thinning procedure* $\mathsf{Thinning} = \mathsf{Thinning}_r^R$ *maps the set of configurations in* B_R *to itself. Let* ζ *be such a configuration. If* $\zeta \notin \mathsf{Fine}$, *then set* $\mathsf{Thinning}(\zeta) = \zeta$.

If $\zeta \in \mathsf{Fine}$*, let* $\mathsf{Thinning}(\zeta)$ *be the configuration in* B_R *of the following form:*

$$\mathsf{Thinning}(\zeta)(x) = \begin{cases} \zeta(x) & \textit{if } x \in B_R \setminus \mathsf{Int}(\Gamma_r), \\ \chi_{r,\Gamma_r,|\mathsf{Piv}_{0\leftrightarrow\Gamma_r}|}(x) & \textit{if } x \in \mathsf{Int}(\Gamma_r). \end{cases}$$

We define a coupling of $\mathbb{P}_{\mathrm{norm}}$ with another dynamical process begun by pairing the initial condition with its thinned counterpart. We denote by ω' the process under $\mathbb{P}_{\mathrm{norm}}$, and write ω'' for the process under the measure $\mathbb{P}_{\mathrm{thin}}$ which we now introduce by coupling with $\mathbb{P}_{\mathrm{norm}}$. We set $\mathbb{P}_{\mathrm{thin}}$ by choosing its initial condition $\omega''_0 = \mathsf{Thinning}(\omega'_0)$; if the hexagon S selected for initial closure in the definition of $\mathbb{P}_{\mathrm{norm}}$ lies in the unbounded component of the complement of $\Gamma_r(\omega'_0)$, we set $S'' = S$; otherwise, we choose S'' uniformly among $\mathsf{Piv}_{0\leftrightarrow R}(\omega''_0) \cap \mathsf{Int}(\Gamma_r)$. We define ω''_{0+} by modifying ω''_0 by closing S''. The subsequent evolution of ω'' is made in accordance with the càglàd dynamical updates used in defining ω'. Note that there might be updates that do not have an effect on ω' (the new status coinciding with the old one), and hence are not visible if we see only ω', while do have an effect on ω''; thus ω'' is not entirely measurable w.r.t. ω', even though the extra randomness in ω'' is quite simple.

We denote by $\mathbb{P}_{\mathrm{norm}}$ and $\mathbb{P}_{\mathrm{thin}}$ the above dynamics and its thinned counterpart, and write N and T for the reconnection time $\inf\big\{t > 0 : 0 \overset{\omega_t}{\longleftrightarrow} R\big\}$ under $\mathbb{P}_{\mathrm{norm}}$ and $\mathbb{P}_{\mathrm{thin}}$. We will often use the above coupling of the two càglàd processes, but will not need a separate notation to denote it. The principal result we need is now stated.

Proposition 4.11 (Thinned versus Normal)**.** *As in Definition 4.8, fix* $\epsilon > 0$ *small, and consider all large enough* $r \in \mathbb{N}$*. Then, uniformly in* $R \geq r^{1+2\epsilon}$*, we have* $\frac{\mathbb{E}_{\mathrm{thin}}(T\mathbb{1}_{\mathsf{Fine}})}{\mathbb{E}_{\mathrm{norm}}(N\mathbb{1}_{\mathsf{Fine}})} \to \infty$ *as* $r \to \infty$*.*

Proof of Theorem 1.3, assuming Proposition 4.11. We want to show that there exists a circuit Γ in the annulus $A_{r,r^{1+\epsilon}}$ and two configurations ζ' and ζ'' on $\Gamma \cup \mathsf{Int}(\Gamma)$ with $\Gamma = \Gamma_r(\zeta') = \Gamma_r(\zeta'')$, such that $\mathsf{IIC}_R(\zeta') = \mathsf{IIC}_R(\zeta'')$ for each integer $R \geq r^{1+2\epsilon}$, with the common value having a positive limit as $R \to \infty$, while $\inf_{R \geq r^{1+2\epsilon}} \frac{\mathsf{FETIC}_R(\zeta'')}{\mathsf{FETIC}_R(\zeta')} > 1$.

By Proposition 4.11, we may choose $r \in \mathbb{N}$ so that $\mathbb{E}_{\mathrm{thin}}(T\mathbb{1}_{\mathsf{Fine}}) > 2\,\mathbb{E}_{\mathrm{norm}}(N\mathbb{1}_{\mathsf{Fine}})$ for all R sufficiently high. Hence, there exists a choice of circuit Γ in $A_{r,r^{1+\epsilon}}$, and a configuration ζ' in $\mathsf{Int}(\Gamma) \cup \Gamma$, such that $\Gamma = \Gamma_r(\zeta')$, the second and third conditions for Fine occur, and, setting ζ'' equal to the restriction of $\mathsf{Thinning}(\zeta')$ to $\mathsf{Int}(\Gamma) \cup \Gamma$,

$$\mathbb{E}_{\mathrm{norm}}\Big(N \,\Big|\, \omega_0\big|_{\mathsf{Int}(\Gamma)\cup\Gamma} = \zeta''\Big) > 2\,\mathbb{E}_{\mathrm{norm}}\Big(N \,\Big|\, \omega_0\big|_{\mathsf{Int}(\Gamma)\cup\Gamma} = \zeta'\Big).$$

It is clear that $\mathsf{IIC}_R(\zeta'') = \mathsf{IIC}_R(\zeta')$; moreover, $\mathsf{IIC}'_R(\zeta'') = \mathsf{IIC}'_R(\zeta')$, since the number of pivotals for $\{0 \leftrightarrow R\}$ is left intact by $\mathsf{Thinning}$. Hence, by Lemma 4.4, we have $\mathsf{FETIC}_R(\zeta'') > 2\,\mathsf{FETIC}_R(\zeta')$. □

The rest of the section will be devoted to the proof of Proposition 4.11. Let us start by collecting the main ingredients needed for the proof; these ingredients will then be proved in the remaining subsections.

Thinning will make a difference only if there is enough time before reconnection for the configuration in $\mathsf{Int}(\Gamma_r)$ to change significantly. To this end, as we will see, the events $\{N > 1/r\}$ and $\{T > 1/r\}$ will be important to us. How different are these two events? Although the set of open hexagons in $\mathsf{Thinning}(\zeta)$ is not exactly a subset of its counterpart for ζ, we can compare the thinned and normal reconnection times in this regime under a certain event Good:

$$\mathsf{Good} \cap \big\{N > 1/r\big\} \subseteq \big\{T > 1/r\big\}, \tag{4.3}$$

where Good is defined as follows (and is applied in the above relation to the configuration before thinning):

Definition 4.12. *Let $R, r \in \mathbb{N}$ satisfy $R \geq r^{1+2\epsilon}$ where $\epsilon > 0$ is specified in Definition 4.8. Let ω be a dynamical configuration in B_R. We say that $\omega \in \mathsf{Good}$ if the following conditions are satisfied:*

- *$\omega_0 \in \mathsf{Fine}$, as specified in Definition 4.8;*
- *for each $t \in [0, r^{-1}]$, the inner and outer boundaries of the annulus $A_{r^{1+\epsilon}, r^{1+2\epsilon}}$ are separated by an ω_t-open circuit;*
- *for each $t \in [0, r^{-1}]$, $0 \stackrel{\omega_t}{\longleftrightarrow} r^{1+2\epsilon}$.*

Now, to see (4.3), note that the occurrence of Good implies that 0 is connected to some open circuit $\Gamma = \Gamma(t)$ such that $B_{r^{1+\epsilon}} \subseteq \Gamma$ for all $0 \leq t \leq r^{-1}$. Hence, $N > 1/r$ implies that $r^{1+\epsilon} \not\longleftrightarrow R$ for all $t \in [0, r^{-1}]$ under $\mathbb{P}_{\mathrm{norm}}$. Since the dynamical percolations under $\mathbb{P}_{\mathrm{norm}}$ and $\mathbb{P}_{\mathrm{thin}}$ agree at all positive times in $A_{r^{1+\epsilon}, R}$, we have that $r^{1+\epsilon} \not\longleftrightarrow R$ for all $t \in [0, r^{-1}]$ also under $\mathbb{P}_{\mathrm{thin}}$. Thus, $T > 1/r$ and we obtain (4.3).

The event Good is of course useful only if it is reasonably likely to occur. Proposition 4.26, which is the main result of the upcoming Subsection 4.2, will show that

$$\mathbb{P}_{\mathrm{norm}}\big(\mathsf{Good} \mid N > 1/r\big) \geq c_1 \, .$$

This, (4.3) and $\mathsf{Good} \subseteq \mathsf{Fine}$ imply the following "stochastic quasi-domination" between T and N:

$$\mathbb{P}_{\mathrm{thin}}(T > 1/r, \mathsf{Fine}) \geq \mathbb{P}_{\mathrm{norm}}(N > 1/r, \mathsf{Good}) \geq c_1 \, \mathbb{P}_{\mathrm{norm}}(N > 1/r) \, . \tag{4.4}$$

Although the event $\{N > 1/r\}$ has minute probability when R is large, a large portion of the expectation $\mathbb{E}(N \mathbb{1}_{\mathsf{Fine}})$ is contributed by sample points realizing this event. This can be proved using the size-biasing description of the connection time discussed in Lemma 4.5. Indeed, by some rather general size-biasing arguments, together with the uniform boundedness of the expectation $\mathbb{E}(\widehat{N}_R) < \infty$ (due to Lemmas 4.5 and 4.6 above), alongside the fact that $\mathbb{P}_{\text{norm}}(\mathsf{Fine} \mid N > 1/r) \geq c_1$ (due to $\mathsf{Good} \subseteq \mathsf{Fine}$), it will be proved in Subsection 4.3 that

$$\mathbb{E}_{\text{norm}}\big(N \bigm| N > 1/r, \mathsf{Fine}\big) < C_2 < \infty\,, \tag{4.5}$$

and that

$$\mathbb{P}_{\text{norm}}\big(\widehat{N\mathbb{1}_{\mathsf{Fine}}} > 1/r\big) = \frac{\mathbb{E}_{\text{norm}}(N\mathbb{1}_{N>1/r}\mathbb{1}_{\mathsf{Fine}})}{\mathbb{E}_{\text{norm}}(N\mathbb{1}_{\mathsf{Fine}})} > c_2 > 0\,. \tag{4.6}$$

Finally, as we will prove in Proposition 4.31 of Subsection 4.4, should the dynamics begun under $\mathsf{Thinning}$ result in at least a short reconnection time, $T > 1/r$, then there is a uniformly positive probability that connection will not be reestablished until very much later:

$$\mathbb{P}_{\text{thin}}\big(T > g(r) \bigm| T > 1/r, \mathsf{Fine}\big) > c_3 > 0\,, \tag{4.7}$$

for some $g(r) \to \infty$ as $r \to \infty$.

Proof of Proposition 4.11. From the above assemblage of facts, we find that

$$\begin{aligned}
\mathbb{E}_{\text{thin}}(T\mathbb{1}_{\mathsf{Fine}}) &\geq \mathbb{E}_{\text{thin}}(T\mathbb{1}_{T>1/r}\mathbb{1}_{\mathsf{Fine}}) \\
&= \mathbb{E}_{\text{thin}}\big(T \bigm| T > 1/r, \mathsf{Fine}\big)\,\mathbb{P}_{\text{thin}}(T > 1/r, \mathsf{Fine}) \\
&\geq c_3\, c_1\, g(r)\,\mathbb{P}_{\text{norm}}(N > 1/r, \mathsf{Fine})\,, \qquad \text{by (4.7) and (4.4)} \\
&\geq c_3\, c_1\, g(r)\,\frac{\mathbb{E}_{\text{norm}}(N\mathbb{1}_{N>1/r}\mathbb{1}_{\mathsf{Fine}})}{C_2}\,, \qquad \text{by (4.5)} \\
&\geq c_3\, c_1\, c_2\, g(r)\,\frac{\mathbb{E}_{\text{norm}}(N\mathbb{1}_{\mathsf{Fine}})}{C_2}\,, \qquad \text{by (4.6)}\,.
\end{aligned}$$

Therefore, the ratio $\frac{\mathbb{E}_{\text{thin}}(T\mathbb{1}_{\mathsf{Fine}})}{\mathbb{E}_{\text{norm}}(N\mathbb{1}_{\mathsf{Fine}})}$ tends to infinity as $r \to \infty$, uniformly in $R \geq r^{1+2\epsilon}$, as required. □

We will now start proving the above ingredients.

4.2 Understanding the law $\mathbb{P}_{\text{norm}}(\cdot \mid N > 1/r)$

In this section, $\mathbb{P}$ will denote the law of càglàd dynamical percolation with time $\mathbb{R}$. Recall that $\mathcal{E}_R$ is the set of times such that $0 \leftrightarrow R$, now a union of left-open right-closed intervals.

It is hard to understand the conditioned measure $\mathbb{P}' := \mathbb{P}_{\text{norm}}\big(\cdot \bigm| N \geq 1/r\big)$ directly, because the condition has a tiny probability. We will handle this issue by noticing that, for large enough $s \in \mathbb{Z}^+$, we have $\mathbb{P}\big(\mathcal{E}_R \cap (s/r, (s+1)/r] = \emptyset \bigm| 0 \in \mathcal{E}_R\big) > c > 0$, uniformly in $r > 0$ (see Lemma 4.18), and given the existence of this empty interval, $\gamma := \sup\{\mathcal{E}_R \cap [0, s/r)\}$ is a moment such that the reconnection time from it is at least $1/r$. If s is bounded, then the law of dynamical percolation viewed from such a γ (to be denoted by $\mathbb{P}''$, see Lemma 4.15) turns out to be not very different from the law $\mathbb{P}'$ (see Lemma 4.16). Therefore, once we prove that ω_t has certain good properties with high probability for all $t \in [0, (s+1)/r]$ under $\mathbb{P}\big(\cdot \bigm| 0 \in \mathcal{E}_R,\ \mathcal{E}_R \cap (s/r, (s+1)/r) = \emptyset\big)$, which is already a feasible task, and hence that the dynamical configuration viewed from γ (i.e., the measure $\mathbb{P}''$) is well behaved, we will be able to deduce almost the same for the measure $\mathbb{P}'$; this will be Proposition 4.26, the main goal of this subsection.

Definition 4.13. *Call an element $x \in \mathcal{E}_R$ a* **marker** *if $\big(x, x + r^{-1}\big] \cap \mathcal{E}_R = \emptyset$. Write $\mathcal{M} \subseteq \mathcal{E}_R$ for the set of markers. For $x \in \mathcal{M}$, set $\ell_x \geq r^{-1}$ so that $x + \ell_x$ is the first limit point of $\mathcal{E}_R$ encountered to the right of x. Let $s \in \mathbb{Z}^+$ be a (large) integer to be determined later. For each $x \in \mathcal{M}$, set $L_x = \big[x - sr^{-1}, x - sr^{-1} + \ell_x - r^{-1}\big]$ if $r^{-1} \leq \ell_x < (s+1)r^{-1}$; if $\ell_x \geq (s+1)r^{-1}$, take $L_x = \big[x - sr^{-1}, x\big]$. Define the* **domain of attraction** *$\mathcal{D}_x$ of $x \in \mathcal{M}$ by $\mathcal{D}_x = L_x \cap \mathcal{E}_R$. See Figure 4.2.*

Note that Lemma 4.2 has a straightforward analogue for $\mathbb{P}\big(\cdot \,\big|\, 0 \in \mathcal{M}\big)$, and we have $\mathbb{P}' = \mathbb{P}_{\mathrm{norm}}\big(\cdot \,\big|\, N \geq 1/r\big) = \mathbb{P}\big(\cdot \,\big|\, 0 \in \mathcal{M}\big)$. We now define the measure $\mathbb{P}''$ on dynamical configurations on B_R that will be our main tool for understanding $\mathbb{P}'$.

Definition 4.14. *Define the law $\mathbb{P}''$ so that, for any càglàd dynamical percolation configuration ω satisfying $0 \in \mathcal{M}$,*

$$\frac{\mathrm{d}\mathbb{P}''}{\mathrm{d}\mathbb{P}'}(\omega) = Z^{-1}|\mathcal{D}_0|,$$

where $|\cdot|$ is Lebesgue measure, and $Z > 0$ is a normalization chosen to ensure that $\mathbb{P}''$ is indeed a probability measure.

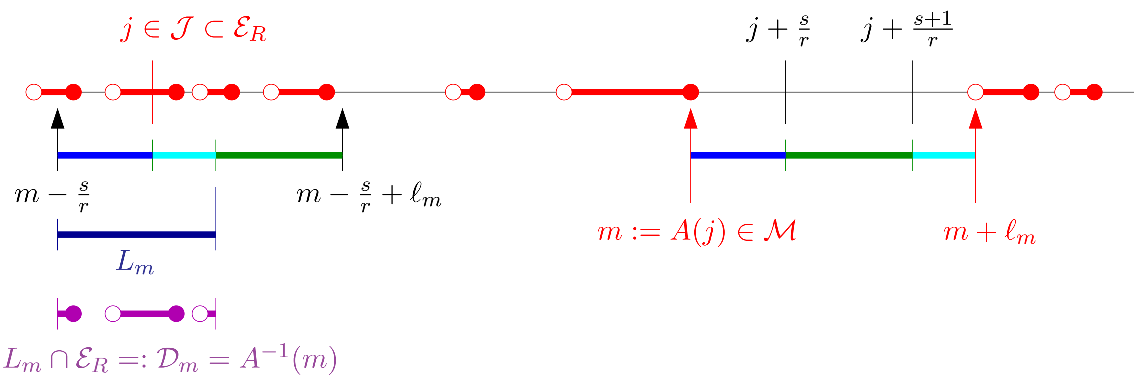

Figure 4.2: The domain of attraction $\mathcal{D}_m$ appearing in the definition of $\mathbb{P}''$, and the map $A : \mathcal{J} \longrightarrow \mathcal{M}$ appearing in the proof of $\mathbb{P}'' = \tilde{\mathbb{P}}$ (Lemma 4.15).

Lemma 4.15. *Let $\tilde{\mathbb{P}}$ denote the following dynamical process. Consider càglàd dynamical percolation $\big\{\omega_t : t \in \mathbb{R}\big\}$ in B_R with ω_0 distributed as IIC_R, and with the update decisions made independently of ω_0. Condition this process on the event that $\mathcal{E}_R \cap \big(sr^{-1}, (s+1)r^{-1}\big) = \emptyset$. Let $\gamma \in [0, sr^{-1}]$ be given by $\gamma = \sup\{\mathcal{E}_R \cap [0, sr^{-1})\}$. Now set $\tilde{\mathbb{P}}$ equal to the conditional law of $\omega(\gamma + \cdot)$. Then $\tilde{\mathbb{P}} = \mathbb{P}''$.*

Proof. Under dynamical percolation on B_R, let $\mathcal{J}$ denote the set of times $j \in \mathcal{E}_R$ such that $\big(j+sr^{-1}, j+(s+1)r^{-1}\big) \cap \mathcal{E}_R = \emptyset$. Consider the map $A : \mathcal{J} \longrightarrow \mathcal{M}$ such that, for each $j \in \mathcal{J}$, $A(j)$ is the largest element of $\mathcal{M}$ preceding $j + sr^{-1}$. Note that $j \in \mathcal{E}_R$ implies that $j \le A(j) \le j + sr^{-1}$. Note further that, for each $m \in \mathcal{M}$, we have $A^{-1}(m) = \mathcal{D}_m$. See Figure 4.2.

Consider now an experiment in which, for $x > 0$, dynamical percolation is sampled conditionally on $\mathcal{J} \cap [0,x] \neq \emptyset$, and an element $\chi \in \mathcal{J} \cap [0,x]$ is chosen with the conditional law of normalized Lebesgue measure on this set. Note that, by $\lim_{x\to\infty} \mathbb{P}(\mathcal{J} \cap [0,x] \neq \emptyset) = 1$, the law of $\omega_{A(\chi)+\cdot}$ (using the randomness in both ω and χ has the limit $\tilde{\mathbb{P}}$ as $x \to \infty$. However, from the previous paragraph we also know that $\omega_{A(\chi)+\cdot}$ has a weak limit whose Radon-Nikodym derivative with respect to dynamical percolation given $0 \in \mathcal{M}$ is $|\mathcal{D}_0|$ up to normalization. □

Lemma 4.16 (Typical events of $\mathbb{P}''$ will appear in $\mathbb{P}'$)**.** *The Radon-Nikodym derivative $\frac{\mathrm{d}\mathbb{P}''}{\mathrm{d}\mathbb{P}'}$ has a second moment that is bounded above by some $B < \infty$ which might depend on the parameter s but not on R. Consequently, $\mathbb{P}'(\mathcal{A}) \ge \mathbb{P}''(\mathcal{A})^2/B$ for any event $\mathcal{A}$.*

Proof. The claim regarding the Radon-Nikodym derivative follows directly from Lemma 4.17 below. The second claim then follows by Cauchy-Schwarz:

$$\mathbb{P}''(\mathcal{A}) = \int \mathbb{1}_{\mathcal{A}}\, \mathrm{d}\mathbb{P}'' = \int \mathbb{1}_{\mathcal{A}} \frac{\mathrm{d}\mathbb{P}''}{\mathrm{d}\mathbb{P}'}\, \mathrm{d}\mathbb{P}' \le \sqrt{\int \mathbb{1}_{\mathcal{A}}^2\, \mathrm{d}\mathbb{P}'}\, \sqrt{\int \Big(\frac{\mathrm{d}\mathbb{P}''}{\mathrm{d}\mathbb{P}'}\Big)^2 \mathrm{d}\mathbb{P}'} \le \sqrt{\mathbb{P}'(\mathcal{A})}\, \sqrt{B}\,,$$

as desired. □

Lemma 4.17. *Let $m_{R,r}$ denote the conditional mean under dynamical percolation of $|\mathcal{E}_R \cap \big(0, r^{-1}\big)|$ given that this intersection is non-empty. Consider dynamical percolation $\mathbb{P}$ on B_R conditionally on $0 \in \mathcal{M}$. Then the Lebesgue measure of the domain of attraction of the origin satisfies*

$$\mathbb{P}\big(|\mathcal{D}_0| \ge c\, m_{R,r} \bigm| 0 \in \mathcal{M}\big) \ge cs^{-2} \tag{4.8}$$

and

$$\mathbb{E}\big(|\mathcal{D}_0|^2 \bigm| 0 \in \mathcal{M}\big) \le Cs^2 m_{R,r}^2\,, \tag{4.9}$$

for constants $C > c > 0$ which do not depend on r, R or s.

Before starting the proof of Lemma 4.17, we need to verify a basic decorrelation result. In light of Lemma 4.15 (describing $\mathbb{P}''$ as $\tilde{\mathbb{P}}$), it is far from surprising that this result will be crucial in understanding the measures $\mathbb{P}''$ and $\mathbb{P}'$.

Lemma 4.18 (Ensuring an empty interval)**.** *There exists a large $s \in \mathbb{Z}^+$ and a small $c > 0$ such that, for each $r \in \mathbb{Z}^+$ and $R > R_0(r)$, the probability that dynamical percolation with initial condition ω_0 distributed according to IIC_R satisfies $\mathcal{E}_R \cap \big(s\, r^{-1}, (s+1)\, r^{-1}\big) = \emptyset$ exceeds c.*

An important element of the proof of Lemma 4.18 is the following claim. It is slightly more convenient to reverse time once again, just for this claim. Recall that $\rho(r) = \inf\{s : s^2\alpha_4(s) \geq r\}$, and keep in mind that its magnitude is known to be $r^{4/3+o(1)}$ for percolation on the faces of $\mathcal{H}$ and to lie between $C^{-1}\, r^{1+\eta}$ and $C\, r^{1/\eta}$ for some $\eta \in (0,1)$ and $0 < C < \infty$ for bond percolation on $\mathbb{Z}^2$.

Lemma 4.19. *There exists $c > 0$ such that the following holds, independently of $r \in \mathbb{N}$. Let $\mathcal{N}$ denote the event that at no time in the interval $[-r^{-1}, 0]$ is there an open crossing of the annulus $A_{\rho(r),\,2\rho(r)}$. For $s > 0$, let $\mathcal{Y}_s$ denote the event that an open crossing of $A_{\rho(r),\,2\rho(r)}$ exists at time sr^{-1}. Then, for all large enough $s > 0$ (without dependence on r), we have $\mathbb{P}\big(\mathcal{N} \cap \mathcal{Y}_s\big) \geq c$.*

Proof. By considering a coupling in which dynamical updates lead always to the closure of hexagons, we know that $\mathbb{P}\big(\mathcal{N}\big) \geq c$ by (1.9), Kesten's result on the near-critical window. Let $\mathcal{N}_0$ denote the time-0 static event that the conditional probability of $\mathcal{N}$ given the time 0 configuration is at least c. We have that $\mathbb{P}(\mathcal{N}_0) \geq c$ by adjusting the value of $c > 0$. Note then that, denoting by f and g the ± 1-indicator functions of $\mathcal{N}_0$ and $\mathcal{Y}_0$, and by $\widehat{f}$ and $\widehat{g}$ their Fourier series, the basic relation (1.13) yields

$$\mathbb{P}\big(\mathcal{N}_0 \cap \mathcal{Y}_s\big) - \mathbb{P}\big(\mathcal{N}_0\big)\mathbb{P}\big(\mathcal{Y}_s\big) = \sum_{S \neq \emptyset} \widehat{f}(S)\widehat{g}(S) \exp\big\{ - sr^{-1}|S|\big\} \, .$$

We apply Cauchy-Schwarz to bound above the absolute value of the right-hand side. Then, the basic relation (1.13) and the decorrelation estimate (1.16) applied to g give the following bound on the resulting term:

$$\bigg(\sum_{S \neq \emptyset} \widehat{f}^2(S)\bigg)^{1/2} \bigg(\sum_{S \neq \emptyset} \widehat{g}^2(S) \exp\big\{ - 2sr^{-1}|S|\big\}\bigg)^{1/2}$$
$$\leq \epsilon_1 + \bigg(\sum_{|S| \geq \epsilon_2 r} \widehat{g}^2(S) \exp\big\{ - 2\epsilon_2 s\big\}\bigg)^{1/2} \leq \epsilon_1 + \exp\big\{ - \epsilon_2 s\big\} \, ,$$

where ϵ_1 depends on the choice of cutoff $\epsilon_2 > 0$ and may be chosen so that $\epsilon_1 \to 0$ as $\epsilon_2 \to 0$. Noting that $\mathbb{P}\big(\mathcal{N}_0\big)\mathbb{P}\big(\mathcal{Y}_s\big) \geq c_1 > 0$, we see that $\mathbb{P}\big(\mathcal{N}_0 \cap \mathcal{Y}_s\big) \geq c_1/2$ by making a suitable choice of ϵ_1, ϵ_2 and s. Note that $\mathbb{P}\big(\mathcal{N} \cap \mathcal{Y}_s\big) \geq c\,\mathbb{P}\big(\mathcal{N}_0 \cap \mathcal{Y}_s\big)$ because $\mathcal{N}$ and $\mathcal{Y}_s$ are conditionally independent given the time-0 configuration. This completes the proof. □

The next lemma relates the restriction of IIC to a dyadic annulus to the percolation configuration in the annulus obtained by conditioning on an open crossing between the annulus' boundaries.

Lemma 4.20 (Localizing the IIC conditioning)**.** *Let $\mathbb{P}_r^R$ denote the law of critical percolation in $A_{r,R}$ given that $r \longleftrightarrow R$, for $0 \leq r < R \leq \infty$ (where the conditional law*

$\mathbb{P}(\cdot \mid r \leftrightarrow \infty)$ on B_r^c is obtained as a weak limit of $\mathbb{P}(\cdot \mid r \leftrightarrow R)$ as $R \to \infty$, constructed by [Kes86]). Then, for each $\epsilon > 0$ there exists $\delta > 0$ such that if $\mathcal{A} \in \sigma\{A_{R,2R}\}$ (i.e., an event measurable in the annulus), then $\mathbb{P}_R^{2R}(\mathcal{A}) \geq \epsilon$ implies that $\mathbb{P}_a^b(\mathcal{A}) \geq \delta$, for all $0 \leq a \leq R/2$ and $4R \leq b \leq \infty$ (in particular, for $\mathsf{IIC} = \mathbb{P}_0^\infty$).

Proof. For ζ a configuration in $A_{R,2R}$ such that $R \leftrightarrow 2R$, let $W_{a,R,b}(\zeta)$ denote the conditional probability that $a \leftrightarrow b$ given the occurrence of the events $\omega\big|_{A_{R,2R}} = \zeta$, $a \leftrightarrow R$ and $2R \leftrightarrow b$. We will argue that for each $\epsilon > 0$ there exists $\delta > 0$ such that, for all large enough $R \in \mathbb{N}$ and $a, b \in \mathbb{N}$ with $0 \leq a \leq R/2$ and $2R \leq b \leq \infty$,

$$\mathbb{P}\Big(W_{a,R,b} \leq \delta \;\Big|\; R \longleftrightarrow 2R\Big) \leq \epsilon\,. \tag{4.10}$$

This easily implies the lemma, as follows. Note that

$$\frac{\mathrm{d}\mathbb{P}_a^b}{\mathrm{d}\mathbb{P}_R^{2R}}(\zeta) = Z_{a,R,b}^{-1}\, W_{a,R,b}(\zeta)\,,$$

where $Z_{a,R,b} = \mathbb{P}(a \leftrightarrow b \mid a \leftrightarrow R, R \leftrightarrow 2R, 2R \leftrightarrow b) \leq 1$. Given $\epsilon > 0$, choose by means of (4.10) an $\epsilon' > 0$ such that $\mathbb{P}_R^{2R}\big(W_{a,R,b} \leq \epsilon'\big) \leq \epsilon/2$ for each $R \in \mathbb{N}$. Thus, if $\mathcal{A} \in \sigma\{A_{R,2R}\}$ satisfies $\mathbb{P}_R^{2R}(\mathcal{A}) \geq \epsilon$, then

$$\mathbb{P}_a^b(\mathcal{A}) = Z_{a,R,b}^{-1} \int_{\mathcal{A}} W_{a,R,b}(\omega)\, \mathrm{d}\mathbb{P}_R^{2R}(\omega) \geq \epsilon'\epsilon/2\,,$$

where the inequality follows from restricting the integral to that part of $\mathcal{A}$ on which $W_{a,R,b} > \epsilon'$. Hence the lemma holds with the choice $\delta = \epsilon'\epsilon/2$.

To prove (4.10), we introduce the function $W_R^\epsilon(\zeta)$ on configurations ζ in $A_{R,2R}$, for $R \in \mathbb{N}$ and $\epsilon \in (0, 1/2)$, which is the conditional probability of $R(1-\epsilon) \longleftrightarrow 2R(1+\epsilon)$ under critical percolation given that $\omega\big|_{A_{R,2R}} = \zeta$.

Lemma 4.21. *For each $\epsilon \in (0, 1/2)$, there exists a constant $c = c_\epsilon > 0$ such that, for each $R, a, b \in \mathbb{N}$ as before and for all configurations ζ in $A_{R,2R}$, we have $W_{a,R,b}(\zeta) \geq c\, W_R^\epsilon(\zeta)$.*

Proof. Let p_1 denote the probability under critical percolation that there exists an open surrounding circuit in the annulus $A_{R(1-\epsilon),R}$, and let p_2 denote the corresponding probability for the annulus $A_{2R,2R(1+\epsilon)}$. Note that $p_1, p_2 \geq c_\epsilon > 0$ for all R by a simple application of RSW. We claim that

$$W_{a,R,b}(\zeta) \geq p_1 p_2 W_R^\epsilon(\zeta)\,. \tag{4.11}$$

Indeed, consider the conditioning appearing in the definition of $W_{a,R,b}(\zeta)$: under the conditional law, the configuration in $A_{R,2R}^c$ stochastically dominates critical percolation, and thus open surrounding circuits appear in the annuli $A_{R(1-\epsilon),R}$ and $A_{2R,2R(1+\epsilon)}$ with probability at least $p_1 p_2$; the presence of such circuits being an increasing event, the conditional law further conditioned on the presence of such circuits has probability at least $W_R^\epsilon(\zeta)$ of realizing $R(1-\epsilon) \longleftrightarrow 2R(1+\epsilon)$. However, the event $R(1-\epsilon) \longleftrightarrow 2R(1+\epsilon)$ and the presence of the two surrounding circuits is enough, alongside the conditions met under the conditional law, to ensure that $0 \longleftrightarrow \infty$. In summary, we obtain (4.11); applying $p_1 p_2 \geq c_\epsilon^2$ completes the proof. □

Lemma 4.22. *For each $\delta > 0$, there exists $\epsilon_0 > 0$ such that, for all large enough $R \in \mathbb{N}$ and all $\epsilon \in (0, \epsilon_0)$,*

$$\mathbb{P}\big(W_R^\epsilon \leq 1-\delta \,\big|\, R \longleftrightarrow 2R\big) \leq \delta\,.$$

Proof. Note that for any $\delta_1 > 0$ there is an $\epsilon_1 > 0$ such that, for all $\epsilon < \epsilon_1$,

$$\begin{aligned}\mathbb{E}\big(W_R^\epsilon \,\big|\, R \longleftrightarrow 2R\big) &= \mathbb{P}\big(R(1-\epsilon) \longleftrightarrow 2R(1+\epsilon) \,\big|\, R \longleftrightarrow 2R\big) \\ &= 1 - \frac{\mathbb{P}\big(R \longleftrightarrow 2R\,,\ \text{but } R(1-\epsilon) \not\longleftrightarrow 2R(1+\epsilon)\big)}{\mathbb{P}(R \longleftrightarrow 2R)} \\ &\geq 1 - \delta_1\,,\end{aligned}$$

because the event $\big\{R \longleftrightarrow 2R\,,\ \text{but } R(1-\epsilon) \not\longleftrightarrow 2R(1+\epsilon)\big\}$ implies that there are three arms from one side of the annulus $A(R(1-\epsilon), 2R(1+\epsilon))$, from radius about ϵR to radius about R, and this event has probability of order ϵ, the 3-arm half-plane probability being of order ϵ^2. See [Wer09, first exercise sheet].

From this bound, applying Markov's inequality to $1 - W_R^\epsilon$, we get that

$$\mathbb{P}\big(1 - W_R^\epsilon \geq \sqrt{\delta_1} \,\big|\, R \longleftrightarrow 2R\big) \leq \sqrt{\delta_1}\,,$$

which implies the lemma immediately. □

Now note that (4.10) follows from Lemmas 4.21 and 4.22 immediately. This completes the proof of Lemma 4.20 (localizing the IIC conditioning) for large enough $R \in \mathbb{N}$; on the other hand, for R bounded, the lemma is trivial. □

Proof of Lemma 4.18. (Ensuring an empty interval.) Let $\mathcal{C} \in \sigma\big\{A_{\rho(r),\,2\rho(r)}\big\}$ denote the static event consisting of configurations ζ satisfying $\rho(r) \leftrightarrow 2\rho(r)$ and such that

$$\mathbb{P}\Big(\rho(r) \leftrightarrow 2\rho(r) \text{ at no time in } [sr^{-1}, (s+1)r^{-1}] \,\Big|\, \omega_0 = \zeta\Big) \geq c\,.$$

By considering the process $\omega\big(sr^{-1} - \cdot\big)$ in Lemma 4.19, we see that

$$\mathbb{P}\Big(\rho(r) \leftrightarrow 2\rho(r) \text{ at time } 0\,,\ \rho(r) \leftrightarrow 2\rho(r) \text{ at no time in } [sr^{-1}, (s+1)r^{-1}]\Big) \geq c\,;$$

in the notation of the statement of Lemma 4.20, we see that $\mathbb{P}_{\rho(r)}^{2\rho(r)}(\mathcal{C}) \geq c$ by reducing the value of $c > 0$. By Lemma 4.20, we infer that for some $\delta > 0$ and for $R \geq 4\rho(r)$, $\mathsf{IIC}_R(\mathcal{C}) > \delta$, as required for the statement of Lemma 4.18. □

Proof of Lemma 4.17. We start by a simple corollary of Lemma 4.18 concerning the density of markers.

Definition 4.23. *Let* $\{I_i = \big(i/r, (i+1)/r\big) : i \in \mathbb{N}\}$ *enumerate the consecutive intervals of length* r^{-1} *rightwards from the origin. Call any such interval* **active** *if it has non-empty intersection with* $\mathcal{E}_R$*. For any* $i \in \mathbb{N}$*, call* I_i **promising** *if* I_i *is an active interval with the property that* $\mathcal{M}$ *intersects* $\cup_{i \leq j \leq i+s} I_j$*.*

Lemma 4.24. *There exists* $c > 0$*, independent of* R*, such that the conditional probability under dynamical percolation* B_R *given that* I_0 *is active that* I_0 *is promising is at least* c*.*

Proof. Let $\mathbb{P}_0$ denote dynamical percolation on $(0, 1/r)$ weighted according to the size $\big|\mathcal{E}_R \cap (0, 1/r)\big|$; under $\mathbb{P}_0$, define τ to be an element of $\mathcal{E}_R \cap (0, 1/r)$ with conditional law given by normalized Lebesgue measure on this set. Under $\mathbb{P}_0$, the law of dynamical percolation at times $\tau + t$, $t \geq 0$ is, by Lemma 3.3, dynamical percolation started from IIC_R. By Lemma 4.18, the conditional probability that $\big(\tau + sr^{-1}, \tau + (s+1)r^{-1}\big) \cap \mathcal{E}_R = \emptyset$ exceeds some R-independent constant $c > 0$. Whenever this disjointness condition is satisfied, there exists an element of $\mathcal{M}$ somewhere in the interval between τ and $\tau + sr^{-1}$, and thus in the interval $\big(0, (s+1)r^{-1}\big)$.

We learn that the $\mathbb{P}_0$-probability that I_0 is promising exceeds an R-independent constant $c > 0$. Lemma 4.24 will follow once we establish this assertion for dynamical percolation conditioned on the interval I_i being active, a measure we label $\mathbb{P}_1$. To make this reduction, it is enough to argue that $\frac{\mathrm{d}\mathbb{P}_0}{\mathrm{d}\mathbb{P}_1}$ has a bounded second moment, in light of the proof of Lemma 4.16, with the roles of $\mathbb{P}''$ and $\mathbb{P}'$ being played by $\mathbb{P}_0$ and $\mathbb{P}_1$. By Lemma 2.6, there exists $C > 0$ such that, for all $R > 0$,

$$\begin{aligned} \int \Big(\frac{\mathrm{d}\mathbb{P}_0}{\mathrm{d}\mathbb{P}_1}\Big)^2 \mathrm{d}\mathbb{P}_1 &= \mathbb{E}\Big(\big|\mathcal{E}_R \cap (0, r^{-1})\big|^2 \,\Big|\, \mathcal{E}_R \cap (0, r^{-1}) \neq \emptyset\Big) \\ &\leq C\bigg(\mathbb{E}\Big(\big|\mathcal{E}_R \cap (0, r^{-1})\big| \,\Big|\, \mathcal{E}_R \cap (0, r^{-1}) \neq \emptyset\Big)\bigg)^2 . \end{aligned}$$

This completes the proof of Lemma 4.24. □

We can now prove (4.9). Let $\big\{m_i : i \in \mathbb{N}^+\big\}$ enumerate the elements of $\mathcal{M} \cap (0, \infty)$ in increasing order. By ergodicity, we have almost surely that

$$\mathbb{E}\big(|\mathcal{D}_0|^2 \,\big|\, 0 \in \mathcal{M}\big) = \lim_n n^{-1} \sum_{i=2}^{n} |\mathcal{D}_{m_i}|^2 , \tag{4.12}$$

where the term with index $i = 1$ has been harmlessly omitted for later notational convenience. Let $\{\lambda_i\}$ (or $\{\alpha_i\}$) enumerate the indices $i \in \mathbb{N}^+$ of promising (or active) intervals I_i in increasing order. For $i \geq 1$, consider the consecutive intervals I_j beginning the interval after that containing m_i and stopping at the one containing m_{i+1}. Among these, there are at most $s+1$ promising intervals, and $\mathcal{D}_{m_{i+1}}$ is contained in the union of these promising intervals. Therefore, $\sum_{i=2}^{n} |\mathcal{D}_{m_i}|^2 \leq (s+1)\sum_{i=2}^{\lambda_{(s+1)n}} |I_i \cap \mathcal{E}_R|^2$. By Lemma 4.24 and the ergodicity Lemma 2.3, $\lambda_n \leq 2c^{-1}\alpha_n$ for all large enough n. Hence, $\sum_{i=2}^{n} |\mathcal{D}_{m_i}|^2 \leq (s+1)\sum_{i=1}^{2c^{-1}\alpha_{(s+1)n}} |I_i \cap \mathcal{E}_R|^2$. By ergodicity again, this upper bound behaves like

$$2c^{-1}(s+1)^2\, n\, \mathbb{E}\big(|\mathcal{E}_R \cap (0,1/r)|^2 \;\big|\; \mathcal{E}_R \cap (0,1/r) \neq \emptyset\big)\,\big(1+o(1)\big)$$

as $n \to \infty$. Applying Lemma 2.6 to $|\mathcal{E}_R \cap (0,1/r)|$ (which is just a scaled version of $\overline{\mu}_R(0,1/r)$) and using (4.12), we obtain (4.9).

To prove (4.8), in light of (4.9), the Paley-Zygmund second moment method says that it suffices to verify that, for some $c > 0$ and all $R, r, s \in \mathbb{N}_+$,

$$\mathbb{E}\big(|\mathcal{D}_0| \;\big|\; 0 \in \mathcal{M}\big) \geq c\, m_{R,r}\,. \tag{4.13}$$

We now verify this inequality. Let $\rho = \lim_n n^{-1}\big|\mathcal{M} \cap (0,n)\big|$ denote the mean number of markers in $[0,1]$, or, alternatively, $\rho = \mathbb{E}(|\mathcal{M} \cap (0,1)|)$.

We claim the following.

Lemma 4.25. *Recall that $\mathcal{J}$ denotes the set of times j such that the event $0 \leftrightarrow R$ occurs at time j and at no time in the interval $\big(j + sr^{-1}, j + (s+1)r^{-1}\big)$. Then $\rho\,\mathbb{E}\big(|\mathcal{D}_0| \;\big|\; 0 \in \mathcal{M}\big) = \mathbb{E}\big(|\mathcal{J} \cap [0,1]|\big)$.*

Proof. Recall that the subset $\mathcal{J}$ of $\mathcal{E}_R$ is partitioned into disjoint classes given by domains of attraction $\mathcal{D}_m$ and thus indexed by the set of markers $m \in \mathcal{M}$.

The quantity $\rho\,\mathbb{E}\big(|\mathcal{D}_0| \;\big|\; 0 \in \mathcal{M}\big)$ is thus the mean Lebesgue measure of the union of the domains of attractions indexed by markers lying in a given unit interval. By the above partition and ergodicity, we arrive at the statement of Lemma 4.25. □

By translation invariance, $\mathbb{E}(|\mathcal{J} \cap [0,1]|) = \lim_n n^{-1}\mathbb{E}(|\mathcal{J} \cap [0,n]|)$; by Lemmas 3.2 and 4.18, there exists $c > 0$ such that, for n sufficiently high,

$$n^{-1}\mathbb{E}(|\mathcal{J} \cap [0,n]|) \geq cn^{-1}\mathbb{E}(|\mathcal{E}_R \cap [0,n]|)\,.$$

By translation invariance again, $n^{-1}\mathbb{E}(|\mathcal{E}_R \cap [0,n]|) = \mathbb{E}(|\mathcal{E}_R \cap [0,1]|)$ which may be written $r m_{R,r}\mathbb{P}(I_0 \,\text{is active})$. To summarise the derivation of (4.13) thus far, the preceding inequality and Lemma 4.25 yield

$$\rho\,\mathbb{E}\big(|\mathcal{D}_0| \;\big|\; 0 \in \mathcal{M}\big) \geq c\, r\, m_{R,r}\, \mathbb{P}(I_0 \,\text{is active}). \tag{4.14}$$

We will show that

$$\rho \,\leq r\,\mathbb{P}(I_0 \,\text{is active})\,; \tag{4.15}$$

note then that (4.14) and (4.15) yield (4.13).

To verify (4.15), recall that a marker is by definition an element of $\mathcal{E}_R$ bordered on the right by an interval of length r^{-1} having no intersection with $\mathcal{E}_R$. Thus, each marker lies in an active interval, and no active interval contains more than one marker. This implies that the mean rate ρ of markers is at most the mean number of active intervals in a given unit interval, a quantity which may be expressed as $r\,\mathbb{P}(I_0 \,\text{is active})$. This verifies (4.15). This completes the derivation of (4.13) and thus of (4.8), which concludes the proofs of Lemmas 4.17 and 4.16 on the Radon-Nikodym derivative $\frac{\mathrm{d}\mathbb{P}''}{\mathrm{d}\mathbb{P}'}$. $\square$

We are now ready to address the main goal of this subsection. Recall the notion of Good from Definition 4.12.

Proposition 4.26 ($\mathbb{P}'$ is well behaved)**.** *There exists $c > 0$ such that, for any $r > r_0$ and $R > R_0(r)$,*

$$\mathbb{P}\big(\omega \in \mathsf{Good} \,\big|\, 0 \in \mathcal{M}\big) \geq c \,.$$

In the proof, we will use the following notion and claim.

Definition 4.27. *Fix $\epsilon > 0$ and $r \in \mathbb{N}$ as in Definition 4.8, and let $R \in \mathbb{N}$ satisfy $R \geq r^{1+2\epsilon}$. We say that a dynamical configuration ω in B_R is $\omega \in \mathsf{VeryGood}$ if the following conditions are satisfied:*

- $0 \overset{\omega_0}{\longleftrightarrow} R$;
- *For each $t \in [0,(s+1)r^{-1}]$, the inner and outer boundaries of the annulus $A_{r^{1+\epsilon},r^{1+2\epsilon}}$ are separated by an ω_t-open circuit;*
- *For each $t \in [0,(s+1)r^{-1}]$, the circuit Γ_r exists and satisfies $\Gamma_r \subseteq B_{r^{1+\epsilon}}$ in ω_t;*
- $|\mathsf{Piv}_{0\leftrightarrow\Gamma_r}| \leq |\mathsf{Piv}_{0\leftrightarrow r^{1+\epsilon}}(\omega_t)| \leq r^{2(1+2\epsilon)}\alpha_4(r^{1+2\epsilon})$ *for all such t.*
- *For each $t \in [0,(s+1)r^{-1}]$, $0 \overset{\omega_t}{\longleftrightarrow} r^{1+2\epsilon}$.*

Lemma 4.28. *For any $\delta > 0$, there exists $r_0 \in \mathbb{N}$ such that, for all $r \geq r_0$ and $R \geq r^{1+2\epsilon}$,*

$$\mathbb{P}\big(\omega \in \mathsf{VeryGood} \,\big|\, 0 \overset{\omega_0}{\longleftrightarrow} R\big) \geq 1 - \delta \,. \tag{4.16}$$

Proof. Let $\mathbb{P}_1$ denote dynamical percolation in $B_{r^{1+2\epsilon}}$ with ω_0 having the distribution $\mathbb{P}\big(\,\cdot\,\big|\, 0 \leftrightarrow R\big)$, and with conditionally independent updates at rate one. Let $\mathbb{P}_2$ denote the asymmetric dynamical process in $B_{r^{1+2\epsilon}}$ with the same initial distribution as in $\mathbb{P}_1$, but with the updates always leading to the closure of hexagons. As usual, we form the obvious coupling $\mathbf{Q}$ of $\mathbb{P}_1$ and $\mathbb{P}_2$ such that the first marginal dominates the second for all $t \geq 0$.

In this new notation, the statement of the lemma is equivalent to: for any $\delta > 0$, there exists $r_0 \in \mathbb{N}$ such that, for all $r \geq r_0$ and $R \geq r^{1+2\epsilon}$,

$$\mathbb{P}_1(\omega \in \mathsf{VeryGood}') \geq 1 - \delta \,, \tag{4.17}$$

where $\mathsf{VeryGood}'$ is given by the second and later conditions defining $\mathsf{VeryGood}$. We claim that, to show (4.17), it is enough that

$$\mathbb{P}_2(\omega \in \mathsf{VeryGood}') \geq 1 - \delta\,. \tag{4.18}$$

To see that (4.18) is enough for (4.17), note that, under the coupling $\mathbf{Q}$, it is clear that if the second ($\mathbb{P}_2$-distributed) marginal satisfies the second, third and fifth conditions of Definition 4.27, then so does the first ($\mathbb{P}_1$-distributed) marginal, because these conditions are monotone. In regard to the fourth condition, write Piv for $\mathsf{Piv}_{0 \leftrightarrow r^{1+\epsilon}}$. Note that if ω_1 and ω_2 are two configurations in $B_{r^{1+\epsilon}}$ such that $\omega_1 \geq \omega_2$ and $0 \leftrightarrow r^{1+\epsilon}$ under ω_2, then $\mathsf{Piv}(\omega_1) \subseteq \mathsf{Piv}(\omega_2)$: indeed, were a hexagon h in $B_{r^{1+\epsilon}}$ to satisfy $h \in \mathsf{Piv}(\omega_1) \setminus \mathsf{Piv}(\omega_2)$, then its closure would disable $0 \leftrightarrow r^{1+\epsilon}$ in ω_1 but not in ω_2, a circumstance which stochastic domination prevents. That is, whenever $0 \xleftrightarrow{\omega_t} r^{1+\epsilon}$ occurs under $\mathbb{P}_2$, we have that $|\mathsf{Piv}(\omega_t^1)| \leq |\mathsf{Piv}(\omega_t^2)|$ (where ω^1 and ω^2 denote the $\mathbb{P}_1$ and $\mathbb{P}_2$ marginals), and thus (4.18) implies (4.17) and hence (4.16).

It remains to verify (4.18). We start with a simple lemma.

Lemma 4.29. *Let $\mathbb{P}_s^{\downarrow}$ denote asymmetric dynamical percolation $\{\omega_t : t \geq 0\}$ with ω_0 having the distribution IIC_s, then closing hexagons at rate one. Then, restricted to the ball B_r, the Radon-Nikodym derivative $\frac{\mathrm{d}\mathbb{P}_R^{\downarrow}}{\mathrm{d}\mathbb{P}_r^{\downarrow}}\big(\omega[0,t]^{B_r}\big)$ is bounded from above uniformly in r, $R \geq r$, $t \geq 0$, and all dynamical configurations $\omega[0,t]^{B_r}$.*

Proof. We claim that

$$\frac{\mathrm{d}\mathbb{P}_R^{\downarrow}}{\mathrm{d}\mathbb{P}_r^{\downarrow}}\big(\omega[0,t]^{B_r}\big) \leq \frac{\mathbb{P}(0 \leftrightarrow r)\mathbb{P}(r+1 \leftrightarrow R)}{\mathbb{P}(0 \leftrightarrow R)}\,, \tag{4.19}$$

with the right hand side understood simply in static critical percolation. From this, the lemma follows by quasi-multiplicativity. For the claim, note that the Radon-Nikodym derivatives with respect to asymmetric dynamical percolation $\mathbb{P}^{\downarrow}$ started from criticality, restricted to B_r, can be written as

$$\frac{\mathrm{d}\mathbb{P}_s^{\downarrow}}{\mathrm{d}\mathbb{P}^{\downarrow}}\big(\omega[0,t]^{B_r}\big) = \frac{\mathbb{P}^{\downarrow}\big(0 \leftrightarrow s \text{ in } \omega_0 \,\big|\, \omega[0,t]^{B_r}\big)}{\mathbb{P}^{\downarrow}(0 \leftrightarrow s \text{ in } \omega_0)}\,,$$

for any $s \geq r$; in particular, for $s \in \{r, R\}$. On the other hand,

$$\mathbb{P}^{\downarrow}\big(0 \leftrightarrow R \text{ in } \omega_0 \,\big|\, \omega[0,t]^{B_r}\big) \leq \mathbb{P}^{\downarrow}\big(0 \leftrightarrow r \text{ in } \omega_0 \,\big|\, \omega[0,t]^{B_r}\big)\, \mathbb{P}^{\downarrow}(r+1 \leftrightarrow R \text{ in } \omega_0)\,.$$

Since the distribution of ω_0 under $\mathbb{P}^{\downarrow}$ is simply critical percolation, from the last two displays follows (4.19). $\square$

Proof of (4.18). Let $\mathbb{P}_3$ denote asymmetric dynamical percolation $\mathbb{P}^{\downarrow}_{r^{1+2\epsilon}}$ in $B_{r^{1+2\epsilon}}$, with the notation of the previous Lemma 4.29. By that lemma, it is enough to verify (4.18) with $\mathbb{P}_3$ in place of $\mathbb{P}_2$. We are going to show that each of the four conditions defining $\mathsf{VeryGood}'$ happens with probability close to 1 if r is large enough.

Let us first look at the four conditions at time zero. The fifth condition (that $0 \leftrightarrow r^{1+2\epsilon}$) is automatically satisfied under $\mathbb{P}_3$. The second and third conditions (open circuits in $A_{r^{1+\epsilon},r^{1+2\epsilon}}$ and in $A_{r^{1+\epsilon},r}$) are satisfied with high probability in critical percolation by RSW along several scales, and also under the conditioning $0 \leftrightarrow r^{1+2\epsilon}$ by FKG. The fourth condition (there are not too many pivotals for $0 \leftrightarrow r^{1+\epsilon}$) follows from standard quasi-multiplicativity arguments. Namely, as illustrated on Figure 4.3, we have

$$\mathbb{P}\big(x \in \mathsf{Piv}_{0\leftrightarrow r^{1+\epsilon}} \;\big|\; 0 \leftrightarrow r^{1+2\epsilon}\big) \asymp \alpha_4\big(\mathrm{dist}(x, \partial B_{r^{1+\epsilon}}) \wedge \mathrm{dist}(0,x)\big)\, \alpha_3\big(\mathrm{dist}(x, \partial B_{r^{1+\epsilon}}), r^{1+\epsilon}\big)\,,$$

which can be summed up over the possible hexagons $x \in B_{r^{1+\epsilon}}$ to get

$$\mathbb{E}\Big(|\mathsf{Piv}_{0\leftrightarrow r^{1+\epsilon}}| \;\Big|\; 0 \leftrightarrow r^{1+2\epsilon}\Big) = O(1)\, r^{2(1+\epsilon)} \alpha_4(r^{1+\epsilon})\,.$$

By quasi-multiplicativity and (1.6), we have

$$\frac{\alpha_4(r^{1+\epsilon})}{\alpha_4(r^{1+2\epsilon})} < C\,(r^{\epsilon})^{2-\eta} \ll r^{2\epsilon}\,,$$

hence Markov's inequality yields

$$\mathbb{P}\Big(|\mathsf{Piv}_{0\leftrightarrow r^{1+\epsilon}}| > r^{2(1+2\epsilon)} \alpha_4(r^{1+2\epsilon}) \;\Big|\; 0 \leftrightarrow r^{1+2\epsilon}\Big) \to 0\,,$$

as $r \to \infty$, as desired.

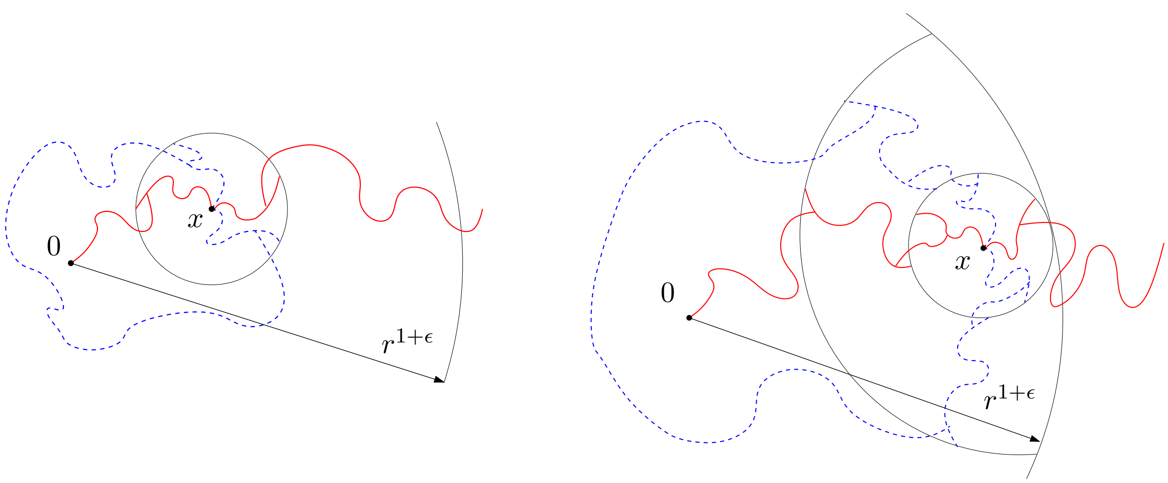

Figure 4.3: Conditioned on $0 \leftrightarrow r^{1+2\epsilon}$, one 4-arm event (first picture) or one 4-arm and one 3-arm events (second picture) are roughly equivalent to x being pivotal for $0 \leftrightarrow r^{1+\epsilon}$.

We now have to prove that the four conditions are also satisfied with high probability at time $t = (s+1)r^{-1}$; then, by the earlier monotonicity argument, we have the result for all $t \in [0, (s+1)r^{-1}]$, as well.

By the exponent bound (1.7) and the choice $(1+2\epsilon)(1-\eta) < 1$ made in Definition 4.8 and onwards, we have that $r^{-1} \ll 1/\big(r^{2(1+2\epsilon)}\alpha_4(r^{1+2\epsilon})\big)$, as $r \to \infty$. Thus, the constant closing of hexagons for time $(s+1)r^{-1}$ keeps the system $B_{r^{1+2\epsilon}}$ well inside the critical window of percolation, established by Kesten, as described in (1.8) and (1.11). Therefore, the above arguments for the second to fourth conditions of Definition 4.27 apply verbatim. The fifth condition can be verified in a similar manner: using (1.11), we have

$$\mathbb{P}^{\downarrow}\Big(0 \overset{\omega_{(s+1)r^{-1}}}{\longleftrightarrow} r^{1+2\epsilon} \;\Big|\; 0 \overset{\omega_0}{\longleftrightarrow} r^{1+2\epsilon}\Big) = 1 - o(1)\,,$$

as $r \to \infty$. This finishes the proof of (4.18) and Lemma 4.28. □

Proof of Proposition 4.26. Whenever (4.16) holds, by Lemma 4.18 we also have that

$$\begin{aligned}
&\mathbb{P}\big(\omega \in \mathsf{VeryGood} \;\big|\; 0 \overset{\omega_0}{\longleftrightarrow} R,\ \mathcal{E}_R \cap (sr^{-1},(s+1)r^{-1}) = \emptyset\big) \\
&\quad \geq \frac{\mathbb{P}\big(\mathcal{E}_R \cap (sr^{-1},(s+1)r^{-1}) = \emptyset \;\big|\; 0 \in \mathcal{E}_R\big) - \mathbb{P}\big(\omega \notin \mathsf{VeryGood} \;\big|\; 0 \in \mathcal{E}_R\big)}{\mathbb{P}\big(\mathcal{E}_R \cap (sr^{-1},(s+1)r^{-1}) = \emptyset \;\big|\; 0 \in \mathcal{E}_R\big)} \\
&\quad \geq 1 - \tfrac{\delta}{c}\,.
\end{aligned} \tag{4.20}$$

Note that if a realization of dynamical percolation in B_R realizes $\mathsf{VeryGood}$, then the process $\omega(\gamma + \cdot)$ identified in Lemma 4.15 realizes Good. By Lemma 4.15 and (4.20), we find then that $\mathbb{P}''(\omega \in \mathsf{Good}) \geq 1 - \delta/c$. Now Lemma 4.16 implies that an appropriate small choice of δ in (4.16) forces $\mathbb{P}'(\omega \in \mathsf{Good}) \geq c'$ for some absolute constant $c' > 0$, concluding the proof of Proposition 4.26. □

4.3 Size-biasing arguments

In this subsection, we will prove the bounds (4.5) and (4.6), used in the proof of Proposition 4.11 at the end of Subsection 4.1. To start with, note that

$$\mathbb{P}\big(\widehat{N} > 1/r\big) = \frac{\mathbb{E}(N\,\mathbb{1}_{N>1/r})}{\mathbb{E}N} > c > 0\,. \tag{4.21}$$

Indeed, by Lemma 4.5, the distribution of $\widehat{N} = \widehat{N}_R$ stochastically dominates that of FET_R, which implies (4.21) trivially.

Deriving (4.5). Our goal is to show that

$$\mathbb{E}\big(N \;\big|\; N > 1/r\big) < C < \infty\,, \tag{4.22}$$

uniformly in r and R for which $R \geq r^{1+2\epsilon}$, since Proposition 4.26 and $\mathsf{Good} \subseteq \mathsf{Fine}$ then imply that, for such values of R and r,

$$\mathbb{E}\big(N \bigm| N > 1/r, \mathsf{Fine}\big) \leq \frac{\mathbb{E}\big(N \bigm| N > 1/r\big)}{\mathbb{P}\big(\mathsf{Fine} \bigm| N > 1/r\big)} \leq c^{-1}\,\mathbb{E}\big(N \bigm| N > 1/r\big) < c^{-1}C < \infty\,,$$

which was the statement of (4.5).

Lemmas 4.5 and 4.6 imply that $\mathbb{E}\big(\widehat{N}\big) < C$ for some constant $C < \infty$ that is independent of R. This, together with the lower bound (4.21), plugged into the next lemma with $X := N$ and $t := 1/r$, implies (4.22).

Lemma 4.30 (Rough size-biasing)**.** *If X is a non-negative random variable, and $0 < t < 1$ is such that $\mathbb{P}(\widehat{X} > t) > c > 0$ and $\mathbb{E}(\widehat{X}) < C < \infty$, then $\mathbb{E}\big(X \bigm| X > t\big) < C' < \infty$, where C' depends only on c and C, and not on t.*

Proof. Note that $\mathbb{E}\big(X \bigm| X > t\big) = \mathbb{P}(\widehat{X} > t)\frac{\mathbb{E}(X)}{\mathbb{P}(X>t)}$. Hence, we need to show that $\mathbb{E}(X) \leq C'\,\mathbb{P}(X > t)$. We will need two ingredients for this:

(A) There exists an absolute constant $A < \infty$ such that

$$\mathbb{E}\big(X \mathbb{1}_{t \geq X}\big) < A\,\mathbb{P}(X > t)\,.$$

(B) For all $b > 0$ there is some $K < \infty$ such that

$$\mathbb{E}\big(X \mathbb{1}_{X > K}\big) < b\,\mathbb{E}\big(X \mathbb{1}_{X > t}\big)\,,$$

and therefore $\mathbb{E}\big(X \mathbb{1}_{X > K}\big) < b'\,\mathbb{E}\big(X \mathbb{1}_{K \geq X > t}\big)$ with $b' = b/(1-b)$.

How would we conclude from here?

$$\begin{aligned}\mathbb{E}(X) &= \mathbb{E}\big(X \mathbb{1}_{t \geq X}\big) + \mathbb{E}\big(X \mathbb{1}_{K \geq X > t}\big) + \mathbb{E}\big(X \mathbb{1}_{X > K}\big)\\ &< A\,\mathbb{P}(X > t) + K(1+b')\,\mathbb{P}(K \geq X > t)\\ &< (A + K(1+b'))\,\mathbb{P}(X > t)\,,\end{aligned}$$

and we are done.

Now, for the proof of (A), let us look at

$$\begin{aligned}C \geq \mathbb{E}(\widehat{X}) = \frac{\mathbb{E}(X^2)}{\mathbb{E}(X)} &= \frac{\mathbb{E}\big(X^2 \mathbb{1}_{X > K}\big) + \mathbb{E}\big(X^2 \mathbb{1}_{K \geq X > t}\big) + \mathbb{E}\big(X^2 \mathbb{1}_{t \geq X}\big)}{\mathbb{E}\big(X \mathbb{1}_{X > K}\big) + \mathbb{E}\big(X \mathbb{1}_{K \geq X > t}\big) + \mathbb{E}\big(X \mathbb{1}_{t \geq X}\big)}\\ &\geq \frac{\mathbb{E}\big(X^2 \mathbb{1}_{X > K}\big)}{\mathbb{E}\big(X^2 \mathbb{1}_{X > K}\big)/K + K\,\mathbb{P}\big(K \geq X > t\big) + \mathbb{E}\big(X \mathbb{1}_{t \geq X}\big)}\,,\end{aligned}$$

hence

$$CK\,\mathbb{P}\big(K \geq X > t\big) + C\,\mathbb{E}\big(X \mathbb{1}_{t \geq X}\big) \geq \left(1 - \frac{C}{K}\right)\mathbb{E}\big(X^2 \mathbb{1}_{X > K}\big)\,,$$

for $K > t$ to be fixed later. Assuming the opposite of (A), we have that $\mathbb{E}\big(X\mathbb{1}_{t\geq X}\big) \geq A\,\mathbb{P}\big(K \geq X > t\big)$, and the last displayed inequality implies that

$$\left(\frac{CK}{A} + C\right)\,\mathbb{E}\big(X\mathbb{1}_{t\geq X}\big) \geq \left(1 - \frac{C}{K}\right)\,\mathbb{E}\big(X^2\mathbb{1}_{X>K}\big) \geq \frac{K}{2}\,\mathbb{E}\big(X\mathbb{1}_{X>K}\big)\,,$$

whenever $K \geq 2C$. Therefore,

$$\begin{aligned} c < \frac{\mathbb{E}\big(X\mathbb{1}_{X>t}\big)}{\mathbb{E}(X)} &\leq \frac{\mathbb{E}\big(X\mathbb{1}_{K\geq X>t}\big) + \mathbb{E}\big(X\mathbb{1}_{X>K}\big)}{\mathbb{E}\big(X\mathbb{1}_{t\geq X}\big)} \\ &\leq \frac{K\,\mathbb{P}\big(K \geq X > t\big)}{\mathbb{E}\big(X\mathbb{1}_{t\geq X}\big)} + \frac{2\left(\frac{CK}{A} + C\right)}{K} \leq \frac{K}{A} + \frac{4C}{K}\,, \end{aligned}$$

whenever $A \geq K$. The first inequality is due to $\mathbb{P}(\widehat{X} > t) > c$. By choosing K then A large enough (depending only on c and C), this gives a contradiction, proving (A).

Now, to prove (B), assume that it is not satisfied for some $b > 0$ and an arbitrarily large $K > 0$. Then

$$C\,\mathbb{E}(X) \geq \mathbb{E}(X^2) \geq \mathbb{E}\big(X^2\mathbb{1}_{X>K}\big) \geq K\,\mathbb{E}\big(X\mathbb{1}_{X>K}\big) \geq bK\,\mathbb{E}\big(X\mathbb{1}_{X>t}\big)\,.$$

For large enough K, this contradicts the bound $\mathbb{P}(\widehat{X} > t) > c > 0$, and we are done. □

Deriving (4.6). Recall that we want to show that $\mathbb{P}\big(\widehat{N\mathbb{1}_{\mathsf{Fine}}} > 1/r\big) > c_2 > 0$, uniformly in r and R. Because of the monotonicity in r, it is enough to prove this for some fixed $r = r_0$ (say, $r_0 = 2$). We obviously have

$$\frac{\mathbb{E}\big(N\mathbb{1}_{N>1/r_0}\mathbb{1}_{\mathsf{Fine}}\big)}{\mathbb{E}\big(N\mathbb{1}_{\mathsf{Fine}}\big)} \geq \frac{r_0^{-1}\,\mathbb{P}\big(N > 1/r_0, \mathsf{Fine}\big)}{\mathbb{E}N}.$$

We have already noted that Proposition 4.26 implies that $\mathbb{P}(\mathsf{Fine} \mid N > 1/r_0) > c > 0$, hence the numerator is at least $c\,r_0^{-1}\,\mathbb{P}(N > 1/r_0)$. For the denominator, in Lemma 4.30 we have proved that $\mathbb{E}N < C'\,\mathbb{P}(N > 1/r_0)$. Thus we get that the ratio is at least $c\,r_0^{-1}/C'$, and we are done. □

4.4 Reconnection from thinned configurations

The missing ingredient in the proof of Proposition 4.11 at the end of Subsection 4.1 is (4.7), namely:

Proposition 4.31 (Things fall apart)**.** *For some $g(r) \to \infty$ as $r \to \infty$, we have that*

$$\mathbb{P}_{\text{thin}}\big(T > g(r) \bigm| T > 1/r\,, \mathsf{Fine}\big) > c_3 > 0\,.$$

The main step in proving this proposition is:

Proposition 4.32 (The centre cannot hold)**.** *Consider dynamical percolation in B_n with an initial condition in which only the hexagons intersecting the x-axis are open. Then, for some function $g : \mathbb{N} \longrightarrow \mathbb{R}^+$ satisfying $g(r) \to \infty$ as $r \to \infty$, the probability that at some time between $1/(2n)$ and $g(2n)$ there exists an open path realizing $0 \leftrightarrow n$ is bounded away from one, uniformly in n.*

Proof of Proposition 4.31 assuming Proposition 4.32. Recall the dynamics $\mathbb{P}_{\mathrm{thin}}$ specified after Definition 4.10, and note that

$$\begin{aligned}\mathbb{P}_{\mathrm{thin}}\big(T \leq g(r) \bigm| T > 1/r \,, \mathsf{Fine}\big) &= \mathbb{P}_{\mathrm{thin}}\big(\, \exists\, t \in [1/r, g(r)] : 0 \overset{\omega_t}{\longleftrightarrow} r \bigm| T > 1/r \,, \mathsf{Fine}\big) \\ &\leq \mathbb{P}_{\mathrm{thin}}\big(\, \exists\, t \in [1/r, g(r)] : 0 \overset{\omega_t}{\longleftrightarrow} r/2 \bigm| T > 1/r \,, \mathsf{Fine}\big) \,.\end{aligned}$$

Under $\mathbb{P}_{\mathrm{thin}}\big(\cdot \mid \mathsf{Fine}\big)$, the starting configuration ω_0 is specified in Definition 4.9; inside $B_{r/2}$, this is a deterministic configuration with only the hexagons intersecting the x-axis being open. Since a point mass trivially satisfies the static FKG inequality, we can apply the dynamical FKG inequality Lemma 1.9 for $\mathbb{P}_{\mathrm{thin}}(\cdot \mid \mathsf{Fine})$ inside $B_{r/2}$. Namely, for any $s \in [0,1]$, consider the dynamical event

$$\mathcal{A}_s := \Big\{ [0,\infty) \overset{\omega}{\longrightarrow} \{0,1\}^{B_{r/2}} \text{ càglàd} : \mathbb{P}\big(T > 1/r \bigm| \omega, \mathsf{Fine}\big) \geq s \Big\}$$

in $B_{r/2}$. This event is decreasing, so that Lemma 1.9 tells us that it is negatively correlated with the increasing event $\{\exists\, t \in [1/r, g(r)] : 0 \overset{\omega_t}{\longleftrightarrow} r/2\}$, that is,

$$\begin{aligned}&\mathbb{P}_{\mathrm{thin}}\Big(\mathcal{A}_s \cap \big\{\exists\, t \in [1/r, g(r)] : 0 \overset{\omega_t}{\longleftrightarrow} r/2\big\} \Bigm| \mathsf{Fine}\Big) \\ \leq\;& \mathbb{P}_{\mathrm{thin}}\big(\mathcal{A}_s \bigm| \mathsf{Fine}\big)\, \mathbb{P}_{\mathrm{thin}}\big(\exists\, t \in [1/r, g(r)] : 0 \overset{\omega_t}{\longleftrightarrow} r/2 \bigm| \mathsf{Fine}\big) \,.\end{aligned}$$

Integrating over $s \in [0,1]$ gives

$$\begin{aligned}&\mathbb{P}_{\mathrm{thin}}\Big(\{T > 1/r\} \cap \big\{\exists\, t \in [1/r, g(r)] : 0 \overset{\omega_t}{\longleftrightarrow} r/2\big\} \Bigm| \mathsf{Fine}\Big) \\ \leq\;& \mathbb{P}_{\mathrm{thin}}\big(T > 1/r \bigm| \mathsf{Fine}\big)\, \mathbb{P}_{\mathrm{thin}}\big(\exists\, t \in [1/r, g(r)] : 0 \overset{\omega_t}{\longleftrightarrow} r/2 \bigm| \mathsf{Fine}\big) \,.\end{aligned}$$

Summarising,

$$\mathbb{P}\big(T \leq g(r) \bigm| T > 1/r \,, \mathsf{Fine}\big) \leq \mathbb{P}_{\mathrm{thin}}\big(\, \exists\, t \in [1/r, g(r)] : 0 \overset{\omega_t}{\longleftrightarrow} r/2 \bigm| \mathsf{Fine}\big) \,.$$

By Proposition 4.32, the right hand side is bounded away from one, uniformly in r. This completes the proof. □

Proof of Proposition 4.32. Let H_n denote the set of hexagons in B_n intersecting the x-axis. The elements of H_n will be labelled $\big\{h_i : i \in \{-n, \ldots, n\}\big\}$ by the x-coordinate of the triangular lattice point at the centre of the hexagon. We let $B_n \setminus H_n = U_n \cup L_n$ decompose $B_n \setminus H_n$ into its two components above and below the x-axis. The domain U_n

has the shape of a half-hexagon, whose inner boundary naturally decomposes into four paths of hexagons, each along a straight line segment: $H_n \cup \ell_n^1 \cup \ell_n^2 \cup \ell_n^3$, where ℓ_n^2 denotes the horizontal path of hexagons on the top side of U_n (so that the "corner" hexagons containing the points given in complex coordinates by $ne^{i\pi/6}$ and $ne^{i\pi/3}$ belong to ℓ_2^n).

We will denote by $\mathbb{P}_{H_n}$ the dynamical percolation process of the proposition, under which only elements of H_n are open at time 0.

Let $\mathcal{C}^{U_n}$ denote the event that there is a closed path in U_n from ℓ_n^1 to ℓ_n^3. For each $i \in \{-n/2, \ldots, n/2\}$, let $\mathcal{S}_{h_i}^{U_n}$ denote the event that there is a closed path γ in U_n from a hexagon bordering h_i to ℓ_n^2. The events $\mathcal{C}^{U_n}$ and $\mathcal{S}_{h_i}^{U_n}$ have counterparts $\mathcal{C}^{L_n}$ and $\mathcal{S}_{h_i}^{L_n}$ defined verbatim after reflection in the x-axis. Finally, define

$$\mathcal{T}_+^n := \{\exists\, i \in \{0, \ldots, n/2\} : h_i \text{ is closed},\ \mathcal{S}_{h_i}^{U_n},\ \mathcal{S}_{h_i}^{L_n}\}$$

and

$$\mathcal{T}_-^n := \{\exists\, i \in \{-n/2, \ldots, 0\} : h_i \text{ is closed},\ \mathcal{S}_{h_i}^{U_n},\ \mathcal{S}_{h_i}^{L_n}\}.$$

Figure 4.4 illustrates that, for any $t \in (0, \infty)$,

$$\{\omega_t \in \mathcal{C}^{U_n} \cap \mathcal{C}^{L_n} \cap \mathcal{T}_+^n \cap \mathcal{T}_-^n\} \subseteq \{0 \not\leftrightarrow n \text{ in } \omega_t\}. \tag{4.23}$$

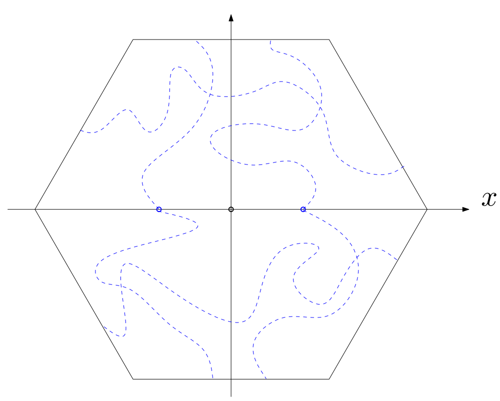

Figure 4.4: The events $\mathcal{C}^{U_n}$, $\mathcal{C}^{L_n}$, $\mathcal{T}_+^n$, $\mathcal{T}_-^n$.

Given (4.23), Proposition 4.32 will easily follow from the next two lemmas.

Lemma 4.33. *For each $t > 0$,*

$$\mathbb{P}_{H_n}\Big(\bigcap_{0<s<t} \{\omega_s \in \mathcal{C}^{U_n} \cap \mathcal{C}^{L_n}\}\Big) \to 1$$

as $n \to \infty$.

Proof. Initially the set of open hexagons in U_n is empty; thus, at time s, it has the law of a Bernoulli percolation $\mathbb{P}_{\frac{1}{2}(1-e^{-s})}$. For $0 < s < t < \infty$, let $\omega_{s,t}$ denote the configuration in which a hexagon is open if the hexagon is open under $\mathbb{P}_{H_n}$ at some time during $[s,t]$. Note then that the marginal law of $\omega_{s,t}$ in U_n is a percolation whose parameter is at most $\frac{1}{2}(1-e^{-s})+\frac{1}{2}(1+e^{-s})(1-e^{-(t-s)})$. For any given $s>0$, the percolation parameter of $\omega_{s,s+e^{-s}/2}$ is subcritical. By a standard subcritical percolation estimate, then, for each $s>0$, $\mathbb{P}_{H_n}\big(\bigcap_{s<t<s+e^{-s}/2}\{\omega_t \in \mathcal{C}^{U_n}\}\big) \to 1$. By a union bound over at most $2se^s$ sets, we see that $\mathbb{P}_{H_n}\big(\bigcap_{0<t<s}\{\omega_t \in \mathcal{C}^{U_n}\}\big) \to 1$. The statement of the lemma follows by symmetry in the x-axis. □

Lemma 4.34. *There exists $c>0$ such that, for all $C>0$ and for all n sufficiently high,*

$$\mathbb{P}_{H_n}\Big(\bigcap_{1/n\leq t\leq C}\{\omega_t \in \mathcal{T}_+^n\}\Big) \geq c\,.$$

Proof. We will argue that, for some $c>0$, and for all n,

$$\mathbb{P}_{H_n}\Big(\bigcap_{1/n\leq t\leq c}\{\omega_t \in \mathcal{T}_+^n\}\Big) \geq c\,, \tag{4.24}$$

and also that, for any $s\in(0,\infty)$,

$$\lim_{n\to\infty}\mathbb{P}_{H_n}\Big(\bigcap_{s\leq t\leq s+e^{-s}/2}\{\omega_t \in \mathcal{T}_+^n\}\Big) = 1\,. \tag{4.25}$$

Note that (4.24) and (4.25) prove the lemma.

Note that each of the percolations ω_t for $s\leq t\leq s+e^{-s}/2$ is stochastically dominated in $U_n\cup L_n$ by $\omega_{s,s+e^{-s}/2}$ which, as we just noted, is a subcritical percolation, of parameter $p_s<1/2$.

Let Q_n denote the set of hexagons in $\mathcal{H}$ that lie in the upper-half plane and that intersect the rectangle with vertices $-n^{1/4}e_1$, $n^{1/4}e_1$, $-n^{1/4}e_1+\frac{n}{2}e_2$ and $n^{1/4}e_1+\frac{n}{2}e_2$. Let $\mathcal{R}_n^+$ denote the event that there exists a closed path in Q_n from a hexagon on the top side of Q_n to one that borders h_0. By [Gri99, Theorem 11.55], for any $p<1/2$, $\liminf_n \mathbb{P}_p(\mathcal{R}_n^+)>0$.

Let $\mathcal{R}_n^-$ denote the event $\mathcal{R}_n^+$ defined after reflection in the y-axis, and let $\mathcal{R}_n = \mathcal{R}_n^+\cap\mathcal{R}_n^-\cap\{h_0 \text{ is closed}\}$. Clearly, $c=\liminf_n \mathbb{P}_p(\mathcal{R}_n)>0$. By partitioning $\{0,\ldots,n/2\}$ into order $n^{1/4}$ disjoint intervals and considering the analogue of $\mathcal{R}_n$ for each one, we see that

$$\mathbb{P}(\exists\, i\in\{0,\ldots,n/2\} : h_i \text{ is closed, } \mathcal{S}_{h_i}^{U_n}\cap\mathcal{S}_{h_i}^{L_n} \text{ under } \omega_{s,s+e^{-s}/2}) \geq 1-\big(1-c_s\big)^{n^{1/4}}\,,$$

where for each $s>0$, $c_s>0$. Hence, we obtain (4.25).

It is a simple matter to verify (4.24). With a probability that is bounded away from zero uniformly in n, some hexagon h_i, $0\leq i\leq n/2$, closes during $[0,1/n]$, and remains closed until at least time one. For some $c>0$, the marginal of $\omega_{0,c}$ in $U_n\cup L_n$ is a subcritical percolation. Thus, $\mathcal{S}_{h_i}^{U_n}\cap\mathcal{S}_{h_i}^{L_n}$ occurs with positive probability under all ω_s for $0\leq s\leq c$. This verifies (4.24) and completes the proof of Lemma 4.34. □

Proof of Proposition 4.32, continued. Note that Lemma 4.34 has a verbatim counterpart for the event $\mathcal{T}_-^n$. Combining these two lemmas with the aid of the dynamical FKG Lemma 1.9 for the process $\mathbb{P}_{H_n}$, and using Lemma 4.33, we find that, for any $C > 0$, the left-hand side of (4.23) is satisfied simultaneously for $1/n \le t \le C$ with probability tending to one as $n \to \infty$. Hence, (4.23) proves the result. $\square$

5 The collapse of the connection near the exceptional set

In this section, we address the question of how quickly the infinite cluster $\mathscr{C}_0$ in dynamical percolation disintegrates as time varies away from a typical exceptional time. In view of Theorems 1.7 and 1.8, we may rephrase the question as how rapidly this collapse occurs at small positive times in dynamical percolation where ω_0 is chosen to have the law IIC. In constructing approximative local times in Section 2, we mentioned that there are several natural measurements for how close a finite cluster $\mathscr{C}_0$ is to being infinite. We write $\textsc{size}(\mathscr{C}_0)$ as a label for any such notion, and consider three possibilities for it: the volume $|\mathscr{C}_0|$, the radius $\sup\{\|x\| : x \in \mathscr{C}_0\}$, or the "helpfulness" (in providing the event $0 \leftrightarrow \infty$) $\textsc{help}(\mathscr{C}_0) = M_{\mathscr{C}_0}(\omega)$ which was defined in (1.2). Using any of these notions of size, one may try to define a static percolation exponent $\sigma_{\textsc{size}}$ that measures the robustness of the infinite cluster $\mathscr{C}_0(\omega_0)$, a dynamical percolation exponent $\delta_{\textsc{size}}$ that measures how the size of $\mathscr{C}_0(\omega_t)$ degrades with time, and then may try to relate the two exponents, a relation that is expected to reflect the fact that the "speed" of the dynamical process is governed by the number of pivotals in critical percolation. We first give a rough heuristic description of such a general scaling relation; however, since the existence of classical critical exponents is known only for $\mathcal{H}$, our actual theorem will reformulate the relation in a way that does not use the existence of exponents, and is valid also for the case of $\mathbb{Z}^2$.

To understand the robustness of the initial infinite cluster $\mathscr{C}_0(\omega_0)$, we measure the size of its restrictions to finite balls. Thus, we define the **static percolation exponents** by

$$\sigma_{\textsc{size}} := \lim_{n\to\infty} \frac{\log \mathbb{E}_{\mathsf{IIC}}\big(\textsc{size}(\mathscr{C}_0 \cap B_n(0))\big)}{\log n}, \qquad \textsc{size} \in \{\textsc{vol}, \textsc{radius}, \textsc{help}\}. \tag{5.1}$$

From [LSW02] and [Kes87b] we know the existence and values of the classical critical exponents

$$\frac{1}{\rho} := \lim_{n\to\infty} \frac{-\log \mathbb{P}\big(\textsc{radius}(\mathscr{C}_0) > n\big)}{\log n} = \frac{5}{48}, \qquad \frac{1}{\delta} := \lim_{n\to\infty} \frac{-\log \mathbb{P}\big(|\mathscr{C}_0| > n\big)}{\log n} = \frac{1}{2\rho - 1} = \frac{5}{91},$$

which imply, with some work, that the exponents (5.1) can be given as

$$\sigma_{\textsc{vol}} = \frac{\delta}{\rho} = 2 - \frac{1}{\rho}, \qquad \sigma_{\textsc{radius}} = \frac{\rho}{\rho} = 1, \qquad \sigma_{\textsc{help}} = \frac{1}{\rho}. \tag{5.2}$$

The first one was established in [Kes86, Theorem (8)]. The second one is a triviality. For the third one, an upper bound on $\mathbb{E}_{\mathsf{IIC}}\big(\textsc{help}(\mathscr{C}_0 \cap B_n)\big)$ follows from (2.2), while a lower bound can be given by the following argument. Under IIC, the smallest open circuit $\Gamma_{n/2}$ that surrounds $B_{n/2}$ is contained in B_n with a uniform probability $c > 0$. When conditioning on ω^{B_n}, let us restrict ourselves to the part of the probability space where $\Gamma_{n/2} \subset B_n$, condition first on $\omega^{\mathsf{Int}(\Gamma_{n/2})}$, and then, for $R > n$, use the bound

$$\begin{aligned}\mathbb{E}_{\mathsf{IIC}}\Big(\mathbb{P}\big(0 \leftrightarrow R \bigm| \omega^{B_n}\big)\Big) &\geq \mathbb{E}_{\mathsf{IIC}}\Big(\mathbb{1}_{\{\Gamma_{n/2} \subset B_n\}}\, \mathbb{E}_{\mathsf{IIC}}\Big(\mathbb{P}\big(0 \leftrightarrow R \bigm| \omega^{B_n}\big) \Bigm| \omega^{\mathsf{Int}(\Gamma_{n/2})}\Big)\Big) \\ &\geq \mathbb{E}_{\mathsf{IIC}}\Big(\mathbb{1}_{\{\Gamma_{n/2} \subset B_n\}}\, \mathbb{P}\big(n/2 \leftrightarrow R\big)\Big) \\ &\geq c\, \mathbb{P}\big(n/2 \leftrightarrow R\big)\end{aligned}$$

to find that

$$\mathbb{E}_{\mathsf{IIC}}\Big(\lim_{R\to\infty} \frac{\mathbb{P}\big(0 \leftrightarrow R \bigm| \omega^{B_n}\big)}{\mathbb{P}(0 \leftrightarrow R)}\Big) \geq \limsup_{R\to\infty} \frac{c\,\mathbb{P}(n/2 \leftrightarrow R)}{\mathbb{P}(0 \leftrightarrow R)} \geq c'\, \mathbb{P}(0 \leftrightarrow n/2)^{-1}\,.$$

In the first inequality, we used quasi-multiplicativity to obtain the uniform boundedness $\mathbb{P}\big(0 \leftrightarrow R \bigm| \omega^{B_n}\big)/\mathbb{P}(0 \leftrightarrow R) \leq C/\mathbb{P}(0 \leftrightarrow n)$ and are thus able to apply the dominated convergence theorem; the second inequality likewise uses quasi-multiplicativity. This concludes the argument for the third equality of (5.2) above.

For the dynamical scaling relation, we will also need the static exponent for the number of pivotals for left-right and annulus crossings:

$$\tau := \lim_{n\to\infty} \frac{\log \mathbb{E}_{p_c}\big|\mathsf{Piv}_{\mathcal{A}(n)}\big|}{\log n} = \lim_{n\to\infty} \frac{\log \mathbb{E}_{p_c}\big|\mathsf{Piv}_{\mathcal{A}(n,2n)}\big|}{\log n} = \frac{3}{4}\,,$$

following from [SmW01], as already mentioned in Subsection 1.4.

Now, we define the **dynamic percolation exponents** by

$$\delta_{\text{SIZE}} = \inf\Big\{y \geq 0 : \liminf_{t\downarrow 0} t^y\, \text{SIZE}(\mathscr{C}_0(\omega_t)) = 0\Big\}\,,$$

starting the process from ω_0 having the law of IIC. Note that this is a reasonable notion of measuring the collapse of the IIC near ω_0: time 0 is a limit point of exceptional times, hence $\text{SIZE}(\mathscr{C}_0(\omega_t))$ is infinite along some sequence $t_n \downarrow 0$, but at typical times the cluster is finite and should indeed get smaller with time, according to the following mechanism.

As we will see, for short times $t > 0$, a fragment of the original infinite cluster $\mathscr{C}_0(\omega_0)$ survives at all times $s \in [0,t]$, with the radius of this fragment determined by the maximal scale on which a pivotal hexagon rings during $[0,t]$. As such, we expect that, for any of the above three notions of size,

$$\tau\, \delta_{\text{SIZE}} = \sigma_{\text{SIZE}}\,. \tag{5.3}$$

In the interests of concision, we will prove this relation only when $\text{SIZE} = \text{RADIUS}$. The next theorem reformulates the relation in this case, in a way that is valid even for the case of $\mathbb{Z}^2$. The rest of the section is devoted to the theorem's proof.

Theorem 5.1. *Consider dynamical percolation $\mathbb{P}_{\mathsf{IIC}}$ with ω_0 having the distribution IIC. For $t > 0$, set $\chi_t = \inf_{s \in [0,t]} \text{RADIUS}\big(\mathscr{C}_0(\omega_s)\big)$. We then have $\log \chi_t \sim \log \rho(1/t)$ $\mathbb{P}$-a.s. as $t \searrow 0$, with $\rho(\cdot)$ introduced in (1.12). In particular, on $\mathcal{H}$, we have $\chi_t = t^{-4/3+o(1)}$.*

Proof. We start by showing the upper bound on the radius, i.e., by proving that the cluster of the origin falls apart fast enough. The following lemma will be a key step.

Lemma 5.2. *There exists $c > 0$ such that the following holds. Let $t \in (0,1)$ and $r \geq \rho(1/t)$. Let ζ denote a configuration in the annulus $A_{r,2r}$. Let $N_r^t(\zeta)$ denote the event that the conditional probability of the inner and outer boundaries of $A_{r,2r}$ not being connected by an open path at time t, given that ω_0 in $A_{r,2r}$ equals ζ, exceeds $c > 0$. Then $\mathsf{IIC}\big(\{\zeta : N_r^t(\zeta)\}\big) \geq c$. Moreover, the same conclusion holds for the measure $\mathsf{IIC}(\cdot \,|\, O$ is open$)$, where O is any given circuit in $A_{r/4,r/2}$ surrounding $B_{r/4}$, and where $c > 0$ may be chosen independently of O.*

Proof. Let $\mathcal{C}$ denote the event that $r \longleftrightarrow 2r$. That

$$\mathbb{P}(\mathcal{C}(\omega_0) \cap \mathcal{C}(\omega_t)^c) \geq c\,, \tag{5.4}$$

where $c > 0$ is uniform in $r \in \mathbb{N}$ and $t \geq 1/(r^2\alpha_4(r))$, is a standard and simple consequence of the discrete Fourier analysis approach to critical percolation, already stated as (1.15).

Note that (5.4) implies the statement of the lemma when ω_0 has the law of critical percolation conditioned to have the crossing. To obtain the same statement when ω_0 has the law IIC, we can apply Lemma 4.20. For the case of $\mathsf{IIC}(\cdot \,|\, O$ is open$)$, we can apply a direct analogue of Lemma 4.20, using $\mathbb{P}_O^\infty = \mathbb{P}(\cdot \mid O \leftrightarrow \infty)$ in place of $\mathbb{P}_a^b$. □

We want to argue that, for any $\epsilon > 0$, we have $\mathbb{P}_{\mathsf{IIC}}$-a.s., for all small enough $t > 0$, that

$$\chi_t \leq \rho(1/t)\, t^{-\epsilon}\,. \tag{5.5}$$

We define an iterative procedure in an effort to prove (5.5). Let $\ell_1 \in \mathbb{N}$ be minimal such that $2^{\ell_1} \geq \rho(1/t)$. Write $\omega_0^{(\ell_1)}$ for ω_0 restricted to $A_1 := A_{2^{\ell_1},2^{\ell_1+1}}$. If $N_{2^{\ell_1}}^t(\omega^{(\ell_1)})$ occurs, and if no open path connects the inner and outer boundaries of A_1 at time t, then the procedure terminates. If one or other of these conditions is unsatisfied, let ℓ_1^* be the minimal $\ell \geq \ell_1+1$ such that $A_{2^\ell,2^{\ell+1}}$ contains an open circuit which encloses B_{2^ℓ}. Set $\ell_2 = \ell_1^* + 2$. Write $A_2 = A_{2^{\ell_2},2^{\ell_2+1}}$ and denote by $\omega_0^{(\ell_2)}$ the configuration ω_0 restricted to A_2. If $N_{2^{\ell_2}}^t(\omega^{(\ell_2)})$ occurs, and if no open path connects the inner and outer boundaries of A_2 at time t, then the procedure terminates. Otherwise, it continues to its next step. The generic step has a similar description to the second one.

Lemma 5.3. *Let $J \geq 1$ denote the index of the step at which the procedure terminates. Then there exists $c > 0$ such that, for each $k \in \mathbb{N}$, $\mathbb{P}\big(\ell_J - \ell_1 \geq k\big) \leq \exp\big\{-ck\big\}$.*

Proof. Note that, by Lemma 5.2, there exists $c > 0$ such that $J = 1$ with $\mathbb{P}_{\mathsf{IIC}}$-probability at least c^2. Under the law $\mathbb{P}_{\mathsf{IIC}}$ given the event that either $N^t_{2^{\ell_1}}(\omega^{(\ell_1)})$ does not occur, or $N^t_{2^{\ell_1}}(\omega^{(\ell_1)})$ occurs and $2^{\ell_1} \stackrel{\omega_t}{\longleftrightarrow} 2^{\ell_1+1}$, note that the conditional distribution of ω_0 in $B^c_{2^{\ell_1+1}}$ stochastically dominates critical percolation. (This statement is true because it is valid for $\mathbb{P}_{\mathsf{IIC}}$ conditionally on an arbitrary configuration in $B_{2^{\ell_1+1}}$ that satisfies $0 \leftrightarrow 2^{\ell_1+1}$ at time zero.) By RSW, FKG and independence on disjoint sets, each dyadic annulus with index at least $\ell_1 + 1$ independently has probability at least $c > 0$ to contain an open circuit disconnecting its boundaries. Thus, conditionally on the value of ℓ_1, the random variable $\ell^*_1 - \ell_1$ is stochastically dominated by a geometric random variable (which we call X_1). Let O_1 denote the innermost of the surrounding open circuits located in $A_{2^{\ell_1},2^{\ell_1+1}}$. Conditionally on ω_0 taking a given form on $O \cup \mathsf{Int}(O)$, the conditional distribution of ω_0 in the exterior of O is given by IIC given that O is open. Thus, we may apply the $\mathsf{IIC}(\,\cdot\,|\,O \text{ is open})$ case of Lemma 5.2 to learn that there is probability at least c that $N_{\ell_2}(\omega^{(\ell_2)})$ occurs. Should this event not occur, or should this event occur alongside the event $2^{\ell_2} \stackrel{\omega_t}{\longleftrightarrow} 2^{\ell_2+1}$, then, as previously, the conditional distribution of $\ell^*_2 - \ell_2$ is stochastically dominated by a geometric random variable, which we call X_2.

In this way, we see that $\ell_J - \ell_1$ is stochastically dominated by $\sum_{i=1}^{G_1-1} X_i + 2(G_1 - 1)$, where G_1 is a geometric random variable and $\{X_i : i \in \mathbb{N}\}$ is an independent sequence of i.i.d. geometric random variables. This completes the proof of Lemma 5.3. □

Note that the inner and outer boundaries of $A_{2^{\ell_J},2^{\ell_J+1}}$ are disconnected at time t. Therefore, by Lemma 5.3, $\chi_t \le \rho(1/t)\, t^{-\epsilon}$ has probability at least $1 - c^{\log_2(t^{-\epsilon})}$. We have thus verified (5.5).

To complete the proof of Theorem 5.1, it remains to argue that, $\mathbb{P}_{\mathsf{IIC}}$-a.s.,

$$\chi_t \ge \rho(1/t)\, t^{\epsilon} \tag{5.6}$$

for all small enough $t > 0$. To prove this, we need the following lemma.

Lemma 5.4. *Let $R \in \mathbb{N}$. For each $\epsilon > 0$, there exists $\delta > 0$ such that if $\mathcal{A} \in \sigma\{B_R\}$ satisfies $\mathsf{IIC}(\mathcal{A}) \ge \epsilon$, then $\mathsf{IIC}_R(\mathcal{A}) \ge \delta$.*

Proof. Recalling the definitions made in (1.1, 1.2), the Bayes' rule computation (1.3), and the quasi-multiplicativity bound (2.2), we have that, for each configuration ζ in B_R realizing $0 \longleftrightarrow R$,

$$\frac{\mathrm{d}\mathsf{IIC}}{\mathrm{d}\mathsf{IIC}_R}(\zeta) = \frac{M_R(\zeta)}{\overline{M}_R(\zeta)} \le C_1 \,,$$

with an absolute constant $C_1 < \infty$. This readily implies the claim. □

Starting dynamical percolation from IIC_R, and using the coupling in which bits always turn off, Kesten's near-critical one-arm stability (1.11) shows that the probability of still having the connection $0 \longleftrightarrow R$ at all times until $\big(R^{2(1-\epsilon)}\alpha_4(R^{1-\epsilon})\big)^{-1}$ is $1 - o(1)$, as

$R \to \infty$. By Lemma 5.4, the same statement holds when the initial condition is IIC-distributed. From this, (5.6) follows readily. This completes the proof of Theorem 5.1. □

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