INTERLACING AND SCALING EXPONENTS FOR THE GEODESIC WATERMELON IN LAST PASSAGE PERCOLATION

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ABSTRACT. In a discrete planar last passage percolation (LPP), random values are assigned independently to each vertex in $\mathbb{Z}^2$, and each finite upright path in $\mathbb{Z}^2$ is ascribed the weight given by the sum of values attached to the vertices of the path. The weight of a collection of disjoint paths is the sum of its members' weights. The notion of a geodesic, namely a path of maximum weight between two vertices, has a natural generalization concerning several disjoint paths. Indeed, a k-geodesic watermelon in $[1, n]^2 \cap \mathbb{Z}^2$ is a collection of k disjoint upright paths contained in this square that has maximum weight among all such collections. While the weights of such collections are known to be important objects, the maximizing paths have remained largely unexplored beyond the $k = 1$ case. For exactly solvable models, such as exponential and geometric LPP, it is well known that for $k = 1$ the exponents that govern fluctuation in weight and transversal distance are $1/3$ and $2/3$; which is to say, the weight of the geodesic on the route $(1, 1) \to (n, n)$ typically fluctuates around a dominant linear growth of the form μn by the order of $n^{1/3}$; and the maximum Euclidean distance of the geodesic from the diagonal typically has order $n^{2/3}$. Assuming a strong but local form of convexity and one-point moderate deviation estimates for the geodesic weight profile—which are available in all known exactly solvable models—we establish that, typically, the k-geodesic watermelon's weight falls below $\mu n k$ by order $k^{5/3} n^{1/3}$, and its transversal fluctuation is of order $k^{1/3} n^{2/3}$. Our arguments crucially rely on, and develop, a remarkable deterministic interlacing property that the watermelons admit. Our methods also yield sharp rigidity estimates for naturally associated point processes. These bounds improve on estimates obtained by applying tools from the theory of determinantal point processes available in the integrable setting.
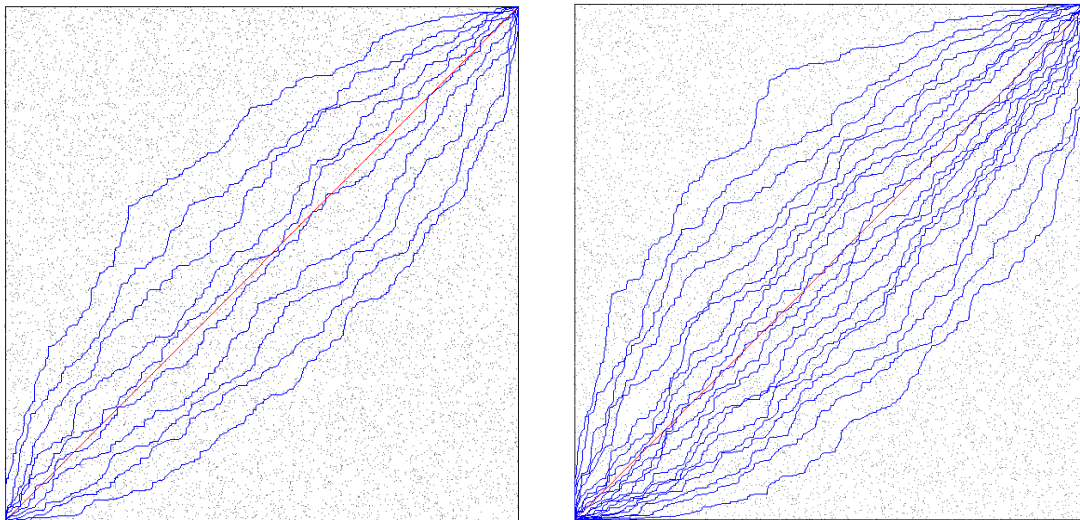
FIGURE 1. Simulations of k-geodesic watermelons in Poissonian LPP; k equals ten on the left and twenty on the right, with $n = 200$ in both. Grey are the points of the underlying Poisson process of intensity one on $[0, n]^2$; blue are the watermelon's curves; and red is the diagonal $y = x$ around which the watermelon fluctuates on the scale $k^{1/3} n^{2/3}$.

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1. INTRODUCTION AND MAIN RESULTS

In a discrete planar last passage percolation (LPP) model, each vertex in $\mathbb{Z}^2$ is independently ascribed a non-negative value sampled from a given law ν. An upright path in $\mathbb{Z}^2$ is a finite nearest-neighbour path on $\mathbb{Z}^2$ each of whose steps is upwards or to the right. To each such path a weight is ascribed by the LPP model, this being the sum of the values assigned to the vertices in $\mathbb{Z}^2$ visited by the path. For $n \in \mathbb{N}$, let X_n denote the maximum weight assigned to an upright path that begins at the vertex $(1, 1)$ and that ends at the vertex (n, n). Any upright path with these endpoints whose weight equals X_n is called a geodesic between $(1, 1)$ and (n, n), and X_n is called the last passage time from $(1, 1)$ to (n, n); these two concepts make sense for any pair of coordinate-wise ordered vertices. Let the transversal fluctuation $\mathrm{TF}(n)$ denote the maximum anti-diagonal fluctuation of any geodesic between $(1, 1)$ and (n, n). That is, $\mathrm{TF}(n)$ equals the minimum of positive real r such that every vertex of any geodesic between $(1, 1)$ and (n, n) has Euclidean distance from the diagonal $y = x$ at most r. Planar LPP models are paradigmatic examples of models predicted to exhibit features of the Kardar-Parisi-Zhang (KPZ) universality class; in particular, the characteristic KPZ exponents of one-third and two-thirds for the weight and transversal fluctuations of geodesics.

A few planar LPP models—for which the law ν has a special form—enjoy *integrable* properties that vastly facilitate their analysis. Two such models are geometric LPP with parameter $p \in (0, 1)$, where $\nu(k) = p^{k-1}(1 - p)$ for $k \in \mathbb{N}$; and exponential LPP, where ν is the exponential law. (By scaling properties of the exponential distribution, the rate of the exponential distribution is irrelevant, and we will consider exponentials of rate one.) For any non-negative ν, the existence of the limiting growth rate μ for the weight maximum, specified as the almost sure limiting value of $n^{-1}X_n$ as $n \to \infty$, is a consequence of a straightforward superadditivity argument; the positivity of μ holds as soon as ν is not degenerate at 0, while the finiteness is guaranteed under mild moment assumptions. For geometric and exponential LPP, μ can be explicitly evaluated. The integrable structure of these and a handful of other models has been crucial in proofs establishing the pair of KPZ scaling exponents.

Incidentally, the first model for which such a program was carried out was Poissonian last passage percolation on the plane, where the statistic of interest L_n is the maximum number of points on an upright path from $(0,0)$ to (n,n) in a rate one Poisson process on $\mathbb{R}^2$; conditionally on the total number of points in $[0,n]^2$, this is Ulam's problem for the longest increasing subsequence in a uniform random permutation. In their seminal work [BDJ99], Baik, Deift, and Johansson showed that $n^{-1/3}(L_n - 2n)$ converges weakly to the GUE Tracy-Widom distribution, namely the high-n limiting law of a scaled version of the largest eigenvalue of an $n \times n$ random matrix picked according to the Gaussian unitary ensemble. Shortly thereafter, Johansson [Joh00b] used this longitudinal fluctuation result to establish the transversal fluctuation exponent of two-thirds in this model, showing that with high probability the smallest strip around the diagonal containing any geodesic has width $n^{2/3+o(1)}$. Similar transversal fluctuation results have been proved for last passage percolation on $\mathbb{Z}^2$ with exponential and geometric passage times [BCS06]; and for the semi-discrete model of Brownian last passage percolation [Ham17].

The Baik-Deift-Johansson theorem was derived by noting that the random statistic L_n has the distribution of the length of the top row of a Young tableau picked according to the Poissonized Plancherel measure of appropriate parameters. The latter observation was obtained via the Robinson-Schensted-Knuth correspondence in [Sch09, Knu70] and was first exploited in [LS77, VK77]. The correspondence extends to other rows of the tableau. Indeed, Greene [Gre82] established that the sum of the lengths of the first k rows of the random Young tableau picked from the same measure has the distribution of the maximum number of Poisson points on k upright paths from $(0,0)$ to (n,n) that are disjoint except at these shared endpoints. Baik, Deift and Johansson conjectured that the scaled lengths of the top k rows converge jointly in distribution to the top k points of a determinantal point process on $\mathbb{R}$ called the Airy point process. The conjecture was proved soon after, independently, by Borodin, Okounkov and Olshanski [BOO00]; Johansson [Joh01]; and Okounkov [Oko00]. By Greene's theorem, then, the highest scaled weight of a set of k disjoint upright paths in Poissonian LPP converges in law to the sum of the top k points of the Airy point process. Other integrable LPP models exhibit a similar correspondence; we discuss it in the case of exponential LPP in Section 10.

We have recounted these fragments of KPZ history to advocate the conceptual importance of systems of k upright disjoint paths with given endpoints that maximize collective weight. We call them *k-geodesic watermelons* and devote this article to a unified geometric treatment of them.

The parameter $k \in \mathbb{N}$ will be positive, with $k \leq n$. Define X_n^k to be the maximum weight of any collection of k disjoint upright paths contained inside the square with opposite corners $(1,1)$ and (n,n); note that the collection attaining the maximum need not be unique, and that $X_n^1 = X_n$. The transversal fluctuation of any such collection of paths is the maximum Euclidean distance between a vertex in $\mathbb{Z}^2$ lying in the paths' union and the diagonal $y = x$. We specify $\overline{\mathrm{TF}}(n,k)$ to equal the maximum value of the transversal fluctuation over such sets of k paths whose weight realises X_n^k, and $\underline{\mathrm{TF}}(n,k)$ to be the minimum value of the transversal fluctuation over the same collection.

Our main result establishes the values of the exponents that govern weight and transversal fluctuations of k-geodesic watermelons in the context of geometric and exponential LPP. We note a simple heuristic to predict the exponents by considering the case that $k = n$, where the k-geodesic watermelon uses all the vertices in $[\![1, n]\!]^2$: then by the strong law of large numbers, X_n^k will be of order $\mathbb{E}[\xi]n^2$, with ξ distributed according to ν, and $\underline{\mathrm{TF}}(n,k) = \overline{\mathrm{TF}}(n,k) = \Theta(n)$. It is reasonable to believe that X_n^k to first-order should be linear in nk, just as X_n is to first-order linear in n. Then if we assume that the fluctuations are governed by exponents α and β as $X_n^k = \mu nk - \Theta(k^\alpha n^{1/3})$ and $\underline{\mathrm{TF}}(n,k) = \overline{\mathrm{TF}}(n,k) = \Theta(k^\beta n^{2/3})$, we obtain, from the $k = n$ case, the prediction that $\alpha = 5/3$ and $\beta = 1/3$. This is what our main result establishes with high probability for all k up to a small constant times n.

Theorem I. *Consider geometric LPP of given parameter $p \in (0, 1)$, or exponential LPP.*

(1) Weight fluctuation: *There exist positive constants C_1, C_2, C, c_1 and c such that*

$$\mathbb{P}\left(X_n^k - \mu n k \notin -k^{5/3} n^{1/3} \cdot \left(C_1, C_2 \right) \right) \leq C e^{-ck^2}$$

for $1 \leq k \leq c_1 n$, where we denote $a \cdot I = \left\{ ax : x \in I \right\}$ for $a \in \mathbb{R}$ and $I \subset \mathbb{R}$.

(2) Transversal fluctuation:

(a) *There exist positive constants M, C, c and c_1 such that, for $1 \leq k \leq c_1 n$,*

$$\mathbb{P}\left(\overline{\mathrm{TF}}(n, k) > M k^{1/3} n^{2/3} \right) \leq C e^{-ck}.$$

(b) *A matching lower bound holds: there exist positive constants C, c, c_1 and δ such that, for $1 \leq k \leq c_1 n$,*

$$\mathbb{P}\left(\underline{\mathrm{TF}}(n, k) < \delta k^{1/3} n^{2/3} \right) \leq C e^{-ck^2}.$$

Since the fundamental advances [BDJ99, Joh00b], much analysis of integrable LPP models has been based on exact formulas for the point-to-point last passage time and for finite-dimensional distributions of the passage time profile from a point to certain special lines [BF08, BFS08]. These integrable techniques extend to the continuous scaling limits, such as the KPZ fixed point [MQR16]. More recently, probabilistic and geometric technique in alliance with integrable input has been brought to bear on LPP problems. The Brownian Gibbs property is a resampling invariance enjoyed by the random ensemble of curves associated to Brownian LPP by the RSK correspondence. It has been used in [Ham17] to analyse the weight of the k-geodesic watermelon on the route from $(0, 0)$ to (n, n) in Brownian LPP, and in [CHH19] to gain strong control on LPP weight profiles and the Airy$_2$ process. The resampling property is also a central tool in the recent advance of [DOV18], which, with the aid of [DV18], constructs the full scaling limit of Brownian LPP. Similar Gibbs properties in other models have been explored in works such as [CD18, CG18, Agg19, Wu19].

The present paper falls within the scope of a separate program of probabilistic and geometric inquiry into KPZ, focused on exponential and Poissonian LPP. By exploiting moderate deviation estimates from integrable probability and aspects of geodesic geometry, the slow bond problem was solved in the preprint [BSS14]. Developing this vein, [BSS19, HS20, BG18] and [Zha19] have offered information about coalescence structure of geodesics; their local fluctuations; and temporal correlation exponents. A combination of geometric and integrable methods, including the use of one-point estimates, has also been crucially used in [FO18, FO19] to establish universality of the GOE Tracy-Widom distribution in point-to-line LPP with general slope and time correlation exponents with generic initial conditions.

This paper pursues the preceding geometric and probabilistic program while adopting a novel geometric perspective: k-geodesic watermelons interlace, each with the next as the parameter $k \in \mathbb{N}$ rises; as we will explain in Section 2, this property is a tool that governs our ideas and the proofs of our results. The technique is robust. Although our main theorem addresses geometric and exponential LPP, its derivation makes very limited use of integrable inputs, holding sway under weak assumptions. Indeed, Theorem I follows directly from a more general result, Theorem II, valid under a rather natural set of assumptions that all known integrable models satisfy, which we state next.

1.1. **Assumptions:** Here we state a set of assumptions and our main result in its general form Theorem II. This form generalizes Theorem I because its hypotheses are the concerned assumptions, which we will show in Appendix A to be satisfied by exponential and geometric LPP.

We recall first that ν is the distribution of the vertex weights and has support contained in $[0, \infty)$. Consider next the limit shape defined by the map $[-1, 1] \to \mathbb{R} : x \mapsto \lim_{r \to \infty} r^{-1} \mathbb{E}[X_{r(1-x), r(1+x)}]$,

where $X_{r(1-x),r(1+x)}$ is the last passage value from $(1,1)$ to $(r(1-x),r(1+x))$. (Note then that $X_{r,r}$ coincides the existing usage X_r; we will write the latter in this special case.) Standard superadditivity arguments yield that the last limit in fact exists and that this map is concave. Recall also that $\mu = \lim_{r\to\infty} r^{-1}\mathbb{E}[X_{r,r}]$ is this map evaluated at zero. Superadditivity also yields that $r^{-1}\mathbb{E}[X_{r(1-x),r(1+x)}]$ for finite r is at most the value of the limiting map at x. An important problem for general LPP models is to bound the non-random fluctuation given by the difference of these two quantities; for example, $\mu r - \mathbb{E}[X_r]$ when $x = 0$, expected to be of fluctuation order $\sqrt{\mathrm{Var}(X_r)} = \Theta(r^{1/3})$. With this context, we state our assumptions.

(1) **No atom at zero and limit shape existence:** The distribution ν is such that $\nu(\{0\}) = 0$ and $\mu < \infty$.

(2) **Strong concavity of limit shape and non-random fluctuations:** There exist positive constants ρ, G_1, G_2, g_1, and g_2 such that, for large enough r and $x \in [-\rho, \rho]$,

$$\mathbb{E}[X_{r(1-x),r(1+x)}] - \mu r \in [-G_1 x^2 r - g_1 r^{1/3}, -G_2 x^2 r - g_2 r^{1/3}].$$

(3) **Moderate and large deviation estimates, uniform in direction:**

(a) Fix any $\delta > 0$, and let $|x| \in [0, 1 - \delta]$. Then, there exist positive finite constants $c = c(\delta)$, $\theta_0 = \theta_0(\delta)$, and $r_0 = r_0(\delta)$ such that, for $r > r_0$ and $\theta > \theta_0$,

$$\mathbb{P}\left(X_{r(1+x),r(1-x)} - \mathbb{E}[X_{r(1+x),r(1-x)}] > \theta r^{1/3}\right) \leq \exp(-c\min(\theta^{3/2}, \theta r^{1/3})) \quad \text{and}$$

$$\mathbb{P}\left(X_{r(1+x),r(1-x)} - \mathbb{E}[X_{r(1+x),r(1-x)}] < -\theta r^{1/3}\right) \leq \exp(-c\theta^{3/2}).$$

(b) There exist convex functions $I_r : [0,\infty) \to \mathbb{R}$ for $r \in \mathbb{N}$ and a constant c' such that $I_r(\theta) \geq c'\min(\theta^{3/2}, \theta r^{1/3})$ for all $\theta \geq 0$, such that

$$\mathbb{P}\left(X_r - \mathbb{E}[X_r] > \theta r^{1/3}\right) \leq \exp(-I_r(\theta)).$$

We will call these, naturally enough, Assumptions 1, 2, and 3. They are expected to hold for a wide class of distributions ν and, in particular, are known to hold for the geometric and exponential cases. It is worth pointing out that a significant portion of our main result, Theorem II ahead, holds without Assumption 3b.

Assumption 2 encodes a non-trivial random fluctuation about a locally strongly concave limit shape. The assumption indicates that, even in the diagonal case, X_r falls short in mean of the linear growth rate μr by the order of typical fluctuation. This phenomenon is associated to the negativity of the mean of the GUE Tracy-Widom distribution. In regard to this assumption, we will say the endpoint of a path starting at $u \in \mathbb{Z}^2$ satisfies the "ρ-condition" if it lies in the interval joining $u + ((1-\rho)r, (1+\rho)r)$ and $u + ((1+\rho)r, (1-\rho)r)$.

It can be shown quite easily that Assumption 3a implies that ν has finite exponential moment, which in particular implies Assumption 1. Assumption 3a is itself expected to hold in a stronger form in general, with the lower tail bound with exponent 3 in place of the weaker $3/2$ as we have assumed. Finally, Assumption 3b is a slightly stronger version of the upper tail bound of 3a when $x = 0$. The existence of the convex functions I_r follows from the superadditivity of the sequence $\{X_r\}_{r \in \mathbb{N}}$ and is not actually an assumption, but the lower bound on I_r is.

We are ready to state Theorem II. But first we remark that much recent progress in understanding the geometry of first passage percolation and other non-integrable models have been conditional results that hinged on similar assumptions on the limit shape and concentration estimates about the limit shape. This approach goes back to Newman's work in the 90s (see e.g. [New95]) where geodesics and fluctuations in FPP were studied under curvature assumptions on the limit shape. More prominent recent examples include the work [Cha13] of Chatterjee where the KPZ relation between the weight and transversal fluctuation exponents were proved assuming in a strong form the existence of these exponents; see also [AD14]. The geometry of geodesics and bi-geodesics has

been addressed under assumptions of strong convexity of the limit shape [DH14] and moderate deviations around it [Ale20]. Similar results have also been obtained in exactly solvable cases where the essential integrable ingredients used were estimates analogous to the ones in the above assumptions. It is thus of much interest to extract the minimal set of assumptions under which one can establish sharp geometric results for LPP models.

Theorem II. *Consider a last passage percolation model on $\mathbb{Z}^2$ that satisfies Assumptions 1, 2 and 3a.*

(1) Weight fluctuation: *There exist positive constants C_1, C_2, C, c_1 and c such that, for $1 \le k \le c_1 n^{1/2}$,*
$$\mathbb{P}\left(X_n^k - \mu nk \notin -k^{5/3}n^{1/3} \cdot (C_1, C_2)\right) \le Ce^{-ck^2}.$$

(2) Transversal fluctuation:
  (a) *There exist positive constants M, C, c and c_1 such that, for $1 \le k \le c_1 n^{1/2}$,*
$$\mathbb{P}\left(\overline{\mathrm{TF}}(n,k) > Mk^{1/3}n^{2/3}\right) \le Ce^{-ck}.$$

  (b) *There exist positive constants C, c, c_1 and δ such that, for $1 \le k \le c_1 n^{1/2}$,*
$$\mathbb{P}\left(\underline{\mathrm{TF}}(n,k) < \delta k^{1/3}n^{2/3}\right) \le Ce^{-ck^2}.$$

If Assumption 3b also holds, there exists $c_1 > 0$ such that the above hold for $k \le c_1 n$.

Remark 1.1. An aspect of Theorem II(1), namely the bound $\mathbb{P}(X_n^k < \mu nk - C_1 k^{5/3}n^{1/3}) \le e^{-ck^2}$, holds in the broader range $k \le c_1 n$ without Assumption 3b: see Theorem 3.1.

A simple argument shows that the tail exponent for this probability is sharp; i.e.,
$$\mathbb{P}\left(X_n^k < \mu nk - C_1 k^{5/3}n^{1/3}\right) \ge e^{-ck^2}$$
for some constant $c > 0$. Indeed, the event that $X_n < \mu n - C_1 k^{2/3}n^{1/3}$ implies that $X_n^k < \mu nk - C_1 k(k^{2/3})n^{1/3}$ for any k, and n correspondingly large enough. The preceding display follows from
$$\mathbb{P}\left(X_n < \mu n - C_1 k^{2/3}n^{1/3}\right) = e^{-\Theta(k^2)},$$
a bound that is known, for example, in Exponential LPP: see [BGHK19, Theorem 2].

Remark 1.2. A weaker form of Assumption 3a, namely that for some $\alpha < 1$, all $\delta > 0$, $|x| \in [0, 1-\delta]$, and $\theta > \theta_0 = \theta_0(\delta)$,
$$\mathbb{P}\left(|X_{r(1+x),r(1-x)} - \mathbb{E}[X_{r(1+x),r(1-x)}]| > \theta r^{1/3}\right) \le \exp(-c\theta^\alpha),$$
for some $c = c(\delta) > 0$, is enough to imply a variant of Theorem II where the tail probability exponents are suitable functions of α; the 5/3 and 1/3 exponents for weight and transversal fluctuations are unchanged. It does not appear to us to be challenging to chase through our arguments to compute the forms of these upper bound tail exponents, although we do not do so.

1.1.1. *A non-determinantal setting: the point-to-line geodesic watermelon.* Though our main result Theorem II is stated for last passage percolation on $\mathbb{Z}^2$, our technique is robust, and an inviting prospect is to adapt our method to other integrable models such as the semi-discrete model of Brownian LPP or the continuum model of Poissonian LPP; the adaptations appear to be for the most part minor, but we have not pursued this direction carefully. However, all four examples—exponential and geometric LPP and these last two—are *determinantal* in the sense that there is an exact representation of the geodesic watermelon weight as the sum of the position of top k particles in a determinantal point process with explicit, albeit complex, formulae available for the distributions. We shall provide a more detailed discussion regarding the determinantal process

connections for exponential and geometric LPP in Section 10. But to illustrate the power of our geometric methods we end by treating a particular "non-determinantal" setting: where although there exist explicit formulae for the one point distribution (the geodesic weight), the weight of the geodesic watermelon is not known to admit any connection to a determinantal process for $k > 1$.

Formally, consider point-to-line LPP with independent vertex weights such that Assumptions 1, 2, and 3 are satisfied. Let Υ_n^k denote a collection of k disjoint paths contained in the triangular region

$$\left\{(x,y) \in \mathbb{Z}^2 : x \geq 1, y \geq 1, x + y \leq 2n\right\},$$

that maximizes the total weight among all such collection of k disjoint paths. Let Z_n^k denote the total weight of the paths in Υ_n^k. Clearly Z_n^1 is the point-to-line last passage time from $(1,1)$ to the line $x + y = 2n$. While it is known in exponential LPP that Z_n^1 has the same distribution as the largest eigenvalue of the Laguerre Orthogonal Ensemble (implicitly in the works [BR01b, Bai02, BR01a] with an explicit statement in [BGHK19]), as far as we are aware there does not exist any representation for Z_n^k for $k > 1$ as a functional of a determinantal point process.

As before we specify the transversal fluctuation of a collection of paths to be the maximum Euclidean distance between a vertex in $\mathbb{Z}^2$ lying in the paths' union and the diagonal $y = x$. We specify $\overline{\mathrm{TF}}^*(n,k)$ to equal the maximum value of the transversal fluctuation over such sets of k paths whose weight realizes Z_n^k, and $\underline{\mathrm{TF}}^*(n,k)$ to be the minimum value of the transversal fluctuation over the same collection of sets.

The following is the analogue of Theorem I in this setting.

Theorem III. *Consider a model, such as geometric or exponential LPP, that satisfies Assumptions 1, 2 and 3a. The statements in Theorem II that invoke Assumption 3a hold after the replacements $X_n^k \to Z_n^k$, $\overline{\mathrm{TF}}(n,k) \to \overline{\mathrm{TF}}^*(n,k)$ and $\underline{\mathrm{TF}}(n,k) \to \underline{\mathrm{TF}}^*(n,k)$ are made. In particular, the results for point-to-line geodesic watermelons hold for $k \leq c_1 n^{1/2}$ for an absolute constant $c_1 > 0$.*

The next two sections develop an overview of the paper's concepts and results. The vital phenomenon of interlacing of geodesic watermelons is surveyed in Section 2, with some of its main consequences being indicated. Section 3 offers an outline to the paper's main proofs. We state some important technical results needed to obtain Theorem II; outline how these results are proved; and explain how they are used, alongside interlacing, to prove Theorem II. Section 3 ends with an indication of the structure of the later sections of the paper, which are devoted to giving the proofs.

2. Watermelon interlacing

In this section, we will specify notation and important definitions concerning geodesic watermelons; state the important interlacing property that they enjoy; and state monotonicity and rigidity results for a natural point process associated to the weights of these watermelons. The results of this

section do not rely on the assumptions introduced in Section 1.1, and, excepting Proposition 2.5, are deterministic.

2.1. **The geodesic watermelon.** Let $\{\xi_v : v \in \mathbb{Z}^2\}$ denote a field of values. For now, we take these values to be deterministic non-negative reals. For $a, b \in \mathbb{Z}$ with $a \leq b$, we denote the integer interval $\{x \in \mathbb{Z} : a \leq x \leq b\}$ by $[\![a, b]\!]$. We let $\mathbb{N} = \{1, 2, \ldots\}$ denote the natural numbers (without zero).

Let two elements $u = (u_1, u_2)$ and $v = (v_1, v_2)$ of $\mathbb{Z}^2$ be such that $u_1 \leq v_1$ and $u_2 \leq v_2$. An upright path from u to v is a function $\gamma : [\![a, b]\!] \to \mathbb{Z}^2$, $\gamma_a = u$ and $\gamma_b = v$, with each increment of γ equalling either $(0, 1)$ or $(1, 0)$; thus, $a, b \in \mathbb{Z}$ satisfy $b - a = (u_2 - u_1) + (v_2 - v_1)$. (We will sometimes omit 'upright': every path is upright.) The weight $\ell(\gamma)$ of γ equals $\sum \{\xi_v : v \in \gamma\}$, where we have abused notation, in a way that we often will, by mistaking γ for its range. The weight of a disjoint collection of upright paths is the sum of the weights of the elements of the collection.

We will denote by $X_{u \to v}$ the maximum weight of all upright paths from u to v. Recall that, for $r \in \mathbb{N}$, X_r is a shorthand for $X_{r,r}$, itself shorthand for the last passage value $X_{(1,1) \to (r,r)}$ for the route from $(1, 1)$ to (r, r).

Let $k \in \mathbb{N}$ be positive. We may wish to specify the k-geodesic watermelon in the square $[\![1, n]\!]^2$ as a maximum weight collection of k disjoint upright paths from $(1, 1)$ to (n, n); but, naturally, we cannot, because such paths meet when they begin and end. The next definition succinctly deals with the need to unpick these points of contact.

Definition 2.1. Let $n \in \mathbb{N}$ and $k \in [\![1, n]\!]$. A k-geodesic watermelon is a maximum weight collection of k disjoint upright paths in $[\![1, n]\!]^2$.

The nomenclature of watermelons is not new. It was introduced in the physics literature to denote certain ensembles of non-intersecting curves, such as non-intersecting Brownian bridges, whose curves bear a faint likeness to the stripes on the surface of watermelon fruit. These bridge systems arise, for example, when describing the weight of collections of disjoint paths in LPP models as the common endpoint of the collection varies. The name 'geodesic watermelon' distinguishes the denoted concept from the existing one, which we might call a *weight* watermelon. We will not allude to the latter, beyond mentioning that [JO20] treats weight watermelons related to the geometric RSK correspondence. We will sometimes write 'k-melon' as shorthand for 'k-geodesic watermelon'.

The two quantities measuring transversal fluctuation of the k-geodesic watermelon, $\overline{\text{TF}}(n, k)$ and $\underline{\text{TF}}(n, k)$, were specified before Theorem I. They respectively equal the maximum and minimum, over the collection of k-geodesic watermelons, of the maximum Euclidean distance to the diagonal $y = x$ of the vertices lying in some path belonging to that k-geodesic watermelon.

The set of vertices in a k-geodesic watermelon may not be unique; it is not in geometric LPP, for example. Further, for a given set of vertices which is a k-geodesic watermelon, the collection of constituent curves is not unique. It is easy to see that a sufficient condition for the geodesic watermelons to be almost surely unique is for ν to have no atoms, as, for example, in the case of exponential LPP.

The next result will help in specifying the curves of a watermelon that we may label and study. The result serves to capture the sense that the watermelon has maximum weight subject to its disentangling coincidences near $(1, 1)$ and (n, n). The proof will be given in Section 9.

Proposition 2.2. *Suppose that $\xi_v \geq 0$ for $v \in [\![1, n]\!]^2$. For any given k-geodesic watermelon ϕ_n^k, there exists a k-geodesic watermelon γ_n^k such that the ith element of γ_n^k starts at $(1, k - i + 1)$ and ends at $(n, n - i + 1)$ for $i \in [\![1, k]\!]$, and the union of the vertices in the curves of γ_n^k contains this union for ϕ_n^k, and coincides with it if $\xi_v > 0$ for all $v \in [\![1, n]\!]^2$.*

Theorem II concerns the vertex sets in geodesic watermelons considered simultaneously, rather than any particular watermelon. For this reason, we do not attempt to single out a melon when there are several distinct watermelon vertex sets. A subtlety regarding non-uniqueness and interlacing will however be addressed in the upcoming Section 2.2.

For any given set of vertices which form a geodesic watermelon, Proposition 2.2 permits us to label the curves of the watermelon, which will aid us in stating the upcoming interlacing result. We wish to do this in a left-to-right manner, and to make this precise we next introduce a partial order on paths.

Definition 2.3. The time-range $\mathcal{R}(\gamma)$ of an upright path γ is the interval of integers t such that $x + y = t$ for some $(x, y) \in \gamma$. For $t \in \mathcal{R}(\gamma)$, the point (x, y) is unique, and we abuse notation by writing $\gamma(t) = x - y$. (In contrast, subscripts were used to express γ as a function on an integer interval earlier in the section.)

Two upright paths γ and ϕ satisfy $\gamma \preceq \phi$ if the set $\mathcal{R}(\gamma) \cap \mathcal{R}(\phi)$ is non-empty and every element t satisfies $\gamma(t) \leq \phi(t)$. If each inequality is strict, we write $\gamma \prec \phi$. Informally, we say that γ is to the left of ϕ.

A vector of upright paths is ordered if its component increase under $\prec$, and weakly ordered if they do so under $\preceq$. For a vector of upright paths γ, note that if its components γ_i are disjoint and every pair has non-disjoint time ranges, then by planarity there is exactly one labelling of the paths which is ordered from left to right. In that case, we shall simply say the collection of paths is ordered.

With this definition, for any given k-geodesic watermelon Γ_n^k identified by Proposition 2.2, we record its elements as the components of the vector

$$\Gamma_n^k = \left(\Gamma_{n,\to}^{k,1}, \ldots, \Gamma_{n,\to}^{k,k} \right), \tag{1}$$

choosing the left-to-right order, so that $\Gamma_{n,\to}^{k,i} \prec \Gamma_{n,\to}^{k,i+1}$ for $i \in [\![1, k-1]\!]$; thus, $\Gamma_{n,\to}^{k,i}$ begins at $(1, k-i+1)$ and ends at $(n, n-i+1)$.

2.2. **Interlacing.** The basic property that drives our proofs is simple and intuitive. Consider exponential LPP, so that all k-geodesic watermelons are almost surely unique. The square $[\![1, n]\!]^2$ is partitioned into two regions, NW and SE, by the geodesic Γ_n^1, which lies in both. Then $\Gamma_{n,\to}^{2,1}$ lies in NW and $\Gamma_{n,\to}^{2,2}$ in SE. Three regions are specified by the boundaries $\Gamma_{n,\to}^{2,1}$ and $\Gamma_{n,\to}^{2,2}$ and the sides of the square, and these contain, one apiece, the three paths in the 3-geodesic watermelon: see Figure 2 ahead. The partial order introduced in Definition 2.3 allows us to put interlacing on a precise footing.

Definition 2.4. A k-vector γ and a $(k+1)$-vector γ' whose components are upright paths *interlace* when their components satisfy $\gamma_i' \preceq \gamma_i \preceq \gamma_{i+1}'$ for $i \in [\![1, k]\!]$.

Figure 2 shows a pair of interlacing geodesic watermelons. We are ready to state our interlacing results, starting with the case where ν is continuous, so that Proposition 2.2 gives a unique k-geodesic watermelon Γ_n^k for every $k \in [\![1, n]\!]$.

Proposition 2.5 (Geodesic watermelons interlace)**.** *Suppose that the law ν is continuous. For $k \in [\![1, n-1]\!]$, the geodesic watermelons Γ_n^k and Γ_n^{k+1} interlace almost surely.*

When the distribution ν has atoms, the set of vertices visited by a geodesic watermelon may be variable. Here is a deterministic interlacing result that is valid in this setting.
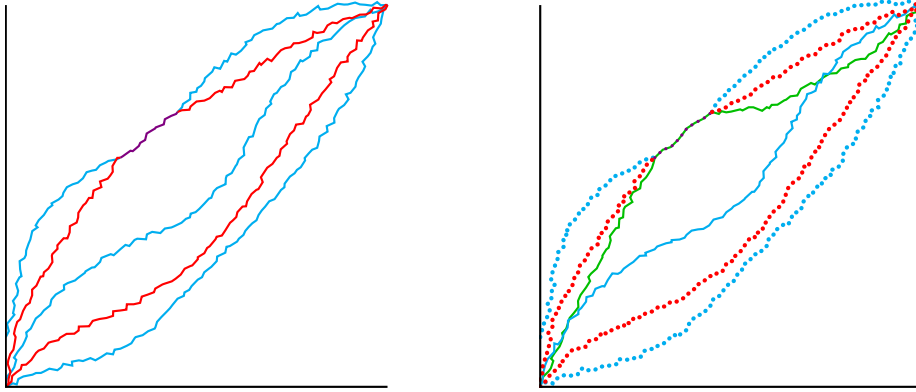
FIGURE 2. On the left is a depiction of the interlacing of k-geodesic watermelons for $k = 2$ (red) and $k = 3$ (blue). Each red curve remains within the region defined by adjacent blue curve boundaries. Observe that the red curves may overlap with the blue ones as shown between the top red and top blue curve; in fact, simulations suggest that a much more significant degree of overlap than depicted is typically present. Interlacing concerns consecutive values of k: the k- and $(k+2)$-melons may cross, as the right figure illustrates for $k = 1$. The geodesic curve in green make uses of the overlapping purple portion of the 2- and 3-melons, while remaining in the envelope defined by the red curves. The middle blue curve cannot use the purple portion, and this causes the green curve to cross over the middle blue curve.

Proposition 2.6. *For $n \in \mathbb{N}$, suppose that the values ξ_v are non-negative for $v \in [\![1, n]\!]^2$. Let $k \in [\![1, n]\!]$, and let ϕ_n^k be any k-geodesic watermelon with starting and ending points on the bottom left and top right sides of $[\![1, n]\!]^2$ as in Proposition 2.2. For $i \in [\![1, n]\!]$, we may find an i-geodesic watermelon γ_n^i such that consecutive terms in the sequence $\{\gamma_n^i : i \in [\![1, n]\!]\}$ interlace, with the union of the vertices in the curves in γ_n^k coinciding with this union for ϕ_n^k.*

2.3. **Monotonicity of watermelon weight increments.** A natural point process $\{Y_{n,i} : i \in [\![1, n]\!]\}$ associated to the sequence $\{X_n^k : k \in [\![1, n]\!]\}$ is obtained by stipulating that

$$X_n^k = \sum_{i=1}^{k} Y_{n,i} \tag{2}$$

for each $k \in [\![1, n]\!]$. This process records increments $Y_{n,i} = X_n^i - X_n^{i-1}$, $i \in [\![2, n]\!]$, in geodesic watermelon weight as the curve number rises, starting out at $Y_{n,1} = X_n^1$, the geodesic weight for the route from $(1, 1)$ to (n, n).

We mention that, in exponential LPP, the process $\{Y_{n,i} : i \in [\![1, n]\!]\}$ equals in law a list in decreasing order of the eigenvalues of an $n \times n$-matrix picked randomly according to the Laguerre Unitary Ensemble (LUE) [AVMW13]. In particular, this random list decreases almost surely. Arguments of the flavour of those that establish interlacing yield this deterministic monotonicity of $\{Y_{n,i} : i \in [\![1, n]\!]\}$ in a more general setting.

Proposition 2.7 (Increment monotonicity: C'est magnifique, mais ce n'est pas Laguerre)**.** *For n a positive integer, suppose that the values ξ_v are non-negative for $v \in [\![1, n]\!]^2$. Let $k \in [\![2, n]\!]$. Then $Y_{n,k-1} \geq Y_{n,k}$. Suppose further that the values $\{\xi_v : v \in [\![1, n]\!]^2\}$ are independently picked according to a continuous law ν. Then the stated inequality is almost surely strict.*

An analogous deterministic statement for semi-discrete LPP is proved by algebraic methods in [DOV18].

3. Technical ingredients and the key ideas in the proofs

Here we describe some important technical tools, relying on the above geometric features, which form key ingredients for our proofs. We will also indicate roughly how these tools are proved. This done, we will outline how the tools will aid in proving Theorem II.

In this section, and indeed in the rest of this paper, Assumptions 1, 2, and 3a will be in force in all statements. When Assumption 3b is used in place of 3a, we will indicate this.

3.1. **The technical tools.** There are three major technical tools, each of which is a result of independent interest. The first two results give existence and non-existence of certain numbers of disjoint paths in $[\![1, n]\!]^2$ with high probability, while the third one gives a quantitative result on paths with high transversal fluctuation.

3.1.1. *First tool: construction of disjoint paths of high weight.* The key starting point is an explicit construction of $m \in [\![1, k]\!]$ disjoint paths achieving, with high probability, a cumulative weight $\mu nm - O(mk^{2/3}n^{1/3})$. This allows by taking $m = k$ to prove the weight lower bound aspect of Theorem II(1).

Theorem 3.1. *There exist $c, c_1, C_1 > 0$, and $k_0 \in \mathbb{N}$ such that, for $k_0 \leq k \leq c_1 n$ and $m \in [\![1, k]\!]$, it is with probability at least $1 - e^{-ckm}$ that there exists a set of m disjoint paths $\{\gamma_i : i \in [\![1, m]\!]\}$ in the square $[\![1, n]\!]^2$, with γ_i for each index i being a path from $(1, k - i + 1)$ to $(n, n - i + 1)$ satisfying $\max_i \mathrm{TF}(\gamma_i) \leq 2mk^{-2/3}n^{2/3}$, such that*

$$\sum_{i=1}^{m} \ell(\gamma_i) \geq \mu nm - C_1 mk^{2/3}n^{1/3}.$$
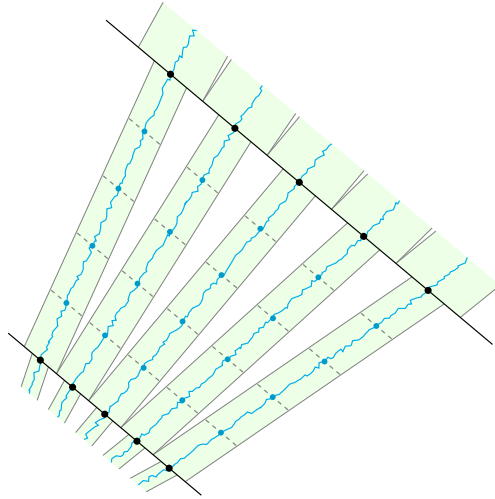
FIGURE 3. The dyadic construction of k disjoint curves with $k = 5$. Here we have focused on a single scale, while the preceding and succeeding scales are partially visible. Observe that in a single scale, each of the k curves passes through k consecutive parallelograms of a particular size (as determined by the scale) and aspect ratio. A more detailed depiction of this part of the construction is provided in Figure 8a

.

The construction, in Section 8, is a multi-scale one where at each dyadic scale we consider geodesics constrained to lie in k disjoint parallelograms with a carefully chosen aspect ratio: see Figure 3. The paths γ_i for $i = 1, 2, \ldots, k$ are now formed by concatenating these constrained

geodesics at different scales. The weight lower bound in Theorem 3.1 can then be obtained by analyzing the typical weight of the constrained geodesics, and a concentration estimate implies that the weight of a concatenation of independent geodesics is typically close to its mean.

3.1.2. *Second tool: several disjoint paths constrained in a strip typically have low weight.* This tool asserts that the event that there exist k curves, each meeting a certain weight lower bound and packed within a thin strip around the diagonal, has very low probability. It will serve to help prove Theorem II(2b), the transversal fluctuation lower bound. Letting

$$U_{n,w} := \left\{ 2 \leq x + y \leq 2n, |x - y| \leq w \right\}, \tag{3}$$

we show that if the width w equals $k^{1/3-o(1)}n^{2/3}$, then it is unlikely that there are k disjoint paths in $U_{n,w}$ each of which has weight at least $\mu n - \Theta(k^{2/3}n^{1/3})$.

Theorem 3.2. *Given $C_3 > 0$, there exist $\delta = \delta(C_3) > 0$ and $c_1 = c_1(C_3) > 0$ such that, for $k \in \mathbb{N}$ and $k \leq c_1 n$, the following holds. Let $\mathcal{E} = \mathcal{E}(n, k, \delta, C_3)$ denote the event that there exist k disjoint paths $\gamma_1, \gamma_2, \ldots, \gamma_k$ contained in $U_{n,\delta k^{1/3}n^{2/3}}$ such that $\ell(\gamma_i) \geq \mu n - C_3 k^{2/3}n^{1/3}$ for $i = 1, 2, \ldots, k$. Then there exists an absolute constant $c > 0$ such that $\mathbb{P}(\mathcal{E}) \leq e^{-ck^2}$.*

This result will be proved in Section 4. How? If there are a large number of disjoint paths in a strip, then some of them must be restricted to some thin region. Our assumptions will imply that any path contained in a thin region has an exponentially-low-in-k probability of having weight shortfall from μn less than $C_3 k^{2/3}n^{1/3}$, and the van den Berg-Kesten (BK) inequality will complete the proof. Recall that the BK inequality bounds the probability of occurrence of a number of events *disjointly* by the product of their individual probabilities.

A similar argument has appeared in [BHS18] to show that the maximum number of disjoint geodesics across the shorter sides of an $n \times n^{2/3}$ rectangle is uniformly tight with good tails. The argument in this paper is a refinement of this approach, requiring more careful entropy calculations to take care of the growing width of $U_{n,\delta k^{1/3}n^{2/3}}$ in k.

3.1.3. *Third tool: the weight shortfall of paths with high transversal fluctuations.* This interesting result will help to prove Theorem II(2a), the upper bound on geodesic watermelon transversal fluctuation. The tool asserts that any path from $(1, 1)$ to (r, r) of transversal fluctuation greater than $sr^{2/3}$ typically suffers a weight loss of order $s^2 r^{1/3}$. The result applies to any path and is not a bound on the probability of the maximal weight path having such a transversal fluctuation. The statement may be expected from known relations between transversal and longitudinal fluctuations (which is a consequence of the curvature assumption in Assumption 2 and one-point fluctuation information in Assumption 3a). While similar results with suboptimal exponents have appeared (see [BSS14, BGZ19]), we obtain the following sharp result, adapting a multi-scale argument from [BSS14]. The proof appears in Section 6.

Theorem 3.3. *Let $X^{r,s,t}$ be the maximum weight over all paths $\Gamma^{r,s,t}$ from the line segment joining $(-tr^{2/3}, tr^{2/3})$ and $(tr^{2/3}, -tr^{2/3})$ to the line segment joining $(r - tr^{2/3}, r + tr^{2/3})$ and $(r + tr^{2/3}, r - tr^{2/3})$ such that $\mathrm{TF}(\Gamma^{r,s,t}) > (s + t)r^{2/3}$, with $t \leq s$. There exist absolute constants r_0, s_0, $c > 0$ and $c_2 > 0$ such that, for $s > s_0$ and $r > r_0$,*

$$\mathbb{P}\left(X^{r,s,t} - \mu r > -c_2 s^2 r^{1/3} \right) < e^{-cs^3}.$$

We now show how the interlacing property and the above results will yield Theorem II.

3.2. **Proof outline of Theorem II.** We already noted how Theorem 3.1 implies the weight lower bound of Theorem II(1). This is a collective weight lower bound, but for the remaining parts of Theorem II we will roughly need a lower bound on individual curve weights of order $\mu n - \Theta(k^{2/3}n^{1/3})$; note that this quantity is k^{-1} times the weight lower bound $\mu nk - \Theta(k^{5/3}n^{1/3})$ of Theorem II(1). This individual curve lower bound is provided in a slightly weaker (but sufficient) sense by an averaging argument, which is a simple but core ingredient of our proofs.

3.2.1. *The averaging argument.* Let $X_n^{j,i}$ be the weight of the ith heaviest curve of a given j-geodesic watermelon. The averaging argument yields that there exists with high probability $j \in [\![k+1, 2k]\!]$ such that, for all $i \leq j$,

$$X_n^{j,i} \geq X_n^{j,j} \geq \mu n - \Theta(k^{2/3}n^{1/3}), \tag{4}$$

and similarly for some $j \in [\![\lfloor k/2 \rfloor, k]\!]$. The argument is powered by a simple but very useful inequality: for $j \in \mathbb{N}$,

$$X_n^{j,j} \geq X_n^j - X_n^{j-1}. \tag{5}$$

This is proved by considering the lowest weight curve and the remaining $j-1$ curves of the j-melon separately. The weight of the latter is of course at most the weight of the $(j-1)$-melon, and this gives the inequality. Notice that while $X_n^{j,j}$ is defined only once a j-geodesic watermelon has been specified, the lower bound in (5) is independent of the choice of j-geodesic watermelon. This allows us to prove that (4) holds for some $j \in [\![k+1, 2k]\!]$ simultaneously for all choices of j-geodesic watermelon with high probability.

Deriving (4) from (5) is a matter of appealing to the lower bound on X_n^{2k} from Theorem II(1) and the following crude upper bound on X_n^k, which is proved using one-point information (Assumption 3a) and the BK inequality (see Section 4); note the change in sign of the fluctuation term when compared with the weight upper bound of Theorem II(1).

Lemma 3.4. *Under Assumption 3a, we may find $c > 0$ such that, if $t > 0$, there exists $k_0 = k_0(t)$ for which $n > n_0$ and $k_0 < k \leq t^{-3/4}n^{1/2}$ imply that*

$$\mathbb{P}\left(X_n^k > \mu nk + tk^{5/3}n^{1/3}\right) \leq \exp\left(-ct^{3/2}k^2\right).$$

If instead Assumption 3b is available, then the upper bound on k may be taken to be $\min(1, t^{-3/2})n$.

The relaxation on the condition on k we obtain under Assumption 3b is important to prove Theorem I with the full claimed range of k. Lemma 3.4 is the source of all future statements which give different conditions k under Assumption 3b. The proof of Lemma 3.4 using Assumption 3b is slightly different than with 3a, and is handled separately in Appendix C, while the proof with 3a appears in Section 4.

With the result of the averaging argument in hand, Theorem II(2) is a straightforward consequence of interlacing and Theorems 3.2 and 3.3, as we briefly describe next.

3.2.2. *Bounding transversal fluctuations.* Equation (4), along with Theorem 3.2, implies the following transversal fluctuation lower bound for the random index j,

$$\mathbb{P}\left(\exists j \in \{\lfloor k/2 \rfloor, \ldots, k\} \text{ such that } \underline{\text{TF}}(n,j) > \delta k^{1/3}n^{2/3}\right) \geq 1 - e^{-ck^2}.$$

This can then be upgraded to Theorem II(2b) by a simple application of interlacing. A similar argument with Theorem 3.3 implies that

$$\mathbb{P}\left(\exists j \in \{k+1, \ldots, 2k\} \text{ such that } \overline{\text{TF}}(n,j) < Ck^{1/3}n^{2/3}\right) \geq 1 - e^{-ck},$$

which when combined with interlacing gives Theorem II(2a).

What remains is the proof of the weight upper bound, Theorem II(1), which will use essentially all of the gathered ingredients.

3.2.3. *Bounding above watermelon weight.* The proof has a few parts. Consider, as we did above, a random $k/2 \le j \le k$ that satisfies (4), so that for all $1 \le i \le j$, $X_n^{j,i}$ is not too light (for any choice of j-geodesic watermelon). By an application of Theorem 3.2 with k replaced by $k/4$, it follows that with probability $1 - e^{-ck^2}$, at least $k/4$ paths in all j-geodesic watermelons exit a strip of width $\Theta(k^{1/3}n^{2/3})$. The interlacing property implies that the latter property holds for any fixed k-geodesic watermelon Γ_n^k as well.

We also know that the cumulative weight of the heaviest $7k/8$ paths of Γ_n^k cannot be too high, by the crude estimate Lemma 3.4 (with k replaced by $7k/8$). This implies that on the event that the k-geodesic watermelon's weight is high, the cumulative weight of the *lightest* $k/8$ paths of Γ_n^k is high. Hence the heaviest path among the latter has unusually high weight which, along with monotonicity, implies the same lower bound for all the heaviest $7k/8$ paths of Γ_n^k.

Since we have already established that at least $k/4$ paths in the ensemble Γ_n^k exit a strip around the diagonal of width $\Theta(k^{1/3}n^{2/3})$, this implies that at least $k/8$ disjoint paths both exit a strip of width $\Theta(k^{1/3}n^{2/3})$ and each have an unusually high weight. The probability that a single curve has both these properties is at most $\exp(-ck)$ by Theorem 3.3, and so the probability that $O(k)$ many disjoint curves have these properties is $\exp(-ck^2)$ by the BK inequality. We note that, without interlacing, the above would give a single curve with unusually high weight and large transversal fluctuation, which would yield the weaker probability bound of $\exp(-ck)$.

3.2.4. *A few further comments in overview.* The proof strategies make it apparent that the arguments have minimal dependence on the precise nature of the model and indeed straightforward modifications will yield Theorem III.

The proofs critically rely on the averaging argument and watermelon interlacing. The latter is a central theme of the paper and is a consequence of non-local geometric aspects of the watermelons as will be clear from the proofs in Section 9. The monotonicity result, Proposition 2.7, will also be proved in Section 9, by similar geometric considerations. The geometric point of view defines our approach, both in technique and outcome. Interlacing is the principal means by which we have expressed this point of view, but in fact the stated monotonicity offers an alternative, but still geometric, route to proving Theorem II. In Section 10, we explore this approach and explain how it leads to certain sharp concentration bounds on each $Y_{n,k}$. We contrast this with weaker concentration bounds obtainable in the integrable setting of exponential LPP by determinantal point process techniques: see Propositions 10.1 and 10.2.

3.3. **Basic tools.** To implement the ideas of the preceding sections, we will require certain basic estimates on geometrically relevant quantities. These include, for example, tail estimates for last passage values where the endpoints are allowed to vary over intervals whose length is of order $n^{2/3}$, or where one endpoint is allowed to vary over the line $x + y = 2n$; and tail and mean estimates for last passage values constrained to remain in certain parallelograms. In this section we state the precise forms of these estimates that we will be using.

Similar estimates have appeared in the literature, for example in the preprint [BSS14], but never under the general assumptions that we adopt. Our proofs follow similar strategies as in these previous works and are provided in Appendix B for completeness. The statements are nonetheless technical and the reader might choose to skip this section at a first reading and only refer back to it as needed later in the paper.

As we prepare to give the statements, we adopt a convention that will prove convenient at several moments in the article: we will refer to measurements taken along the anti-diagonal as "width", and measurements along the diagonal as "height". For example, the parallelogram defined by $\{(x, y) \in \mathbb{Z}^2 : |x - y| \le 2\ell r^{2/3}, 2 \le x + y \le 2r\}$ will be said to have width $2\ell r^{2/3}$ and height r. The

implicit sense of the diagonal direction as vertical will find expression in usages such as 'upper' and 'lower' to refer to more or less advanced diagonal coordinates.

3.3.1. *Interval-to-interval estimates.* We start with estimates for the deviations of the weight when the endpoints are allowed to vary over intervals. For this we define some notation.

Extending the notation (3), let $U = U_{r,\ell r^{2/3}, z r^{2/3}}$ be a parallelogram of height r with one pair of sides parallel to the anti-diagonal, each of width $\ell r^{2/3}$, the lower side centred at $(1,1)$ and the upper side at $(r - z r^{2/3}, r + z r^{2/3})$ for a z such that $|z| \leq r^{1/3}$. Let A and B be the lower and upper sides of U and let $\mathcal{S}(U) = A \times B$. With G_2 and ρ as in Assumption 2, let

$$\widetilde{Z} := \sup_{(u,v) \in \mathcal{S}(U)} \frac{X_{u \to v} - \mu r}{r^{1/3}} + \frac{G_2}{1 + 2\ell^{3/2}} \left(z \wedge \rho r^{1/3} \right)^2.$$

Although we have yet to employ the notation Z without a tilde, it will be used to denote a scaled form for weight of certain LPP paths. The tilde indicates a parabolic adjustment which compensates (and possibly over-compensates) the curvature penalty for off-diagonal movement. The parabolic form of this adjustment is indicated in Assumption 2. This form is guaranteed by Assumption 2 only up to anti-diagonal displacement of order $z = \rho r^{1/3}$; concavity nonetheless guarantees that at least this loss occurs even beyond this point, and this accounts for the $z \wedge \rho r^{1/3}$ term in the specification of $\widetilde{Z}$.

However, Assumption 3a only holds for paths whose endpoints are bounded away from the coordinate axes, and so for extreme anti-diagonal displacement we will need to work with a different object: define, for $\delta > 0$, $\mathbb{L}_{\mathrm{ext}}(\delta) = \left\{ (r + z r^{2/3}, r - z r^{2/3}) : |z| \geq (1 - \delta) r^{1/3} \right\}$, which is an interval of points of extreme slope, and let $\mathbb{L}_{\mathrm{low}} = \{ (x,y) \in \mathbb{Z}^2 : x + y = 2, |x - y| \leq 2 r^{2/3} \}$. Finally define

$$Z^{\mathrm{ext},\delta} = \sup_{u \in \mathbb{L}_{\mathrm{low}}, v \in \mathbb{L}_{\mathrm{ext}}(\delta)} \frac{X_{u \to v} - \mu r}{r^{1/3}},$$

the interval-to-interval passage time between $\mathbb{L}_{\mathrm{low}}$ and $\mathbb{L}_{\mathrm{ext}}(\delta)$.

Proposition 3.5. *Suppose $\delta > 0$ is such that $|z| < (1 - \delta) r^{1/3}$. Then, under Assumptions 2 and 3a, there exist $\theta_0 = \theta_0(\ell, \delta)$, $c = c(\delta) > 0$, and $r_0 = r_0(\ell, \delta)$ such that, for $r > r_0$ and $\theta > \theta_0$,*

$$\mathbb{P}\left(\widetilde{Z} > \theta \ell^{1/2} \right) < \exp\left(-c \min(\theta^{3/2}, \theta r^{1/3}) \right). \tag{6}$$

Further, there exist $\delta > 0$ and $c > 0$ such that, for $\theta > \theta_0$ and $r > r_0$,

$$\mathbb{P}\left(Z^{\mathrm{ext},\delta} > \theta \right) \leq \exp(-cr - c \min(\theta^{3/2}, \theta r^{1/3})). \tag{7}$$

3.3.2. *Point-to-line estimates.* Here we bound the upper tail of point-to-line geodesic weights, using only the parabolic curvature of the limit shape (Assumption 2) and the point-to-point upper tail (Assumption 3a). We define $\mathbb{L}_{\mathrm{left}} = \mathbb{L}_{\mathrm{left}}(r, t) = \{ (x,y) : x + y = 0, |y - x| \leq t r^{2/3} \}$.

Proposition 3.6. *Let X be the maximum weight of all paths Γ from $\mathbb{L}_{\mathrm{left}}$ to the line $x + y = 2r$ whose endpoint does not lie on the line segment connecting $(r - (s+t) r^{2/3}, r + (s+t) r^{2/3})$ and $(r + (s+t) r^{2/3}, r - (s+t) r^{2/3})$. Then under Assumptions 2 and 3a there exist constants $c_1 > 0, c > 0$, s_0, θ_0, and r_0 such that, for $r > r_0$ and either (i) $s = 0$ and $\theta \geq \theta_0$, or (ii) $s > s_0$, $0 \leq t \leq s^2$ and $\theta = 0$, we have that*

$$\mathbb{P}\left(X \geq \mu r + \theta r^{1/3} - c_1 s^2 r^{1/3} \right) \leq \exp\left(-c\left(\min(\theta^{3/2}, \theta r^{1/3}) + s^3 \right) \right).$$

3.3.3. *Lower tail of constrained point-to-point.* Next we come to a crucial ingredient of Theorem 3.1, an upper bound on the lower tail of the weight of the point-to-point geodesic constrained to not exit a given parallelogram, and a lower bound on the mean of the same. We obtain precisely the tail exponent of one that will be used in Theorem 3.1; the optimal exponent, however, is expected to be three, matching the optimal exponent of the point-to-point weight lower tail. Even given the optimal lower tail exponent of three as input via Assumption 3a, the argument we give for Proposition 3.7 would only yield the exponent $3/2$ for the constrained lower tail.

With notation for parallelograms as above, let $U = U_{r,\ell r^{2/3}, zr^{2/3}}$. Set $u = (1,1)$ and $u' = (r - (z+h)r^{2/3}, r + (z+h)r^{2/3})$ where $|h| \leq \ell/2$. Then define $X^U_{u \to u'}$ to be the maximum weight over all paths from u to u' that are constrained not to exit U. Recall G_1 from Assumption 2. We have the following lower tail and mean estimates for $X^U_{u \to u'}$.

Proposition 3.7 (Lower tail and mean of constrained point-to-point). *Let $L_1, L_2 > 0$ and $K > 0$ be fixed. Let z and ℓ be such that $|z| \leq K$ and $L_1 \leq \ell \leq L_2$. There exist positive constants $r_0(K, L_1, L_2)$ and $\theta_0(K, L_1, L_2)$, and an absolute constant $c > 0$, such that, for $r > r_0$ and $\theta > \theta_0$,*

$$\mathbb{P}\left(X^U_{u \to u'} \leq \mu r - \theta r^{1/3}\right) \leq \exp\left(-c\ell\theta\right). \tag{8}$$

As a consequence, there exists $C = C(K, L_1, L_2)$ such that, for $r > r_0$,

$$\mathbb{E}[X^U_{u \to u'}] \geq \mu r - G_1 z^2 r^{1/3} - C r^{1/3}. \tag{9}$$

3.4. **Organization.** In this section, we have signposted the location of several upcoming aspects. With some repetition, and for the sake of convenience, we now summarise the structure of the rest of the paper, which is principally devoted to proving Theorem II. There are four elements: lower and upper bounds on the transversal fluctuation of geodesic watermelons; and upper and lower bounds on the watermelons' weight. The respective arguments are offered in Sections 5, 6, 7 and 8. The first of the four elements is aided by the 'thin strip means low weight' Theorem 3.2. First, in Section 4, we will recall the BK inequality and use it to prove this theorem. The deterministic interlacing results assembled in Section 2 are important inputs, and, in Section 9, we prove them, so that the proof of Theorem II is completed in this section. Section 10 compares concentration bounds for the weight increments $Y_{n,k}$ specified in (2) obtained by our geometric methods and by determinantal point process techniques (in the case of exponential LPP). The proof of the point-to-line result, Theorem III, is provided in Section 11.

There are three appendices: Appendix A is devoted to proving that exponential and geometric LPP satisfy the Assumptions; Appendix B to proving the three basic tools, Propositions 3.5, 3.6 and 3.7, laid out in Section 3.3; and Appendix C to proving Lemma 3.4 under the stronger Assumption 3b. The proofs in Appendix B have been deferred because they roughly mimic publicly available LPP arguments; we judge the flow of the arguments at large to be aided by the few proof deferrals that we have mentioned. The appendices render the article independent of LPP inputs beyond fairly well-known assertions such as the expression of the cardinality of a determinantal point process as a sum of independent Bernoulli random variables, and an estimate on the mean of the eigenvalue count in a given interval for the Laguerre unitary ensemble.

4. DISJOINT PATHS AND THE VAN DEN BERG-KESTEN INEQUALITY

In this section, we prove Theorem 3.2, our assertion that several disjoint high-weight paths may not typically coexist in a narrow strip. The proof is founded on the notion that at least one among k disjoint paths packed into the strip $U_{n, \delta k^{1/3} n^{2/3}}$ is forced to inhabit a thin region of width $\delta k^{-2/3} n^{2/3}$. Here in outline are the principal steps that we will follow to realize the proof.

(1) We will argue that it is with probability decaying exponentially in k that a path existing in a given thin region—of length n and of width roughly $k^{-2/3}n^{2/3}$—has weight larger than $\mu n - \Theta(k^{2/3}n^{1/3})$.

(2) The van den Berg-Kesten (BK) inequality bounds above the probability of events occurring in a certain *disjoint* fashion by the product of the probabilities of the events. This inequality permits us to conclude that the probability that there exist k disjoint paths whose weight satisfies the above lower bound, each contained in a specified thin region, is exponentially small in k^2.

(3) A union bound will then be taken indexed by a collection of k-tuples of thin regions whose elements capture all possible routes for the k paths. The collection will be selected by means of a grid-based discretization; the cardinality of the collection will be small enough that the upper bound on probability for a given k-tuple will not be significantly undone by taking the union bound.

To implement this three-step plan, we need to make precise the notion of 'thin region' in Step 1, and this in fact involves specifying the grid used in Step 3. This we do in a first subsection. But first we remind the reader of our norm that statements by default suppose Assumptions 1, 2 and 3a and that we indicate when Assumption 3b is needed in place of 3a.

4.1. **Specifying a grid that classifies paths running in a narrow strip.**

The grid will be called G. It will comprise anti-diagonal planar intervals whose elements track the progress of any path of transversal fluctuation at most $\delta k^{1/3}n^{2/3}$.

Recall that width and height refer to anti-diagonal and diagonal length. Beyond the height n, path number k and strip thinness parameter $\delta > 0$, the grid will be constructed by the use of two positive parameters η and ε. We hope that the explanation we next offer alongside Figure 4 will make their roles clear.

The strip of width $\delta k^{1/3}n^{2/3}$ and height n will be divided into cells. Each cell will have width $\eta\varepsilon n^{2/3}$ and height $\varepsilon^{3/2}n$. The width is the product of η and the two-thirds' power of the height, so η is a measure of the thinness of the cell, judged on the KPZ scale; we will choose this parameter to be a small constant, independently of k and n. We will choose ε to have order $k^{-2/3}$, so that a union of intervals at consecutive heights may play the role of a 'thin region' indicated in the three-point summary. Each column of the grid contains N cells, where $N = n/(\varepsilon^{3/2}n) = \varepsilon^{-3/2}$; and each row contains M cells, where $M = 2\delta k^{1/3}n^{2/3}/(\eta\varepsilon n^{2/3}) = 2\delta k^{1/3}\eta^{-1}\varepsilon^{-1}$.

To begin the definition, let $\varepsilon > 0$ and $\eta > 0$; we will specify the conditions on these values presently. We suppose $n \in \mathbb{N}$ to be large enough that $\varepsilon\eta n^{2/3} \geq 1$, in order that cell width be at least the difference of x- or y-coordinates between consecutive elements in $\mathbb{Z}^2$ along an anti-diagonal line. Set $M, N \in \mathbb{N}$ according to

$$N = \lfloor \varepsilon^{-3/2} \rfloor \quad \text{and} \quad M = 2 \cdot \lceil \delta k^{1/3}\eta^{-1}\varepsilon^{-1} \rceil. \tag{10}$$

The grid G is a set of planar anti-diagonal intervals $\mathbb{L}_{ij}$:

$$G = \left\{ \mathbb{L}_{ij} : i \in [\![0, N]\!], j \in [\![0, M]\!] \right\},$$

where $\mathbb{L}_{ij}$ connects the points

$$(v_i - h_j, v_i + h_j) \quad \text{and} \quad (v_i - h_{j+1}, v_i + h_{j+1}),$$

with $v_i = \lfloor i\varepsilon^{3/2}n \rfloor$ and $h_j = \lfloor (\delta k^{1/3} - j\eta\varepsilon)n^{2/3} \rfloor$ for $i \in [\![0, N-1]\!]$; for $i = N$ we take $v_i = n$. Thus the last row of cells has greater height than the first $N - 1$ rows', by a factor lying in $[1, 2)$, but this will not affect our arguments. Note that each element of the grid belongs to the rectangle $\{|x - y| \leq 2\delta k^{1/3}n^{2/3}, 0 \leq x + y \leq 2n\}$. The grid is partitioned into subsets at given height by setting $G_i = \left\{ \mathbb{L}_{ij} : j \in [\![0, M]\!] \right\}$ for $i \in [\![0, N]\!]$. To avoid cumbersome expressions, we employ, in

the remainder of this section, a notational abuse by which (x, y) for $x, y \in \mathbb{R}$ will actually denote the rounded integer lattice point $(\lfloor x \rfloor, \lfloor y \rfloor)$.
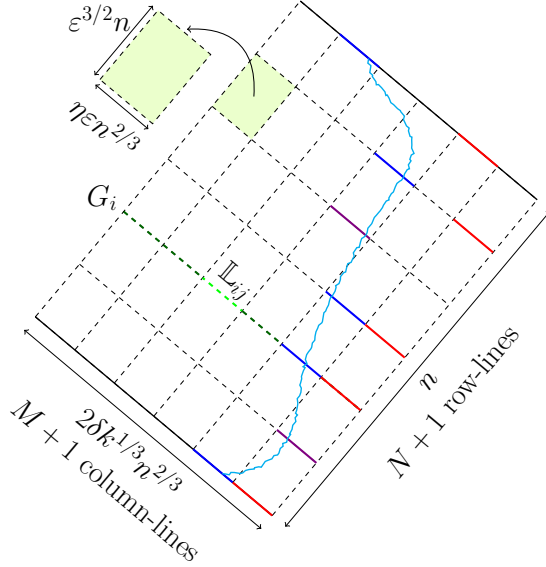

FIGURE 4. The grid G. There are $N + 1$ row lines and $M + 1$ column lines, and the width and height of a single cell are $\eta \varepsilon n^{2/3}$ and $\varepsilon^{3/2} n$ respectively. The width of the grid as a whole is $2\delta k^{1/3} n^{2/3}$. The role of the grid is to be a discretization of paths from the bottom side to the top. On the right, such a path is depicted in sky blue along with its corresponding discretization, which is the sequence of intervals, one on each grid line, which are coloured blue (or purple). Also depicted is a sequence of red intervals (when a red interval coincides with a blue interval, it is coloured purple). These red intervals satisfy the property that on each grid line, the red interval coincides with, or is to the right of, the blue interval on the same grid line. This is a property that is satisfied by the discretization of any path from bottom to top which is disjoint from and to the right of the sky blue path.

4.2. **Bounding the maximum weight of paths between grid elements at consecutive heights.** A *grid journey* is an $(N + 1)$-vector $\{L_i : i \in [\![0, N]\!]\}$ such that $L_i \in G_i$ for each index i. A grid journey plays in the implementation the conceptual role of a thin region in the outline. Indeed, for any grid journey $\mathscr{L}$, we write $X_{\mathscr{L}}$ for the supremum of the weights of paths contained in $[\![1, n]\!]^2$ that intersect every one of the $N + 1$ components of $\mathscr{L}$. In Step 1, we will bound the upper tail of $X_{\mathscr{L}}$. To be ready to do so, we first gain control, in Lemma 4.1, on the mean of the weight of the heaviest path that begins in a given element of G_i and that ends in a given element of G_{i+1}. Since the height discrepancy is $r = \varepsilon^{3/2} n$, Assumption 2 indicates that the weight of the geodesic running between *given* endpoints in these intervals has leading term μr, with a negative correction whose order is $r^{1/3}$. When the endpoints are varied by at most $\eta r^{2/3}$, a Gaussian-order perturbation is transmitted to the maximum weight. This change of order $\eta^{1/2} r^{1/3}$ is less than the order of the negative mean provided that η is small enough (and it is this consideration that determines our upper bound η_0 on the parameter η). This is how we will prove Lemma 4.1.

For $\eta > 0$, $r > 0$ and $|s| \le r$, let $\mathbb{L}_{\mathrm{low}}$ be the line segment joining $(-\eta r^{2/3}, \eta r^{2/3})$ and $(\eta r^{2/3}, -\eta r^{2/3})$ and let $\mathbb{L}_{\mathrm{up}}$ be the line segment joining $(r - s - \eta r^{2/3}, r + s + \eta r^{2/3})$ and $(r -$

$s + \eta r^{2/3}, r + s - \eta r^{2/3})$. Define Z by

$$Z = \sup \left\{ r^{-1/3}\big(X_{u \to v} - \mu r\big) : u \in \mathbb{L}_{\text{low}}, v \in \mathbb{L}_{\text{up}} \right\},$$

so that we seek control on the mean of Z.

Lemma 4.1. *Under Assumption 2, there exist positive constants c_1, r_0 and η_0 such that, for $\eta < \eta_0$, $|s| \leq r$ and $r > r_0$, $\mathbb{E}[Z] \leq -2c_1$.*

The basic strategy of the argument is to back away from $\mathbb{L}_{\text{low}}$ and $\mathbb{L}_{\text{up}}$ on either side and consider a point-to-point weight; such ideas have appeared in the literature: see for instance [BSS14].

Proof. Let m_1 and m_2 be the points on $\mathbb{L}_{\text{low}}$ and $\mathbb{L}_{\text{up}}$ which are closest to the average of the ends of the respective intervals; we make this specification since these averages need not be lattice points.

The ℓ^1 distance of these two points is $2r$. Let $u^* \in \mathbb{L}_{\text{low}}$ and $v^* \in \mathbb{L}_{\text{up}}$ be such that $X_{u^* \to v^*}$ equals $\mu r + Zr^{1/3}$. For $\eta > 0$, let $\phi_1 = m_1 - (\eta^{3/2}r, \eta^{3/2}r)$ and $\phi_2 = m_2 + (\eta^{3/2}r, \eta^{3/2}r)$. Then we have

$$X_{\phi_1 \to u^*} + X_{u^* \to v^*} + X_{v^* \to \phi_2} \leq X_{\phi_1 \to \phi_2}.$$

Note that the ℓ^1 distance between ϕ_1 and ϕ_2 lies in $(2 + 4\eta^{3/2})r + [-2, 0]$. First we consider the case that this pair of points satisfies the ρ-condition in Assumption 2. We get that for large r,

$$\mathbb{E}[X_{\phi_1 \to \phi_2}] \leq \mu(1 + 2\eta^{3/2})r - g_2\big((1 + 2\eta^{3/2})r - 2\big)^{1/3} \leq \mu(1 + 2\eta^{3/2})r - g_2 r^{1/3}.$$

To get an upper bound on $\mathbb{E}[X_{u^* \to v^*}]$, we need a lower bound on $\mathbb{E}[X_{\phi_1 \to u^*}]$, which will also apply to $\mathbb{E}[X_{v^* \to \phi_2}]$ by symmetry. Note that u^* is independent of the environment outside U, so $\mathbb{E}[X_{\phi_1 \to u^*}] \geq \inf_{u \in \mathbb{L}_{\text{low}}} \mathbb{E}[X_{\phi_1 \to u}]$. We bound this below using Assumption 2, and note that that expression is minimized if u is either endpoint of $\mathbb{L}_{\text{low}}$. So we take $u = (-\eta r^{2/3}, \eta r^{2/3})$. This gives

$$\mathbb{E}[X_{\phi_1 \to u}] \geq \mu\eta^{3/2}r - \mu - G_1\eta^{1/2}r^{1/3} - g_1\eta^{1/2}r^{1/3},$$

for r such that $\eta^{3/2}r > r_0$ for a large r_0. Thus we get an upper bound on $\mathbb{E}[X_{u^* \to v^*}]$:

$$\mathbb{E}[X_{u^* \to v^*}] - \mu r \leq 2\mu - g_2 r^{1/3} + 2(G_1 + g_1)\eta^{1/2}r^{1/3} \leq -2c_1 r^{1/3}$$

for $c_1 = g_2/4$, η small enough, and r large enough. This completes the proof when the ρ-condition in Assumption 2 is satisfied.

If the ρ-condition is not satisfied, we have, using the monotonicity of the upper bound in Assumption 2 for all x and a simple calculation,

$$\mathbb{E}[X_{u^* \to v^*}] \leq \mathbb{E}[X_{\phi_1 \to \phi_2}] \leq (\mu + 2\mu\eta^{3/2} - \rho^2 G_2)r.$$

In this case the proof is completed by noting that, for η sufficiently small, the coefficient of r is strictly smaller than μ. $\qquad\square$

The upper bound of η_0 from Lemma 4.1 will be imposed on η, in order that the mean of Z be negative. A further upper bound on η will later be needed; namely,

$$\eta e c_1 C_3^{-1} < 1, \tag{11}$$

where c_1 is as in Lemma 4.1 and C_3 is as in Theorem 3.2. Let $\eta < \eta_0$ satisfy (11).

The next three subsections in turn carry out the outlined three-step plan for proving Theorem 3.2.

4.3. **Step one: bounding above the weight of the heaviest path on a given grid route.**
We rigorously take Step 1 by stating and proving the next result. The tail of $\exp(-\Theta(\varepsilon^{-3/2}))$ we
obtain here is crucial, as this will become $\exp(-\Theta(k))$ when we later set ε to be of order $k^{-2/3}$.

Proposition 4.2. *Let c_1 be as in Lemma 4.1 and let $\eta > 0$ be fixed as after (11). There exist
positive c_3 and C such that, when $\varepsilon > 0$ and $n \in \mathbb{N}$ satisfy $n \geq C\varepsilon^{-3/2}$,*

$$P\left(X_{\mathscr{L}} > \mu n - c_1\varepsilon^{-1}n^{1/3}\right) \leq \exp\left\{-c_3\varepsilon^{-3/2}\right\}.$$

Proof. Set $Z_i = \sup_{u \in \overline{L}_i, v \in L_{i+1}} \varepsilon^{-1/2}n^{-1/3}\left(X_{u \to v} - \mu\varepsilon^{3/2}n\right)$ for $i \in [\![0, N-1]\!]$, where $\overline{L}_i$ is the anti-
diagonal interval which is the union of the two line segments obtained by displacing L_i by $(1, 0)$
and $(0, 1)$; this choice is made so that the Z_i are determined by disjoint regions of the noise field and
so are independent. Then $X_{\mathscr{L}} - \mu n \leq \varepsilon^{1/2}n^{1/3}\sum_{i=0}^{N-1} Z_i$. As noted, the collection $\left\{Z_i : [\![0, N-1]\!]\right\}$
is independent; and, by Lemma 4.1, the elements' means lie uniformly to the left of zero. We will
verify that these random variables have exponential upper tails and apply Bernstein's inequality
to $\sum Z_i$ in order to obtain the proposition.

If we set $C > r_0$ in the statement that we are seeking to prove, then a choice of $r = \varepsilon^{3/2}n$ in
Lemma 4.1 will satisfy the hypothesis that $r \geq r_0$. Applying this lemma with η as specified, we
find that

$$\mathbb{E}[Z_i] \leq -2c_1$$

in view of translation invariance of the noise field $\left\{\xi_v : v \in \mathbb{Z}^2\right\}$. With η now fixed, we claim that
for $\theta > \theta_0$,

$$\mathbb{P}(Z_i > \theta\eta^{1/2}) \leq \exp(-c\theta).$$

This is because $Z_i \leq \tilde{Z}_i$ (where $\tilde{Z}_i$ is defined analogously to $\tilde{Z}$ in Proposition 3.5) in the case
that L_i and L_{i+1} form opposite sides of a parallelogram whose midpoint-to-midpoint anti-diagonal
displacement is at most $(1-\delta)\varepsilon^{3/2}n$ for the $\delta > 0$ fixed in the second part of Proposition 3.5; while
$Z_i \leq Z_i^{\text{ext},\delta}$ (again defined analogously to $Z^{\text{ext},\delta}$ in Proposition 3.5) in the other case. Applying
Proposition 3.5 completes the claim. Note that we have again used that $r = \varepsilon^{3/2}n > r_0$ by setting
$C > r_0$ in the statement of Proposition 4.2.

The above displayed bound establishes that Z_i has exponential upper tails and so satisfies the
conditions of Bernstein's inequality (see equation 2.18 and Theorem 2.13 of [Wai19]). Applying the
latter and since $N = \varepsilon^{-3/2}$ (see (10)), we get, for some $c_3 > 0$,

$$\mathbb{P}\left(X_{\mathscr{L}} > \mu n - c_1\varepsilon^{-1}n^{1/3}\right) \leq \mathbb{P}\left(\sum_{i=0}^{N-1} Z_i > -c_1\varepsilon^{-3/2}\right) \leq \mathbb{P}\left(\sum_{i=0}^{N-1}(Z_i - \mathbb{E}[Z_i]) > c_1\varepsilon^{-3/2}\right)$$

$$\leq \exp\left(-c_3\varepsilon^{-3/2}\right). \qquad \square$$

4.4. **Step 2: the rarity of k high weight paths in a narrow strip.** The BK inequality is the
principal tool enabling the second step. The inequality provides an upper bound on the probability
of a number of events occurring *disjointly* in terms of the individual events' probabilities. To state
it we need a precise definition of disjointly occurring events; this is taken from [AGH18], which also
proves the inequality in the setting of infinite spaces that we require.

Definition 4.3. *For $d, n \in \mathbb{N}$, let $A_i \subseteq \mathbb{R}^d$ be Borel measurable for $i \in [\![1, n]\!]$. For $\omega \in \mathbb{R}^d$ and
$K \subseteq [\![1, d]\!]$, define the cylinder set $\mathrm{Cyl}(K, \omega) = \{\omega' : \omega'_i = \omega_i, i \in K\}$. Also define, for $A \subseteq \mathbb{R}^d$,*

$$[A]_K := \{\omega : \mathrm{Cyl}(K, \omega) \subseteq A\} \quad \text{and} \quad \prod_{i=1}^n A_i := \bigcup_{J_1,\ldots,J_n} \bigcap_{i=1}^n [A_i]_{J_i},$$

where the union is over disjoint subsets $J_1, \ldots, J_n$ of $\{1, \ldots, d\}$.

With the notation established, we may state the BK inequality.

Proposition 4.4 (BK inequality, Theorem 7 of [AGH18])**.** *Fix $n \in \mathbb{N}$ and let $A_i \subseteq \mathbb{R}^d$ be Borel measurable for $i \in [\![1, n]\!]$. Under any complete product probability measure ν on $\mathbb{R}^d$,*

$$\nu \left(\prod_{i=1}^{n} A_i \right) \leq \prod_{i=1}^{n} \nu(A_i).$$

Note that we may take the A_i to be the same event A in this bound, in which case $\prod_{i=1}^{n} A_i$ is the event of the presence of n disjoint instances of A. Many of our applications will take this form. The condition that ν is a complete probability measure is a technical one which does not affect our arguments, as we may assume that our vertex weight distribution ν is complete.

To state the formal realization of Step 2 of the outline, we define a k-*disjoint grid journey* to be a collection of grid journeys $\{\mathscr{L}_m : m \in [\![1, k]\!]\}$ such that there exist k disjoint curves $\gamma_1, \ldots, \gamma_k$, ordered from left to right, with γ_m intersecting each component of $\mathscr{L}_m$ for each $m \in [\![1, k]\!]$.

This can be expressed equivalently as the following constraint on the components of the $\mathscr{L}_m$, which we label as $(L_0^m, \ldots, L_N^m)$: if $L_i^m = \mathbb{L}_{i,j_{i,m}}$, then $j_{i,m} \geq j_{i,m-1}$ for each $i \in [\![0, N]\!]$, $m \in [\![1, k]\!]$. This means that, for every grid line of G of slope -1, the interval chosen for $\mathscr{L}_m$ coincides with or is to the right of that for $\mathscr{L}_{m-1}$ for every $m \in [\![1, k]\!]$. See the red and blue intervals depicted in Figure 4.

We set $\varepsilon > 0$ for the rest of the proof of Theorem 3.2 to be

$$\varepsilon = \min(1, c_1 C_3^{-1}) \cdot k^{-2/3}, \tag{12}$$

where c_1 is as in Proposition 4.2. Thus with both η and ε fixed, the grid cells of G have been fully specified.

With these definitions, we may state the next result, which obtains the $\exp(-ck^2)$ bound promised for Step 2 on the probability of there existing k disjoint paths, each constrained to be in a thin region and each of which is not too light.

Proposition 4.5. *Let $C_3 > 0$ be given and ε as in (12). There exist positive finite c_3 and C such that, for any $n \in \mathbb{N}$ and $k \in \mathbb{N}$ satisfying $n > Ck$ and any k-disjoint grid journey $\{\mathscr{L}_m : m \in [\![1, k]\!]\}$,*

$$\mathbb{P} \left(\prod_{m=1}^{k} \left\{ X_{\mathscr{L}_m} > \mu n - C_3 k^{2/3} n^{1/3} \right\} \right) \leq e^{-c_3 k^2}.$$

Proof. In view of Proposition 4.2 and the value of ε, there exists $C < \infty$ such that, for $m \in [\![1, k]\!]$ and $n > Ck$,

$$\mathbb{P} \left(X_{\mathscr{L}_m} > \mu n - c_1 \varepsilon^{-1} n^{1/3} \right) \leq e^{-c_3 \varepsilon^{-3/2}}.$$

By (12), we may apply $\varepsilon < c_1 C_3^{-1} k^{-2/3}$ on the left and $\varepsilon < k^{-2/3}$ on the right. This yields

$$\mathbb{P} \left(X_{\mathscr{L}_m} > \mu n - C_3 k^{2/3} n^{1/3} \right) \leq e^{-c_3 k}. \tag{13}$$

By the BK inequality Proposition 4.4 and (13), we obtain Proposition 4.5. $\square$

4.5. **Step 3: Bounding the number of k-disjoint grid journeys.** Following Step 3, we plan a union bound over the collection of k-disjoint grid journeys $\{\mathscr{L}_m : m \in [\![1, k]\!]\}$. For this, we need a bound on the cardinality of this collection, which we record next.

Lemma 4.6. *Let ε and η be as set earlier. Then there exists $\delta = \delta(C_3, \eta) > 0$ such that the number of k-disjoint grid journeys is bounded by $e^{\frac{1}{2} c_3 k^2}$ for $k \geq 1$, with c_3 as in Proposition 4.5.*

This lemma permits us to finish the proposed proof.

Proof of Theorem 3.2. We set δ to be as in Lemma 4.6 and ε as in Proposition 4.5. Let $\mathcal{E} = \mathcal{E}(n, k, \delta, C_3)$ be the event in the statement of Theorem 3.2. Note that

$$\mathcal{E} \subseteq \bigcup \left(\prod_{m=1}^{k} \left\{ X_{\mathscr{L}_m} > \mu n - C_3 k^{2/3} n^{1/3} \right\} \right),$$

where the union is over the collection of k-disjoint grid journeys $\{\mathscr{L}_m : m \in [\![1, k]\!]\}$. Lemma 4.6 says that the cardinality of this set is $\exp(\frac{1}{2} c_3 k^2)$, while Proposition 4.5 says that the probability of a single member of the union is at most $\exp(-c_3 k^2)$. Taking a union bound thus yields

$$\mathbb{P}\left(\mathcal{E}\right) < e^{-c_3 k^2 + \frac{1}{2} c_3 k^2} = e^{-\frac{1}{2} c_3 k^2},$$

if $n \geq C \varepsilon^{-3/2} = Ck$ and $k \in \mathbb{N}$. This proves Theorem 3.2 with $c = \frac{1}{2} c_3$. $\square$

One task remains.

Proof of Lemma 4.6. We first observe a bound on the number of ways to select the collection of intervals from the ith gridline G_i that constitutes the union of the ith component of $\mathscr{L}_m$ over $m \in [\![1, k]\!]$ for a k-disjoint grid journey $\{\mathscr{L}_m : m \in [\![1, k]\!]\}$. Raising this bound to the power N will then yield a bound on the cardinality of the set of k-disjoint grid journeys.

Assign to any collection of k intervals from G_i the M-vector whose j^{th} coordinate records the number of times $\mathbb{L}_{ij}$ was picked. The M-vector assigned has components which sum to k, and the cardinality of the set of such vectors is $\binom{k+M-1}{k}$. Not all such vectors can be achieved by the specified map, as we are ignoring the constraint imposed by the selection of intervals in G_{i-1}, and so this binomial coefficient is an upper bound. This then gives an upper bound of

$$\left[\binom{k+M-1}{k} \right]^N \tag{14}$$

on the number of k-disjoint grid journeys. Now to bound this quantity we recall that $M = 2\lceil \delta k^{1/3} \eta^{-1} \varepsilon^{-1} \rceil$. Given $\alpha > 0$, we set δ to be

$$\delta = 2^{-1} \eta (1 + c_1^{-1} C_3)^{-1} \alpha.$$

Then, since $\varepsilon^{-1} < (1 + c_1^{-1} C_3) k^{2/3}$ from the statement of Proposition 4.5, we get $M \leq \alpha k$ from (10). We recall the well-known fact [Ash90, eq. (4.7.4)] that for $0 < \beta < 1/2$ and $n \geq 1$,

$$\binom{n}{\beta n} \leq \exp\left(nH(\beta)\right), \text{ where } H(x) = -x \log x - (1-x) \log(1-x),$$

for $x \in [0, 1]$, is the entropy function. We use this to bound (14):

$$\left[\binom{k+M-1}{k} \right]^N \leq \left[\binom{(1+\alpha)k}{\alpha k} \right]^N \leq \exp\left[(1+\alpha)kN \cdot H\left(\frac{\alpha}{1+\alpha}\right) \right].$$

Using that $N = \varepsilon^{-3/2}$ and $\varepsilon = \min(1, c_1 C_3^{-1}) k^{-2/3}$, we set α small enough that the exponent is smaller than $\frac{1}{2} c_3 k^2$, since $H(x) \to 0$ as $x \to 0$. This bound holds for all $k \geq 1$, yielding

$$\left[\binom{k+M-1}{k} \right]^N \leq \exp\left(\frac{1}{2} c_3 k^2 \right).$$ $\square$

4.6. **A crude upper bound of the watermelon weight: proving Lemma 3.4.** Theorem II(1) implies that $X_n^k - \mu n k$ is typically at most a negative quantity of order $k^{5/3}n^{1/3}$. We will first need a cruder upper bound, Lemma 3.4, where this order is asserted without any claim about the sign of the difference. We begin by restating this application of the BK inequality.

Lemma 3.4. *Under Assumption 3a, we may find $c > 0$ such that, if $t > 0$, there exists $k_0 = k_0(t)$ for which $n > n_0$ and $k_0 < k \le t^{-3/4}n^{1/2}$ imply that*

$$\mathbb{P}\left(X_n^k > \mu n k + t k^{5/3} n^{1/3} \right) \le \exp\left(-c \min(t, t^{3/2}) k^2 \right).$$

If instead Assumption 3b is available, then the upper bound on k may be taken to be n.

As mentioned in Section 3, we will prove here Lemma 3.4 under only Assumption 3a, which limits the range of k up to $O(n^{1/2})$. The stronger conclusion of the lemma under Assumption 3b has a slightly more complicated argument, and this is provided in Appendix C.

Proof of Lemma 3.4 under Assumption 3a. For any fixed k-geodesic watermelon $\mathcal{W}$, let $X_{n,\mathcal{W}}^{k,j}$ be the weight of its jth heaviest curve, imitating notation introduced in Section 2. Define the event A_j by

$$A_j = \left\{ \exists \mathcal{W} : X_{n,\mathcal{W}}^{k,i} > \mu n + \frac{1}{4}t j^{-2/3} k^{4/3} n^{1/3} \text{ for all } i \in [\![1,j]\!] \right\}$$

$$= \left\{ \exists \mathcal{W} : X_{n,\mathcal{W}}^{k,j} > \mu n + \frac{1}{4}t j^{-2/3} k^{4/3} n^{1/3} \right\},$$

where the second equality is because of the monotonicity relation $X_{n,\mathcal{W}}^{k,i} \ge X_{n,\mathcal{W}}^{k,i+1}$ for all $i \in [\![1,k]\!]$. We start by claiming that, for $j \in [\![1,k]\!]$,

$$\mathbb{P}\left(A_j \right) \le \exp\left(-c t^{3/2} k^2 \right). \tag{15}$$

This is because the probability on the left-hand side is bounded by the probability that there exist j disjoint curves, each with weight at least $\mu n + \frac{1}{4}t j^{-2/3} k^{4/3} n^{1/3}$. This probability, by Assumption 3a and the BK inequality (Proposition 4.4), is bounded by

$$\left(\exp(-c t^{3/2} k^2 j^{-1}) \right)^j = \exp\left(-c t^{3/2} k^2 \right),$$

for large enough n, if $t k^{2/3}$ is large enough. For this bound, we also need $\frac{1}{4}t j^{-2/3} k^{4/3} n^{1/3} \le n$, which is provided by the assumed upper bound on k. We now *claim* that

$$\left\{ X_n^k > \mu n k + t k^{5/3} n^{1/3} \right\} \subseteq \bigcup_{j=1}^{k} A_j.$$

Indeed, suppose that A_j^c occurs for each $j \in [\![1,k]\!]$. For each $j \in [\![1,k]\!]$, by definition, on A_j^c, $X_{n,\mathcal{W}}^{k,j} \le \mu n + \frac{1}{4}t j^{-2/3} k^{4/3} n^{1/3}$ for all k-geodesic watermelons $\mathcal{W}$. This then gives, for any $\mathcal{W}$, that

$$X_n^k = \sum_{j=1}^{k} X_{n,\mathcal{W}}^{k,j} \le \mu n k + \frac{1}{4}t k^{4/3} n^{1/3} \sum_{j=1}^{k} j^{-2/3} \le \mu n k + t k^{5/3} n^{1/3},$$

which completes the proof of the claim. The derivation of Lemma 3.4 is concluded by applying (15), using the union bound, and reducing c in (15) to $c/2$ (which is possible for k large enough depending on only t). $\qquad\square$

5. Not too thin: Bounding below the transversal fluctuation

The aim of this section is to prove the lower bound on the transversal fluctuation exponent for the geodesic, i.e., Theorem II(2b). In view of Theorem 3.2, this would be immediate if we could show that, with high probability, $X_n^{k,k} \geq \mu n - C k^{2/3} n^{1/3}$ for some $C > 0$. However, in the absence of such an estimate so far, we follow the strategy outlined in Section 3, relying on an averaging argument and geodesic watermelon interlacing. More precisely, the interlacing result Proposition 2.5 implies that it is sufficient to show the existence of some $j \in [\![1, k]\!]$ such that no j-geodesic watermelon is contained in the strip $U_{n, \delta k^{1/3} n^{2/3}}$. To this end, we shall establish an averaged version of the lower bound of $X_n^{k,k}$; i.e., with high probability, that $X_n^{j,j} \geq \mu n - C k^{2/3} n^{1/3}$ for some $j \in [\![\lfloor \frac{k}{2} \rfloor, k]\!]$. This, together with the above observation and Theorem 3.2, will complete the proof.

We note in the above discussion that $X_n^{j,j}$ is only well-defined after specifying the j-geodesic watermelon, unlike X_n^j—and this we have not done in the case where ν has atoms. However, using a uniform lower bound on $X_n^{j,j}$ which holds over all possible choices of the curves of the j-geodesic watermelon, we will be able to show that the averaging statement holds for $\underline{X}_n^{j,j}$, defined for $j \in [\![1, n]\!]$ as

$$\underline{X}_n^{j,j} \;=\; \min_{\mathcal{W}} X_{n,\mathcal{W}}^{j,j},$$

where $X_{n,\mathcal{W}}^{j,j}$ is the weight of the lightest curve of the j-geodesic watermelon $\mathcal{W}$, and the minimum is over all j-geodesic watermelons.

We now state the averaging result.

Lemma 5.1. *For C_1 as in Theorem II(1), and under Assumption 3a (respectively 3b), there exist positive constants c_1, $c > 0$, C and n_0 such that, for $k \geq 1$ and $n \geq n_0$ for which $k \leq c_1 n^{1/2}$ (resp. $k \leq c_1 n$),*

$$\mathbb{P}\left(\bigcup_{j=\lfloor \frac{k}{2} \rfloor}^{k} \left\{ \underline{X}_n^{j,j} > \mu n - 4 C_1 k^{2/3} n^{1/3} \right\} \right) \geq 1 - C e^{-ck^2}.$$

Proof. Recall from (5) that, for every j, $\underline{X}_n^{j,j} \geq X_n^j - X_n^{j-1}$. Indeed, any j-geodesic watermelon is comprised of a lightest curve and $j - 1$ other curves that together weigh at most as much as the $(j-1)$-geodesic watermelon. See Figure 5. We see then that, for any k,

$$\mathbb{P}\left(\bigcup_{j=\lfloor k/2 \rfloor}^{k} \left\{ \underline{X}_n^{j,j} > \mu n - 4 C_1 k^{2/3} n^{1/3} \right\} \right) \geq \mathbb{P}\left(\bigcup_{j=\lfloor k/2 \rfloor}^{k} \left\{ X_n^j - X_n^{j-1} > \mu n - 4 C_1 k^{2/3} n^{1/3} \right\} \right).$$

We now observe that

$$\mathbb{P}\left(\bigcap_{j=\lfloor k/2 \rfloor}^{k} \left\{ X_n^j - X_n^{j-1} \leq \mu n - 4 C_1 k^{2/3} n^{1/3} \right\} \right)$$
$$\leq \mathbb{P}\left(X_n^k - X_n^{\lfloor k/2 \rfloor - 1} \leq \mu n (k + 1 - \lfloor k/2 \rfloor) - 2 C_1 k^{5/3} n^{1/3} \right).$$

From Lemma 3.4 with $t = 2^{5/3} C_1$, with the valid k range depending on whether Assumption 3b is in force in that statement, we have that $\mathbb{P}\left(X_n^{\lfloor k/2 \rfloor - 1} > \mu n (\lfloor k/2 \rfloor - 1) + C_1 k^{5/3} n^{1/3} \right) \leq e^{-ck^2}$. Thus,

$$\mathbb{P}\left(X_n^k - X_n^{\lfloor k/2 \rfloor - 1} \leq \mu n (k + 1 - \lfloor k/2 \rfloor) - 2 C_1 k^{5/3} n^{1/3} \right) \leq \mathbb{P}\left(X_n^k \leq \mu n k - C_1 k^{5/3} n^{1/3} \right) + e^{-ck^2}.$$
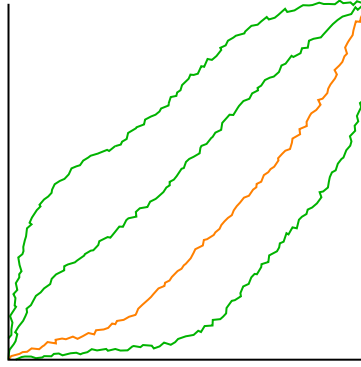
FIGURE 5. The basic argument for the inequality $X_n^{k,k} \geq X_n^k - X_n^{k-1}$. Here $k = 4$. The four curves shown are the curves of the k-melon, and the orange curve is the curve of lowest weight among the four, and so has weight $X_n^{k,k}$. The remaining green curves form a collection of three disjoint curves, and hence have cumulative weight bounded above by X_n^{k-1}. Thus $X_n^{k,k} + X_n^{k-1} \geq X_n^k$, and rearranging gives the inequality.

The weight lower bound of Theorem II(1) tells us that the first term is bounded by e^{-ck^2}. By the three preceding displays, we find that

$$\mathbb{P}\left(\bigcup_{j=\lfloor k/2 \rfloor}^{k} \left\{ \underline{X}_n^{j,j} > \mu n - 4C_1 k^{2/3} n^{1/3} \right\} \right) \geq 1 - 2e^{-ck^2},$$

so we are done.   □

We are now ready to complete the proof of Theorem II(2b).

Proof of Theorem II(2b). Under Assumption 3a (resp. 3b) and c_1 as in Lemma 5.1, fix k sufficiently large and n such that $k \leq c_1 n^{1/2}$ (resp. $k \leq c_1 n$). Let C_1 be as in Theorem II(1). Let $\delta = \delta(2^{2/3} \cdot 4C_1)$ be as in Theorem 3.2 and let $\delta' = 2^{-1/3}\delta$. Let A_k denote the event that there exist $\lfloor k/2 \rfloor$ disjoint paths contained in $U_{n,\delta' k^{1/3} n^{2/3}}$, each of which has weight at least $\mu n - 4C_1 k^{2/3} n^{1/3}$. Further, let B_k denote the event from Lemma 5.1:

$$B_k := \left\{ \bigcup_{j=\lfloor \frac{k}{2} \rfloor}^{k} \left\{ \underline{X}_n^{j,j} > \mu n - 4C_1 k^{2/3} n^{1/3} \right\} \right\}.$$

Clearly, by Theorem 3.2 and Lemma 5.1, we have $\mathbb{P}(A_k^c \cap B_k) \geq 1 - e^{-ck^2}$ for some $c > 0$.

Observe next that on $A_k^c \cap B_k$, there exists $j \in \{\lfloor \frac{k}{2} \rfloor, \ldots, k\}$ such that all the curves of all j-geodesic watermelons have weight at least $\mu n - 4C_1 k^{2/3} n^{1/3}$, and hence some of them must exit $U_{n,\delta' k^{1/3} n^{2/3}}$. By the interlacing result Proposition 2.5, the same is true for all k-geodesic watermelons. This completes the proof of Theorem II(2b) with δ there replaced by δ'.   □

6. NOT TOO WIDE: BOUNDING ABOVE THE TRANSVERSAL FLUCTUATION

Our objective in this section is to prove Theorem II(2a). We derive this result using Theorem 3.3, whose proof appears at the end of the section. The basic idea is same as in Section 5. To show that the k-geodesic watermelon has transversal fluctuation of order $k^{1/3} n^{2/3}$ with large probability, we shall rely on the interlacing property and show that, with large probability, there exists $j \in$

$[\![k, 2k]\!]$ such that the j-geodesic watermelon has transversal fluctuations at most of order $k^{1/3}n^{2/3}$. Admitting Theorem 3.3 for now, we may prove Theorem II(2a).

Proof of Theorem II(2a). Let C_1 be as in the Lemma 5.1 and let

$$B'_k = \bigcup_{j=k}^{2k} \left\{ \underline{X}_n^{j,j} > \mu n - 7C_1 k^{2/3} n^{1/3} \right\};$$

note that $4 \times 2^{2/3} < 7$, so that Lemma 5.1 with $2k$ in place of k implies that $\mathbb{P}(B'_k) \geq 1 - e^{-ck^2}$.

Let $A'_k = A'_k(C'')$ denote the event that there exists a path from $(1,1)$ to (n,n) that exits $U_{n,C''k^{1/3}n^{2/3}}$ (recall the notation from (3)) and has weight at least $\mu n - 7C_1 k^{2/3} n^{1/3}$. Now choose $C'' > 0$ (possible by invoking Theorem 3.3 with s a multiple of $k^{1/3}$) such that

$$\mathbb{P}(A'_k) \leq e^{-c''k}$$

for some $c'' > 0$, for all k sufficiently large and all n sufficiently large, depending on k. Clearly, it now suffices to show that, on $B'_k \cap (A'_k)^c$, no k-geodesic watermelon exits $U_{n,C''k^{1/3}n^{2/3}}$. By definition of B'_k, there exists $j \in [\![k, 2k]\!]$ such that all paths of all j-geodesic watermelons have weight at least $\mu n - 7C_1 k^{2/3} n^{1/3}$, and $(A'_k)^c$ ensures that all these paths are contained in $U_{n,C''k^{1/3}n^{2/3}}$. The proof is completed by interlacing, invoking Proposition 2.5, with the parameter M in Theorem II(2a) being C''. □

Now we turn to the proof of Theorem 3.3. We will follow the argument that yields Theorem 11.1 of [BSS14], but merely suppose Assumptions 2 and 3a. We first prove a companion result concerning paths that have large transversal fluctuation at the midpoint. The idea is that the weight of such a path is less than the sum of the weights of two point-to-line paths whose endpoints lie outside a central interval. We will make use of the upper tail estimate for such point-to-line weights, Proposition 3.6.

For an upright path Γ from $(1,1)$ to (r,r), and $x \in [\![1,r]\!]$, let $\Gamma(x)$ denote the unique integer such that $(x - \Gamma(x), x + \Gamma(x))$ lies on Γ. For $t \geq 1$, let

$$\mathbb{L}_{\text{left}}(r,t) = \{(x,y) : x + y = 2, |y - x| < tr^{2/3}\} \qquad \text{and}$$

$$\mathbb{L}_{\text{right}}(r,t) = \{(x,y) : x + y = 2r, |y - x| < tr^{2/3}\}$$

be line segments of length $2tr^{2/3}$ of slope -1 through $(1,1)$ and (r,r) respectively. For the sake of brevity, and in the hope that there is little scope for confusion, we shall omit the two arguments from now on.

Proposition 6.1. *Let $X_{\text{mid}}^{r,s,t}$ be the maximum weight of all paths Γ from $\mathbb{L}_{\text{left}}$ to $\mathbb{L}_{\text{right}}$ such that $|\Gamma(r/2)| > (s+t)r^{2/3}$. Then there exist constants $r_0, s_0, c > 0$ and $\tilde{c}_2 > 0$ (all independent of t) such that for $s > s_0$, $r > r_0$ and $0 < t \leq s^2$,*

$$\mathbb{P}\left(X_{\text{mid}}^{r,s,t} - \mu r > -\tilde{c}_2 s^2 r^{1/3} \right) < e^{-cs^3}.$$

Proof. Let X_1 be the maximum weight of all paths Γ from $\mathbb{L}_{\text{left}}$ to the line $x+y = r$ whose midpoint is outside the interval joining $(r/2-(s+t)r^{2/3}, r/2+(s+t)r^{2/3})$ and $(r/2+(s+t)r^{2/3}, r/2-(s+t)r^{2/3})$. Let X_2 be the same for paths starting on the line $x + y = r$ outside the mentioned interval and ending on $\mathbb{L}_{\text{right}}$. Then

$$\left\{ X_{\text{mid}}^{r,s,t} > \mu r - \tilde{c}_2 s^2 r^{1/3} \right\} \subseteq \bigcup_{i=1}^{2} \left\{ X_i > \mu \frac{r}{2} - 0.5\tilde{c}_2 s^2 r^{1/3} \right\}.$$

From here, we apply Proposition 3.6 with $\theta = 0$, which yields

$$\mathbb{P}\left(X_i > \mu\frac{r}{2} - 0.5\tilde{c}_2 s^2 r^{1/3}\right) \leq e^{-cs^3}$$

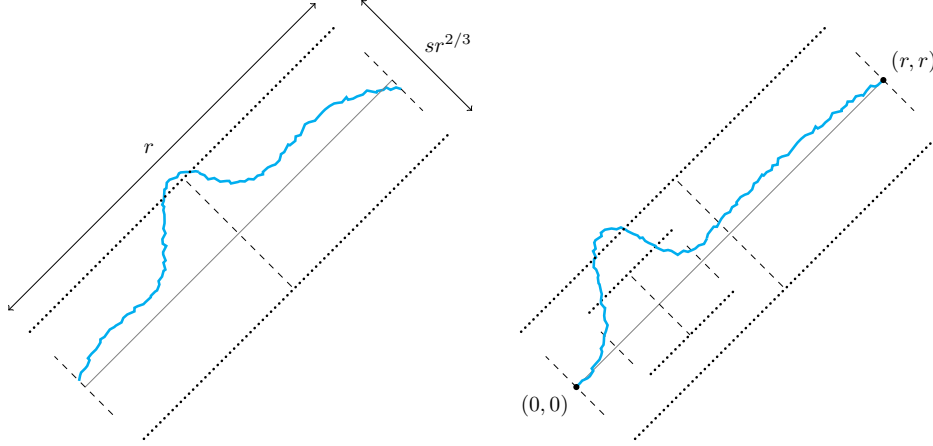for $i \in \{1, 2\}$; a union bound now completes the proof of Proposition 6.1. □

FIGURE 6. On the left is depicted the event whose probability is bounded in Proposition 6.1, namely that a path from $\mathbb{L}_{\text{left}}$ to $\mathbb{L}_{\text{right}}$ has high transversal fluctuation at the midpoint, but does not suffer an appropriately high loss in weight. On the right, we see how this allows us to infer a similar bound for the event that the curve has high transversal fluctuation (not necessarily at the midpoint) and an unusually high weight. We argue that for any such curve there exists a pair of intervals on a dyadic scale such that the path passes through both, but avoids the longer interval at their midpoint; this is the collection of smaller three intervals shown in the right panel, which can be thought of as a smaller scale version of the left panel. The loss suffered by this portion of the curve cannot be compensated by the remaining portions.

We now use a multi-scale argument to extend the result at the midpoint to transversal fluctuations at any point. Roughly, we will dyadically place points on the diagonal $x = y$ between $(1, 1)$ and (r, r). Suppose that some path has transversal fluctuation greater than $(s+t)r^{2/3}$ at some point while satisfying the weight lower bound. Then a consecutive pair of points in the dyadic division exists such that the intervening path is not too light and has a midpoint whose transversal fluctuation has order $(s + t)r^{2/3}$. The probability of this circumstance is bounded by Proposition 6.1. Bounds on path weights for each side of this dyadic interval will finish the proof.

We remind the reader from the statement of Theorem 3.3 that $X^{r,s,t}$ refers to the maximum weight over all paths $\Gamma^{r,s,t}$ from $\mathbb{L}_{\text{left}}$ to $\mathbb{L}_{\text{right}}$ with transversal fluctuation greater than $(s + t)r^{2/3}$ at some point. We now fix $\Gamma^{r,s,t}$ to be the leftmost such path (which is uniquely defined by the weight-maximization property of the path) whose weight is $X^{r,s,t}$.

Proof of Theorem 3.3. We may assume that $s \leq r^{1/3}$, as otherwise the theorem is trivial. We may also simplify by assuming that $\Gamma^{r,s,t}(x) > (s + t)r^{2/3}$ for some $x \in [1, r]$ (the case where $\Gamma^{r,s,t}(x) < -(s + t)r^{2/3}$ is symmetric). Let A be the event $\{X^{r,s,t} > \mu r - c_2 s^2 r^{1/3}\}$ for c_2 to be specified later. For $j \geq 1$, let S_j be the dyadic points, i.e.,

$$S_j = \left\{\ell 2^{-j} r : \ell \in [\![0, 2^j]\!]\right\}.$$

For $j \geq 1$, let T_j be the event defined by

$$T_j = \left\{ \Gamma^{r,s,t}(x) < (s_j + t)r^{2/3} \quad \forall x \in S_j \right\},$$

where

$$s_j = \frac{s}{M} \prod_{i=1}^{j-1}(1 + 2^{-i/3}) \quad \text{and} \quad M = 2 \cdot \prod_{i=1}^{\infty}(1 + 2^{-i/3}) < \infty, \tag{16}$$

so that $s_j < \frac{1}{2}s$ for all j. We fix j_0 so that the separation between points in S_{j_0} is $\frac{s}{10}r^{2/3}$, i.e., j_0 is such that $2^{-j_0}r = \frac{s}{10}r^{2/3}$. The next lemma asserts that the defined dyadic breakup is fine enough to capture any path which has a high transversal fluctuation at some point.

Lemma 6.2. *We have that $\bigcap_{j=1}^{j_0} T_j = \emptyset$.*

Proof. By definition of $\Gamma^{r,s,t}$, there is a $x \in [1, r]$ such that $\Gamma^{r,s,t}(x) > (s + t)r^{2/3}$. Let $\bar{x}$ be the smallest point of S_{j_0} bigger than x. Observe that since $\Gamma^{r,s,t}$ is an upright path, its y-coordinate $x + \Gamma^{r,s,t}(x)$ is increasing in x, so that $x + \Gamma^{r,s,t}(x) \leq \bar{x} + \Gamma^{r,s,t}(\bar{x})$. From the definition of S_{j_0}, we also have that $\bar{x} \leq x + \frac{s}{10}r^{2/3}$. This implies

$$\Gamma^{r,s,t}(x) \leq (\bar{x} - x) + \Gamma^{r,s,t}(\bar{x}) \leq \left(\frac{s}{10} + s_{j_0} + t\right) r^{2/3},$$

the last inequality holding on T_{j_0}. However $s_{j_0} + \frac{s}{10} + t < \frac{s}{2} + \frac{s}{10} + t < s + t$, a contradiction. $\qquad\square$

Let T_0 be the whole probability space. From Lemma 6.2 we have

$$\mathbb{P}(A) = \mathbb{P}\left(\bigcup_{j=0}^{j_0} T_j^c \cap A\right) \leq \sum_{j=1}^{j_0} \mathbb{P}\left(T_{j-1} \cap T_j^c \cap A\right).$$

We set $r > r_0$ and $s > s_0$, where r_0 and s_0 are obtained from Proposition 6.1. Then Proposition 6.1 establishes that $\mathbb{P}(T_1^c \cap A) \leq e^{-cs^3}$, and so the proof of Theorem 3.3 is completed by the next lemma. $\qquad\square$

Lemma 6.3. *Let r and s be as before. There exists $c > 0$ such that, for $j \geq 2$,*

$$\mathbb{P}\left(T_{j-1} \cap T_j^c \cap A\right) \leq 2^{-j}e^{-cs^3}.$$

Proof. We split $T_j^c = \bigcup_{h=0}^{2^j} T_{j,h}^c$ based on at which dyadic point $h2^{-j}r$ of S_j we have $\Gamma^{r,s,t}(h2^{-j}r) > (s_j + t)r^{2/3}$.

Let $\mathbb{L}_1$ be the line segment of length $2(s_{j-1}+t)r^{2/3}$ of slope -1 centred at $((h-1)2^{-j}r, (h-1)2^{-j}r)$, and let $\mathbb{L}_2$ be the same centred at $((h+1)2^{-j}r, (h+1)2^{-j}r)$.

Now, on $T_{j-1} \cap T_{j,h}^c \cap A$, there exists a path Γ_1 from $\mathbb{L}_{\text{left}}$ to $\mathbb{L}_1$, a path Γ_2 from $\mathbb{L}_1$ to $\mathbb{L}_2$, and a path Γ_3 from $\mathbb{L}_2$ to $\mathbb{L}_{\text{right}}$ with the following properties: (i) $\Gamma_2(h2^{-j}r) > (s_j + t)r^{2/3}$ and (ii) $\ell(\Gamma_1) + \ell(\Gamma_2) + \ell(\Gamma_3) > \mu r - c_2 s^2 r^{1/3}$.

We will first show that, due to condition (i), Γ_2 must suffer a large weight loss, via Proposition 6.1. Then we will show using Proposition 3.5 that Γ_1 and Γ_3 are unlikely to be able to make up this loss sufficiently well for (ii) to occur. The basic reason for the large weight loss of Γ_2 is that the loss from the transversal fluctuation is much amplified by its being defined on the scale $2^{-(j-1)}r$ instead of r.

We set the parameters for the application of Proposition 3.5. In order to avoid confusion, the parameter values will be distinguished by tildes. So set $\tilde{r} = 2^{-(j-1)}r$. As the endpoint of Γ_2 can lie anywhere on an interval of length $2(s_{j-1} + t)r^{2/3}$, set $\tilde{t}$ such that $\tilde{t}(\tilde{r})^{2/3} = (s_{j-1} + t)r^{2/3}$; i.e., $\tilde{t} = (s_{j-1} + t)2^{2(j-1)/3}$. The path Γ_2 at its midpoint it must be at least $(s_j + t)r^{2/3}$ away from the

diagonal, so the minimum transversal fluctuation the path undergoes is $((s_j + t) - (s_{j-1} + t))r^{2/3} = s_{j-1}2^{-(j-1)/3}r^{2/3}$ (using (16)). Accordingly we set $\tilde{s} = s_{j-1}2^{(j-1)/3}$, so that $\tilde{s}(\tilde{r})^{2/3} = (s_j - s_{j-1})r^{2/3}$.

To apply Proposition 6.1 with these parameters, we require $\tilde{t} \leq (\tilde{s})^2$. Since the hypotheses of Theorem 3.3 include that $t \leq s$, and since $s_j \in [s/M, s/2]$ for all j, the requirement is implied if

$$\frac{3s}{2} \cdot 2^{2(j-1)/3} \leq \frac{s^2}{M^2} \cdot 2^{2(j-1)/3},$$

which clearly holds for all large enough s. So, making use of distributional translational invariance of the environment and applying Proposition 6.1, we obtain

$$\mathbb{P}\left(\ell(\Gamma_2) > \mu\tilde{r} - \tilde{c}_2(\tilde{s})^2(\tilde{r})^{1/3}\right) \leq \exp\left(-c(\tilde{s})^3\right) ;$$

i.e.,

$$\mathbb{P}\left(\ell(\Gamma_2) > \mu\tilde{r} - \tilde{c}_2 s_{j-1}^2 2^{(j-1)/3} r^{1/3}\right) \leq \exp\left(-c2^{(j-1)}s^3\right) .$$

This yields

$$\mathbb{P}\left(\sum_{i=1}^{3} \ell(\Gamma_i) > \mu r - c_2 s^2 r^{2/3}\right) \leq \mathbb{P}\left(\ell(\Gamma_2) > \mu\tilde{r} - \tilde{c}_2 2^{(j-1)/3} s_{j-1}^2 r^{1/3}\right)$$

$$+ \mathbb{P}\left(\ell(\Gamma_1) + \ell(\Gamma_3) > \mu(r - \tilde{r}) + (\tilde{c}_2 2^{(j-1)/3} s_{j-1}^2 - c_2 s^2)r^{1/3}\right) .$$

We must address the second term. Set $c_2 = \tilde{c}_2/2M^2$, so that $\tilde{c}_2 s_{j-1}^2/2 > c_2 s^2$. After dividing $\mathbb{L}_1$ and $\mathbb{L}_2$ into segments of length $r^{2/3}$, applying Proposition 3.5, and taking a union bound over the segments, we get that the second probability is bounded by $\exp\left(-c2^{(j-1)/2}s^3\right)$.

For large enough s, these two quantities are bounded by $4^{-j}e^{-cs^3}$, and taking a union bound over the 2^j values of h, we obtain Lemma 6.3. □

Theorem 3.3 readily implies that the transversal fluctuation of the geodesic is of order $n^{2/3}$, with the optimal tail exponent of three.

Corollary 6.4. *There exist constants s_0 and n_0 such that for $s > s_0$ and $n > n_0$,*

$$\mathbb{P}\left(\mathrm{TF}(\Gamma_n) > sn^{2/3}\right) \leq e^{-cs^3}.$$

Proof. We have

$$\mathbb{P}\left(\mathrm{TF}(\Gamma_n) > sn^{2/3}\right) \leq \mathbb{P}\left(\mathrm{TF}(\Gamma_n) > sn^{2/3}, X_n > \mu n - c_2 s^2 n^{1/3}\right) + \mathbb{P}\left(X_n \leq \mu n - c_2 s^2 n^{1/3}\right)$$

$$\leq e^{-cs^3} + e^{-cs^3},$$

the last inequality from Theorem 3.3 for the first term and Assumption 3a for the second. □

We end this section by recording a version of Theorem 3.3 for point-to-line paths which will be used in Section 11.

Theorem 6.5. *Let $\Gamma_{\mathrm{line}}^{r,s}$ be the maximum weight path among all paths Γ from $(1,1)$ to the line $x + y = 2r$ with $\mathrm{TF}(\Gamma) > sr^{2/3}$, and let $X_{\mathrm{line}}^{r,s}$ be its weight. Under Assumptions 2 and 3a, there exist absolute constants r_0, s_0, $c > 0$ and $c_3 > 0$ such that, for $s_0 < s < r^{1/3}$ and $r > r_0$,*

$$\mathbb{P}\left(X_{\mathrm{line}}^{r,s} - \mu r > -c_3 s^2 r^{1/3}\right) \leq e^{-cs^3}.$$

Proof. Let x be such that $\Gamma_{\mathrm{line}}^{r,s}$ has endpoint $(r-x, r+x)$. Let A denote the event that $|x| < sr^{2/3}/2$. On the event A, we have the claimed bound by applying Theorem 3.3 with $t = s/2$. On A^c, the claim arises by applying Proposition 3.6 with $t = 0$, $\theta = 0$, and $s/2$ in place of s in its statement. □

7. NOT TOO HEAVY: BOUNDING ABOVE THE WATERMELON WEIGHT

Our aim in this section is to complete the proof of the weight upper bound of Theorem II(1) and to provide the proof of Proposition 10.1. Recall that for the former our rough aim is to prove that there exist $C_2 > 0$ and $c > 0$ such that, for appropriate ranges of k and n,

$$\mathbb{P}\left(X_n^k > \mu n k - C_2 k^{5/3} n^{1/3}\right) \leq e^{-ck^2}. \tag{17}$$

We start by proving that with high probability at least $k/4$ curves of the k-geodesic watermelon must exit a strip $U = U_{n,\frac{1}{2}\delta k^{1/3} n^{2/3}}$ around the diagonal of width $\frac{1}{2}\delta k^{1/3} n^{2/3}$, with δ as in Theorem 3.2. We will resort to proving an averaged version, as in Lemma 5.1; i.e., we will show that there exists $j \in [\![\lfloor k/2 \rfloor, k]\!]$ such that $k/4$ curves of the j-melon exit U. This will suffice by the interlacing guaranteed by Proposition 2.5.

For $j \in [\![1, n]\!]$ and $\delta > 0$, define $\underline{E}_n^j(\delta)$ to be the minimum of the number of curves of a j-geodesic watermelon Γ_n^j which *exit* U, minimized over all j-geodesic watermelons Γ_n^j.

Lemma 7.1. *Let $C_1 > 0$ be as given in Theorem II(1), and let $\delta = \delta(4C_1)$ be as in Theorem 3.2. There exists $c_1 > 0$ such that, under Assumption 3a (resp. 3b), for $k \leq c_1 n^{1/2}$ (resp. $k \leq c_1 n$),*

$$\mathbb{P}\left(\bigcup_{j=\lfloor k/2 \rfloor}^{k} \left\{\underline{E}_n^j(\delta) > \frac{k}{4}\right\}\right) \geq 1 - e^{-ck^2}.$$

Proof. Recall from Section 5 the definition of $\underline{X}_n^{j,j}$ as the minimum weight of the lightest curve of a j-geodesic watermelon $\mathcal{W}$ over all such $\mathcal{W}$. We will bound below

$$\mathbb{P}\left(\bigcup_{j=\lfloor k/2 \rfloor}^{k} \left\{\underline{X}_n^{j,j} > \mu n - 4C_1 k^{2/3} n^{1/3}\right\} \cap \left\{\underline{E}_n^j(\delta) > \frac{k}{4}\right\}\right),$$

which clearly suffices. We have

$$\bigcap_{j=\lfloor k/2 \rfloor}^{k} \left(\left\{\underline{X}_n^{j,j} \leq \mu n - 4C_1 k^{2/3} n^{1/3}\right\} \cup \left\{\underline{E}_n^j(\delta)) \leq \frac{k}{4}\right\}\right)$$

$$\subseteq \left(\bigcap_{j=\lfloor k/2 \rfloor}^{k} \left\{\underline{X}_n^{j,j} \leq \mu n - 4C_1 k^{2/3} n^{1/3}\right\}\right) \cup \left(\bigcup_{j=\lfloor k/2 \rfloor}^{k} \left\{\underline{E}_n^j(\delta) \leq \frac{k}{4}, \underline{X}_n^{j,j} > \mu n - 4C_1 k^{2/3} n^{1/3}\right\}\right).$$

Focus on the right-hand side of this display. Lemma 5.1 says that the probability of the first term in parentheses is bounded by e^{-ck^2}. For the second term, note that, on each inner event, there exist $3k/4$ curves contained in $U_{n,\frac{1}{2}\delta k^{1/3} n^{2/3}}$ whose weight is at least $\mu n - 4C_1 k^{2/3} n^{1/3}$. Theorem 3.2 then implies that the probability of the inner event is bounded by e^{-ck^2} for all j; and a union bound, accompanied by a reduction in the value of c, completes the proof. $\qquad\square$

Proof of the weight upper bound in Theorem II(1). Let B_1 denote the event that for no $j \in [\![\lfloor k/2 \rfloor, k]\!]$ are there at least $k/4$ curves in every j-geodesic watermelon that exit the strip of width $\frac{1}{2}\delta k^{1/3} n^{2/3}$. By Lemma 7.1,

$$\mathbb{P}(B_1) \leq e^{-ck^2}. \tag{18}$$

By interlacing Proposition 2.6, we see that when B_1^c occurs, at least $k/4$ of the curves of every k-geodesic watermelon must exit the strip of width $\frac{1}{2}\delta k^{1/3}$: see Figure 7.
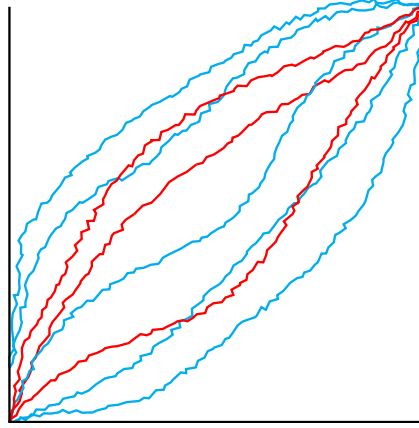
FIGURE 7. An illustration of how curves of Γ_n^j exiting a strip force the same for curves of Γ_n^k for $j = 3$ (red) and $k = 5$ (blue), where the two ensembles are elements of an interlacing family of gedoesic watermelons. The underlying observation is that interlacing allows one to match each curve of Γ_n^j with one curve of Γ_n^k which has larger transversal fluctuation. Perhaps contrary to intuition, interlacing does *not* imply that a single curve of Γ_n^j exiting a strip forces at least $k - j$ curves of Γ_n^k to exit the same strip. This can also be seen in the figure, as there is only a single curve of Γ_n^k which has a larger transversal fluctuation on the left than Γ_n^j even though $k - j = 2$.

Let B_2 denote the event that there exists an upright path Γ from $(1,1)$ to (n,n) for which $\ell(\Gamma) > \mu n - c' k^{2/3} n^{1/3}$ and $\mathrm{TF}(\Gamma) > \frac{1}{2}\delta k^{1/3} n^{2/3}$. By Theorem 3.3, there exists $c' > 0$ such that

$$\mathbb{P}(B_2) \leq e^{-ck}. \tag{19}$$

Set $A = \left\{ X_n^k > \mu n k - \frac{1}{16} c' k^{5/3} n^{1/3} \right\}$ and

$$B_3 = \left\{ X_n^{\lfloor 7k/8 \rfloor} > \mu n \cdot \lfloor 7k/8 \rfloor + \frac{1}{16} c' k^{5/3} n^{1/3} \right\},$$

so that, by Lemma 3.4 under Assumption 3a (resp. 3b), for $k \leq c_1 n^{1/2}$ (resp. $k \leq c_1 n$),

$$\mathbb{P}(B_3) \leq e^{-ck^2}. \tag{20}$$

We now fix some k-geodesic watermelon Γ_n^k, and let $X_n^{k,j}$ be the weight of the jth heaviest curve of Γ_n^k for each $j \in [\![1, k]\!]$. The weight of the $\lfloor 7k/8 \rfloor$ heaviest curves of Γ_n^k must be at most the weight of the $\lfloor 7k/8 \rfloor$-geodesic watermelon. Thus, when $A \cap B_3^c$ occurs,

$$X_n^{\lfloor 7k/8 \rfloor} + \left(X_n^{k, \lfloor 7k/8 \rfloor + 1} + \ldots + X_n^{k,k} \right) \geq X_n^k > \mu n k - \frac{1}{16} c' k^{5/3} n^{1/3}$$

$$\implies X_n^{k, \lfloor 7k/8 \rfloor + 1} + \ldots + X_n^{k,k} > \frac{1}{8} \mu n k - \frac{1}{8} c' k^{5/3} n^{1/3}.$$

By the ordering of $\{ X_n^{k,i} : i \in [\![1, k]\!] \}$, this then implies that $X_n^{k, \lfloor 7k/8 \rfloor + 1} > \mu n - c' k^{2/3} n^{1/3}$. Again by the ordering, this bound applies to $X_n^{k,1}, \ldots, X_n^{k, \lfloor 7k/8 \rfloor}$ as well. This means we have $\lfloor 7k/8 \rfloor$ disjoint curves Γ, each with $\ell(\Gamma) > \mu n - c' k^{2/3} n^{1/3}$.

Thus, on $A \cap B_1^c \cap B_3^c$ and by the pigeonhole principle, we must have at least $\lfloor k/8 \rfloor$ disjoint curves Γ, each satisfying $\mathrm{TF}(\Gamma) > \frac{1}{2}\delta k^{1/3} n^{2/3}$ and $\ell(\Gamma) > \mu n - c' k^{2/3} n^{1/3}$. By the BK inequality and (19), the probability of this occurrence is seen to be at most $\exp\left(-ck \cdot k \right) = \exp\left(-ck^2 \right)$. Noting

the bounds (18) and (20) on $\mathbb{P}(B_1)$ and $\mathbb{P}(B_3)$, we complete the proof of the weight upper bound of Theorem II(1), by taking C_2 in its statement to be c'.   □

8. Not too light: Bounding below the watermelon weight

In this section, we construct collections of disjoint paths in the square $[\![1, n]\!]^2$ that achieve a certain weight with high probability and so prove Theorem 3.1. This construction is one of the principal new tools developed in this paper.

Recall that to prove Theorem 3.1, for some $c > 0$, $c_1 > 0$, $k \leq c_1 n$, and $m \leq k$, we must construct m disjoint paths $\gamma_1, \ldots, \gamma_m$ such that $\sum_{i=1}^m \ell(\gamma_i) \geq \mu n m - C_1 m k^{2/3} n^{1/3}$ and $\max_i \mathrm{TF}(\gamma_i) \leq 2m k^{-2/3} n^{1/3}$, with probability at least $1 - \exp(-cmk)$.

8.1. **The construction in outline.** The construction leading to Theorem 3.1 may be explained in light of Theorem I(2) on the width of the k-geodesic watermelon having order $k^{1/3} n^{2/3}$, even if the latter assertion is a consequence of the theorem rather than a means for deriving it. Indeed, that k watermelon curves coexist in a strip of width $k^{1/3} n^{2/3}$ suggests that, at least around the mid-height $n/2$, adjacent curves will separated on the order of $k^{-2/3} n^{2/3}$. We will demand this separation for the m curves in our construction. The curves will begin near $(1, 1)$ and end near (n, n) at unit-order distance, so we must guide them apart to become separated during their mid-lives.

We will index the life of paths in the square $[\![1, n]\!]^2$ according to distance along the diagonal interval, indexed so that $[a, b]$ refers to the region between the lines $x + y = 2a$ and $x + y = 2b$. The diagonal interval $[1, n]$ that indexes the whole life of paths in the construction will be divided into five consecutive intervals called *phases* that carry the names take-off, climb, cruise, descent and landing. By the start of the middle, cruise, phase, the sought separation has been obtained, and it will be maintained there. This separation is gained during take-off and climb, and it is undone in a symmetric way during descent and landing.

Take-off is a short but intense phase that takes the curves at unit-order separation on the tarmac to a consecutive separation of order $k^{-1/3} n^{1/3}$ in a duration (or height) of order $k^{2/3} n^{1/3}$. Climb is a longer and gentler phase, of duration roughly $n/3$, in which separation expands dyadically until it reaches the scale $k^{-2/3} n^{2/3}$. Cruise is a stable phase of rough duration $n/3$.

The shortfall in weight of the m paths in Theorem 3.1 relative to the linear term $\mu n m$ has order $m k^{2/3} n^{1/3}$. The weight shortfall in each phase is the difference in total weight contributed by the curve fragments in the phase and the linear term given by the product of μm and the duration of the phase. The weight shortfall will be shown to have order $m k^{2/3} n^{1/3}$ for each of the five phases.

Take-off is a phase where gaining separation is the only aim. No attempt is made to ensure that the constructed curves have weight, and the trivial lower bound of zero on weight is applied. The weight shortfall is thus at most $\mu m \cdot \Theta(1) k^{2/3} n^{1/3}$.

The climb phase is where a rapid increase in separation is obtained, via a doubling of separation across dyadic scales. A careful calculation is needed to bound above the shortfall by $m k^{2/3} n^{1/3}$ for the climb (and for the descent) phase. The calculation will estimate the loss incurred across the dyadic scales using the parabolic loss in weight of Assumption 2. This assumption plays a crucial role due to the rapid increase in separation of the curves, which causes a corresponding increase in anti-diagonal separation of the paths being considered. The climb and descent phases are at the heart of the construction.

Cruise must maintain consecutive separation of order $k^{-2/3} n^{2/3}$ for a duration of roughly $n/3$. We construct paths that travel through an order of k consecutive boxes of height $k^{-1} n$ and width $k^{-2/3} n^{2/3}$. As the KPZ one-third exponent for energy predicts, each passage of a path across a box incurs weight shortfall of order $k^{-1/3} n^{1/3}$. Cruise weight shortfall is thus of order $m \cdot k \cdot k^{-1/3} n^{1/3} = m k^{2/3} n^{1/3}$.

Now we turn to giving the details of each phase. As in Section 4, we adopt a rounding convention for coordinates which are not integers, but, in contrast, we will be ignoring the resulting terms of ± 1. More precisely, all expressions for coordinates of points should be rounded down, but extra terms of ± 1 which thus arise in non-coordinate quantities, such as expected values of weights, will be absorbed into constants without explicit mention.

8.2. The construction in detail.

8.2.1. *Take-off.* In this phase, the m curves will travel from $(1,1),\ldots,(1,m)$ to the line $x+y = 2k^{2/3}n^{1/3}$. Since we will make no non-trivial claim about the weight of take-off curves, we may choose these m curves to be any disjoint upright paths, with the ith starting at $(1,i)$ and ending at

$$\left(k^{2/3}n^{1/3} - \left(\frac{m}{2} - i\right) k^{-1/3}n^{1/3},\ k^{2/3}n^{1/3} + \left(\frac{m}{2} - i\right) k^{-1/3}n^{1/3}\right). \tag{21}$$

The next statement suffices to show that these m disjoint paths exist; we omit the straightforward proof.

Lemma 8.1. *Suppose given m starting points $\{(1,i) : i \in [\![1,m]\!]\}$; and m ending points $\{(x_i, y_i) : i \in [\![1,m]\!]\}$ on the line $x+y = r$, for some $r \geq m+1$, with $1 \leq x_1 < x_2 < ... < x_m \leq m$. Then there exist m disjoint paths such that the ith connects $(1,i)$ to (x_{m-i+1}, y_{m-i+1}).*

The hypothesis of this lemma that $2k^{2/3}n^{1/3} \geq m+1$ is satisfied when $n \geq 1$ because $m \leq k \leq n$.

8.2.2. *Climb.* This phase concerns the construction of curves as they pass through diagonal coordinates between $k^{2/3}n^{1/3}$ and a value h that we will specify in (26). In Lemma 8.2, we will learn that $cn \leq h \leq n/2$; the climb phase thus has duration $\Theta(n)$, since k is supposed to be at most a small constant multiple of n.

During climb, the order of separation rises from $k^{-1/3}n^{1/3}$ to $k^{-2/3}n^{2/3}$. Separation will double during each of several segments into which the phase will be divided. The number N of segments is chosen to satisfy

$$2^N \in k^{-1/3}n^{1/3} \cdot [1, 2). \tag{22}$$

A depiction of one of the segments is shown in Figure 8a.

In order that the constructed curves remain disjoint and incur a modest weight shortfall, we will insist that they pass through a system of disjoint parallelograms whose geometry respects KPZ scaling: the width of each parallelogram, and the anti-diagonal offset between its lower and upper sides, will have the order of the two-thirds power of the parallelogram's height.

Segments $[2\ell_{j-1}, 2\ell_j]$ will be indexed by $j \in [\![1, N]\!]$, with

$$\ell_j := \ell_{j-1} + 3^{-1}2^{3(j-1)/2}k^{1/2}n^{1/2}, \tag{23}$$

and $\ell_0 = k^{2/3}n^{1/3}$. Climb begins where takeoff ends, at the diagonal coordinate $2\ell_0$: see (21).

By level j, we mean $\{(x,y) \in \mathbb{Z}^2 : x+y = 2\ell_j\}$. The ith curve will intersect level j at a unique point $(\ell_j - p_{i,k}^{(j)}, \ell_j + p_{i,k}^{(j)})$, where $p_{i,k}^{(j)}$ is inductively defined by

$$p_{i,k}^{(j)} := p_{i,k}^{(j-1)} + \left(\frac{m}{2} - i\right) 2^{j-1}k^{-1/3}n^{1/3} \tag{24}$$

from initial data $p_{i,k}^{(0)} := (m/2 - i)k^{-1/3}n^{1/3}$ that is chosen consistently with (21).

We indicated that separation would double during each segment in climb from an initial value of $k^{-1/3}n^{1/3}$. This is what our definition ensures: the separation $\text{sep}_k^{(j)}$ between positions on the jth level is given by

$$\text{sep}_k^{(j)} := p_{i,k}^{(j)} - p_{i-1,k}^{(j)} = 2^j k^{-1/3}n^{1/3}, \tag{25}$$
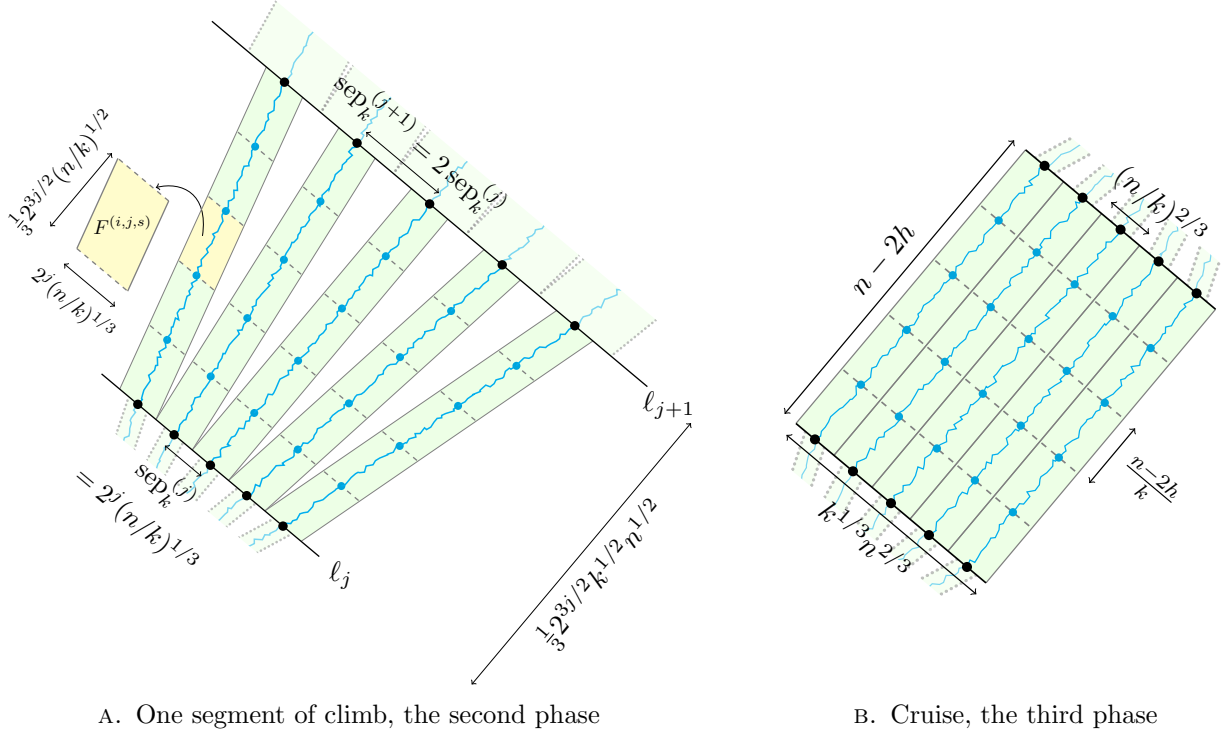
A. One segment of climb, the second phase   B. Cruise, the third phase

FIGURE 8. In Panel A is a depiction of climb, between levels j and $j + 1$. Here $m = k = 5$ and the highlighted flight corridor (small parallelogram) is $F^{(i,j,s)}$ for $i = 1$ (first curve) and $s = 3$ (third sublevel). Observe that any large green parallelogram with opposite sides on ℓ_j and ℓ_{j+1} does *not* have an on-scale aspect ratio as $(\mathrm{sep}_k^{(j)})^{3/2} = 2^{3j/2}k^{-1/2}n^{1/2}$ is a factor of k short of matching the separation along the diagonal between the two lines. However, after dividing the large parallelogram into k smaller parallelograms $F^{(i,j,s)}$, the aspect ratio becomes on-scale as the small parallelogram's height is exactly $\frac{1}{3}2^{3j/2}k^{-1/2}n^{1/2}$. Also depicted in lower opacity is how the construction continues on a larger and smaller scale in the succeeding and preceding levels. In Panel B is depicted the simpler cruise phase, and how the second phase connects to it on either side. Here h is the distance along the diagonal occupied by the first and second phases on the lower side.

where the latter equality is due to (24); and $\mathrm{sep}_k^{(0)} = p_{i,k}^{(0)} - p_{i-1,k}^{(0)} = k^{-1/3}n^{1/3}$.

For indices i that differ from the midpoint value $m/2$ by a unit order, the curve of index i has an anti-diagonal displacement of order $2^{j-1}k^{-1/3}n^{1/3}$ in its traversal between levels $j-1$ and j; see Figure 9 for a depiction of anti-diagonal displacement. The $(3/2)^{\text{th}}$ power $2^{3(j-1)/2}k^{-1/2}n^{1/2}$ would seem a natural candidate for the value of ℓ_j. Our definition in (23) includes a further factor of k, reflecting the greater separation available for curves at the edge, for which the index i is close to zero or m. This dilation by k influences the form of construction in the climb phase, and we will discuss it further soon.

The height h that marks the end of the climb phase may now be set:

$$h := \ell_N. \tag{26}$$

Flight corridors. We have indicated that the curves under construction will be forced to pass through certain disjoint parallelograms. The latter regions will be called *flight corridors*. We may consider a flight corridor delimited by levels j and $j+1$ of width $\mathrm{sep}_k^{(j)}$. The width $2^j k^{-1/3}n^{1/3}$ and
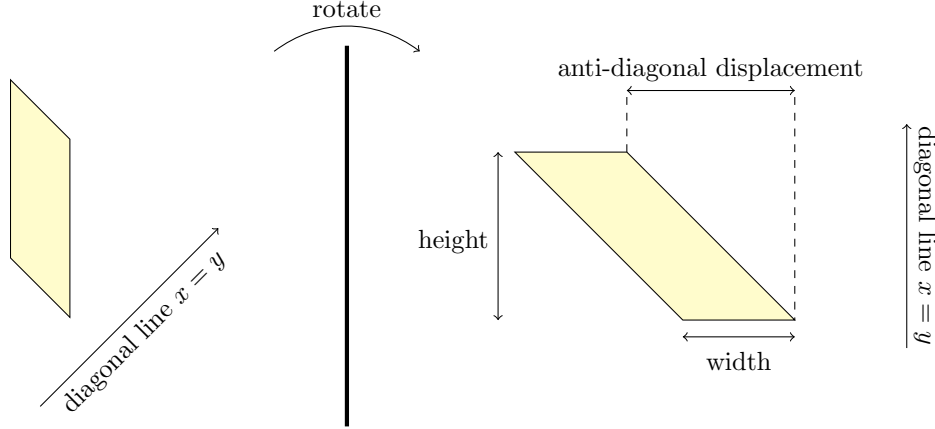
FIGURE 9. In order that the weight loss of the heaviest path constrained to lie in a parallelogram be on-scale, we need the anti-diagonal displacement and width of the parallelogram to both be of order $(\text{height})^{2/3}$. On the left is the parallelogram in the original coordinates, which has been rotated on the right for clarity so that the diagonal $x = y$ is visually vertical.

height $\frac{1}{3}2^{3j/2}k^{1/2}n^{1/2}$ would, however, violate the relation width $\approx \text{height}^{2/3}$ due to a mismatch in the exponent of k. The culprit is the extra factor of k in the definition (23).

The desired aspect ratio and anti-diagonal displacement conditions for all the curves will be obtained by use of a system of k consecutive flight corridors for each curve. Let $s \in [\![1, k]\!]$. Consider the planar line segment that runs from

$$\left(1 - \frac{s-1}{k}\right)\left(\ell_j - p_{i,k}^{(j)},\, \ell_j + p_{i,k}^{(j)}\right) + \frac{s-1}{k}\left(\ell_{j+1} - p_{i,k}^{(j+1)},\, \ell_{j+1} + p_{i,k}^{(j+1)}\right) + (1, 0)$$
$$\text{to} \tag{27}$$
$$\left(1 - \frac{s}{k}\right)\left(\ell_j - p_{i,k}^{(j)},\, \ell_j + p_{i,k}^{(j)}\right) + \frac{s}{k}\left(\ell_{j+1} - p_{i,k}^{(j+1)},\, \ell_{j+1} + p_{i,k}^{(j+1)}\right).$$

The sth flight corridor $F^{(i,j,s)}$ between levels j and $j+1$ for the ith curve consists of those $(x, y) \in \mathbb{Z}^2$ that are displaced from some element in the just indicated line segment by a vector $(t, -t)$ for some t with absolute value at most $2^{-1}\text{sep}_k^{(j)} - 1$. Thus the flight corridors associated to the ith curve for different values of i are disjoint. See Figure 8. The addition of $(1, 0)$ in the first line of (27) is to ensure that the consecutive flight corridors of the same curve are disjoint. The sth subpath of the ith curve from level j to $j+1$ is defined to be a path of maximum weight between the two planar points just displayed that remains in the flight corridor $F^{(i,j,s)}$. Its weight will be denoted by $X_{n,k,2}^{(i,j,s)}$. Note that the sth subpath's ending point in $F^{(i,j,s)}$ and the $(s+1)^{\text{th}}$ subpath's starting point in $F^{(i,j,s+1)}$ are adjacent to each other, which allows us to concatenate these curves.

The height of $F^{(i,j,s)}$ is $k^{-1}\left(\ell_{j+1} - \ell_j\right) = 3^{-1}2^{3j/2}k^{-1/2}n^{1/2}$ and its width is $\text{sep}_k^{(j)} = 2^j k^{-1/3}n^{1/3}$; so the flight corridor satisfies the relation height $= \frac{1}{3} \cdot \text{width}^{3/2}$.

Anti-diagonal displacement at opposite ends of the flight corridor is highest for curves 1 and m. For these paths, this displacement between levels j to $j+1$ is

$$k^{-1}(p_{i,k}^{(j+1)} - p_{i,k}^{(j)}) = \frac{1}{2}2^j m k^{-4/3}n^{1/3}$$

in view of (24). As the height gain in $F^{(i,j,s)}$ is $3^{-1}2^{3j/2}k^{-1/2}n^{1/2}$, and $m \le k$, the anti-diagonal displacement for these corridors is at most $C \cdot (\text{height})^{2/3}$.

8.2.3. *Cruise.* Curves enter cruise at separation $k^{-2/3}n^{2/3}$, and must be maintained there for a duration of order n: see Figure 8b. More precisely, and recalling that $h = \ell_N$, the ith curve enters cruise at

$$\left(h - \left(\frac{m}{2} - i \right) k^{-2/3}n^{2/3}, h + \left(\frac{m}{2} - i \right) k^{-2/3}n^{2/3} \right).$$

We set $\ell_0' = h$, and $\ell_j' = \ell_{j-1}' + \frac{n-2h}{k} = h + (n - 2h) \cdot \frac{j}{k}$ for $j \in [\![1, k]\!]$. We will demand that the ith curve visits each point

$$\left(\ell_j' - \left(\frac{m}{2} - i \right) k^{-2/3}n^{2/3}, \ell_j' + \left(\frac{m}{2} - i \right) k^{-2/3}n^{2/3} \right). \tag{28}$$

Between consecutive points, this curve will be be constrained to remain in a flight corridor that comprises those vertices in $\mathbb{Z}^2$ that are displaced from the planar line segment that interpolates the pair of points by a vector $(t, -t)$ for some t with absolute value at most $2^{-1}k^{-2/3}n^{2/3}$, which will be made to be disjoint by a unit displacement of the bottom side, as in (27). Let $X_{n,k,3}^{(i,j)}$ be the weight of the ith curve in the jth corridor; namely, the maximum weight of paths that interpolate the endpoint pair and remain in the corridor.

Momentarily, we will verify that $h = \Theta(n)$. The new flight corridor thus satisfies the relation

$$\text{width} \in \left[C^{-1} \cdot \text{height}^{2/3}, C \cdot \text{height}^{2/3} \right]$$

for a positive constant C. The corridor has no anti-diagonal displacement.

Lemma 8.2. *There exist positive c_1 and c such that if $k < c_1 n$, then $h = \ell_N$ satisfies $cn \le h \le n/2$ for large enough n.*

Proof. This inference follows by noting that

$$h = \ell_0 + \sum_{j=0}^{N-1} 3^{-1}2^{3j/2}k^{1/2}n^{1/2} = \ell_0 + \frac{1}{3(2\sqrt{2} - 1)} \left(2^{3N/2} - 1 \right) k^{1/2}n^{1/2} \tag{29}$$

$$= k^{2/3}n^{1/3} + \frac{1}{3(2^{3/2} - 1)} \left(n - (kn)^{1/2} \right). \qquad \square$$

8.2.4. *Descent and landing.* These two phases are specified symmetrically with climb and take-off.

8.3. **Bounding below expected weight in the construction.** We will argue, in two steps, that the constructed curves attain the sought weight $\mu nm - \Theta(mk^{2/3}n^{1/3})$ with high probability. In this section, we will show that, under Assumptions 2 and 3a, the expected total curve weight is at least $\mu nm - Cmk^{2/3}n^{1/3}$ for a constant $C > 0$. In the next, the desired bound on curve weight will be obtained by showing that this weight concentrates around its mean due to the independence of contributions from the various flight corridors.

We will control the contribution from a given flight corridor using the lower bound on the expectation of the constrained point-to-point weight from Proposition 3.7. That proposition will permit us to derive the next result, the conclusion of the present section.

Proposition 8.3. *Let X denote the sum of the weights of the m curves in our construction. There exist $n_0 \in \mathbb{N}$, $C > 0$ and $c_1 > 0$ such that, for $n > n_0$, $k < c_1 n$ and $m \le k$, $\mathbb{E}[X] \ge \mu nm - Cmk^{2/3}n^{1/3}$.*

Proof. The values of C and c may change from line to line but they do not depend on n, m, or k.

We start with the weight of a curve fragment between two levels during climb. That is, we find a lower bound on $\sum_{s=1}^{k} \mathbb{E}[X_{n,k,2}^{(i,j,s)}]$, where recall that $X_{n,k,2}^{(i,j,s)}$ the weight of the heaviest path that travels between the points (27) without exiting the flight corridor $F^{(i,j,s)}$.

In specifying the climb phase, we noted that the condition in Proposition 3.7 that the anti-diagonal displacement of the flight corridor is of order $r^{2/3}$ holds. Thus, using Proposition 3.7 with the settings $r = k^{-1}(\ell_{j+1} - \ell_j)$ and $zr^{2/3} = k^{-1}(p_{i,k}^{(j+1)} - p_{i,k}^{(j)})$, we obtain

$$\mathbb{E}[X_{n,k,2}^{(i,j,s)}] \geq \frac{\mu}{k}(\ell_{j+1} - \ell_j) - C\frac{\left(p_{i,k}^{(j+1)} - p_{i,k}^{(j)}\right)^2}{k(\ell_{j+1} - \ell_j)} - \frac{c}{k^{1/3}}(\ell_{j+1} - \ell_j)^{1/3}. \tag{30}$$

We simplify the first two terms of (30) using the expressions from (23) and (24) to obtain

$$\frac{\mu}{3k}2^{3j/2}k^{1/2}n^{1/2} - C\frac{\frac{1}{k^2}\left(\frac{m}{2} - i\right)^2 2^{2j}k^{-2/3}n^{2/3}}{\frac{1}{3k}2^{3j/2}k^{1/2}n^{1/2}}$$

$$= \frac{\mu}{3}2^{3j/2}k^{-1/2}n^{1/2} - C\left(\frac{m}{2} - i\right)^2 2^{j/2}k^{-13/6}n^{1/6}.$$

For the last term of (30) we have, from (23),

$$\frac{c}{k^{1/3}}\left(\ell_2^{(j+1)} - \ell_2^{(j)}\right)^{1/3} = \frac{c}{k^{1/3}} \cdot 2^{j/2}k^{1/6}n^{1/6} = c \cdot 2^{j/2}k^{-1/6}n^{1/6},$$

where we have absorbed the constant factor of $3^{-1/3}$ into the value of c. Combining the three preceding displays and summing over s and i gives a lower bound on the expected weight of all the curves between levels j and $j+1$:

$$\sum_{i=1}^{m}\sum_{s=1}^{k}\mathbb{E}[X_{n,k,2}^{(i,j,s)}] \geq \frac{\mu}{3} \cdot 2^{3j/2}mk^{1/2}n^{1/2} - C \cdot 2^{j/2}m^3 k^{-7/6}n^{1/6} - c2^{j/2}mk^{5/6}n^{1/6}$$

$$\geq \frac{\mu}{3} \cdot 2^{3j/2}mk^{1/2}n^{1/2} - C \cdot 2^{j/2}mk^{5/6}n^{1/6};$$

here we used that $m^3 \leq mk^2$ to bound the second term in the latter inequality. Summing from $j = 0$ to $N - 1$ gives a lower bound for the total weight of the climb phase of

$$\sum_{i=1}^{m}\sum_{j=0}^{N-1}\sum_{s=1}^{k}\mathbb{E}[X_{n,k,2}^{(i,j,s)}] \geq \frac{\mu m}{3} \cdot \sum_{j=0}^{N-1}2^{3j/2}k^{1/2}n^{1/2} - C \cdot 2^{N/2}mk^{5/6}n^{1/6}$$

$$= \mu(h - k^{2/3}n^{1/3}) \cdot m - Cmk^{4/6}n^{1/3} = \mu hm - Cmk^{2/3}n^{1/3}, \tag{31}$$

where $2^{N/2} = k^{-1/6}n^{1/6}$ from (22) and the expression of h from (29) were used.

As for cruise, an easy computation from Proposition 3.7 yields that

$$\mathbb{E}[X_{n,k,3}^{(i,j)}] \geq \frac{\mu(n-2h)}{k} - ck^{-1/3}(n-2h)^{1/3} \geq \frac{\mu(n-2h)}{k} - ck^{-1/3}n^{1/3}. \tag{32}$$

We write $\overline{X}_{n,k,2}^{(i,j,s)}$ for the descent counterpart of the climb flight corridor weight maximum. The bound on mean valid for climb holds equally for descent. Combining climb, descent and cruise weight bounds with the zero bound for take-off and landing, the total weight X in our construction is seen to satisfy

$$X \geq \sum_{i,j,s}\left(X_{n,k,2}^{(i,j,s)} + \overline{X}_{n,k,2}^{(i,j,s)}\right) + \sum_{i,j}X_{n,k,3}^{(i,j)}. \tag{33}$$

Taking expectation, and applying (31) and (32), we obtain

$$\mathbb{E}\left[\sum_{i,j,s}\left(X_{n,k,2}^{(i,j,s)} + \overline{X}_{n,k,2}^{(i,j,s)}\right) + \sum_{i,j}X_{n,k,3}^{(i,j)}\right] \geq \mu nm - Cmk^{2/3}n^{1/3},$$

the conclusion of Proposition 8.3. □

8.4. **One-sided concentration of the construction weight.** We now know that the expected weight of the construction is correct. To prove Theorem 3.1, we argue that the weight is unlikely to fall much below its mean. We will stochastically dominate the summands in the expression for X from (33) with independent exponential random variables of varying parameters, and use the following concentration result from [Jan18] for sums of such variables. We denote by $\mathrm{Exp}(\lambda)$ the exponential distribution with rate λ.

Proposition 8.4 (Theorem 5.1 (i) of [Jan18]). *Let* $W = \sum_{i=1}^{n} W_i$ *where* $W_i \sim \mathrm{Exp}\,(a_i)$ *are independent. Define*

$$\nu := \mathbb{E}W = \sum_{i=1}^{n} \mathbb{E}W_i = \sum_{i=1}^{n} \frac{1}{a_i}, \quad a_* := \min_{i} a_i.$$

Then for $\lambda \geq 1$,

$$\mathbb{P}\left(W \geq \lambda\nu\right) \leq \lambda^{-1} e^{-a_* \nu(\lambda - 1 - \log \lambda)}.$$

Proof of Theorem 3.1. Let $K \geq 0$ and $0 < L_1 \leq L_2$. As before, let $U = U_{r, \ell r^{2/3}, zr^{2/3}}$ be a parallelogram with width $\ell r^{2/3}$, height r, and anti-diagonal displacement $zr^{2/3}$, where $|z| \leq K$ and $L_1 \leq \ell \leq L_2$. Let X_r^U denote the weight of the heaviest midpoint-to-midpoint path that lies in U. By Proposition 3.7, and $\mathbb{E}[X_r^U] \leq \mathbb{E}[X_r] \leq \mu r$ (which is due to Assumption 2), there exists $c_4 = c_4(L_1, L_2, K) > 0$ such that, for $r > r_0 = r_0(L_1, L_2, K)$ and $\theta > \theta_0 = \theta_0(L_1, L_2, K)$,

$$\mathbb{P}\left(X_r^U - \mathbb{E}[X_r^U] < -\theta r^{1/3}\right) \leq e^{-c_4 \theta};$$

or

$$\mathbb{P}\left(X_r^U - \mathbb{E}[X_r^U] < -\theta\right) \leq \exp(-c_4 \theta / r^{1/3}), \tag{34}$$

the latter for θ larger than $\theta_0 r^{1/3}$ and $r > r_0$. Equation (34) says that we have the stochastic domination

$$-\left(X_r^U - \mathbb{E}[X_r^U]\right) \leq_{\mathrm{sd}} \mathrm{Exp}(c_4 / r^{1/3}) + \theta_0 r^{1/3},$$

where, in an abuse of notation, $\mathrm{Exp}(\lambda)$ denotes a random variable with the exponential distribution of rate λ, and $X \leq_{\mathrm{sd}} Y$ denotes that the distribution of X is stochastically dominated by that of Y. Thus, in our construction, we have a coupling

$$-\left(X_{n,k,2}^{(i,j,s)} - \mathbb{E}[X_{n,k,2}^{(i,j,s)}]\right) \leq W_2^{(i,j,s)} + \theta_0 2^{j/2} k^{-1/6} n^{1/6},$$

$$-\left(\overline{X}_{n,k,2}^{(i,j,s)} - \mathbb{E}[\overline{X}_{n,k,2}^{(i,j,s)}]\right) \leq \overline{W}_2^{(i,j,s)} + \theta_0 2^{j/2} k^{-1/6} n^{1/6}, \quad \text{and} \tag{35}$$

$$-\left(X_{n,k,3}^{(i,j)} - \mathbb{E}[X_{n,k,3}^{(i,j)}]\right) \leq W_3^{(i,j)} + \theta_0 k^{-1/3} n^{1/3}.$$

Here the random variables on the right-hand side are independent and distributed as

$$W_2^{(i,j,s)} \sim \mathrm{Exp}\left(c_4 2^{-j/2} k^{1/6} n^{-1/6}\right),$$

$$\overline{W}_2^{(i,j,s)} \sim \mathrm{Exp}\left(c_4 2^{-j/2} k^{1/6} n^{-1/6}\right), \quad \text{and} \tag{36}$$

$$W_3^{(i,j)} \sim \mathrm{Exp}\left(c_4 k^{1/3} n^{-1/3}\right).$$

By our construction, K, L_1, and L_2 in Proposition 3.7 may be chosen independently of i and j. Thus, whatever the dependence of c_4 and θ_0 on i and j, they are uniformly bounded away from 0 and ∞ respectively, and so may be assumed to be constant. We set

$$W := \sum_{i,j,s} \left(W_2^{(i,j,s)} + \overline{W}_2^{(i,j,s)}\right) + \sum_{i,j} W_3^{(i,j)}.$$

We wish to use Proposition 8.4 on the sum W of independent exponential random variables, and so we must estimate Proposition 8.4's parameters. We have

$$\mathbb{E}\left[\overline{W}_2^{(i,j,s)}\right] = \mathbb{E}\left[W_2^{(i,j,s)}\right] = c_4^{-1}2^{j/2}k^{-1/6}n^{1/6} \qquad \text{and} \qquad \mathbb{E}\left[W_3^{(i,j)}\right] = c_4^{-1}k^{-1/3}n^{1/3}.$$

Summing the expressions of the last display over the indices and using that $2^{N/2} = k^{-1/6}n^{1/6}$ from (22) gives that the total mean ν satisfies

$$\nu := \mathbb{E}[W] \leq 8c_4^{-1}mk^{2/3}n^{1/3}. \tag{37}$$

Noting that the coefficients of θ_0 in (35) are the same as the mean of the corresponding exponential random variable up to a factor of c_4, similarly summing these coefficients shows the stochastic domination

$$-(X - \mathbb{E}[X]) \leq W + 8\theta_0 mk^{2/3}n^{1/3}.$$

Using that $2^N = k^{-1/3}n^{1/3}$ from (22), we see that the minimum a^* of the rates of the exponential random variables defined in (36) is given by

$$a^* = \min(c_4 2^{-N/2}k^{1/6}n^{-1/6}, c_4 k^{1/3}n^{-1/3}) = c_4 k^{1/3}n^{-1/3}. \tag{38}$$

Let $S = \max(c_4^{-1}, \theta_0)$. By Proposition 8.3, with this result contributing the value of C, and the stochastic domination (35),

$$\mathbb{P}\Big(X < \mu nm - (C + 20S)mk^{2/3}n^{1/3}\Big) = \mathbb{P}\Big(-(X - \mathbb{E}[X]) > 20Smk^{2/3}n^{1/3}\Big)$$

$$\leq \mathbb{P}\left(W \geq 20Smk^{2/3}n^{1/3} - 8\theta_0 mk^{2/3}n^{1/3}\right).$$

Since $20S - 8\theta_0 \geq 12S \geq 12c_4^{-1}$, the latter quantity is bounded using Proposition 8.4, (37), and (38):

$$\mathbb{P}\left(W > 12c_4^{-1}mk^{2/3}n^{1/3}\right) \leq \mathbb{P}\left(W > \frac{12}{8}\nu\right) \leq \frac{2}{3}\exp\left(-a^*\nu\left(\frac{3}{2} - 1 - \log\frac{3}{2}\right)\right)$$

$$\leq \exp\left(-cmk\right).$$

Numerical evaluation of the exponent shows that we may take $c = 3/4$.

In view of the paragraphs after (27) and (28), flight corridors during climb, descent and cruise lie within the strip around the diagonal of width $2mk^{-2/3}n^{2/3}$; thus, the transversal fluctuations of the constructed curves in these phases also satisfy this bound. By setting $C_1 = C + 20S$, we complete the proof of Theorem 3.1. $\qquad \square$

Remark 8.5. The argument just given shows that, for positive constants c and x_0 and $x \geq x_0$,

$$\mathbb{P}\Big(X < \mu nm - (C + x)mk^{2/3}n^{1/3}\Big) \leq \exp\left(-cmkx\right).$$

9. COMBINATORICS OF WATERMELON GEOMETRY

Here we prove three geometric results that we have invoked several times; namely, Propositions 2.2, 2.5, and 2.7. The arguments are geometric and combinatorial, rather than probabilistic, and, we believe, of independent interest. Proposition 2.2 asserts that a k-geodesic watermelon exists whose curves begin and end at certain prescribed points on the boundary of the square $[\![1, n]\!]^2$.

We begin by defining a partial order on paths which we will need in the proof of Proposition 2.2. Having a partial order will allow us to consider maximal elements, which will have nice geometric properties.

Given two disjoint upright paths γ and γ' in $[\![1, n]\!]^2$, we say γ' is vertically above γ, denoted $\gamma \preceq_v \gamma'$, if

$$\exists x \in [\![1, n]\!] \text{ such that } \max\{y : (x, y) \in \gamma\} \leq \min\{y : (x, y) \in \gamma'\},$$

where $\min \emptyset = -\infty$ and $\max \emptyset = \infty$. Because we want this to be a partial order, we additionally impose reflexivity, i.e., $\gamma \preceq_{\mathrm{v}} \gamma$ for all upright paths γ. That this partial order is anti-symmetric, i.e., $\gamma \neq \gamma'$ implies that not both $\gamma \preceq_{\mathrm{v}} \gamma'$ and $\gamma' \preceq_{\mathrm{v}} \gamma$ hold, follows from the disjointness and upright nature of the two curves. In fact, two disjoint curves are comparable if and only if their projections on to the x-axis are not disjoint; we will refer to these projections as the x-*projections* of the curves.

Proof of Proposition 2.2. The proof is an induction on k. The inductive hypothesis asserts that, for any m and n, and given a collection of $k \leq \min(m, n)$ disjoint upright paths $\{\phi_i : i \in [\![1, k]\!]\}$ in $[\![1, m]\!] \times [\![1, n]\!]$, there exist k disjoint upright paths $\gamma_1, \ldots, \gamma_k$ such that

(1) $\cup_{j=1}^{k} \phi_j \subseteq \cup_{j=1}^{k} \gamma_j$; and
(2) for $i \in [\![1, k]\!]$, γ_i starts at $(1, k - i + 1)$ and ends at $(m, n - i + 1)$.

Taking $m = n$ and $\phi_1, \ldots, \phi_k$ to be the collection of curves in the given k-geodesic watermelon ϕ_n^k, it is easy to see that this statement implies Proposition 2.2: the set of k disjoint paths $\{\gamma_i : i \in [\![1, k]\!]\}$ is a k-geodesic watermelon, because its weight is at least that of ϕ_n^k, in view of $\xi_v \geq 0$ for $v \in \mathbb{Z}^2$.

Now we prove the inductive hypothesis. The base case $k = 1$ is easy to verify. Assuming this statement at index $k - 1$, we derive it at index k. Relabel the given curves if required so that ϕ_1 is the topmost, and, if there is a tie, the leftmost. More precisely, consider the set of maximal elements of $\{\phi_i : i \in [\![1, k]\!]\}$ under the $\preceq_{\mathrm{v}}$ order; as they are disjoint and mutually incomparable, the maximal elements must have disjoint x-projections, and we label the maximal element with leftmost x-projection as ϕ_1. Now remove the top row and the left column; which is to say, consider the restriction of $\phi_2, \ldots, \phi_k$ to $[\![2, m]\!] \times [\![1, n - 1]\!]$. This is a collection of $k - 1$ disjoint upright paths because $\phi_j, j \in [\![2, k]\!]$, are disjoint and upright. Applying the assumed form of the inductive hypothesis to this collection, we obtain paths $\gamma_2', \ldots, \gamma_k'$, with γ_i' from $(2, k - i + 1)$ to $(m, n - i + 1)$, such that $\cup_{j=2}^{k} \gamma_j'$ contains $\cup_{j=2}^{k} \phi_j \cap ([\![2, m]\!] \times [\![1, n - 1]\!])$ for $i \in [\![2, k]\!]$. We select $\gamma_2', \ldots, \gamma_k'$ to be *maximally to the right*; i.e., such that there is no collection of disjoint curves $\tilde{\gamma}_2, \ldots, \tilde{\gamma}_k$ that satisfy the three displayed conditions with $\gamma_i' \preceq \tilde{\gamma}_i$ for all $i \in [\![2, k]\!]$ and $\gamma_j' \neq \tilde{\gamma}_j$ for some $j \in [\![2, k]\!]$. Finally we add the vertex $(1, k - i + 1)$ to γ_i' and call the resulting path γ_i, for each $i \in [\![2, k]\!]$.

It remains to construct γ_1. Starting from $(1, k)$, γ_1 first follows the vertical segment to $(1, y_0)$, where y_0 is the y-coordinate of the top vertex of $\cup_{j=2}^{k} \phi_j$ on the $x = 1$ line; if no such vertex exists, we set $y_0 = k$. We now follow the vertices of ϕ_1 which are not elements of $\cup_{j=2}^{k} \gamma_j$ till we reach the last such vertex (x_1, y_1). This is possible because Lemma 9.2 ahead asserts that the leftmost vertex of $\phi_1 \setminus \cup_{j=2}^{k} \gamma_j$ is reachable from $(1, y_0)$ and because Lemma 9.1, also ahead, says that no vertex of $\phi_1 \setminus \cup_{j=2}^{k} \gamma_j$ is to the right of γ_2, which implies that the vertices of $\phi_1 \setminus \cup_{j=2}^{k} \gamma_j$ form a union of path fragments which may be joined without intersecting γ_2. Now γ_1 follows the vertical segment from (x_1, y_1) to (x_1, n), if $y_1 < n$, and then horizontally to (m, n).

The displayed condition (2) is thus satisfied by $\{\gamma_i : i \in [\![1, k]\!]\}$. It also immediately follows from the construction that the vertices of ϕ_1 and the restriction of $\cup_{j=2}^{k} \phi_j$ to $[\![1, m]\!] \times [\![1, n - 1]\!]$ are contained in $\cup_{j=1}^{k} \gamma_j$; the only vertices left to be accounted for are the vertices of $\cup_{j=2}^{k} \phi_j$ on the line $y = n$. But since ϕ_1 is the leftmost maximal element, all these vertices are included in the final horizontal segment of γ_1. Indeed, if not, there would be a vertex of ϕ_{j_0} on the line $y = n$ with x-coordinate strictly smaller than x_1 for some $j_0 \in [\![2, k]\!]$. Then either $\phi_1 \preceq_{\mathrm{v}} \phi_{j_0}$ or, if these paths are not comparable, ϕ_1 is not the leftmost maximal element; either way, a contradiction. Thus $\{\gamma_j : j \in [\![1, k]\!]\}$ satisfies the inductive hypothesis at index k, as we sought to show. $\square$

Lemma 9.1. *Any vertex of ϕ_1 that lies to the right of γ_2 belongs to $\cup_{i=2}^{k} \gamma_i$.*

Proof. In search of a contradiction, suppose the contrary. Vertices not lying in $\cup_{i=2}^{k}\gamma_i$ will be called uncovered. Let j be the smallest index such that an uncovered vertex (x', y') of ϕ_1 lies between γ_j and γ_{j+1}.

We claim the following: No vertex of $\cup_{j=2}^{k}\phi_j$ lies in the quadrant whose south-east corner is (x', y'), i.e., $\{(x, y) : x \leq x', y \geq y'\}$. To prove this, suppose to the contrary that there is a vertex of ϕ_{j_0} contained in this quadrant for some $j_0 \in [\![2, k]\!]$. Consider the maximal element $\bar{\phi}$ of some chain starting from ϕ_{j_0}. The union of the x-projections of the elements of this chain forms an interval. Let I be the x-projection of $\bar{\phi}$, which is an interval. We have two cases: (i) $\max I < x'$ and (ii) $\max I \geq x'$. Case (i) contradicts that ϕ_1 is the leftmost maximal element. In case (ii), the union of the x-projections of the elements of the chain contains x' and hence, by the disjointness of $\{\phi_i : i \in [\![1, k]\!]\}$, there is an element of the chain which is larger in the $\preceq_v$ order than ϕ_1, which is again a contradiction.

Consider the alternative path γ'_j formed by concatenating four curves: the portion of γ_j up to and including the rightmost vertex on the line $y = y'$; the horizontal segment from this point to (x', y'); the vertical segment from (x', y') to the lowest vertex of γ_j on the line $x = x'$; and the portion of γ_j from this latter point onwards. Note that replacing γ_j by γ'_j maintains the displayed conditions (1) and (2) in the proof of Proposition 2.2 due to the above claim which ensures that no vertex in $\cup_{j=2}^{k}\phi_j \cap ([\![2, m]\!] \times [\![1, n-1]\!])$ becomes uncovered. However, the set $\{\gamma_i : i \in [\![2, k]\!]\}$ was constructed to be maximally to the right, in contradiction to the last inference. $\qquad\square$

Lemma 9.2. *The leftmost vertex of $\phi_1 \setminus \cup_{j=2}^{k}\gamma_j$ is reachable from $(1, y_0)$, as defined in the proof of Proposition 2.2, by an upright path.*

Proof. Let the leftmost lowermost vertex of $\phi_1 \setminus \cup_{j=2}^{k}\gamma_j$ be (x', y'). We first note that Lemma 9.1 implies that $y' \geq k$. So we only need to consider the case of the definition of y_0 where there is at least one vertex of $\cup_{j=2}^{k}\phi_j$ on the $x = 1$ line.

We must prove that $y' \geq y_0$, as obviously $x' \geq 1$. Without loss of generality, assume that $(x_0, y_0) \in \phi_2$. Consider the maximal elements of all chains that ϕ_2 is a member of, and consider the leftmost such maximal element $\bar{\phi}$. We claim that $\bar{\phi} = \phi_1$, which clearly implies that $y' \geq y_0$.

To prove this claim, note that the union of the x-projections of the elements of this chain form a single interval; call it I. Recall how we labelled ϕ_1 and note that it precludes ϕ_1's x-projection from being disjoint from I, and so there must be an element of the chain, say ϕ_{j_0}, whose x-projection is not disjoint from that of ϕ_1. Now since ϕ_{j_0} and ϕ_1 are disjoint, and by the labelling of ϕ_1, we must have $\phi_{j_0} \preceq_v \phi_1$. But this implies that $\bar{\phi} = \phi_1$, completing the proof. $\qquad\square$

9.1. **Proof of melon interlacing, Proposition 2.5.** With Proposition 2.2, which allows us to pick the curves of geodesic watermelons to begin and end at the bottom-left and top-right corners of $[\![1, n]\!]^2$, in hand, we now move to showing Proposition 2.5, which says that we may also pick the curves such that they interlace. Instead of assuming that ν is continuous as in that proposition, we will prove the following deterministic statement, whose hypotheses hold almost surely when ν has no atoms. For a subset $A \subseteq [\![1, n]\!]^2$ of vertices, we first define its weight by

$$\ell(A) := \sum_{v \in A} \xi_v.$$

Proposition 9.3. *Suppose the vertex weights $\{\xi_v : v \in [\![1, n]\!]^2\}$ are non-negative and such that no two distinct subsets $A, B \subseteq [\![1, n]\!]^2$ have equal weight. Then the consecutive terms in the unique sequence of geodesic watermelons $\{\gamma_n^j : j \in [\![1, n]\!]\}$ produced by Proposition 2.2 interlace.*

The sequence of geodesic watermelons produced by Proposition 2.2 is unique because the set of vertices of the k-geodesic watermelon is unique for every $k \in [\![1, n]\!]$ under the hypotheses of Proposition 9.3.

Proof of Proposition 2.5. This is immediate since when ν has no atoms and has support contained in $[0, \infty)$, the hypotheses of Proposition 9.3 are almost surely satisfied. □

After proving Proposition 9.3, we will upgrade it to Proposition 2.6, which handles the case that ν has atoms, by a perturbation argument. We next introduce some notation to facilitate the proof of Proposition 9.3.

Recall the partial order $\preceq$ introduced in Definition 2.3, where informally $\gamma \preceq \gamma'$ means γ is to the left of γ'. Then as in (1), for any k, let $\phi_{n,\to}^{k,i}$ be the curves of ϕ_n^k produced by Proposition 2.2, ordered from left to right, i.e., $\phi_{n,\to}^{k,1} \prec \phi_{n,\to}^{k,2} \prec \ldots \prec \phi_{n,\to}^{k,k}$. For a set of vertices A, we call $\{\gamma_1, \ldots, \gamma_m\}$ a *path fragment decomposition* of A if each γ_i is a connected component of A (under the connectivity structure of $\mathbb{Z}^2$) as well as an upright path and $A = \cup_{i=1}^m \gamma_i$; note that not all sets have a path fragment decomposition. We then call the γ_i *path fragments*.

For $j \in [\![1, k-1]\!]$, let R_j^k be the region between $\phi_{n,\to}^{k,j}$ and $\phi_{n,\to}^{k,j+1}$, defined by

$$R_j^k = \left\{ (x,y) \in [\![1,n]\!]^2 : \phi_{n,\to}^{k,j} \preceq \gamma_{(x,y)} \preceq \phi_{n,\to}^{k,j+1} \right\},$$

where $\gamma_{(x,y)}$ is the singleton path at (x,y). Recall that an order relation between two paths is well-defined only if their time ranges are not disjoint. Since the elements of $\{\phi_{n,\to}^{k,j} : j \in [\![1,k]\!]\}$ do not share the same time range, for certain (x,y), it may be that only one of the inequalities $\phi_{n,\to}^{k,j} \preceq \gamma_{(x,y)}$ and $\gamma_{(x,y)} \preceq \phi_{n,\to}^{k,j+1}$ is well-defined (regardless of whether it is true or false). In these cases, the definition of R_j^k above will only require that inequality for which the corresponding time-ranges are not disjoint to be true, and will disregard the remaining inequality which is not well-defined. We also define $R_0^k = \{(x,y) \in [\![1,n]\!]^2 : \gamma_{(x,y)} \preceq \phi_{n,\to}^{k,1}\}$ and $R_k^k = \{(x,y) \in [\![1,n]\!]^2 : \phi_{n,\to}^{k,k} \preceq \gamma_{(x,y)}\}$. We note that, by the ordering of $\{\phi_{n,\to}^{k,j} : j \in [\![1,k]\!]\}$,

$$R_{j_1}^k \cap R_{j_2}^k = \emptyset \quad \text{if} \quad |j_1 - j_2| \geq 2. \tag{39}$$

We also note that ϕ_n^k interlacing with ϕ_n^{k-1} is equivalent to $\phi_{n,\to}^{k,j} \subseteq R_{j-1}^{k-1}$ for each $j \in [\![1,k]\!]$. We say that $\phi_{n,\to}^{k,j}$ is in its *home region* if $\phi_{n,\to}^{k,j} \subseteq R_{j-1}^{k-1}$. We define the *wilderness region* W_j^k (for $\phi_{n,\to}^{k,j}$) to be the complement of R_{j-1}^{k-1}, i.e.,

$$W_j^k := [\![1,n]\!]^2 \setminus R_{j-1}^{k-1}.$$

The idea of the proof of Proposition 9.3 is the following. Consider the case that ϕ_n^k and ϕ_n^{k-1} do not interlace, and look at the excursions of $\phi_{n,\to}^{k,j}$ into the respective wilderness regions W_j^k for each $j \in [\![1,k]\!]$. For each excursion, we identify a candidate path along the home region border, i.e., a subpath of $\phi_{n,\to}^{k-1,j-1}$ or $\phi_{n,\to}^{k-1,j}$, which has the same endpoints as the excursion. We then simultaneously swap all the excursions with the candidate paths. This results in two new collections of k and $k-1$ disjoint paths (which do not necessarily interlace) which cover the same set of vertices with the same multiplicities. The weight maximality property of ϕ_n^k and ϕ_n^{k-1} will then imply that the two new collections of paths are also $(k-1)$- and k-geodesic watermelons, contradicting the uniqueness of the watermelons guaranteed by the hypothesis. See Figure 10.

To implement this plan, we define for each $j \in [\![1,k]\!]$ the set WE_j^k of *wilderness excursions* to be the path fragment decomposition of $\phi_{n,\to}^{k,j} \cap W_j^k$. The first step is to identify the border candidate paths—subpaths of the paths of ϕ_n^{k-1}—which will be swapped with the excursions into the wilderness—into elements of WE_j^k.

Lemma 9.4. *For every $\gamma \in \mathrm{WE}_j^k$, there exists a path $C_j(\gamma)$, which we call a candidate path, such that $C_j(\gamma) \subseteq \phi_{n,\to}^{k-1,j-1}$ (if γ is to the left of $\phi_{n,\to}^{k-1,j-1}$) or $C_j(\gamma) \subseteq \phi_{n,\to}^{k-1,j}$ (if γ is to the right of $\phi_{n,\to}^{k-1,j}$). Further, $(\phi_{n,\to}^{k,j} \cup C_j(\gamma)) \setminus \gamma$ and $(\phi_{n,\to}^{k-1,i} \cup \gamma) \setminus C_j(\gamma)$ are upright paths, where $i \in \{j-1, j\}$ is chosen so that $C_j(\gamma) \subseteq \phi_{n,\to}^{k-1,i}$; that is, the sets obtained by swapping the wilderness excursion with the candidate path are upright paths.*
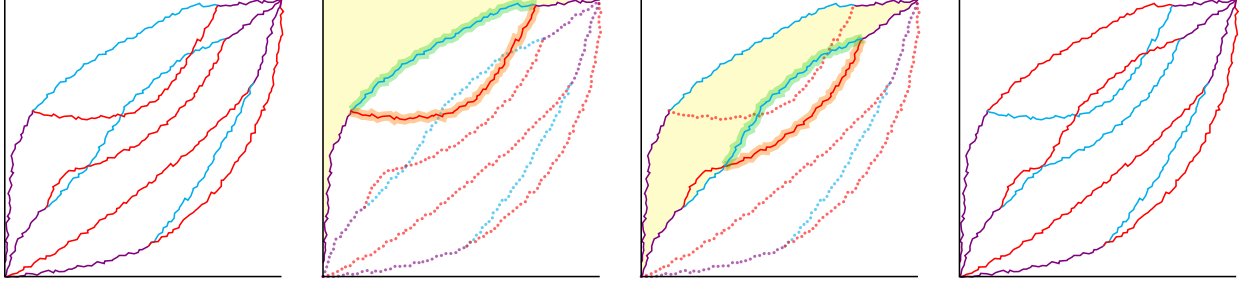
FIGURE 10. A depiction of the argument for Proposition 9.3. The red curves comprise the supposed unique 4-geodesic watermelon ϕ_n^4 and the blue curves comprise the unique 3-geodesic watermelon ϕ_n^3; vertices which are covered by both types of curves are coloured violet. Note that the left two red curves are not in their home regions (the home regions of $\phi_{n,\to}^{4,1}$ and $\phi_{n,\to}^{4,2}$ are shaded in light yellow in the second and third panels respectively), and so ϕ_n^4 is not interlacing with ϕ_n^3; we look for a contradiction. In the middle two panels, the excursions of ϕ_n^4 in their wilderness regions are highlighted in orange, and the border subsets of ϕ_n^3 they will be swapped with are highlighted in green. In the final panel we have swapped each orange excursion with the corresponding green path to result in two new collections of three or four disjoint paths, γ_n^3 and γ_n^4 (which do not necessarily interlace). But $\sum_{j=3,4} \ell(\gamma_n^j) = \sum_{j=3,4} \ell(\phi_n^j)$ since the same vertices are covered the same number of times; thus the weight maximizing property of ϕ_n^3 and ϕ_n^4 implies that $\ell(\gamma_n^j) = \ell(\phi_n^j)$ for $j = 3, 4$, contradicting the uniqueness of ϕ_n^3 and ϕ_n^4.

Proof. Let $\gamma \in \mathrm{WE}_j^k$ be a given wilderness excursion, and let u be its startpoint and v be its endpoint. Note by planarity; the ordering of the paths $\{\phi_{n,\to}^{k-1,j} : j \in [\![1, k-1]\!]\}$; and the starting and ending points of the curves of ϕ_n^k and ϕ_n^{k-1} guaranteed by Proposition 2.2 that both u and v are adjacent to $\phi_{n,\to}^{k-1,j}$ or both are adjacent to $\phi_{n,\to}^{k-1,j-1}$.

Suppose both are adjacent to $\phi_{n,\to}^{k-1,j}$; the other case is argued in the same way. Then γ must be to the right of $\phi_{n,\to}^{k-1,j}$. The points on $\phi_{n,\to}^{k-1,j}$ adjacent to u and v must also lie on $\phi_{n,\to}^{k,j}$; call these points $\underline{u}$ and $\overline{v}$. Then define $C_j(\gamma) \subseteq \phi_{n,\to}^{k-1,j}$ to be the portion of $\phi_{n,\to}^{k-1,j}$ lying between $\underline{u}$ and $\overline{v}$, excluding these two points. It is easy to see that this path satisfies the claims made in Lemma 9.4. $\square$

Having identified which subsets of vertices to swap, we define the resulting new set of k curves. For each $j \in [\![1, k]\!]$, define

$$\gamma_{n,\to}^{k,j} = \phi_{n,\to}^{k,j} \cup \left(\bigcup_{\gamma \in \mathrm{WE}_j^k} C_j(\gamma) \right) \setminus \left(\bigcup_{\gamma \in \mathrm{WE}_j^k} \gamma \right).$$

This is an upright path by Lemma 9.4. We will define an analogous collection of $k - 1$ curves by considering the other half of the swap after the next lemma, which asserts that these just defined k curves are in fact disjoint.

Lemma 9.5. $\{\gamma_{n,\to}^{k,j} : j \in [\![1, k]\!]\}$ *is a disjoint collection of k curves.*

Proof. Suppose to the contrary that, for some $j_1, j_2 \in [\![1, k]\!]$ with $j_1 \neq j_2$, there is a vertex v such that $v \in \gamma_{n,\to}^{k,j_1} \cap \gamma_{n,\to}^{k,j_2}$. Since $\phi_{n,\to}^{k,j_1} \cap \phi_{n,\to}^{k,j_2} = \emptyset$, there are two remaining cases:

(1) there exist $\gamma_1 \in \mathrm{WE}_{j_1}^k$ and $\gamma_2 \in \mathrm{WE}_{j_2}^k$ such that $v \in C_{j_1}(\gamma_1) \cap C_{j_2}(\gamma_2)$; or
(2) there exists $\gamma \in \mathrm{WE}_{j_1}^k$ such that $v \in C_{j_1}(\gamma) \cap \phi_{n,\to}^{k,j_2}$ and $v \notin \bigcup\{\gamma' : \gamma' \in \mathrm{WE}_{j_2}^k\}$ (without loss of generality).

We note immediately that Case 1 cannot occur as $C_{j_1}(\gamma_1)$ and $C_{j_2}(\gamma_2)$ are subsets of distinct curves of γ_n^{k-1}, which are disjoint. We next show that Case 2 leads to a contradiction.

There are again two cases, depending on whether $C_{j_1}(\gamma)$ is a subset of $\phi_{n,\to}^{k-1,j_1}$ or of $\phi_{n,\to}^{k-1,j_1-1}$. We handle the first as the argument for the second is similar. In that case, by planarity, $\gamma \subseteq \phi_{n,\to}^{k,j_1}$ is to the right of $\phi_{n,\to}^{k-1,j_1}$. Since $v \in C_{j_1}(\gamma) \cap \phi_{n,\to}^{k,j_2}$, it follows that $j_2 \leq j_1 - 1$; for planarity and the ordering of $\{\phi_{n,\to}^{k,j} : j \in [\![1,k]\!]\}$ precludes $j_2 \geq j_1 + 1$. But this implies that $v \in W_{j_2}^k = [\![1,n]\!]^2 \setminus R_{j_2-1}^{k-1}$ since $v \in \phi_{n,\to}^{k-1,j_1} \subseteq R_{j_1}^{k-1}$ and $R_{j_1}^{k-1} \cap R_{j_2-1}^{k-1} = \emptyset$ when $j_2 \leq j_1 - 1$ by (39).

We have arrived at a contradiction since this implies that $v \in \cup\{\gamma' : \gamma' \in \mathrm{WE}_{j_2}^k\}$.   □

We now define the new set of $k-1$ disjoint curves by doing the swap in the other way: for each $j \in [\![1, k-1]\!]$, define

$$\gamma_{n,\to}^{k-1,j} = \phi_{n,\to}^{k-1,j} \cup \left(\bigcup_{i=j}^{j+1} \bigcup_{\substack{\gamma \in \mathrm{WE}_i^k: \\ C_i(\gamma) \subseteq \phi_{n,\to}^{k-1,j}}} \gamma \right) \setminus \left(\bigcup_{i=j}^{j+1} \bigcup_{\substack{\gamma \in \mathrm{WE}_i^k: \\ C_i(\gamma) \subseteq \phi_{n,\to}^{k-1,j}}} C_i(\gamma) \right).$$

From Lemma 9.4, we see that each $\gamma_{n,\to}^{k-1,j}$ is an upright path. We also need the analogue of Lemma 9.5, that $\{\gamma_{n,\to}^{k-1,j} : j \in [\![1, k-1]\!]\}$ is a collection of $k-1$ disjoint curves. The proof bears some similarities to that of Lemma 9.5 but we provide it here for completeness. For convenience, we first define

$$A_j = \bigcup_{i=j}^{j+1} \bigcup_{\substack{\gamma \in \mathrm{WE}_i^k: \\ C_i(\gamma) \subseteq \phi_{n,\to}^{k-1,j}}} \gamma.$$

Lemma 9.6. $\{\gamma_{n,\to}^{k-1,j} : j \in [\![1, k-1]\!]\}$ *is a disjoint collection of $k-1$ curves.*

Proof. Suppose to the contrary that, for some $j_1, j_2 \in [\![1, k-1]\!]$ with $j_1 \neq j_2$, there is a vertex v such that $v \in \gamma_{n,\to}^{k-1,j_1} \cap \gamma_{n,\to}^{k-1,j_2}$. Since $\phi_{n,\to}^{k-1,j_1} \cap \phi_{n,\to}^{k-1,j_2} = \emptyset$, there are two remaining cases:

(1) $v \in A_{j_1} \cap A_{j_2}$; or
(2) $v \in \phi_{n,\to}^{k-1,j_1} \cap A_{j_2}$ and $v \notin \bigcup\{C_i(\gamma) : i \in [\![j_1, j_1 + 1]\!], \gamma \in \mathrm{WE}_i^k, C_i(\gamma) \subseteq \phi_{n,\to}^{k-1,j_1}\}$ (without loss of generality).

We immediately note that Case 1 cannot occur since any $\gamma \in \cup_{i=1}^{j}\mathrm{WE}_i^k$ can belong to at most one of the A_j, and distinct elements of $\cup_{i=1}^{j}\mathrm{WE}_i^k$ are disjoint. We next show that Case 2 leads to a contradiction.

Since $v \in A_{j_2}$, there are two possibilities: there exists $\gamma \in \mathrm{WE}_i^k$ such that $v \in \gamma$ and $C_i(\gamma) \subseteq \phi_{n,\to}^{k-1,j_2}$ for $i = j_2$ or $j_2 + 1$. We deal with the case that $i = j_2$; the argument for the other is similar.

That $C_{j_2}(\gamma) \subseteq \phi_{n,\to}^{k-1,j_2}$ implies that γ is to the right of $\phi_{n,\to}^{k-1,j_2}$ from Lemma 9.4. Thus $v \in \phi_{n,\to}^{k-1,j_1}$ is to the right of $\phi_{n,\to}^{k-1,j_2}$, yielding that $j_1 \geq j_2 + 1$ since $j_1 \neq j_2$.

Now $\phi_{n,\to}^{k,j_2+1}$ is to the right of $\phi_{n,\to}^{k,j_2}$, and so also of γ. Since $v \in \gamma$ lies on $\phi_{n,\to}^{k-1,j_1}$, which is the left boundary of $R_{j_1}^{k-1}$, we see that $\phi_{n,\to}^{k,j_2+1}$ intersects R_j^{k-1} for some $j \geq j_1 \geq j_2 + 1$, which for all such j is to the right of the home region $R_{j_2}^{k-1}$ for $\phi_{n,\to}^{k,j_2+1}$. This implies that $\phi_{n,\to}^{k,j_1}$ also intersects its wilderness region, and that it is to the right of its home region, by the ordering of $\{\phi_{n,\to}^{k,j} : j \in [\![1,k]\!]\}$ by considering the anti-diagonal line through v. But then by the definition of the candidate paths, there must exist $\tilde{\gamma} \in \mathrm{WE}_{j_1}^k$ such that $v \in C_{j_1}(\tilde{\gamma})$, which is a contradiction.   □

Finally we record a sufficient condition for there to be no wilderness path fragments which we will use shortly; note that there being no wilderness path fragments is equivalent to ϕ_n^k and ϕ_n^{k-1} interlacing.

Lemma 9.7. *If $\cup_{j=1}^k \mathrm{WE}_j^k \subseteq \phi_n^{k-1}$, then $\cup_{j=1}^k \mathrm{WE}_j^k = \emptyset$.*

The proof idea is to assume the contrary and to then consider the leftmost path $\phi_{n,\to}^{k,j}$ of ϕ_n^k which has a wilderness excursion. This excursion must be a subset of some path of ϕ_n^{k-1}. By considering an anti-diagonal line $x + y = \ell$ through a vertex of this wilderness excursion, we see that each path of ϕ_n^k to the right of $\phi_{n,\to}^{k,j}$ must have a wilderness excursion which intersects this anti-diagonal line and which must be a subset of a distinct path of ϕ_n^{k-1}. But this is a contradiction as there are not enough paths in ϕ_n^{k-1} to accommodate all these paths of ϕ_n^k.

Proof of Lemma 9.7. Suppose $\cup_{j=1}^k \mathrm{WE}_j^k \neq \emptyset$. Then there is a minimum j_0 such that $\phi_{n,\to}^{k,j_0} \not\subseteq R_{j_0-1}^{k-1}$. Let $v \in \phi_{n,\to}^{k,j_0} \setminus R_{j_0-1}^{k-1}$. That $\phi_{n,\to}^{k,j_0}$ is a path implies that $\phi_{n,\to}^{k,j_0} \subseteq \phi_n^{k-1}$, and therefore there is an $i_0 \geq 1$ such that $\phi_{n,\to}^{k,j_0} = \phi_{n,\to}^{k-1,j_0+i_0}$ on the unique anti-diagonal line $x + y = \ell$ that passes through v. The ordering of $\{\phi_{n,\to}^{k,j} : j \in [\![1,k]\!]\}$ implies that, for each $i \geq 1$, $\phi_{n,\to}^{k,j_0+i}$ equals a distinct curve of ϕ_n^{k-1}, again on the line $x + y = \ell$, because the vertex of $\phi_{n,\to}^{k,j_0+i}$ on this line cannot lie in $R_{j_0+i-1}^k$. This is a contradiction as there are $k - 1 - j_0 - i_0$ curves of ϕ_n^{k-1} to the right of $\phi_{n,\to}^{k-1,j+i_0}$ but $k - j_0 - 1$ to the right of $\phi_{n,\to}^{k,j_0}$. $\square$

With the two new collections of k and $k-1$ disjoint curves, we may implement the last step of the outline of the proof of Proposition 9.3 by showing that γ_n^k and γ_n^{k-1} are new geodesic watermelons, contradicting the uniqueness of geodesic watermelons as implied by the hypothesis that distinct sets have distinct weights.

Proof of Proposition 9.3. We first note that ϕ_n^k and ϕ_n^{k-1} interlacing is equivalent to $\cup_{j=1}^k \mathrm{WE}_j^k = \emptyset$, and we will show the latter for $k \in [\![1,n]\!]$.

Fix $k \in [\![1,n]\!]$. The weight maximality property of ϕ_n^k and ϕ_n^{k-1}, combined with Lemmas 9.5 and 9.6, implies that

$$\ell(\gamma_n^k) \leq \ell(\phi_n^k) \quad \text{and} \quad \ell(\gamma_n^{k-1}) \leq \ell(\phi_n^{k-1}). \tag{40}$$

But in the specification of γ_n^k and γ_n^{k-1} compared to ϕ_n^k and ϕ_n^{k-1}, we have only swapped vertices. Thus we have

$$\ell(\gamma_n^k) + \ell(\gamma_n^{k-1}) = \ell(\phi_n^k) + \ell(\phi_n^{k-1}).$$

Combining the last display with (40) implies that

$$\ell(\gamma_n^{k-1}) = \ell(\phi_n^{k-1}) \quad \text{and} \quad \ell(\gamma_n^k) = \ell(\phi_n^k).$$

Then by the hypothesis that distinct sets have distinct weights, $\gamma_n^k = \phi_n^k$ and $\gamma_n^{k-1} = \phi_n^{k-1}$, where the equalities refer to the vertex sets. This can only happen if $\cup_{j=1}^k \mathrm{WE}_j^k = \emptyset$. This is because $\gamma_n^{k-1} = \phi_n^{k-1}$ implies that $\cup_{j=1}^k \mathrm{WE}_j^k \subseteq \phi_n^{k-1}$, which allows us to reach the conclusion of Lemma 9.7. This completes the proof of Proposition 9.3. $\square$

Next we turn to proving Proposition 2.6, which handles the case where ν may have atoms. Its proof proceeds by introducing a deterministic perturbation which makes the given k-geodesic watermelon ϕ_n^k the unique such melon; adding a second random perturbation to make all distinct subsets have distinct weights; invoking Proposition 9.3 to get interlacing; and showing that the k-geodesic watermelons in the perturbed environment are also k-geodesic watermelons in the original environment.

Proof of Proposition 2.6. Consider the perturbed environment obtained from $[\![1,n]\!]^2$ as follows: each point v in ϕ_n^k is deterministically given weight $\xi_v + \varepsilon/4n^2$, where ξ_v is the original weight of v and $\varepsilon > 0$ will be chosen later; the remaining points in $[\![1,n]\!]^2$ have weight unchanged. We now further randomly perturb weights, by increasing the weight of each $v \in [\![1,n]\!]^2$ by an independent random variable distributed uniformly on $[0, \varepsilon/8n^2]$. Then it is clear that, for every $\varepsilon > 0$, ϕ_n^k is the

vertices of the unique k-geodesic watermelon in the perturbed environment and that every distinct subset of vertices in this environment almost surely has distinct weight.

Applying Proposition 9.3 gives a sequence of interlacing geodesic watermelons $\{\widetilde{\gamma}_n^j : j \in [\![1, n]\!]\}$ (which is ε-dependent) in the perturbed environment, and the uniqueness at level k implies that the set of vertices of $\widetilde{\gamma}_n^k$ is the same as that of ϕ_n^k.

We set ε to be the minimum gap, over $j \in [\![1, n]\!]$, between the j-geodesic melon's weight and the weight of the best collection of j disjoint paths which is not a j-geodesic melon, both in the unperturbed environment. It is clear that $\varepsilon > 0$.

We now argue that $\{\widetilde{\gamma}_n^j : j \in [\![1, n]\!]\}$, which is a sequence of geodesic watermelons in the perturbed environment, must also be a sequence of geodesic watermelons in the original environment. If we let $\widetilde{\ell}(\cdot)$ denote the perturbed length, then it is easy to see that, for any subset of vertices $A \subseteq [\![1, n]\!]^2$,

$$\widetilde{\ell}(A) - \frac{3\varepsilon}{8} \le \ell(A) \le \widetilde{\ell}(A). \tag{41}$$

Let γ_n^j be a j-geodesic melon in the unperturbed environment and suppose by way of contradiction that for some $j \in [\![1, n]\!]$, $\widetilde{\gamma}_n^j$ is not a j-geodesic melon in the unperturbed environment. Then observing $\widetilde{\ell}(\gamma_n^j) < \widetilde{\ell}(\widetilde{\gamma}_n^j)$ and applying (41) with $A = \gamma_n^j$ for the first inequality and $A = \widetilde{\gamma}_n^j$ for the final inequality yields

$$\ell(\gamma_n^j) \le \widetilde{\ell}(\gamma_n^j) < \widetilde{\ell}(\widetilde{\gamma}_n^j) \le \ell(\widetilde{\gamma}_n^j) + \frac{3\varepsilon}{8}.$$

But the definition of ε implies that this inequality does not hold. So $\widetilde{\gamma}_n^j$ must be a j-geodesic melon in the original environment for every $j \in [\![1, n]\!]$. □

9.2. **Proving Proposition 2.7.** We must prove that the sequence $Y_{n,k} = X_n^k - X_n^{k-1}$ is non-decreasing in $k > 1$, for any given set of non-negative weights. Note that $Y_{n,k} \le Y_{n,k-1}$ is equivalent to $X_n^k + X_n^{k-2} \le 2X_n^{k-1}$. To prove the latter, we will decompose the $2(k-1)$ curves of γ_n^k and γ_n^{k-2} into two sets of $k - 1$ disjoint curves and then invoke the maximality of the weight X_n^{k-1}. This is easy when $k = 3$, as Figure 11 illustrates.

The case of general k will be proved by induction. We will establish a stronger inductive hypothesis, Lemma 9.8. We remind the reader of the abuse of notation we have adopted till now where a path is sometimes identified with the set of vertices in its range. It serves the purpose of stating and proving Lemma 9.8 to adopt a further abuse, where we will sometimes regard a path as a *multi*set of vertices in its range. A multiset in this instance is an $\mathbb{N}$-valued map on $[\![1, n]\!]^2$, i.e, a subset of $[\![1, n]\!]^2$ where the elements may have multiplicities. We use the symbol $\equiv$ to denote equality of two multisets. So, for example, it will hold for two paths γ_1 and γ_2 that $(\gamma_1 \cup \gamma_2) \setminus \gamma_1 \equiv \gamma_2$, even if the two paths have portions of overlap.

Lemma 9.8. *Suppose $\gamma_1 \preceq \ldots \preceq \gamma_k$ and $\gamma_1' \preceq \ldots \preceq \gamma_{k-2}'$ are such that all $2k - 2$ curves have the same time-range (recall Definition 2.3) and no vertex is counted with multiplicity strictly greater than two in $\bigcup_{i=1}^{k} \gamma_i \cup \bigcup_{i=1}^{k-2} \gamma_i'$. Then there exist two collections A_1 and A_2 of $k - 1$ disjoint curves each such that the curves have the same time range as before and*

$$A_1 \cup A_2 \equiv \bigcup_{i=1}^{k} \gamma_i \cup \bigcup_{i=1}^{k-2} \gamma_i',$$

with $\gamma_1 \in A_1$ and $\gamma_k \in A_2$.

Here is how we will prove Lemma 9.8: regard the union of the paths as a subset of $[\![1, n]\!]^2$, peel off the left boundary and label it as one path in A_1, and peel off the right boundary and label it as a path in A_2. Now peel off the *right* boundary of the remaining set of vertices and label it as a path in A_1, and do likewise with the left boundary and A_2. Continue iteratively in this fashion,
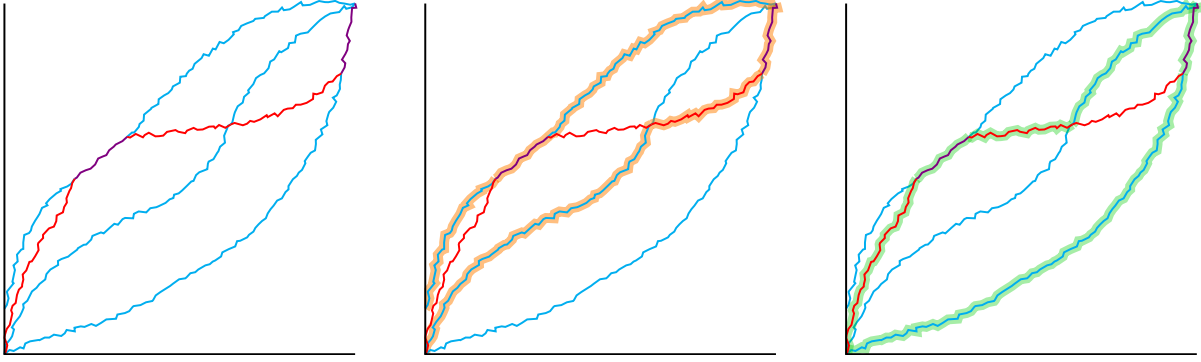
FIGURE 11. The base case of Lemma 9.8. In the first panel are the γ curves in blue and γ' curve in red (with overlap in purple), which in our application of Lemma 9.8 will be the 3-melon and 1-melon. In the remaining two panels are highlighted how these four paths can be decomposed into a pair of two disjoint paths while respecting multiplicities. The rightmost orange path in the second panel is $\max(\gamma'_1, \gamma_2)$.

switching whether the left boundary is assigned to A_1 or A_2, till all the vertices have been depleted. This will yield the desired decomposition: see Figure 11 to see that this is how the orange curves have been assigned to A_1 and the green curves to A_2.

To render a proof from this idea, we will need to define the max and min of two curves γ and γ' in order to specify precisely what we called peeling off the right and left boundaries. In each excursion between γ and γ', we will retain the right path fragment in the max, the left fragment in the min, and the overlapping portion in both. If the curves are disjoint, the right path is the max. In the notation introduced in Definition 2.3, for $t \in \mathcal{R}(\gamma) \cap \mathcal{R}(\gamma')$,

$$\max(\gamma, \gamma')(t) = \max(\gamma(t), \gamma'(t))$$
$$\min(\gamma, \gamma')(t) = \min(\gamma(t), \gamma'(t)),$$

where the max and min on the right are with respect to the normal $\leq$ ordering. Note that $\min(\gamma, \gamma') \preceq \gamma, \gamma' \preceq \max(\gamma, \gamma')$. We also record without proof the following simple statement.

Lemma 9.9. *Let γ_1, γ'_1, γ_2, and γ'_2 be upright paths with $\mathcal{R}(\gamma_1) = \mathcal{R}(\gamma_2)$, $\mathcal{R}(\gamma'_1) = \mathcal{R}(\gamma'_2)$, and $\gamma_i \preceq \gamma'_i$ for $i = 1, 2$. Then*

  (1) $\gamma_1 \cup \gamma_2 \equiv \max(\gamma_1, \gamma_2) \cup \min(\gamma_1, \gamma_2)$;
  (2) $\max(\gamma_1, \gamma_2) \preceq \max(\gamma'_1, \gamma'_2)$ and $\min(\gamma_1, \gamma_2) \preceq \min(\gamma'_1, \gamma'_2)$; and
  (3) $\max(\gamma_1, \gamma_2)$ and $\min(\gamma_1, \gamma_2)$ are upright paths.

Proof of Lemma 9.8. The proof is by induction on k. We first verify the base case of $k = 3$; see Figure 11. We take $A_1 = \{\gamma_1, \max(\gamma'_1, \gamma_2)\}$ (highlighted orange in the second panel of Figure 11). We take $A_2 = \{\min(\gamma'_1, \gamma_2), \gamma_3\}$ (highlighted green in the third panel there). We now show that A_1 is a pair of disjoint curves; the argument of the same for A_2 is similar. We observe that

$$\gamma_1 \preceq \gamma_2 \preceq \max(\gamma'_1, \gamma_2) \quad \text{and} \quad \gamma_1 \preceq \gamma'_1 \preceq \max(\gamma'_1, \gamma_2).$$

Thus any vertex in both γ_1 and $\max(\gamma'_1, \gamma_2)$ must be a member of γ_1, γ_2, and γ'_1, contradicting the multiplicity assumption. This completes the case $k = 3$.

Now we move to the inductive step. The idea is similar to the base case's. Consider the set

$$B \equiv \bigcup_{i=2}^{k-1} \gamma_i \cup \bigcup_{i=1}^{k-2} \gamma'_i;$$

i.e., remove the left and right boundaries γ_1 and γ_k from the original collection. We claim that this set satisfies the inductive hypotheses for $k-1$. To see this, define $\widetilde{\gamma}_i = \min(\gamma_{i+1}, \gamma'_i)$ for $i \in [\![1, k-2]\!]$ and $\widetilde{\gamma}_{k-1} = \max(\gamma_{k-1}, \gamma'_{k-2})$, and define $\widetilde{\gamma}'_i = \max(\gamma_{i+1}, \gamma'_i)$ for $i \in [\![1, k-3]\!]$. Lemma 9.9 asserts that each of these is an upright path, and that their union is B. It is also immediate from Lemma 9.9 that the ordering relations of $\widetilde{\gamma}_i$ and $\widetilde{\gamma}'_i$ are satisfied, and the multiplicity condition is inherited from the original γ_i and γ'_i curves. Thus B satisfies the inductive hypotheses for $k-1$.

We therefore obtain two sets, A'_1 and A'_2, each with $k-2$ disjoint curves, such that

$$\widetilde{A}_1 \cup \widetilde{A}_2 \equiv \bigcup_{i=2}^{k-1} \gamma_i \cup \bigcup_{i=1}^{k-2} \gamma'_i\,,$$

$\min(\gamma_2, \gamma'_1) = \widetilde{\gamma}_1 \in \widetilde{A}_1$, and $\max(\gamma_{k-1}, \gamma'_{k-2}) = \widetilde{\gamma}_{k-1} \in \widetilde{A}_2$. We now form A_1 and A_2 by defining

$$A_1 = \widetilde{A}_2 \cup \gamma_1 \quad \text{and} \quad A_2 = \widetilde{A}_1 \cup \gamma_k\,.$$

All we have to verify is that γ_1 is disjoint with the curves in $\widetilde{A}_2$ and the same for γ_k and $\widetilde{A}_1$. We will show the first as the argument for the second is similar.

Suppose there is $\gamma \in \widetilde{A}_2$ such that there exists $v \in \gamma \cap \gamma_1$. Notice that

$$\gamma_1 \preceq \min(\gamma_2, \gamma'_1) \preceq \gamma\,.$$

Thus if v is an element of the left-hand side and right-hand side paths in this display, then it must also be an element of the middle path. But then v must have multiplicity at least three in $\bigcup_{i=1}^{k} \gamma_i \cup \bigcup_{i=1}^{k-2} \gamma'_i$ as the three paths in the last display belong to three disjoint collections of paths, namely $\bigcup_{i=1}^{k} \gamma_i \cup \bigcup_{i=1}^{k-2} \gamma'_i \setminus (\widetilde{A}_1 \cup \widetilde{A}_2)$, $\widetilde{A}_1$, and $\widetilde{A}_2$ respectively. This is a contradiction, completing the induction step and the proof of Lemma 9.8.   □

Remark 9.10. In fact, the hypotheses that $\{\gamma_i : i \in [\![1, k]\!]\}$ and $\{\gamma'_i : i \in [\![1, k-2]\!]\}$ are ordered; $\gamma_1 \preceq \gamma'_1$; and $\gamma'_{k-2} \preceq \gamma_k$ are all not needed for the conclusion of Lemma 9.8 to hold, with the modification that we require the left boundary of the collection to lie in A_1 and the right boundary in A_2, instead of γ_1 and γ_k. Without these hypotheses, the set B in the proof above is taken to be the original collection with the left boundary and right boundary curves removed; that the remaining curves can be decomposed as needed to apply the induction hypothesis is proved by a separate induction argument. As the argument becomes more involved and we do not require this generality for our application, we have chosen to not make this more general statement.

Now we use Lemma 9.8 to complete the proof of Proposition 2.7.

Proof of Proposition 2.7. That $Y_{n,1} \geq Y_{n,2}$ is equivalent to $2X_n^1 \geq X_n^2$, which is obvious from the definition of the two quantitites.

To prove $Y_{n,k-1} \geq Y_{n,k}$ for $k \geq 3$, as we noted, it is sufficient to prove $X_n^{k-2} + X_n^k \leq 2X_n^{k-1}$. Let $\gamma_n^k = \{\gamma_{n,\to}^{k,j} : j \in [\![1, k]\!]\}$ and $\gamma_n^{k-2} = \{\gamma_{n,\to}^{k-2,j} : j \in [\![1, k-2]\!]\}$ be k- and $(k-2)$-geodesic watermelons as given by Proposition 2.2. We extend them by horizontal line segments so that γ_i is $\gamma_{n,\to}^{k,i}$ extended to begin at $(i - k + 1, k - i + 1)$ and end at $(n + i - 1, n - i + 1)$ for $i \in [\![1, k]\!]$, and γ'_i is $\gamma_{n,\to}^{k-2,i}$ extended to begin at $(i - k + 3, k - i - 1)$ and end at $(n + i - 1, n - i + 1)$ for $i \in [\![1, k-2]\!]$; this is done so that all the paths have the same time-range.

We apply Lemma 9.8 to these curves; the hypothesis on the ordering of the curves and the multiplicity assumption is satisfied by the disjointness of the curves in geodesic watermelons and an invocation of interlacing (Proposition 9.3). Then we obtain two collections of $k-1$ disjoint curves A_1 and A_2. We intersect these curves with $[\![1, n]\!]^2$ to get A'_1 and A'_2, which are still two collections of $k-1$ disjoint curves and whose union is the same as the union of γ_n^k and γ_n^{k-2},

regarded as multisets. The maximum weight property of $(k-1)$-geodesic watermelons completes the proof by implying that

$$X_n^{k-2} + X_n^k = \ell(A_1') + \ell(A_2') \leq 2X_n^{k-1}. \qquad \square$$

10. Connections to determinantal point processes and eigenvalue rigidity

We begin this section by showing how the just proved deterministic monotonicity of $\{Y_{n,i} : i \in [\![1, n]\!]\}$ can be used, somewhat surprisingly, to quickly obtain a concentration of measure result (which we expect to be optimal) for the same point process.

Proposition 10.1. *Suppose that Assumptions 1, 2, and 3a (resp. 3b) hold. There exist positive constants C_1, C_2 and c_1, such that, for some $c > 0$, and for $1 \leq k \leq c_1 n^{1/2}$ (resp. $1 \leq k \leq c_1 n$),*

$$\mathbb{P}\left(Y_{n,k} - \mu n \notin -k^{2/3} n^{2/3} \cdot (C_1, C_2)\right) \leq e^{-ck^2}.$$

Proof. By Proposition 2.7, $\{Y_{n,i} : i \in [\![1, n]\!]\}$ are ordered. Thus, taking C_1 as in Theorem II(1),

$$\mathbb{P}\left(Y_{n,k} < \mu n - 4C_1 k^{2/3} n^{1/3}\right) \leq \mathbb{P}\left(\sum_{i=k}^{2k-1} Y_{n,i} < \mu n k - 4C_1 k^{5/3} n^{1/3}\right).$$

The latter quantity is bounded by

$$\mathbb{P}\left(\sum_{i=1}^{2k-1} Y_{n,i} < \mu n(2k-1) - 2^{5/3} C_1 k^{5/3} n^{1/2}\right) + \mathbb{P}\left(\sum_{i=1}^{k-1} Y_{n,i} > \mu n(k-1) + (4 - 2^{5/3})C_1 k^{5/3} n^{1/3}\right).$$

Note that $\sum_{i=1}^{2k-1} Y_{n,i} = X_n^{2k-1}$ and $\sum_{i=1}^{k-1} Y_{n,i} = X_n^{k-1}$. Thus the preceding display is equal to

$$\mathbb{P}\left(X_n^{2k-1} < \mu n(2k-1) - C_1 (2k)^{5/3} n^{1/3}\right) + \mathbb{P}\left(X_n^{k-1} > \mu n(k-1) + (4 - 2^{5/3})C_1 k^{5/3} n^{1/3}\right);$$

the first term is bounded by e^{-ck^2} by the construction in Theorem 3.1, and the second term by the crude upper bound in Lemma 3.4 (noting that $4 - 2^{5/3} > 0$). Both these inferences hold for ranges of k and n depending on whether Assumption 3a or 3b is in force, as in the statement of Proposition 10.1.

Now we turn to the other bound. Since the sequence $\{Y_{n,i} : i \in [\![1, n]\!]\}$ is ordered,

$$\mathbb{P}\left(Y_{n,k} > \mu n - C_2 k^{2/3} n^{1/3}\right) \leq \mathbb{P}\left(\sum_{i=1}^{k} Y_{n,i} > \mu n k - C_2 k^{5/3} n^{1/2}\right) = \mathbb{P}\left(X_n^k > \mu n k - C_2 k^{5/3} n^{1/2}\right)$$

$$\leq e^{-ck^2},$$

where Theorem II(1) was used in the last line, with C_2 as in the latter, and where k and n are again supposed to belong to certain ranges depending on whether Assumption 3a or 3b is in force. $\square$

Integrable methods also yield a lot of information about the process $\{Y_{n,i} : i \in [\![1, n]\!]\}$. For instance, in exponential LPP, this process has the distribution of $\lambda_1 \geq \cdots \geq \lambda_n$, the eigenvalues of the LUE [AVMW13], i.e., the Hermitian matrix X^*X where X is an $n \times n$ matrix of i.i.d. standard complex Gaussian entries. This can be used to show that the edge, given by $\{Y_{n,i} : i \in [\![1, k]\!]\}$ with k fixed, converges weakly in the limit of high n to the k uppermost elements in the Airy point process, after appropriate centering and scaling. Determinantal techniques prove the following weaker counterpart to Proposition 10.1.

Proposition 10.2. *There exist $C_5 > C_6 > 0, c > 0$ and $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and all $n \geq n_0(k)$ we have*

$$\mathbb{P}\left(\lambda_k \notin (4n - C_5 k^{2/3} n^{1/3}, 4n - C_6 k^{2/3} n^{1/3})\right) \leq e^{-ck}.$$

The basic argument for Proposition 10.2 uses the representation of the number of points of a determinantal point process in a given interval as the sum of independent Bernoulli random variables [HKPV09, Theorem 4.5.3] and then applies standard concentration inequalities for such sums. However, we also need a sharp estimate on the mean of this sum of Bernoulli random variables, which is available in the exponential LPP case from the literature on the LUE. The corresponding estimate does not seem to be available for geometric LPP, where the relevant determinantal process is the less-studied Meixner ensemble.

Proof of Proposition 10.2. We first prove that

$$\mathbb{P}(\lambda_k < 4n - C_5 k^{2/3} n^{1/3}) \leq e^{-ck}.$$

The argument for the other side is analogous; we shall point out the steps at the end. Let us denote the scaled eigenvalues λ_i/n by $\tilde{\lambda}_i$. Thus it suffices to prove that

$$\mathbb{P}(\tilde{\lambda}_k < 4 - C_5(k/n)^{2/3}) \leq e^{-ck}. \tag{42}$$

Let $I_k = [4 - C_5 k^{2/3} n^{-2/3}, \infty)$. The event that $\tilde{\lambda}_k < 4 - C_5 k^{2/3} n^{-2/3}$ is the same as $N_{I_k} < k$, where N_I is the number of scaled eigenvalues lying in the interval I. So it suffices to prove

$$\mathbb{P}(N_{I_k} < k) \leq e^{-ck}.$$

The eigenvalues of the LUE form a determinantal point process, and for such point processes, for any interval I, N_I can be expressed as the sum of independent Bernoulli random variables [HKPV09, Theorem 4.5.3]. Following an argument in [Dal13], we combine this fact with a Bernstein-type inequality to arrive at

$$\mathbb{P}\left(|N_{I_k} - \mathbb{E}N_{I_k}| > t\right) \leq 2\exp\left(-\frac{t^2}{2\sigma_k^2 + t}\right),$$

where $\sigma_k^2 = \mathrm{Var}(N_{I_k})$. Since N_{I_k} is a sum of Bernoulli random variables, $\sigma_k^2 \leq \mathbb{E}[N_{I_k}]$, so we have

$$\mathbb{P}\left(|N_{I_k} - \mathbb{E}N_{I_k}| > t\right) \leq 2\exp\left(-\frac{t^2}{2\mathbb{E}[N_{I_k}] + t}\right). \tag{43}$$

(We mention in passing that the empirical distribution of $\tilde{\lambda}_i$ converges to the Marchenko-Pastur distribution: see [MP67].) What we require is a rate for this convergence of the empirical dsitribution of $\tilde{\lambda}_i$ to the Marchenko-Pastur distribution; this is given by [GT05, Theorem 1.4], which states that that there exists an absolute constant $C_7 > 0$ independent of n, k, C_5 such that

$$\left|\mathbb{E}[N_{I_k}] - n\int_{4 - C_5 k^{2/3} n^{-2/3}}^{4} \mu_{\mathrm{MP}}(x)\, dx\right| < C_7,$$

where the Marchenko-Pastur density function $\mu_{\mathrm{MP}}(x)$ equals $\frac{1}{2\pi x}\sqrt{x(4-x)}$ on $[0, 4]$. Straightforward calculus yields that for $0 < y < 1/5$, and for $0 < b_1 < b_2$,

$$b_1 y^{3/2} \leq \int_{4-y}^{4} \frac{1}{2\pi x}\sqrt{x(4-x)}\, dx \leq b_2 y^{3/2}.$$

Now putting $y = C_5 k^{2/3} n^{-2/3}$, $k > C_7$, we find that, for a large enough C_5 and $n \geq C_8 k$ (as $y < 1/5$ is needed for the last estimate),

$$\mathbb{E}[N_{I_k}] \in [R_1 k, R_2 k],$$

for some numbers R_1 and R_2 (although one can precisely compute the number up to smaller order terms by exactly evaluating the integral). Now, manipulating, and using (43),

$$\mathbb{P}(N_{I_k} < k) = \mathbb{P}\big(N_{I_k} - \mathbb{E}[N_{I_k}] < k - \mathbb{E}[N_{I_k}]\big) \leq \mathbb{P}\big(|N_{I_k} - \mathbb{E}[N_{I_k}]| > k\big)$$
$$\leq 2\exp\left(-\frac{k^2}{2\mathbb{E}[N_{I_k}] + k}\right)$$

for large enough k and $n > k$. This last quantity is bounded by $2e^{-ck}$, as required.

For proving $\mathbb{P}(\lambda_k > 4n - C_6 k^{2/3} n^{1/3}) \leq e^{-ck}$, we similarly define the interval $\widetilde{I}_k = [4 - C_6 k^{2/3} n^{-2/3}, \infty)$ and note that this probability is the same as $\mathbb{P}(N_{\widetilde{I}_k} > k)$. We follow the same steps as before, writing $N_{\widetilde{I}_k}$ as a sum of independent Bernoulli random variables and using a Bernstein-type inequality. The only difference is that, for this side, we use $\mathbb{E}[N_{\widetilde{I}_k}] < k/2$ for some C_6, which follows using the same estimate of $\mathbb{E}[N_{\widetilde{I}_k}]$ as before for $k > 4C_7$. The upper bound with these choices is $2e^{-k/6}$. We omit the remaining details. $\qquad\square$

We wish to point out that while straightforward applications of tools from determinantal point processes, as above, lead to quantitatively weaker concentration estimates for $Y_{n,k}$ even in case of the integrable model of exponential LPP, it may be possible to obtain such estimates by using more refined techniques from random matrix theory and determinantal point processes. We have not explored this direction.

10.1. **Another route to Theorems I and II using the lower bound of $Y_{n,k}$.** The proof of Theorem II that we have presented entails obtaining a high probability lower bound on $X_n^{j,j}$ for a random j by averaging (4), and applying interlacing to relate j and k. Since $X_n^{k,k} \geq Y_{n,k}$ by (5), there is another route to Theorems I and II, which uses the lower bound of $Y_{n,k}$ provided by Propositions 10.1 and 10.2, allowing the direct application of Theorems 3.2 and 3.3 to prove Theorem II(2), bypassing averaging and interlacing. The upper bound on X_n^k follows the argument given in Section 7, but again invokes the lower bound on $X_n^{k,k}$ directly.

Note that the lower bound on $Y_{n,k}$ in Proposition 10.1 relies only on the construction in Theorem 3.1, the upper bound in Lemma 3.4, and the key monotonicity property in Proposition 2.7.

For exponential LPP, a more direct approach that yields weaker bounds invokes Proposition 10.2 to obtain the required lower bound on $Y_{n,k}$ and proceeds as in the preceding paragraph. The monotonicity result in this case is an immediate consequence of the correspondence with LUE [AVMW13].

11. Point-to-line LPP

In this section we prove Theorem III. We start by noting that the proof of interlacing for Γ_n^k applies verbatim to Υ_n^k.

Proof of Theorem III. We will first bound the weight fluctuations, proving the analogue of Theorem II(1), and then provide the arguments to verify the versions of Theorem II(2) concerning transversal fluctuations.

Weight fluctuations: The lower bound follows immediately from the weight lower bound of Theorem II(1) as the point-to-point k-geodesic watermelon's weight is stochastically dominated by the point-to-line k-geodesic watermelon's weight. So we need to only prove the upper bound.

In fact the proofs of all the ingredient lemmas and propositions that went into the proof of the weight upper bound of Theorem II(1) apply verbatim to the point-to-line watermelon by replacing

X_n^k with Z_n^k and $\underline{X}_n^{k,k}$ with $\underline{Z}_n^{k,j}$. The only ingredient for which this is not true is Theorem 3.3, which must be replaced by its point-to-line version Theorem 6.5.

We will now follow the steps of the proof of the weight upper bound of Theorem II(1). We first observe the analogue of Lemma 3.4 for the point-to-line watermelon weight:

$$\mathbb{P}\left(Z_n^k > \mu n k + t k^{5/3} n^{1/3} \right) \leq e^{-ct^{3/2}k^2}, \tag{44}$$

for $k < t^{-3/4} n^{1/2}$. This requires the one-point input for the point-to-line single geodesic weight, which is provided from Assumption 3a by Proposition 3.6 with $t = s = 0$. With (44) and the weight lower bound of Theorem II(1), we obtain the analogue of Lemma 5.1, which holds for large enough n, a $C < \infty$, and $k < C^{-3/4} n^{1/2}$:

$$\mathbb{P}\left(\bigcup_{j=\lfloor \frac{k}{2} \rfloor}^{k} \left\{ \underline{Z}_n^{j,j} > \mu n - C k^{2/3} n^{1/3} \right\} \right) \geq 1 - e^{-ck^2}. \tag{45}$$

Let $C_3 > 0$. Similarly to Lemma 7.1's statement, let $\underline{E}_n^{j,*}(\delta)$ is the minimum number of curves of Υ_n^j, over all j-geodesic point-to-line watermelons, which exit the strip $U_{n,\frac{1}{2}\delta k^{1/3} n^{2/3}}$ with δ as given in Theorem 3.2. (Note that Theorem 3.2 applies equally well without any modification to its statement in this situation.) Thus we obtain the analogue of Lemma 7.1:

$$\mathbb{P}\left(\bigcup_{j=\lfloor k/2 \rfloor}^{k} \left\{ \underline{E}_n^{j,*}(\delta) > \frac{k}{4} \right\} \right) \geq 1 - e^{-ck^2}. \tag{46}$$

At this point the proof of the weight upper bound of Theorem II(1) applies essentially verbatim, but we reproduce it here for the reader's benefit.

Let B_1 be the complement of the event whose probability is bounded below in (46), so $\mathbb{P}(B_1) \leq e^{-ck^2}$. On B_1^c, by interlacing, at least $k/4$ of the curves of any k-geodesic point-to-line watermelon Υ_n^k must exit the strip of width $\frac{1}{2}\delta k^{1/3}$. By Theorem 6.5, there is a c' such that, if B_2 is defined as

$$B_2 = \left\{ \exists \Gamma : \ell(\Gamma) > \mu n - c' k^{2/3} n^{1/3}, \mathrm{TF}(\Gamma) > \frac{1}{2} \delta k^{1/3} n^{2/3} \right\},$$

where Γ is a upright path from $(1,1)$ to the line $x + y = 2n$, then

$$\mathbb{P}(B_2) \leq e^{-ck}. \tag{47}$$

Consider the events $A = \left\{ Z_n^k > \mu n k - \frac{1}{16} c' k^{5/3} n^{1/3} \right\}$ and $B_3 = \{ Z_n^{\lfloor 7k/8 \rfloor} > \mu n \lfloor 7k/8 \rfloor + \frac{1}{16} c' k^{5/3} n^{1/3} \}$. By (44), $\mathbb{P}(B_3) \leq e^{-ck^2}$. Fix some k-geodesic point-to-line watermelon Υ_n^k, and let $Z_n^{k,j}$ be the weight of its jth heaviest curve for each $j \in [\![1, k]\!]$. Now on $A \cap B_3^c$ we have

$$Z_n^{\lfloor 7k/8 \rfloor} + \left(Z_n^{k,\lfloor 7k/8 \rfloor+1} + \ldots + Z_n^{k,k} \right) \geq Z_n^k > \mu n k - \frac{1}{16} c' k^{5/3} n^{1/3}$$

$$\implies Z_n^{k,\lfloor 7k/8 \rfloor+1} + \ldots + Z_n^{k,k} > \frac{1}{8} \mu n k - \frac{1}{8} c' k^{5/3} n^{1/3} .$$

By definition, the $Z_n^{k,i}$ are ordered; thus, we learn that $Z_n^{k,\lfloor 7k/8 \rfloor+1} > \mu n - c' k^{2/3} n^{1/3}$. Again by the ordering, this bound applies to $Z_n^{k,1}, \ldots, Z_n^{k,\lfloor 7k/8 \rfloor}$ as well. This means we have $\lfloor 7k/8 \rfloor$ disjoint curves Γ, each with $\ell(\Gamma) > \mu n - c' k^{2/3} n^{1/3}$. Thus, on $A \cap B_1^c \cap B_3^c$, we must have at least $\lfloor k/8 \rfloor$ disjoint curves Γ, each satisfying $\mathrm{TF}(\Gamma) > \frac{1}{2} \delta k^{1/3} n^{2/3}$ and $\ell(\Gamma) > \mu n - c' k^{2/3} n^{1/3}$. By the BK inequality Proposition 4.4 and the bound on $\mathbb{P}(B_2)$ in (47), the probability of this occurring is bounded by $\exp\left(-ck \cdot k\right) = \exp\left(-ck^2\right)$. Noting the bounds on $\mathbb{P}(B_1)$ and $\mathbb{P}(B_3)$ completes the proof.

Transversal fluctuations: We first prove the exponent lower bound, analogous of Theorem II(2b); namely, that there exist $C < \infty$, $c > 0$ and $\delta > 0$ such that, for $k > k_0$ and $n > Ck$,

$$\mathbb{P}\left(\underline{\mathrm{TF}}^*(n,k) < \delta k^{1/3} n^{2/3}\right) \leq e^{-ck^2}. \tag{48}$$

The proof proceeds in exactly the same way as that of Theorem II(2b). Let C be as in (45). Let $\delta = \delta(2^{2/3}C)$ as in Theorem 3.2 and let $\delta' = 2^{-1/3}\delta$. Let A_k denote the event that there exist $\lfloor k/2 \rfloor$ disjoint paths contained in $U_{n,\delta'k^{1/3}n^{2/3}}$, each of which has weight at least $\mu n - Ck^{2/3}n^{1/3}$. Further, let B_k denote the event from (45):

$$B_k := \bigcup_{j=\lfloor \frac{k}{2} \rfloor}^{k} \left\{ \underline{Z}_n^{j,j} > \mu n - Ck^{2/3}n^{1/3} \right\}.$$

Clearly, by Theorem 3.2 and (45), we have $\mathbb{P}(A_k^c \cap B_k) \geq 1 - e^{-ck^2}$ for some $c > 0$.

Observe next that, on $A_k^c \cap B_k$, there exists $j \in \{\frac{k}{2}, \ldots, k\}$ such that each of the curves of any j-geodesic point-to-line watermelon Υ_n^j has weight at least $\mu n - Ck^{2/3}n^{1/3}$, and hence some of them must exit $U_{n,\delta'k^{1/3}n^{2/3}}$. By the interlacing result for point-to-line watermelons, the same must be true for any k-geodesic point-to-line watermelon. This completes the proof of (48) with δ in the statement replaced by δ'.

Now we turn to the exponent upper bound, analogous to Theorem II(2a)); namely, that there exist $C'' < \infty$, $c > 0$ and k_0 such that, for $k > k_0$ and $n > k$,

$$\mathbb{P}\left(\overline{\mathrm{TF}}^*(n,k) > C''k^{1/3}n^{2/3}\right) \leq e^{-ck}, \tag{49}$$

which immediately implies the upper exponent bound. This proof proceeds exactly as does Theorem II(2a)'s.

Let C' be as in (45) with $2k$ in place of k, and B_k' denote the large probability event from there:

$$B_k' := \bigcup_{j=k}^{2k} \left\{ \underline{X}_n^{j,j} > \mu n - C'k^{2/3}n^{1/3} \right\}.$$

Let $A_k' = A_k'(C'')$ denote the event that there exists a path from $(1,1)$ to the line $x + y = 2n$ that exits $U_{n,C''k^{1/3}n^{2/3}}$ and has weight at least $\mu n - C'k^{2/3}n^{1/3}$. Now choose $C'' > 0$ (possible by Theorem 6.5) such that $\mathbb{P}(A_k') \leq e^{-c''k}$, for some $c'' > 0$ for all k and all n sufficiently large. Clearly now it suffices to show that, on $B_k' \cap (A_k')^c$, no k-geodesic point-to-line watermelon exits $U_{n,C''k^{1/3}n^{2/3}}$. By the definition of B_k', there exists $j \in [\![k, 2k]\!]$ such that all paths of all j-geodesic point-to-line watermelons have weight at least $\mu n - C'k^{2/3}n^{1/3}$, and $(A_k')^c$ ensures that all these paths are contained in $U_{n,C''k^{1/3}n^{2/3}}$. The proof of (49) is completed by invoking the interlacing of geodesic point-to-line watermelons. $\square$

REFERENCES

[AD14]   Antonio Auffinger and Michael Damron. A simplified proof of the relation between scaling exponents in first-passage percolation. *Ann. Probab.*, 42(3):1197–1211, 2014.

[Agg19]   Amol Aggarwal. Universality for lozenge tiling local statistics. *arXiv preprint arXiv:1907.09991*, 2019.

[AGH18]   Richard Arratia, Skip Garibaldi, and Alfred W. Hales. The van den Berg–Kesten–Reimer operator and inequality for infinite spaces. *Bernoulli*, 24(1):433–448, 2018.

[Ale20]   Kenneth Alexander. Geodesics, bigeodesics, and coalescence in first passage percolation in general dimension. *arXiv preprint arXiv:2001.08736*, 2020.

[Ash90]   Robert B. Ash. *Information theory*. Dover Publications, Inc., New York, 1990. Corrected reprint of the 1965 original.

[AVMW13] Mark Adler, Pierre Van Moerbeke, and Dong Wang. Random matrix minor processes related to percolation theory. *Random Matrices: Theory and Applications*, 2(04):1350008, 2013.

[Bai02] Jinho Baik. Painlevé expressions for LOE, LSE, and interpolating ensembles. *International Mathematics Research Notices*, 2002(33):1739–1789, 2002.

[BCS06] Marton Balazs, Eric Cator, and Timo Seppäläinen. Cube root fluctuations for the corner growth model associated to the exclusion process. *Electron. J. Probab.*, 11:1094–1132, 2006.

[BDJ99] Jinho Baik, Percy Deift, and Kurt Johansson. On the distribution of the length of the longest increasing subsequence of random permutations. *Journal of the American Mathematical Society*, 12(4):1119–1178, 1999.

[BDM$^+$01] Jinho Baik, Percy Deift, Ken McLaughlin, Peter Miller, and Xin Zhou. Optimal tail estimates for directed last passage site percolation with geometric random variables. *arXiv preprint math/0112162*, 2001.

[BF08] Alexei Borodin and Patrik Ferrari. Large time asymptotics of growth models on space-like paths I: PushASEP. *Electronic Journal of Probability*, 13:1380–1418, 2008.

[BFS08] Alexei Borodin, Patrik L Ferrari, and Tomohiro Sasamoto. Large time asymptotics of growth models on space-like paths II: PNG and parallel TASEP. *Communications in Mathematical Physics*, 283(2):417–449, 2008.

[BG18] Riddhipratim Basu and Shirshendu Ganguly. Time correlation exponents in last passage percolation. *arXiv preprint arXiv:1807.09260*, 2018.

[BGHK19] Riddhipratim Basu, Shirshendu Ganguly, Milind Hegde, and Manjunath Krishnapur. Lower deviations in β-ensembles and law of iterated logarithm in last passage percolation. *arXiv preprint arXiv:1909.01333*, 2019.

[BGZ19] Riddhipratim Basu, Shirshendu Ganguly, and Lingfu Zhang. Temporal correlation in last passage percolation with flat initial condition via Brownian comparison. *arXiv preprint arXiv:1912.04891*, 2019.

[BHS18] Riddhipratim Basu, Christopher Hoffman, and Allan Sly. Nonexistence of bigeodesics in integrable models of last passage percolation. *arXiv preprint arXiv:1811.04908*, 2018.

[BOO00] Alexei Borodin, Andrei Okounkov, and Grigori Olshanski. Asymptotics of Plancherel measures for symmetric groups. *Journal of the American Mathematical Society*, 13(3):481–515, 2000.

[BR00] Jinho Baik and Eric M Rains. Limiting distributions for a polynuclear growth model with external sources. *Journal of Statistical Physics*, 100(3-4):523–541, 2000.

[BR01a] Jinho Baik and Eric M Rains. Algebraic aspects of increasing subsequences. *Duke Mathematical Journal*, 109(1):1–65, 2001.

[BR01b] Jinho Baik and Eric M Rains. Symmetrized random permutations. *Random Matrix Models and their Applications*, 40:1–19, 2001.

[BSS14] Riddhipratim Basu, Vladas Sidoravicius, and Allan Sly. Last passage percolation with a defect line and the solution of the slow bond problem. *arXiv preprint arXiv:1408.3464*, 2014.

[BSS19] Riddhipratim Basu, Sourav Sarkar, and Allan Sly. Coalescence of geodesics in exactly solvable models of last passage percolation. *J. Math. Phys.*, 60, 2019.

[CD18] Ivan Corwin and Evgeni Dimitrov. Transversal fluctuations of the ASEP, stochastic six vertex model, and Hall-Littlewood Gibbsian line ensembles. *Communications in Mathematical Physics*, 363(2):435–501, 2018.

[CG18] Ivan Corwin and Promit Ghosal. KPZ equation tails for general initial data. *arXiv preprint arXiv:1810.07129*, 2018.

[Cha13] Sourav Chatterjee. The universal relation between scaling exponents in first-passage percolation. *Annals of Mathematics*, pages 663–697, 2013.

[CHH19] Jacob Calvert, Alan Hammond, and Milind Hegde. Brownian structure in the KPZ fixed point. *arXiv preprint arXiv:1912.00992*, 2019.

[Dal13] Sandrine Dallaporta. Eigenvalue variance bounds for covariance matrices. *arXiv preprint arXiv:1309.6265*, 2013.

[DH14] Michael Damron and Jack Hanson. Busemann functions and infinite geodesics in two-dimensional first-passage percolation. *Communications in Mathematical Physics*, 325(3):917–963, 2014.

[DOV18] Duncan Dauvergne, Janosch Ortmann, and Bálint Virág. The directed landscape. *arXiv preprint arXiv:1812.00309*, 2018.

[DV18] Duncan Dauvergne and Bálint Virág. Basic properties of the Airy line ensemble. *arXiv preprint arXiv:1812.00311*, 2018.

[FO18] Patrik L Ferrari and Alessandra Occelli. Universality of the GOE Tracy-Widom distribution for TASEP with arbitrary particle density. *Electronic Journal of Probability*, 23, 2018.

[FO19] Patrik L Ferrari and Alessandra Occelli. Time-time covariance for last passage percolation with generic initial profile. *Mathematical Physics, Analysis and Geometry*, 22(1):1, 2019.

[Gre82] Curtis Greene. An extension of Schensted's theorem. In *Young Tableaux in Combinatorics, Invariant Theory, and Algebra*, pages 39–50. Elsevier, 1982.

[GT05]   Friedrich Götze and Alexander Tikhomirov. The rate of convergence for spectra of GUE and LUE matrix ensembles. *Open Mathematics*, 3(4):666–704, 2005.
[Ham17]   Alan Hammond. Exponents governing the rarity of disjoint polymers in Brownian last passage percolation. *arXiv preprint arXiv:1709.04110*, 2017.
[HKPV09]   John Ben Hough, Manjunath Krishnapur, Yuval Peres, and Bálint Virág. *Zeros of Gaussian analytic functions and determinantal point processes*, volume 51. American Mathematical Soc., 2009.
[HS20]   Alan Hammond and Sourav Sarkar. Modulus of continuity for polymer fluctuations and weight profiles in Poissonian last passage percolation. *Electronic Journal of Probability*, 25, 2020.
[Jan18]   Svante Janson. Tail bounds for sums of geometric and exponential variables. *Statistics & Probability Letters*, 135:1–6, 2018.
[JO20]   Samuel GG Johnston and Neil O'Connell. Scaling limits for non-intersecting polymers and Whittaker measures. *Journal of Statistical Physics*, 179(2):354–407, 2020.
[Joh00a]   Kurt Johansson. Shape fluctuations and random matrices. *Communications in Mathematical Physics*, 209(2):437–476, 2000.
[Joh00b]   Kurt Johansson. Transversal fluctuations for increasing subsequences on the plane. *Probability Theory and Related Fields*, 116(4):445–456, 2000.
[Joh01]   Kurt Johansson. Discrete orthogonal polynomial ensembles and the Plancherel measure. *Annals of Mathematics*, 153(1):259–296, 2001.
[Knu70]   Donald Knuth. Permutations, matrices, and generalized Young tableaux. *Pacific Journal of Mathematics*, 34(3):709–727, 1970.
[LR10]   Michel Ledoux and Brian Rider. Small deviations for beta ensembles. *Electron. J. Probab.*, 15:1319–1343, 2010.
[LS77]   BF Logan and LA Shepp. A variational problem for random Young tableaux. *Adv. Math*, 26:206, 1977.
[MP67]   Vladimir A Marčenko and Leonid Andreevich Pastur. Distribution of eigenvalues for some sets of random matrices. *Mathematics of the USSR-Sbornik*, 1(4):457, 1967.
[MQR16]   Konstantin Matetski, Jeremy Quastel, and Daniel Remenik. The KPZ fixed point. *arXiv preprint arXiv:1701.00018*, 2016.
[New95]   Charles M Newman. A surface view of first-passage percolation. In *Proceedings of the International Congress of Mathematicians*, pages 1017–1023. Springer, 1995.
[Oko00]   Andrei Okounkov. Random matrices and random permutations. *International Mathematics Research Notices*, 2000(20):1043–1095, 2000.
[QR19]   Jeremy Quastel and Daniel Remenik. How flat is flat in random interface growth? *Transactions of the American Mathematical Society*, 371(9):6047–6085, 2019.
[Sch09]   Craige Schensted. Longest increasing and decreasing subsequences. In *Classic Papers in Combinatorics*, pages 299–311. Springer, 2009.
[VK77]   Anatolii Moiseevich Vershik and Sergei Vasil'evich Kerov. Asymptotics of the Plancherel measure of the symmetric group and the limiting form of Young tableaux. In *Doklady Akademii Nauk*, volume 233, pages 1024–1027. Russian Academy of Sciences, 1977.
[Wai19]   Martin J. Wainwright. *High-Dimensional Statistics: A Non-Asymptotic Viewpoint*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2019.
[Wu19]   Xuan Wu. Tightness of discrete Gibbsian line ensembles with exponential interaction Hamiltonians. *arXiv preprint arXiv:1909.00946*, 2019.
[Zha19]   Lingfu Zhang. Optimal exponent for coalescence of finite geodesics in exponential last passage percolation. *arXiv preprint arXiv:1912.07733*, 2019.

Appendix A. Exponential & geometric LPP satisfy the assumptions

In this appendix we cite the results which show that exponential and geometric LPP satisfy Assumptions 1, 2, and 3.

We start with the foundational result on the Tracy-Widom fluctuations of the maximal path weight, due to Johansson [Joh00a], and continue with moderate deviations estimates and expectation asymptotics for the maximal path weight.

Recall that $X_{n,\lfloor hn \rfloor}$ is the last passage value from $(1,1)$ to $(n, \lfloor hn \rfloor)$.

Theorem A.1. *Suppose that the vertex weight law in LPP is exponential of mean one, and let $h > 0$. As $n \to \infty$,*

$$\frac{X_{n,\lfloor hn \rfloor} - (1 + \sqrt{h})^2 n}{h^{-1/6}(1 + \sqrt{h})^{4/3} n^{1/3}} \xrightarrow{d} F_{\mathrm{TW}} .$$

Suppose instead that the weight law is geometric with parameter $p \in (0, 1)$. Then, as $n \to \infty$,

$$\frac{X_{n,\lfloor hn \rfloor} - \omega(h, p) n}{\sigma(h, p) n^{1/3}} \xrightarrow{d} F_{\mathrm{TW}},$$

where

$$\omega(h, p) = \frac{(1 + \sqrt{hp})^2}{1 - p} \quad and \quad \sigma(h, p) = \frac{p^{1/6} h^{-1/6}}{1 - p} (\sqrt{h} + \sqrt{p})^{2/3} (1 + \sqrt{hp})^{2/3}.$$

Here $\xrightarrow{d}$ denotes convergence in distribution and F_{TW} is the GUE Tracy-Widom distribution.

Proof. These results appear in [Joh00a], the first as Theorem 1.6 there and the second as Theorem 1.2. $\qquad\square$

The next three statements establish that exponential and geometric LPP satisfy Assumptions 2 and 3a.

Theorem A.2 (Moderate deviation estimate). *Consider exponential LPP. Fix $\psi > 1$ and let $h \in [\psi^{-1}, \psi]$. There exist $t_0 = t_0(\psi), n_0 = n_0(\psi)$ and $c = c(\psi) > 0$ such that, for $n > n_0$ and $t > t_0$,*

$$\mathbb{P}\left(X_{n,\lfloor hn \rfloor} - (1 + \sqrt{h})^2 n > t n^{1/3} \right) \leq \exp\left(-c \min(t^{3/2}, t n^{1/3}) \right) .$$

For the lower tail, for $t > t_0$,

$$\mathbb{P}\left(X_{n,\lfloor hn \rfloor} - (1 + \sqrt{h})^2 < -t n^{1/3} \right) \leq \exp\left(-c t^3 \right) .$$

Similarly, in geometric LPP of parameter $p \in (0, 1)$, the above two displays hold with $\omega(h, p)$ (as in Theorem A.1) in place of $(1 + \sqrt{h})^2$.

Proof. For exponential LPP, the lower tail for $t > t_0$ and the upper tail for $t_0 < t < n^{2/3}$ are provided by Theorem 2 of [LR10]. For the remaining case of $t \geq n^{2/3}$ in the upper tail, see [Joh00a].

For geometric LPP, the upper tail bound is proved in [Joh00a] (combining Corollary 2.4 and equation (2.22) there), while the lower tail bound is implied by [BDM$^+$01, Theorem 1.1]. $\qquad\square$

Theorem A.3 (Expected point-to-point weight). *Fix $\psi > 1$ and let $h \in [\psi^{-1}, \psi]$. There exist $c_1 = c_1(\psi) > 0$, $c_2 = c_2(\psi) > 0$, and $n_0 = n_0(\psi)$ such that, for $n > n_0$, in exponential LPP,*

$$\frac{\mathbb{E}[X_{n,\lfloor hn \rfloor}] - (1 + \sqrt{h})^2 n}{h^{-1/6}(1 + \sqrt{h})^{4/3} n^{1/3}} \in (-c_1, -c_2),$$

while for geometric LPP of parameter $p \in (0, 1)$,

$$\frac{\mathbb{E}[X_{n,\lfloor hn \rfloor}] - \omega(h, p) n}{\sigma(h, p) n^{1/3}} \in (-c_1, -c_2),$$

with $\omega(h, p)$ and $\sigma(h, p)$ as in Theorem A.1.

Proof. This follows from Theorem A.1 and Theorem A.2, and that the GUE Tracy-Widom distribution has negative mean. $\qquad\square$

Assumption 2 can be verified in exponential and geometric LPP by replacing n by $n - xn^{2/3}$ and setting h to be $(n + xn^{2/3})/(n - xn^{2/3})$ in Theorem A.3.

We now note why the GUE Tracy-Widom distribution has negative mean, by pulling together known results to which Ivan Corwin has drawn our attention.

Lemma A.4 (Negative mean of GUE Tracy-Widom). *Let X_{TW} be distributed according to the GUE Tracy-Widom distribution. Then we have $\mathbb{E}[X_{\mathrm{TW}}] < 0$.*

Proof. Remark 1.15 and equation 1.29 of [QR19] show that X_{BR} strictly stochastically dominates $4^{1/3}X_{\mathrm{GUE}}$, where the law of X_{BR} is the Baik-Rains distribution (the cumulative distribution function of X_{BR} is labeled F^0_{stat} in [QR19, Equation 1.29]). Now [BR00, Proposition 2.1] asserts that $\mathbb{E}[X_{\mathrm{BR}}] = 0$, completing the proof. $\qquad\square$

APPENDIX B. PROOFS OF BASIC TOOLS

This appendix contains some consequences of Assumptions 1, 2, and 3a that were used in the main text.

- Section B.1 proves Proposition 3.5, the upper tail bound on interval-to-interval weights;
- Section B.2 proves Proposition 3.6, concerning upper tails for point-to-line weights; and
- Section B.3 proves Proposition 3.7, concerning the lower tail and mean of the constrained point-to-point weights.

B.1. **Interval-to-interval estimates.**

Proof of Proposition 3.5. We start with the bound (6) on $\widetilde{Z}$, and we treat only the case that $|z| \leq \rho r^{1/3}$; when $|z| > \rho r^{1/3}$, our argument works by setting $z = \rho r^{1/3}$ and using the concavity of the limit shape posited in Assumption 2.

By considering the event that $\sup_{(u,v)\in\mathcal{S}(U)} X_{u\to v}$ is large and two events defined in terms of the environment outside of U, we find a point-to-point path which has large length. To define these events, first define points ϕ_{low} and ϕ_{up} on either side of A and B:

$$\phi_{\mathrm{low}} := \left(-\ell^{3/2}r, -\ell^{3/2}r\right)$$
$$\phi_{\mathrm{up}} := \left((1 + \ell^{3/2})r - zr^{2/3}, (1 + \ell^{3/2})r + zr^{2/3}\right).$$

Let u^* and v^* be points on A and B where the suprema in the definition of $\widetilde{Z}$ are attained, and set

$$E_{\mathrm{low}} = \left\{X_{\phi_{\mathrm{low}}\to u^*-(0,1)} > \mu\ell^{3/2}r - \frac{\theta\ell^{1/2}}{3}r^{1/3}\right\} \quad \text{and} \quad E_{\mathrm{up}} = \left\{X_{v^*+(0,1)\to\phi_{\mathrm{up}}} > \mu\ell^{3/2}r - \frac{\theta\ell^{1/2}}{3}r^{1/3}\right\}.$$

Using Assumption 2 to bound the expectation in going from the second to the third line of the following, we find that

$$\mathbb{P}\left(\sup_{(u,v)\in\mathcal{S}(U)} X_{u\to v} > \mu r - G_2\frac{z^2 r^{1/3}}{1 + 2\ell^{3/2}} + \theta\ell^{1/2}r^{1/3}, E_{\mathrm{low}}, E_{\mathrm{up}}\right) \tag{50}$$

$$\leq \mathbb{P}\left(X_{\phi_{\mathrm{low}}\to\phi_{\mathrm{up}}} \geq \mu(1 + 2\ell^{3/2})r - G_2\frac{z^2 r^{1/3}}{1 + 2\ell^{3/2}} + \frac{\theta\ell^{1/2}}{3}r^{1/3}\right)$$

$$\leq \mathbb{P}\left(X_{\phi_{\mathrm{low}}\to\phi_{\mathrm{up}}} \geq \mathbb{E}\left[X_{\phi_{\mathrm{low}}\to\phi_{\mathrm{up}}}\right] + \frac{\theta\ell^{1/2}}{3}r^{1/3}\right) \leq \exp\left(-c\min(\theta^{3/2}, \theta r^{1/3})\right),$$

for $c > 0$ independent of ℓ. We used Assumption 2 for the former inequality of the final line, and Assumption 3a and the stationarity of the random environment for the latter.

Let us denote conditioning on the environment U by the notation $\mathbb{P}(\cdot \mid U)$. By this we mean we condition on the weights of vertices interior to U as well as those on the lower and upper sides A and B. Then we see

$$\mathbb{P}\left(\sup_{(u,v)\in\mathcal{S}(U)} X_{u\to v} > \mu r - G_2 \frac{z^2 r^{1/3}}{1+2\ell^{3/2}} + \theta\ell^{1/2} r^{1/3}, E_{\text{low}}, E_{\text{up}} \,\middle|\, U\right)$$

$$= \mathbb{P}\left(\sup_{(u,v)\in\mathcal{S}(U)} X_{u\to v} > \mu r - G_2 \frac{z^2 r^{1/3}}{1+2\ell^{3/2}} + \theta\ell^{1/2} r^{1/3} \,\middle|\, U\right) \cdot \mathbb{P}\left(E_{\text{low}} \mid U\right) \cdot \mathbb{P}\left(E_{\text{up}} \mid U\right).$$

So with (50), all we need is a lower bound on $\mathbb{P}\left(E_{\text{low}} \mid U\right)$ and $\mathbb{P}\left(E_{\text{up}} \mid U\right)$. This is straightforward using independence of the environment between U and the regions above and below it:

$$\mathbb{P}\left(E_{\text{lower}}^c \mid U\right) \leq \sup_{u\in A} \mathbb{P}\left(X_{\phi_{\text{low}}\to u-(0,1)} \leq \mu\ell^{3/2}r - \frac{\theta\ell^{1/2}}{3} r^{1/3}\right) \leq \frac{1}{2}$$

for large enough θ and r (depending on ℓ), using Assumption 3a. A similar upper bound holds for $\mathbb{P}\left(E_{\text{upper}}^c \mid U\right)$. Together this gives

$$\mathbb{P}\left(\sup_{(u,v)\in\mathcal{S}(U)} X_{u\to v} > \mu r - G_2 \frac{z^2 r^{1/3}}{1+2\ell^{3/2}} + \theta\ell^{1/2} r^{1/3}, E_{\text{low}}, E_{\text{up}} \,\middle|\, U\right)$$

$$\geq \frac{1}{4} \cdot \mathbb{P}\left(\sup_{(u,v)\in\mathcal{S}(U)} X_{u\to v} > \mu r - G_2 \frac{z^2 r^{1/3}}{1+2\ell^{3/2}} + \theta\ell^{1/2} r^{1/3}\right),$$

and taking expectation on both sides, combined with (50), gives the bound (6) of Proposition 3.5.

We now treat the bound (7) on $Z^{\text{ext},\delta}$. Observe that

$$\left\{Z^{\text{ext},\delta} > \theta\right\} \subseteq \bigcup_{u\in\mathbb{L}_{\text{low}}} \left(\left\{X_{u\to(\delta r, 2r)} > \mu r + \theta r^{1/3}\right\} \cup \left\{X_{u\to(2r, \delta r)} > \mu r + \theta r^{1/3}\right\}\right).$$

We bound the probability of $\bigcup_{u\in\mathbb{L}_{\text{low}}} \left\{X_{u\to(\delta r, 2r)} > \mu r + \theta r^{1/3}\right\}$; the full bound then follows by a symmetric argument and a union bound.

The point $(\delta r, 2r)$ can be written as $(\tilde{r} - \tilde{z}, \tilde{r} + \tilde{z})$ with $\tilde{r} = (1+\delta/2)r$ and $\tilde{z} = (1-\delta/2)r$. By the definitions, we have that $\tilde{z}/\tilde{r} \geq \rho$ if δ is sufficiently small. So by the concavity guaranteed by Assumption 2 and by making δ small enough, we see that

$$\mathbb{E}\left[X_{u\to(\delta r, 2r)}\right] \leq \mu\tilde{r} - G_2\rho^2\tilde{r} \leq \mu r + \frac{\mu\delta}{2} r - G_2\rho^2\left(1+\frac{\delta}{2}\right) r \leq (\mu-\eta)r$$

for some $\eta > 0$. With such a value of δ fixed, note that $(\delta r, 2r)$ is such that we may apply Assumption 3a; we find by so doing that

$$\mathbb{P}\left(X_{u\to(\delta r, 2r)} \geq \mu r + \theta r^{1/3}\right) \leq \mathbb{P}\left(X_{u\to(\delta r, 2r)} - \mathbb{E}[X_{u\to(\delta r, 2r)}] \geq \eta r + \theta r^{1/3}\right)$$

$$\leq \exp\left(-cr - c\min(\theta^{3/2}, \theta r^{1/3})\right).$$

Taking a union bound over the $O(r^{2/3})$ many u in $\mathbb{L}_{\text{low}}$ and reducing the value of c to absorb this factor into the exponent complete the proof of Proposition 3.5. $\square$

B.2. **Point-to-line estimates.**

Proof of Proposition 3.6. Note that we may assume $s \leq r^{1/3}$. Let ρ and G_1 be as in Assumption 2. Let Γ be the leftmost path with endpoint not on the line segment connecting $(r - (s+t)r^{2/3}, r + (s+t)r^{2/3})$ and $(r+(s+t)r^{2/3}, r-(s+t)r^{2/3})$ whose weight is X; this is well-defined by the weight maximization property of Γ. Let p_{left} be the coordinates of the starting point of Γ on $\mathbb{L}_{\text{left}}$, and

p_{right} be the coordinates of the endpoint of Γ on the line $x + y = 2r$. Let $y_j = (s + t + j)r^{2/3}$, $\mathbb{L}_j$ be the line segment joining the points

$$(r - y_j, r + y_j) \quad \text{and} \quad (r - y_{j+1}, r + y_{j+1}),$$

and let A_j be the event that $p_{\text{right}} \in \mathbb{L}_j$ and $(p_{\text{left}}, p_{\text{right}})$ are such that the slope of the line connecting them is not extreme enough to apply the second part of Proposition 3.5, i.e., it holds that $|(p_{\text{left}} - p_{\text{right}})_y|/(p_{\text{left}} + p_{\text{right}})_x \leq 1 - \delta/2$, for $j = 0, \ldots, r^{1/3}$, with δ as in that proposition's second part. Finally, let A be the event that $(p_{\text{left}}, p_{\text{right}})$ satisfy $|(p_{\text{left}} - p_{\text{right}})_y|/(p_{\text{left}} + p_{\text{right}})_x > 1 - \delta/2$. Then clearly the whole probability space equals

$$\bigcup_{j=0}^{r^{1/3}} A_j \cup A.$$

Thus we have

$$\mathbb{P}\Big(X - \mu r > \theta r^{1/3} - 0.5c_1 s^2 r^{1/3}\Big) \leq \sum_{j=0}^{r^{1/3}} \mathbb{P}\Big(X > \mu r + \theta r^{1/3} - 0.5c_1 s^2 r^{1/3}, A_j\Big)$$
$$+ \mathbb{P}\Big(X > \mu r + \theta r^{1/3} - 0.5c_1 s^2 r^{1/3}, A\Big). \tag{51}$$

We will bound the two terms using the next two lemmas.

Lemma B.1. *In the notation of Proposition 3.6 and under Assumptions 2 and 3a, there exist $c > 0$, $c_1 > 0$, and r_0 such that, for $r > r_0$ and $j = 0, 1, \ldots, r^{1/3}$,*

$$\mathbb{P}\Big(X > \mu r + \theta r^{1/3} - 0.5c_1 s^2 r^{1/3}, A_j\Big) \leq \exp\Big(-c(\min(\theta^{3/2}, \theta r^{1/3}) + s^3 + j^3)\Big).$$

Lemma B.2. *In the notation of Proposition 3.6 and under Assumptions 2 and 3a, there exist constants $c > 0$, $c_1 > 0$, θ_0, s_0, and r_0 such that, for $r > r_0$, $s > s_0$, and $\theta_0 < \theta < r^{2/3}$,*

$$\mathbb{P}\Big(X > \mu r + \theta r^{1/3} + \theta r^{1/3} - 0.5c_1 s^2 r^{1/3}, A\Big) \leq \exp\Big(-c(\min(\theta^{3/2}, \theta r^{1/3}) + s^3)\Big).$$

Before proving these lemmas, we show how the proof of Proposition 3.6 is completed using them. Lemma B.1 says that each individual summand in the first term of (51) is bounded by $\exp\big(-c(\min(\theta^{3/2}, \theta r^{1/3}) + s^3 + j^3)\big)$, while Lemma B.2 says that the second term is bounded by $\exp\big(-c(\min(\theta^{3/2}, \theta r^{1/3}) + s^3)\big)$. So we have, by summing over j,

$$\mathbb{P}\Big(X > \mu r + \theta r^{1/3} - 0.5c_1 s^2 r^{1/3}\Big) \leq C \exp\Big(-c(\min(\theta^{3/2}, \theta r^{1/3}) + s^3)\Big)$$

for some $C < \infty$. Reducing the value of c completes the proof. $\qquad\qquad\square$

In seeking to prove Lemmas B.1 and B.2, we wish to show that when the endpoint of a particular geodesic is sufficiently extreme, it suffers a weight loss with high probability. Lemma B.1 addresses the case that the slope between the points is bounded away from 0 and ∞, while Lemma B.2 addresses when the slope between the points is extreme.

Proof of Lemma B.1. We fix j. We divide $\mathbb{L}_{\text{left}}$ into segments of size at most $r^{2/3}$ each, indexed by i as $\mathbb{L}_{\text{left}}^i$. Thus there are at most t segments.

In the notation of Proposition 3.5, we have that z is bounded uniformly away from r, and so we may bound X on A_j by using Assumption 2 and Proposition 3.5. We set $c_1 = G_2/3$ with G_2 as in Assumption 2. Then,

$$\mathbb{P}\Big(X > \mu r + \theta r^{1/3} - 0.5c_1 s^2 r^{1/3}, A_j\Big) \leq \mathbb{P}\Bigg(\sup_{\substack{y \in \mathbb{L}_{\text{left}}, \\ w \in \mathbb{L}_j}} X_{y \to w} > \mu r + \theta r^{1/3} - 0.5c_1 s^2 r^{1/3}\Bigg)$$

$$\leq \sum_{i=1}^{t} \mathbb{P}\left(\sup_{\substack{y \in \mathbb{L}_{\mathrm{left}}^{i}, \\ w \in \mathbb{L}_j}} X_{y \to w} > \mu r + \theta r^{1/3} - 0.5 c_1 s^2 r^{1/3} \right).$$

Each summand in the last quantity is in turn bounded by

$$\mathbb{P}\left(\sup_{\substack{y \in \mathbb{L}_{\mathrm{left}}^{i}, \\ w \in \mathbb{L}_j}} \left(X_{y \to w} - \mu r - \frac{1}{3} G_2 (s+j)^2 r^{1/3} \right) > \theta r^{1/3} - 0.5 c_1 s^2 r^{1/3} + \frac{1}{3} G_2 (s+j)^2 r^{1/3} \right)$$

$$\leq \exp\left(-c(\min(\theta^{3/2}, \theta r^{1/3}) + s^3 + j^3) \right).$$

The last inequality is via the first part of Proposition 3.5, and holds for both (i) $s = 0$ and θ sufficiently large, as well as for (ii) $\theta = 0$ and $s > s_0$ (with the earlier mentioned assumption that $s \leq r^{1/3}$). Using that $t \leq s^2$ and reducing the value of c complete the proof. $\qquad\square$

Proof of Lemma B.2. Here the endpoints are not bounded uniformly away from the coordinate axes, and so we will make use of the second part of Proposition 3.5, which yields

$$\mathbb{P}\left(X > \mu r + \theta r^{1/3} - 0.5 c_1 s^2 r^{1/3}, A \right) \leq \exp\left(-c(r + \min(\theta^{3/2}, \theta r^{1/3})) \right)$$

$$\leq \exp\left(-0.5 c(r + \min(\theta^{3/2}, \theta r^{1/3}) + s^3) \right).$$

The last inequality is from the fact that $s \leq r^{1/3}$ (as otherwise the statement is trivial) and $\theta < r^{2/3}$, and again holds for both choices of parameters (i) and (ii) in Proposition 3.6. $\qquad\square$

B.3. **Lower tail and mean of constrained point-to-point.**

Proof of Proposition 3.7. We first derive the lower tail bound (8). Fix $J = \theta^{1/2}/\ell$ and define $u_j = \left(J^{-1} \cdot j(r - z - hr^{2/3}), J^{-1} \cdot j(r + z + hr^{2/3}) \right)$ for $j = 0, \dots, J$. By the stationarity of the random field and the union bound, we have

$$\mathbb{P}\left(X_{u \to u'}^{U} \leq \mu r - \theta r^{1/3} \right) \leq J \cdot \mathbb{P}\left(X_{u \to u_1}^{U} \leq \frac{1}{J} \mu r - \frac{\theta}{J} r^{1/3} \right).$$

We also have

$$\mathbb{P}\left(X_{u \to u_1}^{U} \leq \frac{1}{J} \mu r - \frac{\theta}{J} r^{1/3} \right) \leq \mathbb{P}\left(X_{u \to u_1} \leq \mu \frac{r}{J} - \frac{\theta}{J^{2/3}} \cdot \left(\frac{r}{J} \right)^{1/3} \right)$$

$$+ \mathbb{P}\left(X_{u \to u_1} > \mu \frac{r}{J} - \frac{\theta}{J^{2/3}} \cdot \left(\frac{r}{J} \right)^{1/3}, \mathrm{TF}(\Gamma_{u, u_1}) > \ell J^{2/3} \left(\frac{r}{J} \right)^{2/3} \right)$$

$$\leq \exp\left(-c\theta^{3/2}/J \right) + \exp\left(-c\ell^3 J^2 \right)$$

$$\leq 2 \exp\left(-c\ell\theta \right),$$

for sufficiently large θ depending on K. For the second-to-last inequality we have used Assumption 3a for the first term and Theorem 3.3 for the second. The parameters (marked here with tildes to avoid confusion) for the application of Theorem 3.3 are as follows: $\tilde{r} = r/J$, $\tilde{t} = KJ^{2/3}/J = K/J^{1/3}$, and $\tilde{s} = \theta^{1/2}/J^{1/3}$. It is easy to check that the conditions of Theorem 3.3 are met with these parameter choices for sufficiently large θ. For Theorem 3.3 to apply we also need $\ell J^{2/3} \geq \tilde{s} = \theta^{1/2}/J^{1/3}$, which is satisfied with equality by our choice of J. This completes the proof of the lower tail estimate.

For the lower bound (9) on $\mathbb{E}[X_{u \to u'}^{U}]$, we have

$$\mathbb{E}[X_{u \to u'}^{U}] = \mathbb{E}[X_{u \to u'}] - \mathbb{E}[X_{u \to u'} - X_{u \to u'}^{U}]$$

$$\geq \mu r - G_1 z^2 r^{1/3} - g_1 r^{1/3} - \mathbb{E}[X_{u \to u'} - X_{u \to u'}^{U}],$$

using the lower bound for the first term from Assumption 2. Note that the second term is the expectation of a positive random variable. We claim that the expectation is bounded above by $Cr^{1/3}$ for some $C < \infty$. This follows from the tail probability formula for expectation:

$$r^{-1/3}\mathbb{E}[X_{u \to u'} - X^U_{u \to u'}] = \int_0^\infty \mathbb{P}\left(X_{u \to u'} - X^U_{u \to u'} > tr^{1/3}\right) dt.$$

Bounding the integrand by $\mathbb{P}(X_{u \to u'} > \mu r + 0.5tr^{1/3}) + \mathbb{P}(X^U_{u \to u'} < \mu r - 0.5tr^{1/3})$ and using the bounds from Assumption 3a and the tail bound from above to show that this expression is integrable, with the bound on the integral being independent of r, complete the proof. $\square$

Appendix C. Proof of a crude upper bound under a convexity assumption

In this appendix we complete the proof of Lemma 3.4 under the upper tail convexity hypothesis, Assumption 3b.

Proof of Lemma 3.4 under Assumption 3b. As in the earlier proof of Lemma 3.4, for any fixed k-geodesic watermelon $\mathcal{W}$, let $X^{k,j}_{n,\mathcal{W}}$ be the weight of its jth heaviest curve. For $J \subseteq [\![1,k]\!]$ and $\bar{\delta} = (\delta_1, \ldots, \delta_k)$ such that $\delta_j \geq 0$ for each $j \in J$ and $\delta_j = 0$ for $j \in [\![1,k]\!] \setminus J$, define

$$A_{\bar{\delta},J} = \left\{ \exists \mathcal{W} : X^{k,j}_{n,\mathcal{W}} > \mu n + \delta_j n^{1/3} \text{ for all } j \in J \right\}.$$

We claim that if $\sum_{j \in J} \delta_j = \lceil tk^{5/3} \rceil$, then, for an absolute constant $c > 0$ and $c_1 = c_1(t) > 0$, and for $k \leq t^{-3/2}n$,

$$\mathbb{P}(A_{\bar{\delta},J}) \leq \exp\left(-ct^{3/2}k^2\right). \tag{52}$$

This follows from the BK inequality and Assumption 3b, which gives that

$$\mathbb{P}\left(A_{\bar{\delta},J}\right) \leq \mathbb{P}\left(\exists \text{ disjoint paths } \{\gamma_j : j \in J\} \text{ with } \ell(\gamma_j) \geq \mu n + \delta_j n^{1/3} \ \ \forall i = j \in J\right)$$

$$\leq \exp\left(-\sum_{j \in J} I_n(\delta_j)\right).$$

Since I_n is convex and $\sum_{j \in J} \delta_j = \lceil tk^{5/3} \rceil$, we get

$$\sum_{j \in J} I_n(\delta_j) \geq kI_n\left(\frac{1}{k}\sum_{j \in J} \delta_j\right) = kI_n\left(k^{-1}\lceil tk^{5/3}\rceil\right) \geq c' \min(t^{3/2}k^2, tk^{5/3}n^{1/3})$$

$$\geq c't^{3/2}k^2$$

as $I_n(x) \geq c' \min(x, x^{3/2})$ for all $x \geq 0$ from Assumption 3b, and $k \in [\![1, \lfloor \min(1, t^{-3/2})n \rfloor]\!]$. Let D be the set of $\bar{\delta}$ defined by

$$D = \left\{ \bar{\delta} : \sum_{j=1}^k \delta_j = \lceil tk^{5/3} \rceil, \delta_j \in \mathbb{N} \cup \{0\} \ \ \forall j \in [\![1,k]\!] \right\}.$$

Now we observe $\mathbb{P}\left(X^k_n > \mu nk + tk^{5/3}n^{1/3}\right) = \mathbb{P}\left(\exists \mathcal{W} : \sum_{j=1}^k (X^{k,j}_{n,\mathcal{W}} - \mu n)n^{-1/3} > tk^{5/3}\right)$. By considering the ceiling of each summand in the event of this last probability, we obtain that the last expression is bounded by

$$\mathbb{P}\left(\exists \mathcal{W} : \sum_{j=1}^k \lceil (X^{k,j}_{n,\mathcal{W}} - \mu n)n^{-1/3}\rceil > tk^{5/3}\right)$$

$$\leq \mathbb{P}\left(\bigcup_{\bar{\delta} \in D} \left\{\exists \mathcal{W} \text{ and } J \subseteq [\![1, k]\!] : \lceil (X_{n,\mathcal{W}}^{k,j} - \mu n)n^{-1/3}\rceil > \delta_j \ \ \forall j \in J\right\}\right)$$

$$\leq 2^k \cdot |D| \cdot \exp\left(-ct^{3/2}k^2\right),$$

where the first inequality is seen by considering J to be the set of indices j such that $\lceil(X_{n,\mathcal{W}}^{k,j} - \mu n)n^{-1/3}\rceil > 0$ and taking δ_j to be an integer at most the latter quantity such that $\sum_{j\in J} \delta_j = \lceil tk^{5/3}\rceil$ for $j \in J$, and $\delta_j = 0$ for $j \in [\![1, k]\!] \setminus J$. The second inequality follows by the union bound; the bound (52) on $\mathbb{P}(A_{\bar{\delta}})$; and the cardinality of the number of subsets of $[\![1, k]\!]$. The cardinality of D is the number of non-negative solutions to $x_1 + \ldots + x_k = tk^{5/3}$, which is $\binom{tk^{5/3}+k-1}{k-1} \leq \binom{2(tk^{5/3}\vee k)}{k}$. Using the trivial bound $\binom{n}{k} \leq n^k$ and simplifying, we obtain that $|D| \leq \exp(\frac{1}{2}ct^{3/2}k^2)$ for k larger than some absolute k_0 depending on t, so the proof is complete by reducing the value of c. $\square$

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