HAUSDORFF DIMENSIONS FOR SHARED ENDPOINTS OF DISJOINT GEODESICS IN THE DIRECTED LANDSCAPE

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Abstract. Within the Kardar–Parisi–Zhang universality class, the space-time Airy sheet is conjectured to be the canonical scaling limit for last passage percolation models. In recent work [23] of Dauvergne, Ortmann, and Virág, this object was constructed and shown to be the limit after parabolic correction of one such model: Brownian last passage percolation. This limit object, called the directed landscape, admits geodesic paths between any two space-time points (x, s) and (y, t) with $s < t$. In this article, we examine fractal properties of the set of these paths. Our main results concern exceptional endpoints admitting disjoint geodesics. First, we fix two distinct starting locations x_1 and x_2, and consider geodesics traveling $(x_1, 0) \to (y, 1)$ and $(x_2, 0) \to (y, 1)$. We prove that the set of $y \in \mathbb{R}$ for which these geodesics coalesce only at time 1 has Hausdorff dimension one-half. Second, we consider endpoints $(x, 0)$ and $(y, 1)$ between which there exist two geodesics intersecting only at times 0 and 1. We prove that the set of such $(x, y) \in \mathbb{R}^2$ also has Hausdorff dimension one-half. The proofs require several inputs of independent interest, including (i) connections to the so-called *difference weight profile* studied in [10]; and (ii) a tail estimate on the number of disjoint geodesics starting and ending in small intervals. The latter result extends the analogous estimate proved for the prelimiting model in [31].

Contents

Key words and phrases. Brownian last passage percolation, geodesics, polymers, Airy sheet, directed landscape.
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1. Introduction

1.1. **Random growth models and Kardar–Parisi–Zhang universality.** The Kardar–Parisi–Zhang (KPZ) universality class is a broad collection of one-dimensional random growth models sharing the asymptotic features exhibited by solutions to a stochastic PDE known as the KPZ equation [18, 19, 27]. The models known or believed to belong to this collection, including asymmetric exclusion processes, first and last passage percolation, and directed polymers in random media, are characterized by the combination of local growth driven by white noise and a smoothing effect from some notion of surface tension. The resulting growth interface $h(t,x)$ manifests a triple $(1, \frac{1}{3}, \frac{2}{3})$ of exponents: at time t^1, the deviations of $h(t,x)$ from its mean are of order $t^{1/3}$, and fluctuations of this same order are observed when x is varied on the scale of $t^{2/3}$. Furthermore, once $h(t,x)$ is properly centered and rescaled according to these exponents, a universal limit emerges as $t \to \infty$ [40].

For most models, the picture just described is conjectural even if representing the consensus view. Nevertheless, recent developments have set on rigorous footing the convergence of at least one prelimiting model, namely *Brownian last passage percolation* (LPP), to a well-defined scaling limit. Depending on the level of information one seeks to retain as $t \to \infty$, various limiting objects can be discussed, in the same way that a standard normal random variable, a multivariate Gaussian, and Brownian motion can all arise from a common central limit theorem. And just as Brownian motion, the most general scaling limit in this list, possesses interesting fractal properties, the analogous KPZ scaling limit invites inquiries into its own fractal geometry. This object was introduced in [23] and named the *directed landscape*. Before we introduce precise notation, let us describe which geometric features this paper will investigate.

The directed landscape is a random function which assigns a passage time $\mathcal{L}(x,s;y,t)$ between any two space-time points (x,s) and (y,t) with $s<t$. It respects the usual last passage composition rule,

$$\mathcal{L}(x,s;y,t) = \sup_{z\in\mathbb{R}}[\mathcal{L}(x,s;z,r) + \mathcal{L}(z,r;y,t)] \quad \text{for all } r \in (s,t),$$

which allows one to define the passage time $\mathcal{L}(\gamma)$ of any particular path $\gamma : [s,t] \to \mathbb{R}$. Then $\mathcal{L}(x,s;y,t)$ is equal to the largest $\mathcal{L}(\gamma)$ among paths satisfying $\gamma(s) = x$ and $\gamma(t) = y$, and a path achieving this maximum is called a *geodesic*. Typically geodesics are unique, and those with a shared endpoint typically coalesce before reaching that endpoint. We are interested in the exceptional cases violating these properties.

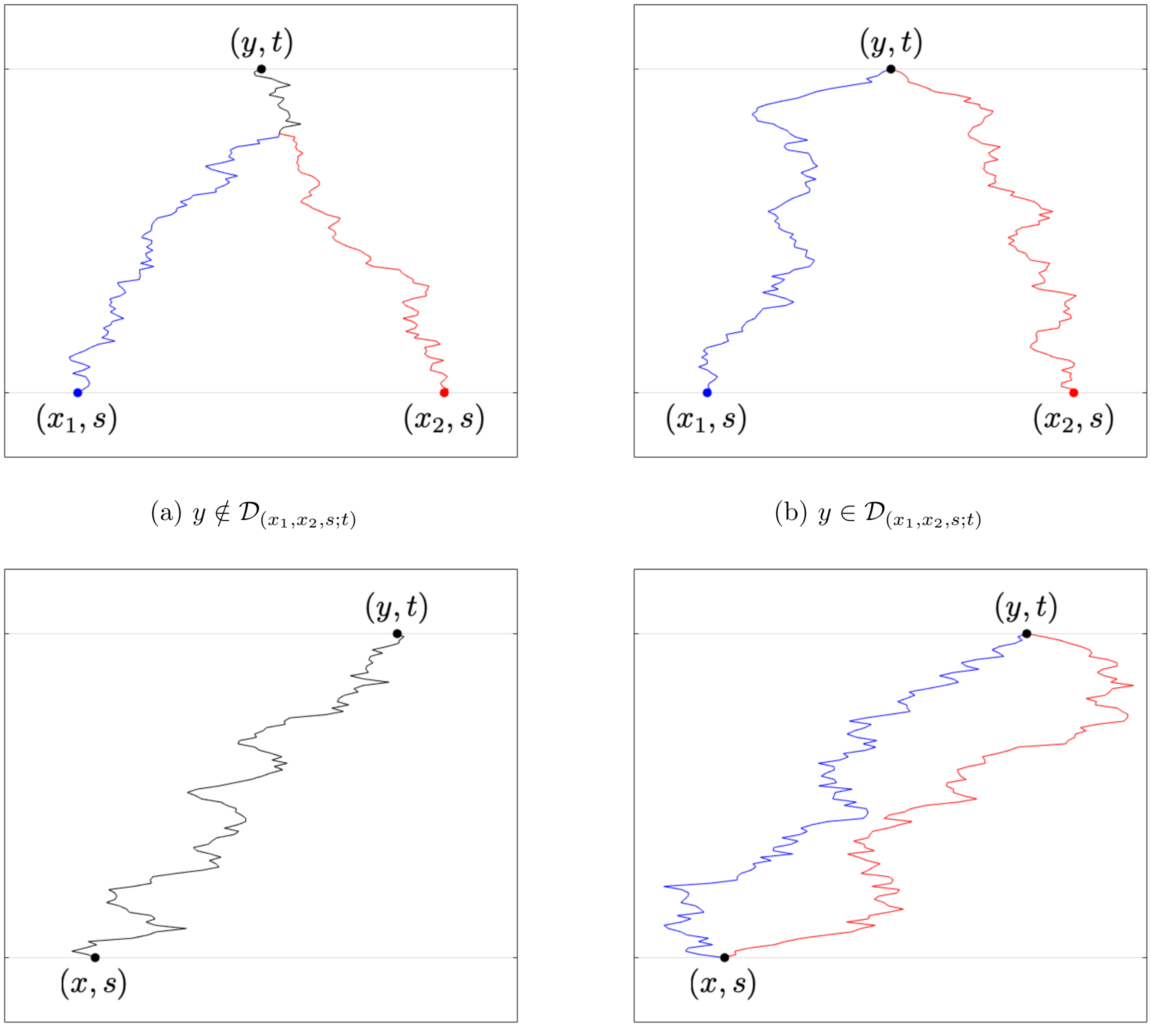
Our first consideration is of the following scenario. Fixing the starting locations x_1, x_2 and the time interval $[s,t]$, let $\mathcal{D}_{(x_1,x_2,s;t)} \subset \mathbb{R}$ be the set of terminal locations $y \in \mathbb{R}$ for which there exist geodesics $x_1 \to y$ and $x_2 \to y$ whose only point of intersection is the endpoint itself; see Figure 1b. These exceptional endpoints form a random Cantor-like set for which we have the following result.

Theorem 1. *The Hausdorff dimension of $\mathcal{D}_{(x_1,x_2,s;t)}$ is almost surely $\frac{1}{2}$.*

There is an analogous bivariate scenario. Fixing now only the interval $[s,t]$, let $\mathcal{D}_{s,t} \subset \mathbb{R}^2$ be the set of (x,y) for which there exist two geodesics $x \to y$ that intersect only at the endpoints; see Figure 1d. Interestingly, this second exceptional set can be related to the first by associating to $\mathcal{L}$ certain random measures with fractal supports. In developing this connection, we also obtain the following.

Theorem 2. *The Hausdorff dimension of $\mathcal{D}_{s,t}$ is almost surely $\frac{1}{2}$.*

We will restate and in fact expand upon these two results in Theorems 1.9 and 1.10, after having properly defined the relevant objects. We first define the model of Brownian LPP, and then turn our attention to its scaling limit, the directed landscape.



(a) $y \notin \mathcal{D}_{(x_1,x_2,s;t)}$

(b) $y \in \mathcal{D}_{(x_1,x_2,s;t)}$

(c) $(x,y) \notin \mathcal{D}_{s,t}$

(d) $(x,y) \in \mathcal{D}_{s,t}$

Figure 1. Exceptionality of the sets $\mathcal{D}_{(x_1,x_2,s;t)}$ and $\mathcal{D}_{s,t}$. Time is visualized in the vertical direction, and space in the horizontal direction. The curves shown are graphs of geodesics as functions of the vertical coordinate. For fixed x_1, x_2, y, s, t, we will almost surely witness the scenario in (a), in which the coalescence of two geodesics sharing a terminal location y happens before the terminal time t. Similarly, for fixed x, y, s, t, we will almost surely witness the scenario in (c), in which there is a unique geodesic associated to the pair of space-time points (x,s) and (y,t).

1.2. **Prelimiting model: Brownian last passage percolation.** Let $B(\cdot,k) : \mathbb{R} \to \mathbb{R}$, $k \in \mathbb{Z}$, denote independent two-sided Brownian motions supported on a common probability space equipped with probability measure $\mathbb{P}$. To each pair of real numbers $x \le y$ together with any pair of integers $i \le j$, we associate a passage time

$$M(x,i;y,j) := \sup\left\{ \sum_{k=i}^{j} [B(z_{k+1},k) - B(z_k,k)] \;\middle|\; x = z_i \le z_{i+1} \le \cdots \le z_j \le z_{j+1} = y \right\}. \quad (1.1)$$

That is, $M(x,i;y,j)$ is the largest number that can be obtained by partitioning the interval $[x,y]$ into $j-i+1$ ordered subintervals $[z_i, z_{i+1}], [z_{i+1}, z_{i+2}], \dots, [z_j, z_{j+1}]$ and then summing the Brownian increments incurred by traversing the subinterval $[z_k, z_{k+1}]$ using the k^{th} Brownian motion. This model was first studied in [25] as the limit of a queueing problem, and we refer to M as *Brownian LPP*.

1.2.1. *Unscaled coordinates: staircases and a variational formula on functions.* By compactness and the continuity of the Brownian motions, there is at least one partition that achieves the supremum in (1.1). As another perspective, the candidate partitions are in bijection with right-continuous, non-decreasing functions $\varphi : [x,y) \to [\![i,j]\!]$, where $[\![i,j]\!]$ denotes the integer interval $\{i, i+1, \dots, j\}$. Namely, $\varphi(z) = k$ precisely when $z \in [z_k, z_{k+1})$. Therefore, we can express M formally as

$$M(x,i;y,j) \coloneqq \sup_{\varphi} \int_x^y \mathrm{d}B(z, \varphi(z)). \tag{1.2}$$

Each φ can be associated to a "staircase" path in $\mathbb{R}^2$ starting at (x,i), ending at (y,j), and consisting of alternating horizontal and vertical line segments; see Figure 2a.

1.2.2. *Scaled coordinates: standardized passage times, zigzags, and polymers.* For any $(x,i;y,j)$, the distribution of $M(x,i;y,j)$ can be inferred from that of $M(0,0;n,n)$ by Brownian rescaling, namely

$$M(x,i;y,j) \overset{\text{dist}}{=} M(0,0;y-x,j-i) \overset{\text{dist}}{=} \sqrt{\frac{y-x}{j-i}} M(0,0;j-i,j-i). \tag{1.3}$$

It is thus natural to study the quantity $M(0,0;n,n)$, which has the remarkable property of agreeing in distribution with the largest eigenvalue of an $(n+1)\times(n+1)$ matrix sampled from the Gaussian unitary ensemble (GUE) with entry variance n [9, 26, 38]. Therefore, the mean of $M(0,0;n,n)$ is asymptotic to $2n$ as $n \to \infty$, and its fluctuations about this mean are of order $n^{1/3}$, in agreement with KPZ scaling. More precisely, if we define

$$W_n \coloneqq \frac{M(0,0;n,n) - 2n}{n^{1/3}}, \tag{1.4}$$

then W_n converges in law as $n \to \infty$ to the GUE Tracy–Widom distribution [4, Theorem 3.1.4].

In order to recover a scaling limit for the joint process $(M(0,0;n+y,n))_{y \ge -n}$, one must also know on which scale to vary y to induce fluctuations of order $n^{1/3}$ relative to $M(0,0;n,n)$. An equivalent question is, by how much does a staircase achieving $M(0,0;n,n)$ in (1.2) deviate from the straight line connecting $(0,0)$ and (n,n)? KPZ considerations put forward the answer of $n^{2/3}$, leading us to introduce the scaled coordinates

$$(x,s)_n \coloneqq (sn + 2n^{2/3}x, \lfloor sn \rfloor), \quad x, s \in \mathbb{R}, \tag{1.5}$$

so that x and s can be regarded as unit-order. Generalizing (1.4), we define the standardized passage time

$$W_n(x,s;y,t) \coloneqq \frac{M((x,s)_n;(y,t)_n) - 2(t-s)n - 2n^{2/3}(y-x)}{n^{1/3}}. \tag{1.6}$$

Remark 1.1. The definition (1.6) makes sense whenever $\lfloor sn \rfloor \le \lfloor tn \rfloor$ and $sn + 2n^{2/3}x \le tn + 2n^{2/3}y$. When $s < t$, these requirements will be satisfied for all n sufficiently large. Therefore, for any

$$u \in \mathbb{R}^4_\uparrow \coloneqq \{(x,s;y,t) \in \mathbb{R}^4 : s < t\},$$

the quantity $W_n(u)$ is well-defined for all n larger than some $n_0 = n_0(u)$. Henceforth, whenever we consider $W_n(u)$ or any other object that depends on both u and n, it will be implicitly assumed that $n \ge n_0(u)$. For any compact $K \subset \mathbb{R}^4_\uparrow$, the choice of n_0 can made uniformly over $u \in K$.

The scale prescribed by definitions (1.5) and (1.6) has reshaped the original geometry of Brownian LPP via the linear transformation R_n mapping $(2n^{2/3}, 0) = (1, 0)_n \mapsto (1, 0)$ and $(n, n) = (0, 1)_n \mapsto (0, 1)$. The images under R_n of the staircases from before are now "zigzags" consisting of horizontal and oblique segments, as seen in Figures 2b and 2c. Given a staircase function φ, let us write $R_n(\varphi)$ for the planar path determined by the associated zigzag. To be completely precise, we make the following definitions.

Definition 1.2. For the right-continuous, non-decreasing $\varphi : [x, y) \to [\![i, j]\!]$ defined by $\varphi(z) = k$ for $z \in [z_k, z_{k+1})$, the *zigzag* $R_n(\varphi) \subset \mathbb{R}^2$ is the image under R_n of the following staircase:

$$\underbrace{\bigcup_{k=i}^{j}([z_k, z_{k+1}] \times \{k\})}_{\text{horizontal segments}} \cup \underbrace{\bigcup_{k=i}^{j-1}(\{z_{k+1}\} \times [k, k+1])}_{\text{vertical segments}}.$$

The reader might find this definition more transparent by simply examining Figure 2c.

Definition 1.3. For $u = (x, s; y, t) \in \mathbb{R}^4_\uparrow$, if φ achieves the supremum in (1.2) for $M((x, s)_n; (y, t)_n)$, then we will say $R_n(\varphi)$ is an *n-polymer* between the *endpoints* (x, s) and (y, t). Let $P_{n,u}$ denote the set of such polymers.

For fixed u, there is almost surely a unique n-polymer in $P_{n,u}$ [30, Lemma 4.6(1)], although there may be random exceptional $u \in \mathbb{R}^4_\uparrow$ admitting more than one.

1.2.3. *Spatial deviations and geodesics.* After the application of R_n, the order $n^{2/3}$ spatial deviations mentioned before are observed as order 1 deviations of polymers from the vertical axis connecting $(0, 0)$ and $(0, 1)$. These fluctuations will be recorded as follows. Given $u = (x, s; y, t) \in \mathbb{R}^4_\uparrow$ and a candidate φ in (1.2) for $M((x, s)_n; (y, t)_n)$, i.e., a right-continuous, non-decreasing $\varphi : [sn + 2n^{2/3}x, tn + 2n^{2/3}y) \to [\![\lfloor sn\rfloor, \lfloor tn\rfloor]\!]$, we consider the function $\Gamma^{(\varphi)}_{n,u} : [s, t] \to \mathbb{R}$ given by

$$\Gamma^{(\varphi)}_{n,u}(r) := \frac{L_{n,u}(r) - \varphi(L_{n,u}(r))}{2n^{2/3}}, \quad r \in [s, t), \qquad \Gamma^{(\varphi)}_{n,u}(t) := \lim_{r\nearrow t} \Gamma^{(\varphi)}_{n,u}(r), \tag{1.7}$$

where

$$L_{n,u}(r) := rn + \frac{t-r}{t-s}2n^{2/3}x + \frac{r-s}{t-s}2n^{2/3}y, \quad r \in [s, t]. \tag{1.8}$$

That is, in unscaled coordinates, the vertical separation between the point $(L_{n,u}(r), L_{n,u}(r))$ and the staircase associated to φ is exactly $\Gamma^{(\varphi)}_{n,u}(r) \cdot 2n^{2/3}$; see Figures 2d and 2e.

Definition 1.4. For $u = (x, s; y, t) \in \mathbb{R}^4_\uparrow$, if φ achieves the supremum in (1.2) for $M((x, s)_n; (y, t)_n)$, then $\Gamma^{(\varphi)}_{n,u}$ will be called an *n-geodesic* between (x, s) and (y, t). Let $G_{n,u}$ denote the set of such geodesics.

Henceforth, the variables x and y are to be thought of as spatial coordinates, despite their initial role as time coordinates in Brownian motions. Instead, s and t are now the temporal coordinates, reflecting our desire to think of $\Gamma^{(\varphi)}_{n,u}$ as an upward moving path in $\mathbb{R}^2$ starting at (x, s) and terminating at (y, t), as illustrated in Figures 2e and 2f. To be completely precise, though, we note that $\Gamma^{(\varphi)}_{n,u}(s)$ and $\Gamma^{(\varphi)}_{n,u}(t)$ are not necessarily exactly equal to x and y, respectively. For instance, the equality with x will only be approximate if $sn \notin \mathbb{Z}$, or if $sn \in \mathbb{Z}$ but $\varphi(sn + 2n^{2/3}x) > sn$. When $\Gamma^{(\varphi)}_{n,u}$ is an n-geodesic, the latter scenario happens with probability zero.

Note that $\Gamma^{(\varphi)}_{n,u} \in G_{n,u}$ if and only if $R_n(\varphi) \in P_{n,u}$, and so n-geodesics and n-polymers are two slightly different ways of obtaining scaled versions of the maximizers in (1.2). The difference between the two objects is highlighted in Figure 2, and we will later discuss how they nonetheless give rise to the same limiting object.

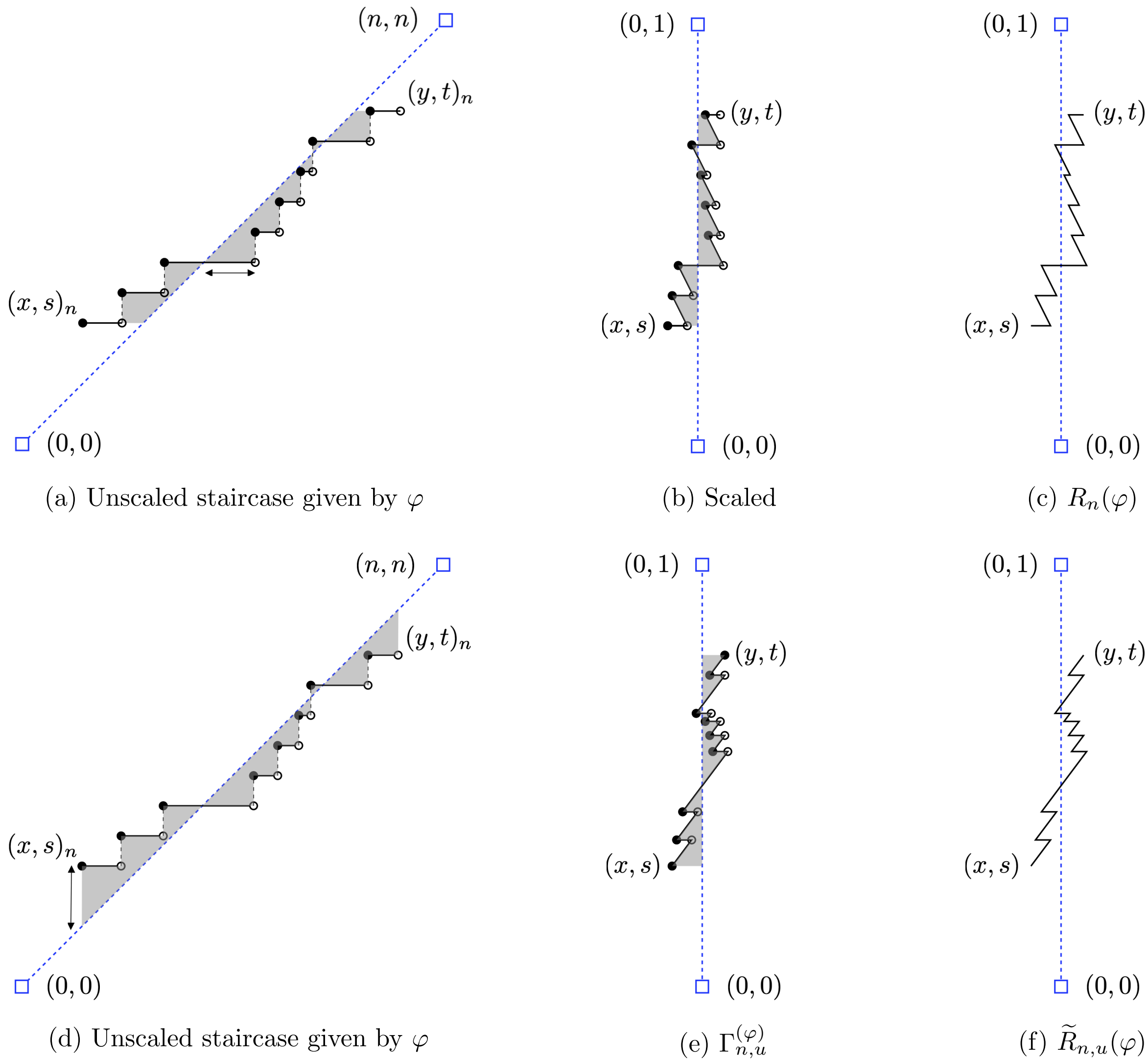


Figure 2. In this example, $x < 0 < y$, $0 < s < t < 1$, and we assume $sn, tn \in \mathbb{Z}$ for clarity. The unscaled staircase in (a) and (d) between $(x,s)_n$ and $(y,t)_n$ is the graph of φ. The horizontal segments are of the form $[z_k, z_{k+1}] \times \{k\}$ and connected via the vertical segments $\{z_{k+1}\} \times [k, k+1]$. When deviations from the diagonal connecting $(0,0)$ and (n,n) are measured as a function of the vertical coordinate and scaled by $2n^{2/3}$, the result is the zigzag in (b), equivalently realized by applying the scaling map R_n to the picture in (a). We will view the zigzag as a closed planar path $R_n(\varphi) \subset \mathbb{R}^2$, depicted in (c). Its horizontal segments have length $\frac{1}{2}n^{-2/3}(z_{k+1} - z_k)$, while its oblique segments have slope $-2n^{-1/3}$ and extend over vertical distances that are multiples of n^{-1}. Alternatively, when the deviations in (d) are regarded as a function of the horizontal coordinate and reparameterized via (1.7), the resulting $\Gamma^{(\varphi)}_{n,u}$ is shown in (e) as a function of the time variable on the vertical axis. The associated planar path $\widetilde{R}_{n,u}(\varphi) \subset \mathbb{R}^2$ is shown in (f). Its oblique segments have slope $(\frac{1}{2}n^{1/3} + \frac{y-x}{t-s})^{-1}$ and traverse a horizontal distance of $\frac{1}{2}n^{-2/3}(z_{k+1} - z_k)$, while the horizontal segments have lengths that are multiples of $\frac{1}{2}n^{-2/3}$.

1.3. **Limiting model: the directed landscape.** Section 1.2.2 saw the appearance of our first canonical object in the KPZ universality class, namely the Tracy–Widom law. It has been proven to arise also in Poissonian LPP [6, 8]; the asymmetric simple exclusion process (ASEP) [42]; the totally asymmetric simple exclusion process (TASEP) [34, 24, 15]; the positive temperature version of Brownian LPP [11, 12] introduced by O'Connell and Yor [37]; the continuum random polymer [3, 12] introduced by Alberts, Khanin, and Quastel [2, 1]; and the fully discrete log-gamma polymer [13] introduced by Seppäläinen [41]. Apart from its universality, the function of the Tracy–Widom law in our current setting is to describe the limiting behavior of any one-point distribution. Indeed, upon knowing that $W_n = W_n(0,0;0,1)$ converges in distribution to a GUE Tracy–Widom random variable, one can infer—from (1.3) and a Taylor expansion of the square root factor—that $W_n(x,s;y,t)$ obeys the same convergence but where the limit has been scaled by $(t-s)^{1/3}$ and then shifted by $-(y-x)^2/(t-s)$.

1.3.1. *Airy$_2$ process and the space-time Airy sheet.* In the same way, any limiting *multi*-point distributions (in a single spatial coordinate) can be deduced from the limiting behavior of the one-parameter process $y \mapsto W_n(0,0;y,1)$. For this object and its counterparts in other one-dimensional growth models within the KPZ universality class, the distributional limit is a parabolically shifted *Airy process*, our second canonical object. In the setting of Brownian LPP, this means $y \mapsto W_n(0,0;y,1) + y^2$ converges in law as $n \to \infty$ to a stationary process $y \mapsto \mathcal{A}_2(y)$ known as Airy$_2$, introduced in [39]. While this statement remains for most models a conjecture, it has been proved in the sense of finite-dimensional distributions for Poissonian LPP [39, 16] and TASEP [14, 17, 7], and as a functional limit theorem for geometric LPP [35].

Given those developments, it is only natural that more recent work has been toward understanding the full four-parameter process $(x,s;y,t) \mapsto W_n(x,s;y,t)$. In this case, the relevant limiting object (again after a parabolic correction) was conjectured in [21] to be the so-called *space-time Airy sheet*, whose rigorous construction was left open. One view is that the one-parameter process $y \mapsto W_n(0,0;y,t)$ is Markovian in t and evolves forward from a Dirac delta mass at $t = 0$. If this evolution has a large-n limit, then the noise driving said evolution is the Airy sheet. This perspective was advanced in [36], where the authors allow very general initial conditions—the pre-limiting model being TASEP—and derive determinantal formulas for the transition probabilities of the limiting Markov process.

Very recently in [23], the Airy sheet was constructed directly from a last passage model on the Airy line ensemble, whose top curve is itself the distributional limit of $y \mapsto W_n(0,0;y,1)$. Featured in the paper's analysis is an extension of the Robinson–Schensted–Knuth correspondence that allows the original Brownian LPP to be mapped to a last passage problem involving Brownian motions conditioned to not intersect. It is this collection of non-intersecting Brownian motions that converges in a suitable scaling limit to the Airy line ensemble, hence the prospect—ultimately realized—that Brownian LPP has as its limit a last passage problem on this ensemble respecting said convergence. (A separate work [22] makes progress toward showing the same statement for other classical LPP models.) We will now define the resulting object.

Definition 1.5. The *directed landscape* is a random continuous function $\mathcal{L} : \mathbb{R}^4_\uparrow \to \mathbb{R}$ almost surely satisfying

$$\mathcal{L}(x,s;y,t) = \sup_{z\in\mathbb{R}}[\mathcal{L}(x,s;z,r) + \mathcal{L}(z,r;y,t)] \quad \text{for all } (x,s;y,t) \in \mathbb{R}^4_\uparrow,\ r \in (s,t), \tag{1.9}$$

and having the property that for any disjoint intervals (s_i,t_i), $i = 1,\dots,n$, the following processes on $\mathbb{R}^2$ are i.i.d.:

$$(x,y) \mapsto (t_i - s_i)^{-1/3}\mathcal{L}(x(t_i-s_i)^{2/3}, s_i; y(t_i-s_i)^{2/3}, t_i), \quad i = 1,\dots,n. \tag{1.10}$$

The property (1.9) means $\mathcal{L}$ can be thought of as an "anti-metric" in that it satisfies the reverse triangle inequality. It was shown in [23, Lemma 10.3] that $\mathcal{L}$ exists and has a unique law determined entirely by the law of the two-parameter process $(x, y) \mapsto \mathcal{L}(x, 0; y, 1)$. The space-time Airy sheet is obtained by the parabolic correction $\mathcal{A}(x, s; y, t) := \mathcal{L}(x, s; y, t) + \frac{(x-y)^2}{t-s}$, which then has the property that $y \mapsto \mathcal{A}(x, 0; y, 1)$ is an Airy$_2$ process for any fixed $x \in \mathbb{R}$. It also satisfies space-time stationarity,

$$\mathcal{A}(x, s; y, t) \stackrel{\text{dist}}{=} \mathcal{A}(x + z, s + r; y + z, t + r) \quad \text{for any } z, r \in \mathbb{R}.$$

The following convergence result justifies the consideration of $\mathcal{L}$ as a canonical object in the KPZ universality class.

Theorem A. [23, Theorem 1.5] *There exists a coupling of Brownian LPP and $\mathcal{L}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ so that*

$$W_n(x, s; y, t) = \mathcal{L}(x, s; y, t) + o_n(x, s; y, t),$$

where o_n is a random function admitting, for every compact $K \subset \mathbb{R}^4_\uparrow$, a deterministic constant $a > 1$ such that $\mathbb{E}(a^{\sup_K |o_n|^{3/4}}) \to 1$.

Remark 1.6. We may assume without loss of generality that $\mathcal{F}$ is complete. That is, if $A_0 \in \mathcal{F}$ satisfies $\mathbb{P}(A_0) = 0$, and $A \subset A_0$, then $A \in \mathcal{F}$. Equivalently, if $A_1 \in \mathcal{F}$ satisfies $\mathbb{P}(A_1) = 1$, and $A \supset A_1$, then $A \in \mathcal{F}$. This assumption will be a technical convenience when we check the measurability of certain events.

1.3.2. *Geodesics.* In the fully continuous setting of the directed landscape, the analogues of the polymers from Definition 1.3 are fractal, upward moving paths in $\mathbb{R}^2$ like the one seen in Figure 3. Given a realization of $\mathcal{L}$, each continuous $\gamma : [s, t] \to \mathbb{R}$ has a *length* $\mathcal{L}(\gamma)$ given by

$$\mathcal{L}(\gamma) := \inf_{k \in \mathbb{N}} \inf_{s=t_0<t_1<\cdots<t_k=t} \sum_{i=1}^{k} \mathcal{L}(\gamma(t_{i-1}), t_{i-1}; \gamma(t_i), t_i). \tag{1.11}$$

This definition is in analogy with that of Euclidean length, except that infima replace suprema because of the anti-metric nature of $\mathcal{L}$. If $\gamma(s) = x$ and $\gamma(t) = y$, then the coordinate pairs (x, s) and (y, t) are referred to as the *endpoints* of γ. By taking $k = 1$ in (1.11), it is clear that $\mathcal{L}(\gamma) \le \mathcal{L}(x, s; y, t)$. The limiting version of (1.2) is now

$$\mathcal{L}(u) = \sup_{\substack{\gamma:[s,t]\to\mathbb{R} \\ \gamma(s)=x,\ \gamma(t)=y}} \mathcal{L}(\gamma), \quad u = (x, s; y, t) \in \mathbb{R}^4_\uparrow,$$

where the supremum is taken over continuous functions.

Definition 1.7. Let $u = (x, s; y, t) \in \mathbb{R}^4_\uparrow$ and suppose $\gamma : [s, t] \to \mathbb{R}$ is a continuous function such that $\gamma(s) = x$ and $\gamma(t) = y$. If $\mathcal{L}(\gamma) = \mathcal{L}(u)$, then we say γ is a *geodesic* between (x, s) and (y, t). Let us write G_u to denote the set of all geodesics from (x, s) to (y, t).

The collection of all geodesics was termed the *polymer fixed point* in [21] and is thought to also be universal to the KPZ class. For instance, the authors of [21] suggest the polymer fixed point might also be realized as the zero-temperature limit of the continuum directed random polymer [1], and so their use of "polymer" serves as a nod to positive-temperature models. This usage is consistent with the convention of reserving "polymer" to refer to a positive-temperature object (a sample from a *measure* on paths) while keeping to "geodesic" for a zero-temperature object (a single path of maximal energy). As this paper deals only with zero-temperature models, we have deviated from that convention in order to clearly distinguish between Definitions 1.3 and 1.4.

The existence of geodesics is a consequence of (1.9), since one can consider γ defined by $\gamma(r) = z_r$, where z_r is a suitably chosen maximizer in (1.9). (For instance, see Lemma 3.3.) Typically the maximizer z_r is unique for each $r \in (s, t)$, in which case G_u is a singleton; that is, for fixed $u \in \mathbb{R}^4_\uparrow$, we have $|G_u| = 1$ with probability one [23, Theorem 12.1]. It is also a fact that geodesics are

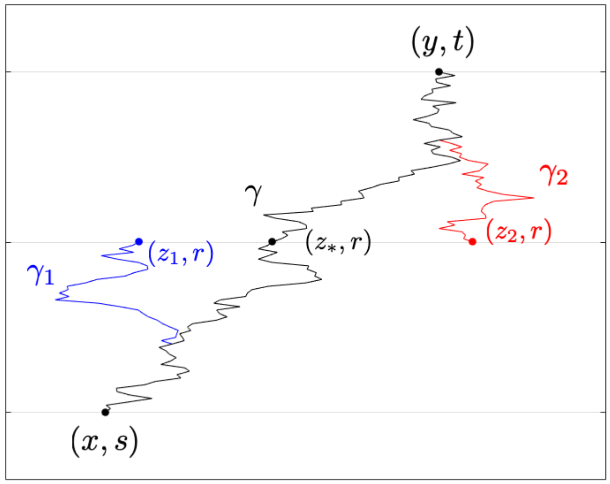


Figure 3. Example geodesics in $\mathcal{L}$. Time is visualized in the vertical direction. Shown above is a geodesic γ between (x, s) and (y, t) that passes through (z_*, r), meaning z_* achieves the supremum in (1.9). For generic $z_1 \neq z_*$, geodesics γ_1 and γ will typically coincide for some random length of time, and then remain disjoint after said time. Similarly, geodesics γ_2 and γ will typically coincide for all times after their first intersection.

typically Hölder-$2/3^-$ continuous in time [23, Theorem 1.7]. Nevertheless, as in the prelimit, there may exist random exceptional points for which these statements are not true.

Remark 1.8. Admittedly, it is an abuse of notation to write $\mathcal{L}(\gamma)$ given that $\mathcal{L}$ was defined to be a continuous function on $\mathbb{R}^4_\uparrow$. Nevertheless, this notational convenience should not lead to any confusion, as we will adhere to the following conventions:

- u always denotes an element of $\mathbb{R}^4_\uparrow$.
- x, y, z, w, p, q are spatial coordinates (typically x, z, p are associated to initial endpoints of geodesics, and y, w, q to terminal endpoints).
- r, s, t are temporal coordinates ($s \leq r \leq t$).
- φ, ϕ are maximizers achieving $M_n(x, i, y, j)$ in (1.2), where $(x, i; y, j)$ will be apparent from context (in particular, φ and ϕ are right-continuous, non-decreasing $\mathbb{Z}$-valued functions).
- γ denotes a continuous function of time, an object in the limiting model.
- Γ denotes the corresponding prelimiting object (generally a discontinuous function).
- $a_j \nearrow a$ means $a_j \leq a_{j+1}$ for all j, and a_j converges to a as $j \to \infty$; similarly for $a_j \searrow a$.

The following result from [23] confirms that the directed landscape retains the limiting information of not only the passage times in Brownian LPP, but also of the maximizing paths comprising the polymer fixed point. Recall that for $u = (x, s; y, t)$ and sufficiently large n, $G_{n,u}$ denotes the set of n-geodesics $\Gamma^{(\varphi)}_{n,u} : [s, t] \to \mathbb{R}$ defined in (1.7), where φ is a maximizer in (1.2).

Theorem B. [23, Theorems 1.8 and 13.5] *In the coupling of Theorem A, there exists an event $\mathcal{P}$ of probability 1 such that the following holds for any $u \in \mathbb{R}^4_\uparrow$. On the almost sure event $\mathcal{P} \cap \{|G_u| = 1\}$, if γ_u is the unique element of G_u, and $\Gamma^{(\varphi_n)}_{n,u} \in P_{n,u}$ for each n, then*

$$\lim_{n\to\infty} \|\Gamma^{(\varphi_n)}_{n,u} - \gamma_u\|_\infty = 0,$$

where $\|\cdot\|_\infty$ denotes the sup-norm on $[s, t]$.

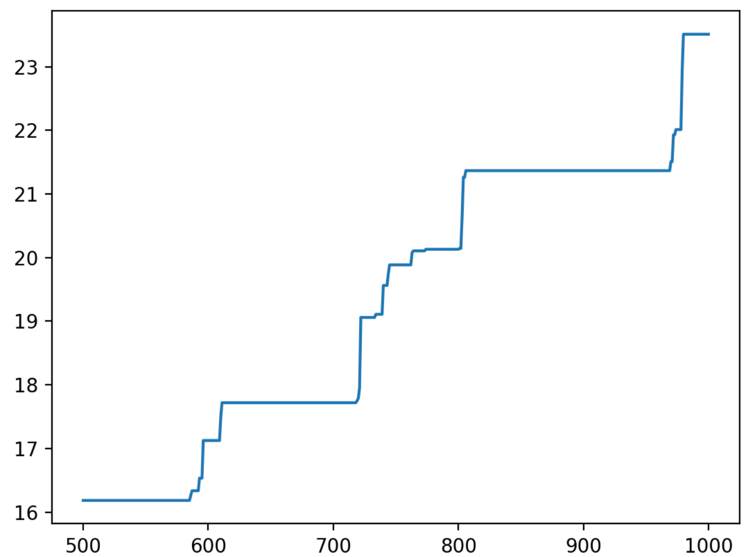


Figure 4. An LPP simulation by Junou Cui, Zoe Edelson, and Bijan Fard. With $n = 500$, the unscaled difference $M(n^{2/3}, 0; y, n) - M(-n^{2/3}, 0; y, n)$ is plotted on the vertical axis against the unscaled location y on the horizontal axis. Once rescaled, this corresponds to the picture in Figure 5 with $x_1 = -\frac{1}{2}$, $x_2 = \frac{1}{2}$, $s = 0$, $t = 1$, and y varying between $-\frac{1}{2}$ and $\frac{1}{2}$.

The event $\mathcal{P}$ is that $\mathcal{L}$ is *proper*, a notion defined in [23, Section 13] and recalled in Section 2.2. We will assume throughout the paper that this event occurs.

1.4. **Main results on the Hausdorff dimensions of exceptional sets.** The present study is motivated by the view that the fractal properties of the polymer fixed point have much to say about the probabilistic structure of the directed landscape. This perspective is very much aligned, for instance, with the interest in coalescence of geodesics for both first and last passage percolation models (the literature is vast; see [5, Chapters 4 and 5] for one starting point). Somewhat similar to the Busemann functions employed in those settings, a novel object called the *difference weight profile* was studied in [10]. Given $x_1 < x_2$ and $s < t$, this is the random map $y \mapsto \mathcal{Z}_{(x_1,x_2,s;t)}(y) \coloneqq \mathcal{L}(x_2, s; y, t) - \mathcal{L}(x_1, s; y, t)$, an almost-surely continuous, non-decreasing function on the real line; see Figure 5. A striking feature, which is suggested by simulation of the prelimit (see Figure 4), is that $\mathcal{Z}_{(x_1,x_2,s;t)}$ is locally constant almost everywhere in the sense of Lebesgue. Indeed, the following theorem was shown in [10]. Recall that $y \in \mathbb{R}$ is a point of local variation for a function $f : \mathbb{R} \to \mathbb{R}$ if there exists no open interval containing y on which f is constant.

Theorem C. [10, Theorem 1.1] *For any fixed $x_1 < x_2$ and $s < t$, the set of local variation for $\mathcal{Z}_{(x_1,x_2,s;t)}$ almost surely has Hausdorff dimension one-half.*

The proof of the upper bound for this result proceeded, at least heuristically, by examining points of coalescence between geodesics emanating from (x_1, s) and (x_2, s), and terminating at a common point (y, t). It was suggested that the set of local variation for $\mathcal{Z}_{(x_1,x_2,s;t)}$ is a subset of those y for which the coalescence point is (y, t) itself. In other words, the supposed superset is the set $\mathcal{D}_{(x_1,x_2,s;t)}$ of locations y for which there exist two geodesics to (y, t), one originating from (x_1, s) and the other from (x_2, s), that are disjoint except at the terminal point (y, t). That is,

$$\mathcal{D}_{(x_1,x_2,s;t)} \coloneqq \Big\{ y \in \mathbb{R} : \exists\, \gamma_1 \in G_{(x_1,s;y,t)}, \gamma_2 \in G_{(x_2,s;y,t)}, \gamma_1(r) < \gamma_2(r) \text{ for all } r \in (s,t) \Big\}, \tag{1.12}$$

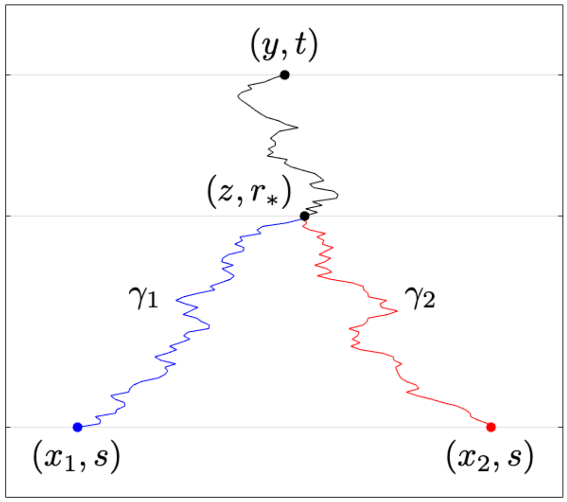


Figure 5. Determining the value of $\mathcal{Z}_{(x_1,x_2,s;t)}(y)$. Assuming geodesics exist, one can (using Lemma 2.2) always find $\gamma_1 \in G_{(x_1,s;y,t)}$ and $\gamma_2 \in G_{(x_2,s;y,t)}$ such that $\gamma_1(r) = \gamma_2(r)$ for all r above some $r_* \in (s,t]$, and $\gamma_1(r) < \gamma_2(r)$ for all r below r_*. If $\gamma_1(r_*) = \gamma_2(r_*) = z$, then $\mathcal{L}(\gamma_1) = \mathcal{L}(x_1,s;z,r_*) + \mathcal{L}(z,r_*;y,t)$ while $\mathcal{L}(\gamma_2) = \mathcal{L}(x_2,s;z,r_*) + \mathcal{L}(z,r_*;y,t)$. Therefore, $\mathcal{Z}_{(x_1,x_2,s;t)}(y) = \mathcal{L}(\gamma_2) - \mathcal{L}(\gamma_1) = \mathcal{L}(x_2,s;z,r_*) - \mathcal{L}(x_1,s;z,r_*)$.

where the inequality $\gamma_1(r) < \gamma_2(r)$ comes from the ordering of $x_1 < x_2$. (By planarity, $\gamma_1(r) > \gamma_2(r)$ only occurs if $\gamma_1(r') = \gamma_2(r')$ for some $r' < r$.) See Figure 1b for an illustration. Our first main result shows that the heuristic from [10] turns out to be correct, and that the exceptional set $\mathcal{D}_{(x_1,x_2,s;t)}$ also has Hausdorff dimension one-half.

Theorem 1.9. *For any fixed $x_1 < x_2$ and $s < t$, the following statements hold almost surely:*

(a) the set of local variation for $\mathcal{Z}_{(x_1,x_2,s;t)}$ is contained in $\mathcal{D}_{(x_1,x_2,s;t)}$; and

(b) the Hausdorff dimension of $\mathcal{D}_{(x_1,x_2,s;t)}$ is equal to $\frac{1}{2}$.

Since $\mathcal{Z}_{(x_1,x_2,s;t)}$ is non-decreasing by [10, Theorem 1.1(1)]—a consequence of planarity—another perspective is that $\mathcal{Z}_{(x_1,x_2,s;t)}$ is the distribution function of a random measure $\mu_{(x_1,x_2,s;t)}$ on the real line, in the sense that

$$\begin{aligned}\mu_{(x_1,x_2,s;t)}([y_1,y_2]) &= \mathcal{Z}_{(x_1,x_2,s;t)}(y_2) - \mathcal{Z}_{(x_1,x_2,s;t)}(y_1)\\ &= \mathcal{L}(x_2,s;y_2,t) + \mathcal{L}(x_1,s;y_1,t) - \mathcal{L}(x_1,s;y_2,t) - \mathcal{L}(x_2,s;y_1,t).\end{aligned} \tag{1.13}$$

In this language, the result of [10] is that the support of $\mu_{(x_1,x_2,s;t)}$ is a random Cantor-like set of Hausdorff dimension one-half, and Theorem 1.9(a) says this set is contained in $\mathcal{D}_{(x_1,x_2,s;t)}$.

A slightly more general perspective is that the final expression in (1.13) defines a measure $\mu_{s,t}$ on $\mathbb{R}^2$, namely

$$\mu_{s,t}([x_1,x_2]\times[y_1,y_2]) = \mu_{(x_1,x_2,s;t)}([y_1,y_2]). \tag{1.14}$$

In other words, by jointly varying the initial location x and the terminal location y, one obtains a measure-theoretic encoding of $\mathcal{L}$ on the time horizon $[s,t]$. Then one can ask if the statements from before regarding $\mu_{(x_1,x_2,s;t)}$ have analogues for $\mu_{s,t}$. Indeed, by modifications to the proof in [10], one can show—as we do in Theorem 4.3—that the support of $\mu_{s,t}$ also has Hausdorff dimension one-half. Furthermore, in place of $\mathcal{D}_{(x_1,x_2,s;t)}$, the related exceptional set is

$$\mathcal{D}_{s,t} := \Big\{(x,y) \in \mathbb{R}^2 : \exists\, \gamma_1 \in G_{(x,s;y,t)}, \gamma_2 \in G_{(x,s;y,t)}, \gamma_1(r) < \gamma_2(r) \text{ for all } r \in (s,t)\Big\}.$$

In words, $\mathcal{D}_{s,t}$ is the set of all pairs (x, y) admitting two geodesics between (x, s) and (y, t) which coincide only at the endpoints; see Figure 1d. We can now state our second main result.

Theorem 1.10. *For any fixed $s < t$, the following statements hold almost surely:*

(a) the support of $\mu_{s,t}$ is contained in $\mathcal{D}_{s,t}$; and
(b) the Hausdorff dimension of $\mathcal{D}_{s,t}$ is equal to $\frac{1}{2}$.

Interestingly, a measure similar to $\mu_{s,t}$ was studied in [32] in the context of planar LPP and positive-temperature directed polymers. Moreover, that measure is also related via its support to exceptional disjointness of geodesics [33]. In fact, Theorem 1.10(a) bears a striking resemblance to [33, Theorem 3.1], except that the latter result obtains *equality* of the two relevant sets. It would be interesting to know if the same is true here. That is, does the support of $\mu_{s,t}$ constitute all of $\mathcal{D}_{s,t}$?

On a technical note, we mention that (1.13) and (1.14) prescribe well-defined measures given the almost sure event $\mathcal{P}$; see Definition 2.3(v). Off of $\mathcal{P}$, one can simply take $\mu_{(x_1,x_2,s;t)}$ and $\mu_{s,t}$ to be the zero measure. Let us henceforth write $\mathrm{Supp}(\cdot)$ for the support of a measure.

Remark 1.11. It might seem natural to also consider the case when only the terminal location y is fixed, while the starting location(s) is allowed to vary. This perspective, however, is an ineffective way of studying the fractal geometry of the directed landscape. Indeed, for any fixed $y \in \mathbb{R}$, it is almost surely the case that $y \notin \mathcal{D}_{(x_1,x_2,s;t)}$ for every $x_1 \leq x_2$; this is shown in Section 5.4. Moreover, for the random exceptional y belonging to some $\mathcal{D}_{(x_1,x_2,s;t)}$, one trivially has $y \in \mathcal{D}_{(x_1',x_2',s;t)}$ for all $x_1' \leq x_1$ and $x_2' \geq x_2$, due to Lemma 2.6. From these observations, it is clear that a more refined description of the exceptional endpoints is obtained only by either fixing $x_1 < x_2$ and varying y, or imposing $x_1 = x_2 = x$ and varying (x, y).

1.5. Comments on the proofs, and key inputs.

Theorems 1.9(b) and 1.10(b) will be proved by realizing $\frac{1}{2}$ as both a lower and an upper bound on the dimension of each set. These two directions will be pursued separately in Sections 4 and 5, respectively.

Given that $\mathrm{Supp}(\mu_{(x_1,x_2,s;t)})$ and $\mathrm{Supp}(\mu_{s,t})$ each have Hausdorff dimension one-half, the lower bounds trivially follow from Theorems 1.9(a) and 1.10(a). Consequently, the work of Section 4 is to prove the claimed containments. Nevertheless, in recognition of the fact that the results appearing in [10] are only for $\mu_{(x_1,x_2,s;t)}$, we do separately check the required statements for $\mu_{s,t}$. On this matter, the lower bound argument carried out in [10]—which used the local Gaussianity of weight profiles [29], which in turn is a consequence of the Brownian Gibbs property [20, 28] enjoyed by the parabolic Airy line ensemble and its prelimit—could be replicated here, but we instead present in Section 4.2 a short proof that the Hausdorff dimension of $\mathrm{Supp}(\mu_{s,t})$ is at least that of $\mathrm{Supp}(\mu_{(x_1,x_2,s;t)})$. This indirect lower bound provides sufficient information to conclude Theorem 1.10(b) from 1.10(a), and (a) and (b) together provide anyway the matching upper bound for $\mathrm{Supp}(\mu_{s,t})$.

Returning to the more central issue of proving the containments claimed in Theorems 1.9(a) and 1.10(a), we can summarize the argument for 1.9(a) in the following broad strokes (the argument for 1.9(b) is similar):

- If $y \notin \mathcal{D}_{(x_1,x_2,s;t)}$, then the leftmost geodesic γ^{L} making the journey $(x_1, s) \to (y, t)$ must have non-trivial intersection with the rightmost geodesic γ^{R} traveling $(x_2, s) \to (y, t)$. An example is shown in Figure 8a.
- We prove in Lemma 4.4 the general fact that if $y_j^{\mathrm{L}} \nearrow y$, then there is a corresponding sequence of geodesics γ_j^{L} traveling $(x_1, s) \to (y_j, t)$ that converge uniformly to γ^{L}; similarly for $y_j^{\mathrm{R}} \searrow y$ and γ^{R}. Therefore, by choosing $y^{\mathrm{L}} < y < y^{\mathrm{R}}$ sufficiently close, we can find $\widetilde{\gamma}^{\mathrm{L}}$ and $\widetilde{\gamma}^{\mathrm{R}}$ approximating γ^{L} and γ^{R} to any desired precision.

- The critical observation, stated as Theorem 1.17, is that if the endpoints of two geodesics belong to a common compact set, then the geodesics cannot approximate each other arbitrarily well without intersecting. Therefore, y^{L} and y^{R} can be chosen so that $\widetilde{\gamma}^{\mathrm{L}}$ intersects γ^{L}, and $\widetilde{\gamma}^{\mathrm{R}}$ intersects γ^{R}.
- By forcing this intersection to occur at a suitably chosen location based on the non-trivial intersection of γ^{L} and γ^{R} (see Figure 8b), we deduce that $\widetilde{\gamma}^{\mathrm{L}}$ and $\widetilde{\gamma}^{\mathrm{R}}$ do themselves intersect.
- The proof is then completed by appealing to the observation—originally made in [10] and stated locally as Lemma 4.5—that $\mathcal{Z}_{(x_1,x_2,s;t)}$ is constant on $[y^{\mathrm{L}}, y^{\mathrm{R}}]$ whenever there are intersecting geodesics $\widetilde{\gamma}^{\mathrm{L}}$ and $\widetilde{\gamma}^{\mathrm{R}}$ making the respective journeys $(x_1, s) \to (y^{\mathrm{L}}, t)$ and $(x_2, s) \to (y^{\mathrm{R}}, t)$. The reason for this constancy is outlined in Figure 7.

Meanwhile, our method for the upper bound in Theorem 1.9(b) is inspired by the approach of [10], and the method for 1.10(b) is again similar. The central idea is captured in Figure 9 and is briefly described as follows:

- Let us restrict our attention to a bounded interval $[-R, R]$. Suppose $y \in \mathcal{D}_{(x_1,x_2,s;t)} \cap [-R, R]$, and let $y - \varepsilon < y_1 < y < y_2 < y + \varepsilon$. Because $y \in \mathcal{D}_{(x_1,x_2,s;t)}$, planarity guarantees that any geodesic γ_1 traveling $(x_1, s) \to (y_1, t)$ is disjoint from any geodesic γ_2 traveling $(x_2, s) \to (y_2, t)$, as shown in Figure 9b. Moreover, this holds for arbitrarily small $\varepsilon > 0$.
- By varying the initial location of γ_1 rightward from x_1 toward x_2, there must be some random point $x \in (x_1, x_2]$ at which γ_1 first intersects γ_2. By choosing $x^{\mathrm{L}} \in (x - \varepsilon, x)$ and $x^{\mathrm{R}} \in (x, x+\varepsilon)$, it can be shown that there exist *disjoint* geodesics traveling $(x^{\mathrm{L}}, s) \to (y_1, t)$ and $(x^{\mathrm{R}}, s) \to (y_2, t)$. This is done in [10, Proposition 3.5] and recalled as Lemma 5.3; the resulting geodesics are displayed in Figure 9c.
- We have now identified two small intervals $I = (x - \varepsilon, x + \varepsilon)$, $J = (y - \varepsilon, y + \varepsilon)$ that admit two disjoint geodesics starting in $I \times \{s\}$ and ending in $J \times \{t\}$. A required input is that as $\varepsilon \to 0$, the likelihood of this event for given x and y is, to leading order, bounded from above by $\varepsilon^{3/2}$. This fact is stated as Corollary 1.16.
- Now we cover $[x_1, x_2]$ and $[-R, R]$ each by order ε^{-1} many intervals I, J of radius ε. The expected number of pairs of these intervals with the above property is at most $\varepsilon^{-2} \cdot \varepsilon^{3/2} = \varepsilon^{-1/2}$. Therefore, the box-counting dimension of $\mathcal{D}_{(x_1,x_2,s;t)} \cap [-R, R]$, which always serves as an upper bound for the Hausdorff dimension, is at most $\frac{1}{2}$.

We anticipate that the arguments just described work quite generally and could potentially be modified to address related questions about the fractal structure of prelimiting models. Nevertheless, our present approach requires a small amount of information derived from integrable properties of Brownian LPP, and unfortunately certain technical aspects of this information significantly complicate the possible pursuance of Theorems 1.9 and 1.10, or versions thereof, in the prelimiting setting. (Specifically, the $\varepsilon^{3/2}$ asymptotic mentioned above does not hold on very small scales depending on n; see Theorem 3.5.) We have thus elected to pursue a more transparent argument by studying the limiting model; in this way, the present work harnesses the preprint [23] with a strong reliance on Theorem B in particular. Considering the putative universality of the directed landscape and the polymer fixed point, we anyway expect this to be a more fruitful line of research.

With that said, the present paper does marry arguments for the limiting model with inputs previously known or verified only in the prelimiting one. As such, there are a number of important facts requiring extension to the limiting setting. Therefore, in Sections 1.5.1 and 1.5.2 below, we state several other new results concerning the directed landscape and the polymer fixed point. These inputs, which are proved in Section 3, may be of independent interest as tools in or inspiration for future works. Section 2 contains several more input facts; these are straightforward statements about geodesics that will not be highlighted here.

1.5.1. *Convergence of polymers.* Owing to the novelty of the construction in [23], our first set of inputs work to build one bridge (of certainly many more) from the directed landscape to previously studied objects. In particular, we focus on Theorem B, which addresses the convergence of n-geodesics in Brownian LPP to their continuous counterparts in the directed landscape. While these particular prelimiting objects, defined via (1.7), enable the treatment of geodesics as functions, they are somewhat less natural from the view of geodesics as planar paths. For this latter perspective, it is desirable to simply consider the polymers from Definition 1.3, i.e., the piecewise linear paths obtained by applying the scaling map R_n to the Brownian LPP staircase paths (recall Figure 2). Indeed, the map R_n is intrinsic to the coupling of Brownian LPP and the directed landscape in the first place.

On a practical level, many estimates for Brownian LPP are stated for n-polymers rather than n-geodesics; for instance, see [28–31]. In order to leverage these results to prove statements in the limiting setting, it will be useful to know that n-polymers share the same limit as n-geodesics. For example, the upcoming Theorem 1.15 asks for probabilistic control on disjointness for geodesics. Unfortunately, disjointness of n-polymers—for which we have the corresponding information—is not equivalent to disjointness of n-geodesics. This is because the time parameterization (1.8) used in (1.7) depends on the spatial coordinates x and y, meaning that distinct $u_1 = (x_1, s; y_1, t)$, $u_2 = (x_2, s; y, t)$ can admit n-geodesics $\Gamma^{(\varphi)}_{(n,u_1)}$, $\Gamma^{(\phi)}_{(n,u_2)}$ that intersect,

$$\varphi(L_{n,u_1}(r)) = \phi(L_{n,u_2}(r)) \quad \text{for some } r \in [s,t],$$

even if the corresponding n-polymers $R_n(\varphi)$, $R_n(\phi)$ do not:

$$\varphi(z) \neq \phi(z) \quad \text{for all } z \in [sn + 2n^{2/3}x_1, tn + 2n^{2/3}y_1) \cap [sn + 2n^{2/3}x_2, tn + 2n^{2/3}y_2).$$

In this and other circumstances, the next theorem and corollary can help translate between the two prelimiting objects.

Let $u = (x, s; y, t) \in \mathbb{R}^4_\uparrow$ and $\varphi : [sn+2n^{2/3}x, tn+2n^{2/3}y) \to [\![\lfloor sn \rfloor, \lfloor tn \rfloor]\!]$ be the right-continuous, non-decreasing function defined by $\varphi(z) = k$ if and only if $z \in [z_k, z_{k+1})$. Recall the definitions of $\Gamma^{(\varphi)}_{n,u}$ and $L_{n,u}$ from (1.7) and (1.8). For each $k \in [\![\lfloor sn \rfloor, \lfloor tn \rfloor]\!]$, let r_k be the unique value in $[s,t]$ such that $L_{n,u}(r_k) = z_k$.

Definition 1.12. Denote by $\widetilde{R}_{n,u}(\varphi) \subset \mathbb{R}^2$ the planar path determined by the graph of $\Gamma^{(\varphi)}_{n,u}$; that is,

$$\widetilde{R}_{n,u}(\varphi) \coloneqq \underbrace{\big\{(z,r) : r \in [s,t], z = \Gamma^{(\varphi)}_{n,u}(r)\big\}}_{\text{oblique segments}} \cup \underbrace{\bigcup_{k=\lfloor sn \rfloor+1}^{\lfloor tn \rfloor} \big\{(z, r_k) : z \in [\Gamma_{n,u}(r_k), \Gamma_{n,u}(r_k-)]\big\}}_{\text{horizontal segments}},$$

where $\Gamma_{n,u}(r_k-) \coloneqq \lim_{r \nearrow r_k} \Gamma_{n,u}(r)$ if $r_k > s$, and $\Gamma_{n,u}(s-) \coloneqq \Gamma_{n,u}(s)$. See Figure 2f.

The theorem below says that the two planar paths $R_n(\varphi)$ and $\widetilde{R}_{n,u}(\varphi)$ are asymptotically equal if φ is a maximizer in (1.2). In particular, n-polymers share the same limit as n-geodesics, but use the language of sets rather than of functions. Recall that the *Hausdorff distance* between two nonempty subsets $\mathcal{X}, \mathcal{Y}$ of a metric space with metric τ is

$$\mathrm{dist}_H(\mathcal{X}, \mathcal{Y}) \coloneqq \max\Big\{\sup_{x\in\mathcal{X}} \inf_{y\in\mathcal{Y}} \tau(x,y), \sup_{y\in\mathcal{Y}} \inf_{x\in\mathcal{X}} \tau(x,y)\Big\}.$$

In stating the next result, we return to the setting of Theorem B. Recall that $\{\Gamma^{(\varphi_n)}_{n,u}\}_{n\geq 1}$ was assumed to be a sequence of n-geodesics, known to converge uniformly to γ_u. Since $\widetilde{R}_{n,u}(\varphi_n)$ is *the graph of* $\Gamma^{(\varphi_n)}_{n,u}$, some definition chasing will show that $\widetilde{R}_{n,u}(\varphi_n)$ converges to *the graph of* γ_u. It is by this logic, carried out in Section 3.1, that (1.16) will follow from (1.15).

Theorem 1.13. *In the coupling of Theorem A, the following holds for any $u \in \mathbb{R}^4_\uparrow$. On the almost sure event $\mathcal{P} \cap \{|G_u| = 1\}$, if γ_u is the unique element of G_u, and $R_n(\varphi_n) \in P_{n,u}$ for each n, then*

$$\limsup_{n\to\infty} \frac{\mathrm{dist}_H(R_n(\varphi_n), \widetilde{R}_{n,u}(\varphi_n))}{n^{-1/3}} < \infty. \tag{1.15}$$

In particular, upon defining $\mathrm{Graph}(\gamma_u) := \{(\gamma_u(r), r) : r \in [s,t]\}$, *we have*

$$\lim_{n\to\infty} \mathrm{dist}_H(R_n(\varphi_n), \mathrm{Graph}(\gamma_u)) = 0. \tag{1.16}$$

From this result, we will obtain the following essential ingredient to the proof of Theorem 1.15. While the statement below does not immediately follow from Theorem 1.13, it will be an easy consequence of the argument we give for (1.16). A short proof is included in Section 3.1.

Corollary 1.14. *Let $u_1 = (x_1, s; y_1, t)$ and $u_2 = (x_2, s; y_2, t)$ with $x_1 < x_2$ and $y_1 < y_2$. For each n, choose any $R_n(\varphi_n) \in P_{n,u_1}$ and $R_n(\phi_n) \in P_{n,u_2}$. On the event $\mathcal{P} \cap \{|G_{u_1}| = 1\} \cap \{|G_{u_2}| = 1\}$, if the unique geodesics $\gamma_1 \in G_{u_1}$ and $\gamma_2 \in G_{u_2}$ are disjoint, then $R_n(\varphi_n) \cap R_n(\phi_n) = \varnothing$ for all n sufficiently large.*

1.5.2. *Estimates for disjoint collections of geodesics.* We will say that two paths $\gamma_1, \gamma_2 : [s,t] \to \mathbb{R}$ are *disjoint* if $\gamma_1(r) \neq \gamma_2(r)$ for all $r \in [s,t]$. While the geodesics on which we will ultimately focus, namely those defining $\mathcal{D}_{(x_1,x_2,s;t)}$ and $\mathcal{D}_{s,t}$, are not strictly speaking disjoint—they coincide at the endpoint(s)—our arguments will exploit their influence on the disjointness of other nearby geodesics. Therefore, our second series of inputs concerns the rarity of certain events involving disjointness. Theorem 1.15 and Corollary 1.16, in particular, are very much in the aim of translating known results about the prelimiting model into ones about the limiting model.

For subsets $A, B, C \subset \mathbb{R}$ and times $s < t$, let $\mathrm{MaxDisjtGeo}^C_{s,t}(A, B)$ denote the maximum size of a collection of disjoint geodesics whose endpoints lie in $(A \cap C) \times \{s\}$ and $(B \cap C) \times \{t\}$. When C is countable (by which we mean having a cardinality that is either finite or countably infinite) the measurability of this random variable is proved in Proposition 3.2. The following tail bound is analogous to, and indeed proved from, [31, Theorem 1.1].

Theorem 1.15. *There exists a positive constant G such that the following holds for all countable $C \subset \mathbb{R}$. For any $\varepsilon > 0$, integer $k \geq 2$, and $u = (x, s; y, t) \in \mathbb{R}^4_\uparrow$ satisfying*

$$\frac{\varepsilon}{(t-s)^{2/3}} \leq G^{-4k^2}, \quad \frac{|x-y|}{(t-s)^{2/3}} \leq \Big(\frac{\varepsilon}{(t-s)^{2/3}}\Big)^{-1/2}\Big(\log\frac{(t-s)^{2/3}}{\varepsilon}\Big)^{-2/3} G^{-k}, \tag{1.17}$$

we have

$$\begin{aligned} &\mathbb{P}\Big(\mathrm{MaxDisjtGeo}^C_{s,t}([x-\varepsilon, x+\varepsilon], [y-\varepsilon, y+\varepsilon]) \geq k\Big) \\ &\leq G^{k^3} \exp\Big\{G^k\Big(\log\frac{(t-s)^{2/3}}{\varepsilon}\Big)^{5/6}\Big\}\Big(\frac{\varepsilon}{(t-s)^{2/3}}\Big)^{(k^2-1)/2}. \end{aligned} \tag{1.18}$$

The restriction that C be countable arises from the possibility that one of the mutually disjoint geodesics is associated to an exceptional $u \in \mathbb{R}^4_\uparrow$ for which $|G_u| \geq 2$. In this scenario, Corollary 1.14 no longer guarantees that the concerned collection of geodesics can be realized from a disjoint collection of polymers in the prelimit, thereby rendering the estimate from [31] inapplicable. When C is countable, however, this hurdle can be avoided by simply assuming the almost sure event in which all geodesics whose spatial endpoints lie in C are unique.

Notwithstanding these technical impediments, we anticipate that Theorem 1.15 is true with $C = \mathbb{R}$. Indeed, the $k = 2$ case admits a simple argument that will allow us to bootstrap to the following statement.

Corollary 1.16. *Let G be the constant from Theorem 1.15. For any $\varepsilon > 0$ and $u = (x,s;y,t) \in \mathbb{R}^4_\uparrow$ satisfying*

$$\frac{\varepsilon}{(t-s)^{2/3}} \leq G^{-16}, \quad \frac{|x-y|}{(t-s)^{2/3}} \leq \Big(\frac{\varepsilon}{(t-s)^{2/3}}\Big)^{-1/2}\Big(\log\frac{(t-s)^{2/3}}{\varepsilon}\Big)^{-2/3} G^{-2}, \tag{1.19}$$

we have

$$\begin{aligned}&\mathbb{P}\Big(\operatorname{MaxDisjtGeo}^{\mathbb{R}}_{s,t}((x-\varepsilon,x+\varepsilon),(y-\varepsilon,y+\varepsilon)) \geq 2\Big)\\ &\leq G^8 \exp\Big\{G^2\Big(\log\frac{(t-s)^{2/3}}{\varepsilon}\Big)^{5/6}\Big\}\Big(\frac{\varepsilon}{(t-s)^{2/3}}\Big)^{3/2}.\end{aligned} \tag{1.20}$$

Our arguments for the upper bounds in Theorems 1.9(b) and 1.10(b) will use Corollary 1.16 directly. Meanwhile, Theorems 1.9(b) and 1.10(b) will require the following application of Corollary 1.16 regarding geodesics that not only start and end nearby one another, but also remain close at all intermediate times.

Theorem 1.17. *On the event $\mathcal{P}$, for any compact $K \subset \mathbb{R}^4_\uparrow$, there is a random $\varepsilon > 0$ such that the following is true. If $u_1 = (x,s;y,t), u_2 = (z,s;w,t) \in K$ admit geodesics $\gamma_1 \in G_{u_1}$, $\gamma_2 \in G_{u_2}$ satisfying $|\gamma_1(r) - \gamma_2(r)| < \varepsilon$ for all $r \in [s,t]$, then γ_1 and γ_2 are not disjoint.*

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2. Preliminary facts concerning geodesics

In this section, we establish some basic facts about paths and geodesics in the directed landscape.

2.1. **New geodesics from old.** We begin with the following lemma concerning subpaths and subgeodesics.

Lemma 2.1. *For any continuous path $\gamma : [s,t] \to \mathbb{R}$ and any partition $s = t_0 < t_1 < \cdots < t_k = t$, the following statements hold.*

(a) We have the concatenation identity

$$\mathcal{L}(\gamma) = \sum_{i=1}^k \mathcal{L}(\gamma\big|_{[t_{i-1},t_i]}). \tag{2.1}$$

(b) If γ is a geodesic, then $\gamma\big|_{[t_{i-1},t_i]}$ is a geodesic for each $i = 1,\dots,k$, and

$$\mathcal{L}(\gamma) = \sum_{i=1}^k \mathcal{L}(\gamma(t_{i-1}),t_{i-1};\gamma(t_i),t_i). \tag{2.2}$$

Proof. First we prove (a). By induction, it suffices to prove the claim in the case $k = 2$ with $s < r < t$. Since any pairing of a partition of $[s,r]$ with a partition of $[r,t]$ induces a partition of $[s,t]$, it is clear that $\mathcal{L}(\gamma) \leq \mathcal{L}(\gamma\big|_{[s,r]}) + \mathcal{L}(\gamma\big|_{[r,t]})$. On the other hand, for any partition of $[s,t]$ not arising in this way (i.e., a sequence $s = t_0 < t_1 < \cdots < t_k = t$ such that $t_{j-1} < r < t_j$ for some j), we have

$$\begin{aligned}\sum_{i=1}^{k} \mathcal{L}(\gamma(t_{i-1}), t_{i-1}; \gamma(t_i), t_i) &\overset{(1.9)}{\geq} \sum_{i=1}^{j-1} \mathcal{L}(\gamma(t_{i-1}), t_{i-1}; \gamma(t_i), t_i) \;+\; \mathcal{L}(\gamma(t_{j-1}), t_{j-1}; \gamma(r), r) \\ &\qquad + \mathcal{L}(\gamma(r), r; \gamma(t_j), t_j) + \sum_{i=j+1}^{k} \mathcal{L}(\gamma(t_{i-1}), t_{i-1}; \gamma(t_i), t_i) \\ &\overset{(1.11)}{\geq} \mathcal{L}(\gamma\big|_{[s,r]}) + \mathcal{L}(\gamma\big|_{[r,t]}).\end{aligned}$$

Hence $\mathcal{L}(\gamma) \geq \mathcal{L}(\gamma\big|_{[s,r]}) + \mathcal{L}(\gamma\big|_{[r,t]})$, which completes the proof of (a).

For (b), we can again appeal to induction and reduce to the case $k = 2$. If γ is a geodesic, then

$$\mathcal{L}(\gamma\big|_{[s,r]}) + \mathcal{L}(\gamma\big|_{[r,t]}) \overset{(2.1)}{=} \mathcal{L}(\gamma) = \mathcal{L}(\gamma(s), s; \gamma(t), t) \overset{(1.9)}{\geq} \mathcal{L}(\gamma(s), s; \gamma(r), r) + \mathcal{L}(\gamma(r), r; \gamma(t), t).$$

Since we always have

$$\mathcal{L}(\gamma\big|_{[s,r]}) \leq \mathcal{L}(\gamma(s), s; \gamma(r), r) \quad \text{and} \quad \mathcal{L}(\gamma\big|_{[r,t]}) \leq \mathcal{L}(\gamma(r), r; \gamma(t), t),$$

the only possibility is that each of the two inequalities in the above display is achieved with equality. That is, $\gamma\big|_{[s,r]}$ and $\gamma\big|_{[r,t]}$ are geodesics, in which case (2.2) follows from (2.1). □

For the arguments to come, it will be useful to have the following notation for concatenating paths. If $\gamma_1 : [s, r'] \to \mathbb{R}$ and $\gamma_2 : [r', t] \to \mathbb{R}$ satisfy $\gamma_1(r') = \gamma_2(r')$, then $\gamma_1 \oplus \gamma_2 : [s, t] \to \mathbb{R}$ will denote the function defined by

$$(\gamma_1 \oplus \gamma_2)(r) := \begin{cases} \gamma_1(r) & \text{if } r \in [s, r'], \\ \gamma_2(r) & \text{if } r \in (r', t]. \end{cases}$$

The following lemma says that if two geodesics intersect twice, then exchanging their segments between these intersections results in another geodesic.

Lemma 2.2. *Suppose $\gamma_1 : [s_1, t_1] \to \mathbb{R}$ and $\gamma_2 : [s_2, t_2] \to \mathbb{R}$ are geodesics. If $\gamma_1(r') = \gamma_2(r')$ and $\gamma_1(r'') = \gamma_2(r'')$ for some $r' < r''$ belonging to $[s_1, t_1] \cap [s_2, t_2]$, then each of the following paths is a geodesic:*

(i) $\gamma_1\big|_{[s_1,r']} \oplus \gamma_2\big|_{[r',r'']} \oplus \gamma_1\big|_{[r'',t_1]}$,
(ii) $\gamma_1\big|_{[s_1,r']} \oplus \gamma_2\big|_{[r',r'']}$,
(iii) $\gamma_2\big|_{[r',r'']} \oplus \gamma_1\big|_{[r'',t_1]}$.

Proof. First notice that (ii) and (iii) follow from (i) by Lemma 2.1(b), and so we just prove (i). Let us write $\gamma = \gamma_1\big|_{[s_1,r']} \oplus \gamma_2\big|_{[r',r'']} \oplus \gamma_1\big|_{[r'',t_1]}$. We have

$$\begin{aligned}\mathcal{L}(\gamma_1) &\overset{(2.2)}{=} \mathcal{L}(x, s_1; \gamma_1(r'), r') + \mathcal{L}(\gamma_1(r'), r'; \gamma_1(r''), r'') + \mathcal{L}(\gamma_1(r''), r''; \gamma_1(t), t) \\ &= \mathcal{L}(x, s_1; \gamma_1(r'), r') + \mathcal{L}(\gamma_2(r'), r'; \gamma_2(r''), r'') + \mathcal{L}(\gamma_1(r''), r''; \gamma_1(t), t) \\ &\overset{\text{Lemma 2.1(b)}}{=} \mathcal{L}(\gamma_1\big|_{[s_1,r']}) + \mathcal{L}(\gamma_2\big|_{[r',r'']}) + \mathcal{L}(\gamma_1\big|_{[r'',t_1]}) \overset{(2.1)}{=} \mathcal{L}(\gamma).\end{aligned}$$

Since γ_1 is a geodesic with the same endpoints as γ, it follows from $\mathcal{L}(\gamma_1) = \mathcal{L}(\gamma)$ that γ is a geodesic. □

2.2. **Typical and atypical properties.** In subsequent proofs, it will be important to know what is entailed in the almost sure event $\mathcal{P}$ from Theorem B.

Definition 2.3. [23, Section 13] The function $\mathcal{L} : \mathbb{R}^4_\uparrow \to \mathbb{R}$ is said to be a *proper landscape*, and we say $\mathcal{P}$ occurs, if the following conditions hold:

(i) $\mathcal{L}$ is continuous;

(ii) for every $R > 0$, there is a constant c such that

$$\left|\mathcal{L}(x,s;y,t) + \frac{(x-y)^2}{t-s}\right| \leq c \quad \text{for all } (x,s;y,t) \in \mathbb{R}^4_\uparrow \cap [-R,R]^4;$$

(iii) for every $(x,s;y,t) \in \mathbb{R}^4_\uparrow$ and $r \in (s,t)$, the supremum in (1.9) is achieved by some $z \in \mathbb{R}$;

(iv) for every compact set $K \subset \mathbb{R}^4_\uparrow$, the values of $z \in \mathbb{R}$ achieving the supremum in (1.9) are uniformly bounded among $(x,s;y,t) \in K$ and $r \in (s,t)$; and

(v) for every $x_1 \leq x_2$, $y_1 \leq y_2$, and $s < t$, we have

$$\mathcal{L}(x_2,s;y_2,t) + \mathcal{L}(x_1,s;y_1,t) - \mathcal{L}(x_1,s;y_2,t) - \mathcal{L}(x_2,s;y_1,t) \geq 0.$$

While conditions (i)–(iv) are various quantifications of tightness, property (v) can be regarded as a deterministic fact about planar geodesic spaces, discussed in [23, Lemma 9.1] and [10, Theorem 1.1(1)]. In particular, the measures $\mu_{(x_1,x_2,s;t)}$ and $\mu_{s,t}$ given by (1.13) and (1.14) are well-defined because of (v).

Remark 2.4. Regarding the above definition, we make the following observations so that they can be referred to later in the paper. When $\mathcal{P}$ occurs:

(a) (i) $\Rightarrow$ $\mathcal{L}$ is bounded on any compact subset of $\mathbb{R}^4_\uparrow$;

(b) (ii) $\Rightarrow$ $\mathcal{L}(x,s;y,t) \to -\infty$ as $t \searrow s$, and for $\varepsilon > 0$, this divergence is uniform over $x,y,s \in [-R,R]$ such that $|x-y| \geq \varepsilon$; and

(c) (iv) $\Rightarrow$ for any compact $K \subset \mathbb{R}^4_\uparrow$, there is a random constant $R > 0$ such that

$$u = (x,s;y,t) \in K,\ \gamma \in G_u \quad \Rightarrow \quad |\gamma(r)| \leq R \quad \text{for all } r \in [s,t].$$

This is because Lemma 2.1(b) implies that for any $\gamma \in G_{(x,s;y,t)}$, the value $z = \gamma(r)$ is a maximizer in (1.9) for every $r \in (s,t)$. See also Lemma 3.3.

Definition 2.5. For $u = (x,s;y,t) \in \mathbb{R}^4_\uparrow$, we say that γ^{L} is the *leftmost* geodesic in G_u if

$$\gamma^{\mathrm{L}}(r) \leq \gamma(r) \quad \text{for all } \gamma \in G_u,\ r \in [s,t].$$

Similarly, γ^{R} is the *rightmost* geodesic in G_u if

$$\gamma(r) \leq \gamma^{\mathrm{R}}(r) \quad \text{for all } \gamma \in G_u,\ r \in [s,t].$$

Typically geodesics are unique, in which case the leftmost and rightmost geodesics are the same. It will be useful to record this and two other types of almost sure events concerning geodesics:

(1) (Existence) By [23, Lemma 13.2], the following event is a superset of $\mathcal{P}$ and thus occurs with probability one:

$$\mathcal{E} \coloneqq \{G_u \text{ contains a leftmost and a rightmost geodesic for every } u \in \mathbb{R}^4_\uparrow\}. \tag{2.3}$$

(2) (Uniqueness) For any fixed $u \in \mathbb{R}^4_\uparrow$, the measurability of the event $\{|G_u| = 1\}$ is argued in [23, Section 13]. Moreover, [23, Theorem 12.1] gives $\mathbb{P}(|G_u| = 1) = 1$.

(3) (Ordering) Finally, consider the event

$$\mathcal{O} \coloneqq \bigcap_{s<t} \bigcap_{x_1<x_2} \bigcap_{y_1<y_2} \{\forall\, \gamma_1 \in G_{(x_1,s;y_1,t)}, \gamma_2 \in G_{(x_2,s;y_2,t)}, \text{ we have } \gamma_1(r) \leq \gamma_2(r)\, \forall\, r \in [s,t]\}. \tag{2.4}$$

That is, whenever $x_1 < x_2$ and $y_1 < y_2$, the geodesics from (x_1, s) to (y_1, t) do not "cross" those from (x_2, s) to (y_2, t). We will soon check in Lemma 2.7 that $\mathcal{O}$ is an almost sure event. One can also allow $x_1 = x_2 = x$ if the times $s < t$ and the value of x are fixed:

$$\mathcal{O}_{(x,s;t)} \coloneqq \bigcap_{y_1<y_2} \{\forall\, \gamma_1 \in G_{(x,s;y_1,t)}, \gamma_2 \in G_{(x,s;y_2,t)}, \text{ we have } \gamma_1(r) \leq \gamma_2(r)\, \forall\, r \in [s,t]\}. \tag{2.5}$$

Before proving almost sure ordering of geodesics, we show that even if violations occur, we can still find geodesics that observe the correct ordering.

Lemma 2.6. *The following statements hold for any $x_1 \leq x \leq x_2$, $y_1 \leq y \leq y_2$, and $s < t$ such that $G_{(x,s;y,t)}$ is nonempty.*

(a) For any $\gamma_1 \in G_{(x_1,s;y_1,t)}$, there is $\gamma \in G_{(x,s;y,t)}$ such that $\gamma_1(r) \leq \gamma(r)$ for all $r \in [s,t]$.
(b) For any $\gamma_2 \in G_{(x_2,s;y_2,t)}$, there is $\gamma \in G_{(x,s;y,t)}$ such that $\gamma(r) \leq \gamma_2(r)$ for all $r \in [s,t]$.
(c) For any $\gamma_1 \in G_{(x_1,s;y_1,t)}$, $\gamma_2 \in G_{(x_2,s;y_2,t)}$ satisfying $\gamma_1(r) \leq \gamma_2(r)$ for all $r \in [s,t]$, there is $\gamma \in G_{(x,s;y,t)}$ such that $\gamma_1(r) \leq \gamma(r) \leq \gamma_2(r)$ for all $r \in [s,t]$.

Proof. The statements (a) and (b) are symmetric, and so we just prove (a). Take any $\gamma_1 \in G_{(x_1,s;y_1,t)}$ and $\widetilde{\gamma} \in G_{(x,s;y,t)}$. If

$$\gamma_1(r) \neq \widetilde{\gamma}(r) \quad \text{for all } r \in [s,t], \tag{2.6}$$

then we must have $\gamma_1(s) = x_1 < x = \widetilde{\gamma}(s)$, and so (2.6) and the continuity of geodesics force $\gamma_1(r) \leq \widetilde{\gamma}(r)$ for all $r \in [s,t]$, as desired. If, on the other hand, $\gamma_1(r) = \widetilde{\gamma}(r)$ for some $r \in [s,t]$, then upon defining the times

$$r' \coloneqq \inf\{r \geq s : \gamma_1(r) = \widetilde{\gamma}(r)\}, \qquad r'' \coloneqq \sup\{r \leq t : \gamma_1(r) = \widetilde{\gamma}(r)\},$$

we have

$$s \leq r' \leq r'' \leq t. \tag{2.7}$$

Furthermore, continuity implies

$$\gamma_1(r') = \widetilde{\gamma}(r'), \qquad \gamma_1(r'') = \widetilde{\gamma}(r''),$$

as well as

$$\gamma_1(r) < \widetilde{\gamma}(r) \quad \text{for all } r \in [s,r') \cup (r'',t].$$

The first of the previous two displays allows us to define the path $\gamma \coloneqq \widetilde{\gamma}\big|_{[s,r']} \oplus \gamma_1\big|_{[r',r'']} \oplus \widetilde{\gamma}\big|_{[r'',t]}$, where if any of the inequalities in (2.7) is an equality, we simply omit the corresponding segment. The second display implies that $\gamma_1(r) \leq \gamma(r)$ for all $r \in [s,t]$. Finally, Lemma 2.2 ensures $\gamma \in G_{(x,s;y,t)}$, thus completing the proof of (a).

For (c), we can apply (a) and (b) in succession upon noting that in the above proof, $\gamma(r) \in \{\widetilde{\gamma}(r), \gamma_1(r)\}$ for all $r \in [s,t]$. Therefore, if $\widetilde{\gamma}$ is taken to be the element of $G_{(x,s;y,t)}$ resulting from (b), then γ as defined above will necessarily satisfy

$$\gamma(r) \leq \widetilde{\gamma}(r) \vee \gamma_1(r) \leq \gamma_2(r) \quad \text{for all } r \in [s,t],$$

in addition to $\gamma(r) \geq \gamma_1(r)$. ☐

Lemma 2.7. *We have $\mathbb{P}(\mathcal{O}) = 1$ and, for any $x \in \mathbb{R}$ and $s < t$, $\mathbb{P}(\mathcal{O}_{(x,s;t)}) = 1$.*

Proof. Let us define the events

$$\mathcal{U}^{\mathbb{Q}} \coloneqq \bigcap_{\substack{r',r''\in\mathbb{Q}\\ r'<r''}} \bigcap_{p,q\in\mathbb{Q}} \{|G_{(p,r';q,r'')}| = 1\}, \quad \mathcal{U}^{\mathbb{Q}}_{(x,s;t)} \coloneqq \{|G_{(x,s;q,t)}| = 1 \text{ for every } q \in \mathbb{Q}\}, \quad x \in \mathbb{R},\, s < t.$$

Since $\mathbb{Q}$ is countable and $\mathbb{P}(|G_u| = 1) = 1$ for any $u \in \mathbb{R}^4_\uparrow$, we have $\mathbb{P}(\mathcal{U}^\mathbb{Q}) = \mathbb{P}(\mathcal{U}^\mathbb{Q}_{(x,s;t)}) = 1$. Therefore, if we can show that $\mathcal{O} \supset \mathcal{U}^\mathbb{Q}$ and $\mathcal{O}_{(x,s;t)} \supset \mathcal{U}^\mathbb{Q}_{(x,s;t)}$, then $\mathcal{O}$ and $\mathcal{O}_{(x,s;t)}$ are necessarily measurable by Remark 1.6, and also $\mathbb{P}(\mathcal{O}) = \mathbb{P}(\mathcal{O}_{(x,s;t)}) = 1$. Let us first prove $\mathcal{U}^\mathbb{Q}_{(x,s;t)} \subset \mathcal{O}_{(x,s;t)}$, as the argument for $\mathcal{U}^\mathbb{Q} \subset \mathcal{O}$ will require only slight modifications.

Assume $\mathcal{U}^\mathbb{Q}_{(x,s;t)}$ occurs. Consider any two values $y_1 < y_2$ and any geodesics $\gamma_1 \in G_{(x,s;y_1,t)}$ and $\gamma_2 \in G_{(x,s;y_2,t)}$. Take any $q \in \mathbb{Q} \cap (y_1, y_2)$ and consider the unique $\gamma \in G_{(x,s;q,t)}$. Given this uniqueness, we can apply Lemma 2.6(a) with $y = y_1$ and $y' = q$, to conclude

$$\gamma_1(r) \leq \gamma(r) \quad \text{for all } r \in [s,t].$$

Analogously, by Lemma 2.6(b) with $y = q$ and $y' = y_2$, we must also have

$$\gamma(r) \leq \gamma_2(r) \quad \text{for all } r \in [s,t].$$

Together, these two displays yield $\gamma_1(r) \leq \gamma_2(r)$ for all $r \in [s,t]$.

Now assume the occurrence of $\mathcal{U}^\mathbb{Q}$. Consider any $x_1 < x_2$, $y_1 < y_2$, $s < t$, and any geodesics $\gamma_1 \in G_{(x_1,s;y_1,t)}$ and $\gamma_2 \in G_{(x_2,s;y_2,t)}$. Since $\gamma_1(s) = x_1 < x_2 = \gamma_2(s)$, continuity guarantees that $\gamma_1(r) < \gamma_2(r)$ for all $r \in [s, s+\varepsilon]$, for some $\varepsilon > 0$. By symmetric reasoning, we may assume $\gamma_1(r) < \gamma_2(r)$ for all $r \in [t-\varepsilon, t]$. Now pick any rational times $r' \in \mathbb{Q} \cap [s, s+\varepsilon]$, $r'' \in \mathbb{Q} \cap [t-\varepsilon, t]$, as well as rational spatial coordinates $p \in (\gamma_1(r'), \gamma_2(r'))$, $q \in (\gamma_1(r''), \gamma_2(r''))$. By assumption, there is a unique $\gamma \in G_{(p,r';q,r'')}$. Moreover, Lemma 2.1(b) ensures that $\gamma_1\big|_{[r',r'']}$ and $\gamma_2\big|_{[r',r'']}$ are again geodesics. Therefore, the same argument as above (using Lemma 2.6) yields

$$\gamma_1(r) \leq \gamma(r) \leq \gamma_2(r) \quad \text{for all } r \in [r', r''].$$

But of course, we also know

$$\gamma_1(r) < \gamma_2(r) \quad \text{for all } r \in [s, r'] \subset [s, s+\varepsilon] \text{ and all } r \in [r'', t] \subset [t-\varepsilon, t].$$

Hence $\gamma_1(r) \leq \gamma_2(r)$ at every $r \in [s,t]$, as desired. □

While Lemmas 2.6 and 2.7 give us control over violations of geodesic ordering, the final result of this section considers violations of geodesic uniqueness. It is not required elsewhere in the paper, but rather included as an incidental result. Given $x \in \mathbb{R}$ and $s < t$, let us consider the set

$$\mathcal{M}_{(x,s;t)} \coloneqq \{y \in \mathbb{R} : |G_{(x,s;y,t)}| \geq 2\}. \tag{2.8}$$

Lemma 2.8. *For any $x \in \mathbb{R}$ and $s < t$, the set $\mathcal{M}_{(x,s;t)}$ is almost surely countable.*

Proof. By Lemma 2.7, we may assume $\mathcal{O}_{(x,s;t)}$ occurs. Suppose $y \in \mathbb{R}$ is such that $G_{(x,s;y,t)}$ contains two distinct elements γ_y and $\widetilde{\gamma}_y$. Without loss of generality, $\gamma_y(r_y) < \widetilde{\gamma}_y(r_y)$ for some $r_y \in (s,t)$, where we may assume by continuity that $r_y \in \mathbb{Q}$. Moreover, we can choose $q_y \in \mathbb{Q}$ such that

$$\gamma_y(r_y) < q_y < \widetilde{\gamma}_y(r_y).$$

Now, if $y_1 < y_2$ and both $|G_{(x,s;y_1,t)}|$ and $|G_{(x,s;y_2,t)}|$ are at least 2, then it must be that $(q_{y_1}, r_{y_1}) \neq (q_{y_2}, r_{y_2})$. Indeed, we would otherwise have

$$\gamma_{y_2}(r_{y_2}) < q_{y_2} = q_{y_1} < \widetilde{\gamma}_{y_1}(r_{y_1}) = \widetilde{\gamma}_{y_1}(r_{y_2}),$$

which is exactly the scenario ruled out by $\mathcal{O}_{(x,s;t)}$. In summary, each y for which $|G_{(x,s;y,t)}| \geq 2$ can be associated uniquely to some element of the countable set $\mathbb{Q} \times \mathbb{Q}$. The claim of the lemma is thus evident. □

3. Proofs of input results

In Section 3.1, we prove Theorem 1.13 and Corollary 1.14. We then use Corollary 1.14 in Section 3.2 to deduce Theorem 1.15 from the corresponding result in [31]. Corollary 1.16 will follow from a brief topological argument. Finally, Section 3.3 gives the proof of Theorem 1.17.

3.1. **Convergence of polymers.** Throughout this section, we fix $u = (x, s; y, t) \in \mathbb{R}^4_\uparrow$ and assume the setting of Theorem B. That is, G_u consists of a single element γ_u, the event $\mathcal{P}$ occurs, and for each n, we have chosen

$$\varphi_n : [sn + 2n^{2/3}x, tn + 2n^{2/3}y) \to [\![\lfloor sn \rfloor, \lfloor tn \rfloor]\!]$$

such that $\Gamma^{(\varphi_n)}_{n,u} \in G_{n,u}$ (equivalently, $R_n(\varphi_n) \in P_{n,u}$). By Theorem B, we have the following uniform convergence of functions on $[s, t]$:

$$\lim_{n\to\infty} \|\Gamma^{(\varphi_n)}_{n,u} - \gamma_u\|_\infty = 0. \tag{3.1}$$

We preface the proof of Theorem 1.13 with the following simple observations about the geometry of n-polymers and n-geodesics. The reader may find Figure 2 to be a useful reference.

Lemma 3.1. *Assume the setting of Theorem B. Then for any $\varepsilon > 0$, there is N such that for all $n \geq N$, we have the following:*

(a) Every horizontal segment in $\widetilde{R}_{n,u}(\varphi_n)$ has length at most ε.
(b) Every oblique segment in $R_n(\varphi_n)$ has horizontal width at most ε.
(c) Every oblique segment in $R_n(\varphi_n)$ has vertical height at most $2\varepsilon n^{-1/3}$.

Proof. Fix $\varepsilon > 0$. Notice that if $\widetilde{R}_{n,u}(\varphi_n)$ has a horizontal segment at height r, then $\Gamma^{(\varphi_n)}_{n,u}(\cdot)$ has a jump discontinuity at time r, where the size of the jump is exactly equal to the length of the horizontal segment. Therefore, to satisfy (a), it suffices to choose N large enough that for all $n \geq N$, every discontinuity of $\Gamma^{(\varphi_n)}_{n,u}$ is no larger than ε. Such an N exists by (3.1) and the (uniform) continuity of γ_u.

Now (b) follows from (a) because every oblique segment in $R_n(\varphi_n)$ corresponds to a horizontal segment in $\widetilde{R}_{n,u}(\varphi_n)$ of the same width. Finally, (c) follows from (b) because the slope of any oblique segment in $R_n(\varphi_n)$ is $-2n^{-1/3}$. □

Proof of Theorem 1.13. First we prove (1.15). Fix any $\varepsilon > 0$. Observe that the horizontal segments in $R_n(\varphi_n)$, minus their rightmost points, consist entirely of points (z, r) of the form

$$(z, r) = R_n(z', \varphi_n(z')), \quad z' \in [sn + 2n^{2/3}x, tn + 2n^{2/3}y). \tag{3.2}$$

Similarly, the oblique segments in $\widetilde{R}_{n,u}(\varphi_n)$, minus their uppermost points, consist entirely of points $(\tilde{z}, \tilde{r})$ of the form

$$\tilde{z} = \Gamma^{(\varphi_n)}_{n,u}(\tilde{r}), \quad L_{n,u}(\tilde{r}) = z' \in [sn + 2n^{2/3}x, tn + 2n^{2/3}y). \tag{3.3}$$

Therefore, these two categories of points are in bijection $(z, r) \leftrightarrow (\tilde{z}, \tilde{r})$ through the unscaled coordinate z'. If we can show that

$$\limsup_{n\to\infty} \sup_{z'} \|(z, r) - (\tilde{z}, \tilde{r})\| = 0, \tag{3.4}$$

then we claim (1.15) holds. Indeed, Lemma 3.1(a) shows that for $n \geq N$, every point in $\widetilde{R}_{n,u}(\varphi_n)$ is within distance ε of some $(\tilde{z}, \tilde{r})$ of the form (3.3). Meanwhile, Lemma 3.1(b,c) shows that every point in $R_n(\varphi_n)$ is within distance $\varepsilon + 2\varepsilon n^{-1/3}$ of some (z, r) of the form (3.2). Consequently, (3.4) leads to

$$\limsup_{n\to\infty} \operatorname{dist}_H(R_n(\varphi_n), \widetilde{R}_{n,u}(\varphi_n)) \leq \limsup_{n\to\infty} \Big[2\varepsilon + 2\varepsilon n^{-1/3} + \sup_{z'} \|(z, r) - (\tilde{z}, \tilde{r})\|\Big] = 2\varepsilon.$$

As ε is arbitrary, (1.15) follows. We now proceed to establish (3.4).

Because of (3.1), there exist random L (depending only on γ_u) and N large enough that

$$|\Gamma^{(\varphi_n)}_{n,u}(r)| \leq L \quad \text{for all } r \in [s, t],\ n \geq N. \tag{3.5}$$

Fix any $z' \in [sn + 2n^{2/3}x, tn + 2n^{2/3}y)$, and consider (z, r) and $(\tilde{z}, \tilde{r})$ as defined through (3.2) and (3.3). In particular,

$$z = \frac{z' - \varphi(z')}{2n^{2/3}} = \frac{L_{n,u}(\tilde{r}) - \varphi_n(L_{n,u}(\tilde{r}))}{2n^{2/3}} = \Gamma^{(\varphi_n)}_{n,u}(\tilde{r}) = \tilde{z},$$

and so

$$\|(z, r) - (\tilde{z}, \tilde{r})\| = |r - \tilde{r}|. \tag{3.6}$$

Now observe that

$$L_{n,u}(\tilde{r}) = z' = 2n^{2/3}\Gamma^{(\varphi_n)}_{n,u}(\tilde{r}) + rn \quad \Rightarrow \quad r = \tilde{r} + 2n^{-1/3}\Big(\frac{t - \tilde{r}}{t - s}x + \frac{\tilde{r} - s}{t - s}y - \Gamma^{(\varphi_n)}_{n,u}(\tilde{r})\Big),$$

from which we can deduce, by (3.5), the uniform bound

$$|r - \tilde{r}| \le 2n^{-1/3}(|x| + |y| + L). \tag{3.7}$$

Together, (3.6) and (3.7) imply (3.4), and so (1.15) has been proved.

Now we turn our attention to showing (1.16), which is clearly implied by following statement:

$$\lim_{n\to\infty} \sup_{(z,r)\in R_n(\varphi_n)} |z - \gamma_u(r)| = 0. \tag{3.8}$$

So let us just establish (3.8). If we denote, for each $r \in [s, t]$, the leftmost and rightmost points of $(\mathbb{R} \times \{r\}) \cap R_n(\varphi_n)$ by

$$a_n(r) \coloneqq \inf\{z \in \mathbb{R} : (z, r) \in R_n(\varphi_n)\}, \qquad b_n(r) \coloneqq \sup\{z \in \mathbb{R} : (z, r) \in R_n(\varphi_n)\},$$

then (3.8) is equivalent to

$$a_n(r) \to \gamma_u(r) \quad \text{and} \quad b_n(r) \to \gamma_u(r) \quad \text{uniformly in } r \in [s, t]. \tag{3.9}$$

To argue (3.9), let us consider the analogous quantities for $\widetilde{R}_{n,u}(\varphi_n)$, namely

$$\tilde{a}_n(r) \coloneqq \inf\{z \in \mathbb{R} : (z, r) \in \widetilde{R}_{n,u}(\varphi_n)\}, \qquad \tilde{b}_n(r) \coloneqq \sup\{z \in \mathbb{R} : (z, r) \in \widetilde{R}_{n,u}(\varphi_n)\},$$

and observe (perhaps with the aid of Figure 2e) that

$$\tilde{a}_n(r) = \Gamma^{(\varphi_n)}_{n,u}(r), \qquad \tilde{b}_n(r) = \begin{cases} \lim_{r' \nearrow r} \Gamma^{(\varphi_n)}_{n,u}(r') & \text{if } r \in (s, t], \\ \tilde{a}_n(s) & \text{if } r = s. \end{cases}$$

By (3.1), we then have

$$\tilde{a}_n(r) \to \gamma_u(r) \quad \text{and} \quad \tilde{b}_n(r) \to \gamma_u(r) \quad \text{uniformly in } r \in [s, t]. \tag{3.10}$$

Now let $\varepsilon > 0$ and choose $\delta \in (0, \varepsilon]$ sufficiently small that

$$|\gamma_u(r') - \gamma_u(r)| \le \varepsilon \quad \text{whenever } r, r' \in [s, t],\ |r - r'| \le \delta.$$

By (3.10) and (1.15), we can select N such that for all $n \ge N$, we have

$$|\tilde{a}_n(r) - \gamma_u(r)| \le \varepsilon \quad \text{and} \quad |\tilde{b}_n(r) - \gamma_u(r)| \le \varepsilon \quad \text{for all } r \in [s, t],$$

as well as

$$\mathrm{dist}_H(R_n(\varphi_n), \widetilde{R}_{n,u}(\varphi_n)) \le \delta.$$

Since $(a_n(r), r) \in R_n(\varphi)$, it follows from the above display that

$$\inf_{(\tilde{z},\tilde{r})\in \widetilde{R}_{n,u}(\varphi_n)} \|(a_n(r), r) - (\tilde{z}, \tilde{r})\| \le \delta \quad \text{for all } r \in [s, t],\ n \ge N,$$

which can be trivially rewritten as

$$\inf_{\substack{(\tilde{z},\tilde{r})\in \widetilde{R}_{n,u}(\varphi_n) \\ \tilde{r}\in[r-\delta, r+\delta]}} \|(a_n(r), r) - (\tilde{z}, \tilde{r})\| \le \delta \quad \text{for all } r \in [s, t],\ n \ge N.$$

On the other hand, for any $(\tilde{z}, \tilde{r}) \in \widetilde{R}_{n,u}(\varphi_n)$ with $|\tilde{r} - r| \leq \delta$, we have

$$\begin{aligned}|\tilde{z} - \gamma_u(r)| &\leq |\tilde{z} - \gamma_u(\tilde{r})| + |\gamma_u(\tilde{r}) - \gamma_u(r)| \\ &\leq |\tilde{a}_n(\tilde{r}) - \gamma_u(\tilde{r})| + |\tilde{b}_n(\tilde{r}) - \gamma_u(\tilde{r})| + |\gamma_u(\tilde{r}) - \gamma_u(r)| \leq 3\varepsilon.\end{aligned}$$

Together, the two previous displays imply

$$|a_n(r) - \gamma_u(r)| \leq \delta + 3\varepsilon \leq 4\varepsilon \quad \text{for all } r \in [s,t],\ n \geq N,$$

and an analogous argument shows

$$|b_n(r) - \gamma_u(r)| \leq \delta + 3\varepsilon \leq 4\varepsilon \quad \text{for all } r \in [s,t],\ n \geq N.$$

As ε is arbitrary, we conclude that (3.9) holds. □

Given the convergence (1.16) from Theorem 1.13 (or equivalently, (3.8)), it is a simple matter to verify Corollary 1.14.

Proof of Corollary 1.14. Recall the notation from the statement of the corollary. It is trivial that

$$\begin{aligned}&\inf\{|z_1 - z_2| : (z_1, r) \in R_n(\varphi_n), (z_2, r) \in R_n(\phi_n), r \in [s,t]\} \\ &\geq \inf_{r\in[s,t]} |\gamma_{u_1}(r) - \gamma_{u_2}(r)| - \sup_{(z_1,r_1)\in R_n(\varphi_n)} |z_1 - \gamma_{u_1}(r)| - \sup_{(z_2,r_2)\in R_n(\phi_n)} |z_2 - \gamma_{u_2}(r_2)|.\end{aligned}$$

Under the hypotheses of the corollary, we have

$$\inf_{r\in[s,t]} |\gamma_{u_1}(r) - \gamma_{u_2}(r)| > 0,$$

while (3.8) gives

$$\lim_{n\to\infty} \sup_{(z_1,r_1)\in R_n(\varphi_n)} |z_1 - \gamma_{u_1}(r_1)| = \lim_{n\to\infty} \sup_{(z_2,r_2)\in R_n(\phi_n)} |z_2 - \gamma_{u_2}(r_2)| = 0.$$

Therefore, for all n sufficiently large, we have

$$\inf\{|z_1 - z_2| : (z_1, r) \in R_n(\varphi_n), (z_2, r) \in R_n(\phi_n), r \in [s,t]\} > 0,$$

meaning that $R_n(\varphi_n)$ and $R_n(\phi_n)$ are disjoint. □

3.2. **Tail estimates for the size of a disjoint collection of geodesics.** Before proving Theorem 1.15, we need to know that the relevant random variable is measurable.

Proposition 3.2. *For any times $s < t$ and subsets $A, B, C \subset \mathbb{R}$ with C countable, the quantity $\mathrm{MaxDisjtGeo}^C_{s,t}(A,B)$ is a measurable random variable almost surely taking values in $\{1, 2, \ldots\} \cup \{\infty\}$.*

Since the proof of Proposition 3.2 will need to consider certain events involving geodesics, it will be useful to have the following description of a geodesic.

Lemma 3.3. [23, proof of Lemma 13.2] *On the almost sure event $\mathcal{P}$ of Theorem B, the following is true for every $u \in \mathbb{R}^4_\uparrow$. If G_u consists of a single element γ_u, then for every $r \in (s,t)$, there is a unique $z_r \in \mathbb{R}$ satisfying*

$$\mathcal{L}(x, s; z_r, r) + \mathcal{L}(z_r, r; y, t) = \sup_{z\in\mathbb{R}} [\mathcal{L}(x, s; z, r) + \mathcal{L}(z, r; y, t)],$$

and $\gamma_u(r) = z_r$.

We use the above characterization of γ_u to prove the next lemma, which constitutes the bulk of the work toward Proposition 3.2. Let $\mathrm{NonInt}_{s,t}(x_1, x_2; y_1, y_2)$ denote the event that every $\gamma_1 \in G_{(x_1,s;y_1;t)}$ is disjoint from every $\gamma_2 \in G_{(x_2,s;y_2,t)}$.

Lemma 3.4. *For any $x_1, x_2, y_1, y_2 \in \mathbb{R}$ and $s < t$, the event $\mathrm{NonInt}_{s,t}(x_1, x_2; y_1, y_2)$ is measurable.*

Proof. Recall that almost surely, both $G_{(x_1,s;y_1;t)}$ and $G_{(x_2,s;y_2,t)}$ are singletons, and $\mathcal{P}$ occurs. Because we have assumed in Remark 1.6 that $\mathcal{F}$ is complete, it suffices to show that the intersection of $\mathrm{NonInt}_{s,t}(x_1,x_2;y_1,y_2)$ with these three almost sure events is measurable. So let us assume henceforth that $\gamma_1 \in G_{(x_1,s;y_1;t)}$ and $\gamma_2 \in G_{(x_2,s;y_2;t)}$ are unique, and that $\mathcal{P}$ occurs; thus γ_1 and γ_2 are as described in Lemma 3.3. In this case,

$$\mathrm{NonInt}_{s,t}(x_1,x_2;y_1,y_2) = \{\gamma_1,\gamma_2 \text{ disjoint}\},$$

and so it suffices to show the measurability of $\Omega \setminus \{\gamma_1,\gamma_2 \text{ disjoint}\}$.

Observe that by compactness and continuity,

$$\Omega \setminus \{\gamma_1,\gamma_2 \text{ disjoint}\} = \bigcap_{m=1}^{\infty} \bigcup_{r\in\mathbb{Q}} \{|\gamma_1(r)-\gamma_2(r)| \le 1/m\}, \tag{3.11}$$

where $\{|\gamma_1(r)-\gamma_2(r)| \le 1/m\}$ is equivalent to the following event, which we call $\mathcal{G}_{r,m}$: For some positive integer R and every $j = 1,2,\dots$, there exist rationals $a_j, b_j \in [-R,R]$ satisfying

$$|a_j - b_j| \le 1/m, \tag{3.12a}$$

$$\mathcal{L}(x_1,s;a_j,r) + \mathcal{L}(a_j,r;y_1,t) > \sup_{z\in\mathbb{R}}[\mathcal{L}(x_1,s;z,r)+\mathcal{L}(z,r;y_1,t)] - 1/j, \tag{3.12b}$$

$$\mathcal{L}(x_2,s;b_j,r) + \mathcal{L}(b_j,r;y_2,t) > \sup_{z\in\mathbb{R}}[\mathcal{L}(x_2,s;z,r)+\mathcal{L}(z,r;y_2,t)] - 1/j. \tag{3.12c}$$

Indeed, if this statement holds for the integer R, then by passing to a subsequence, we may assume $a_j \to a$ and $b_j \to b$ as $j \to \infty$. The condition (3.12a) guarantees $|a-b| \le 1/m$, while (3.12b) and (3.12c) imply through the continuity of $\mathcal{L}$ that

$$\mathcal{L}(x_1,s;a,r) + \mathcal{L}(a,r;y_1,t) = \sup_{z\in\mathbb{R}}[\mathcal{L}(x_1,s;z,r)+\mathcal{L}(z,r;y_1,t)],$$
$$\mathcal{L}(x_2,s;b,r) + \mathcal{L}(b,r;y_2,t) = \sup_{z\in\mathbb{R}}[\mathcal{L}(x_2,s;z,r)+\mathcal{L}(z,r;y_2,t)].$$

Therefore, by Lemma 3.3 we have $\gamma_1(r) = a$ and $\gamma_2(r) = b$, and so $|\gamma_1(r)-\gamma_2(r)| \le 1/m$. Conversely, if $|\gamma_1(r)-\gamma_2(r)| \le 1/m$, then let us assume $\gamma_1(r) \le \gamma_2(r)$ without loss of generality. From Lemma 3.3 (or alternatively (2.2)), we know

$$\mathcal{L}(x_1,s;\gamma_1(r),r) + \mathcal{L}(\gamma_1(r),r;y_1,t) = \sup_{z\in\mathbb{R}}[\mathcal{L}(x_1,s;z,r)+\mathcal{L}(z,r;y_1,t)],$$
$$\mathcal{L}(x_2,s;\gamma_2(r),r) + \mathcal{L}(\gamma_2(r),r;y_2,t) = \sup_{z\in\mathbb{R}}[\mathcal{L}(x_2,s;z,r)+\mathcal{L}(z,r;y_2,t)].$$

By continuity of $\mathcal{L}$, one can choose a sequence of rationals $a_j \searrow \gamma_1(r)$ and $b_j \nearrow \gamma_2(r)$ such that (3.12) is true for each j. Finally, choosing R sufficiently large that $\gamma_1(r),\gamma_2(r),a_1,b_1 \in [-R,R]$, we see that $\mathcal{G}_{r,m}$ has occurred.

We have now argued that on the almost sure event $\{|G_{(x_1,s;y_1;t)}| = 1\} \cap \{|G_{(x_2,s;y_2;t)}| = 1\} \cap \mathcal{P}$, we have $\{|\gamma_1(r)-\gamma_2(r)| \le 1/m\} = \mathcal{G}_{r,m}$. As $\mathcal{G}_{r,m}$ involves checking only countably many conditions on $\mathcal{L}$, it is measurable. Therefore, (3.11) shows that $\Omega \setminus \{\gamma_1,\gamma_2 \text{ disjoint}\}$ is measurable. $\square$

Proof of Proposition 3.2. We wish to show that for any $k \in \{1,2,\dots\}$, the disjointness event $\{\mathrm{MaxDisjtGeo}^C_{s,t}(A,B) \ge k\}$ belongs to the σ-algebra $\mathcal{F}$. First note that the countability of C implies $\bigcap_{p,q\in C}\{|G_{(p,s;q,t)}| = 1\}$ occurs with probability one. Therefore, by Remark 1.6, it suffices to show that

$$\{\mathrm{MaxDisjtGeo}^C_{s,t}(A,B) \ge k\} \cap \bigcap_{p,q\in C} \{|G_{(p,s;q,t)}| = 1\} \cap \mathcal{P} \in \mathcal{F},$$

On the almost sure event $\bigcap_{p,q\in C}\{|G_{(p,s;q,t)}|=1\}\cap\mathcal{P}$, the set under consideration can be expressed as

$$\{\mathrm{MaxDisjtGeo}^C_{s,t}(A,B)\geq k\} = \bigcup_{\substack{p_1,\dots,p_k\in A\cap C\\ q_1,\dots,q_k\in B\cap C}} \bigcap_{1\leq i<j\leq k} \mathrm{NonInt}_{s,t}(p_i,p_j;q_i,q_j).$$

As the union and the intersection in this display take place over countable index sets, Lemma 3.4 completes the proof. □

Recall the definition of an n-polymer from Section 1.2. For intervals $I,J\subset\mathbb{R}$, denote by $\mathrm{MaxDisjtPoly}_n(I,J)$ the maximum size of a collection of disjoint n-polymers having endpoints of the form $(x,0)$ and $(y,1)$ with $x\in I$ and $y\in J$. The following result of [31] will naturally translate into Theorem 1.15, since we know from Section 3.1 that n-polymers converge to (the graph of) geodesics.

Theorem 3.5. [31, Theorem 1.1] *There exists a positive constant G such that the following holds. For any $\delta>0$, integers k and n, and $z,w\in\mathbb{R}$ satisfying*

$$k\geq 2,\quad \delta\leq G^{-4k^2},\quad n\geq G^{k^2}(1+|z-w|^{36})\delta^{-G},\quad |z-w|\leq \delta^{-1/2}(\log\delta^{-1})^{-2/3}G^{-k}, \tag{3.13}$$

we have

$$\mathbb{P}\Big(\mathrm{MaxDisjtPoly}_n([z-\delta,z+\delta],[w-\delta,w+\delta])\geq k\Big)\leq G^{k^3}\exp\big\{G^k(\log\delta^{-1})^{5/6}\big\}\delta^{(k^2-1)/2}.$$

Proof of Theorem 1.15. Let $C\subset\mathbb{R}$ be countable. Fix k, $\varepsilon>0$, and $u=(x,s;y,t)\in\mathbb{R}^4_\uparrow$ satisfying (1.17). Then (3.13) is satisfied with $\delta=\varepsilon/(t-s)^{2/3}$, $z=x/(t-s)^{2/3}$, $w=y/(t-s)^{2/3}$, and n sufficiently large, meaning we will ultimately be able to invoke Theorem 3.5. By the scaling in (1.10), we have

$$\begin{aligned}&\mathbb{P}\Big(\mathrm{MaxDisjtGeo}^C_{s,t}([x-\varepsilon,x+\varepsilon],[y-\varepsilon,y+\varepsilon])\geq k\Big)\\ &=\mathbb{P}\Big(\mathrm{MaxDisjtGeo}^{(t-s)^{-2/3}C}_{0,1}([z-\delta,z+\delta],[w-\delta,w+\delta])\geq k\Big).\end{aligned}\tag{3.14}$$

Now assume of the coupling of Theorem A, and suppose as in Theorem B the almost sure occurrence of $\mathcal{P}$ and of the event $\{|G_u|=1\}$ for every $u=(z,0;w,1)$ with $z,w\in(t-s)^{-2/3}C$. If

$$\mathrm{MaxDisjtGeo}^{(t-s)^{-2/3}C}_{0,1}([z-\delta,z+\delta],[w-\delta,w+\delta])\geq k,$$

then there are $u_i=(z_i,0;w_i,1)$, $i=1,\dots,k$, such that

$$z_i\in[z-\delta,z+\delta]\cap(t-s)^{-2/3}C,\qquad w_i\in[w-\delta,w+\delta]\cap(t-s)^{-2/3}C,$$

admitting disjoint and unique geodesics $\gamma_{u_1},\dots,\gamma_{u_k}:[0,1]\to\mathbb{R}$. By Corollary 1.14, if we select some $R_n(\varphi_n^{(i)})\in P_{n,u_i}$ for each n, then the polymers $R_n(\varphi_n^{(1)}),\dots,R_n(\varphi_n^{(k)})$ must be disjoint for all n sufficiently large. We have thus argued that on the almost sure event

$$\mathcal{P}\cap\bigcap_{z,w\in(t-s)^{-2/3}C}\{|G_{(z,0;w,1)}|=1\},$$

we have

$$\begin{aligned}&\Big\{\mathrm{MaxDisjtGeo}^{(t-s)^{-2/3}C}_{0,1}([z-\delta,z+\delta],[w-\delta,w+\delta])\geq k\Big\}\\ &\subset\bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty}\{\mathrm{MaxDisjtPoly}_n([z-\delta,z+\delta],[w-\delta,w+\delta])\geq k\}.\end{aligned}$$

Hence

$$\mathbb{P}\Big(\mathrm{MaxDisjtGeo}^C_{s,t}([x-\varepsilon,x+\varepsilon],[y-\varepsilon,y+\varepsilon])\geq k\Big)$$

$$
\begin{aligned}
&\overset{(3.14)}{=} \mathbb{P}\Big(\mathrm{MaxDisjtGeo}_{0,1}^{(t-s)^{-2/3}C}([z-\delta,z+\delta],[w-\delta,w+\delta]) \geq k \Big) \\
&\leq \mathbb{P}\Big(\bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{\mathrm{MaxDisjtPoly}_n([z-\delta,z+\delta],[w-\delta,w+\delta]) \geq k\} \Big) \\
&= \lim_{N\to\infty} \mathbb{P}\Big(\bigcap_{n=N}^{\infty} \{\mathrm{MaxDisjtPoly}_n([z-\delta,z+\delta],[w-\delta,w+\delta]) \geq k\} \Big) \\
&\leq \liminf_{n\to\infty} \mathbb{P}\Big(\mathrm{MaxDisjtPoly}_n([z-\delta,z+\delta],[w-\delta,w+\delta]) \geq k \Big) \\
&\leq G^{k^3} \exp\big\{G^k (\log \delta^{-1})^{5/6}\big\} \delta^{(k^2-1)/2},
\end{aligned}
$$

where we have used Theorem 3.5 to obtain the final inequality. □

We now prove that in the case $k = 2$, one can take $C = \mathbb{R}$.

Proof of Corollary 1.16. Assume the occurrence of the geodesic existence/ordering events $\mathcal{E}$ and $\mathcal{O}$ from (2.3) and (2.4). We will argue that on $\mathcal{E} \cap \mathcal{O}$, we have the following equality of events for any $u = (x, s; y, t) \in \mathbb{R}^4_{\uparrow}$ and $\varepsilon > 0$:

$$
\begin{aligned}
&\{\mathrm{MaxDisjtGeo}_{s,t}^{\mathbb{R}}((x-\varepsilon,x+\varepsilon),(y-\varepsilon,y+\varepsilon)) \geq 2\} \\
&= \{\mathrm{MaxDisjtGeo}_{s,t}^{\mathbb{Q}}((x-\varepsilon,x+\varepsilon),(y-\varepsilon,y+\varepsilon)) \geq 2\}.
\end{aligned}
$$

Since the first event is clearly implied by the second, we need only prove the reverse containment. Once this is done, we will have shown (i) that the first event is measurable, as the second is measurable by Proposition 3.2; and (ii) that whenever $\varepsilon > 0$ satisfies (1.19), the first event adheres to the estimate (1.20), since the second event is contained in $\{\mathrm{MaxDisjtGeo}_{s,t}^{\mathbb{Q}}([x-\varepsilon,x+\varepsilon],[y-\varepsilon,y+\varepsilon]) \geq 2\}$, which in turn adheres to (1.18).

So suppose $\mathrm{MaxDisjtGeo}_{s,t}^{\mathbb{R}}((x-\varepsilon,x+\varepsilon),(y-\varepsilon,y+\varepsilon)) \geq 2$. That is, there are $x_1 < x_2$ in $(x-\varepsilon,x+\varepsilon)$ and $y_1 < y_2$ in $(y-\varepsilon,y+\varepsilon)$ admitting geodesics $\gamma_1 \in G_{(x_1,s;y_1,t)}$ and $\gamma_2 \in G_{(x_2,s;y_2,t)}$ that satisfy

$$
\gamma_1(r) < \gamma_2(r) \quad \text{for all } r \in [s,t]. \tag{3.15}
$$

Then select any rationals

$$
\begin{aligned}
p_1 &\in \mathbb{Q} \cap (x-\varepsilon, x_1), \quad & p_2 &\in \mathbb{Q} \cap (x_2, x+\varepsilon), \\
q_1 &\in \mathbb{Q} \cap (y-\varepsilon, y_1), \quad & q_2 &\in \mathbb{Q} \cap (y_2, y+\varepsilon),
\end{aligned}
$$

and any $\widetilde{\gamma}_1 \in G_{(p_1,s;q_1,t)}$ and $\widetilde{\gamma}_2 \in G_{(p_2,s;q_2,t)}$. By geodesic ordering, we have

$$
\widetilde{\gamma}_1(r) \leq \gamma_1(r) \quad \text{and} \quad \gamma_2(r) \leq \widetilde{\gamma}_2(r) \quad \text{for all } r \in [s,t].
$$

In light of (3.15), this implies that $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ are disjoint, meaning

$$
\mathrm{MaxDisjtGeo}_{s,t}^{\mathbb{Q}}((x-\varepsilon,x+\varepsilon),(y-\varepsilon,y+\varepsilon)) \geq 2.
$$

□

3.3. **Geodesics in a common compact set cannot be arbitrarily close.** The proof of a final input remains.

Proof of Theorem 1.17. Let K be a given compact subset of $\mathbb{R}^4_{\uparrow}$. For each $\varepsilon > 0$, define the event

$$
\mathcal{B}_\varepsilon := \left\{ \begin{matrix} \exists\, u_1 = (x_1,s;w_1,t), u_2 = (x_2,s;w_2,t) \in K \\ \gamma_1 \in G_{u_1}, \gamma_2 \in G_{u_2} \end{matrix} : 0 < \gamma_2(r) - \gamma_1(r) < \varepsilon \quad \forall\, r \in [s,t] \right\}. \tag{3.16}
$$

We wish to show $\mathbb{P}\big(\mathcal{P}\cap\bigcap_{\varepsilon>0}\mathcal{B}_\varepsilon\big)=0$. Recall the random number $R>0$ from Remark 2.4(c). By possibly replacing R with a larger deterministic number, we may assume that $K\subset[-R,R]^4$. If we can show that for any integer m, the event $\{R\le m\}\cap\bigcap_{\varepsilon>0}\mathcal{B}_\varepsilon$ is contained in a probability zero event, then measurability will be implied by Remark 1.6, and

$$\mathbb{P}\Big(\mathcal{P}\cap\bigcap_{\varepsilon>0}\mathcal{B}_\varepsilon\Big)=\mathbb{P}\Big(\mathcal{P}\cap\bigcup_{m=1}^{\infty}\Big(\{R\le m\}\cap\bigcap_{\varepsilon>0}\mathcal{B}_\varepsilon\Big)\Big)\le\sum_{m=1}^{\infty}\mathbb{P}\Big(\{R\le m\}\cap\bigcap_{\varepsilon>0}\mathcal{B}_\varepsilon\Big)=0.$$

So let us assume $R\le m$. By compactness of K, there is a deterministic number $\delta>0$ such that whenever $u=(x,s;y,t)\in K$, we have $t-s\ge 6\delta$. Therefore, if γ_1 and γ_2 are as in (3.16), then there is some $r\in[-m,m]\cap 3\delta\mathbb{Z}$ satisfying $[r,r+3\delta]\subset[s,t]$. Furthermore, there are $x,y,z,w\in[-m,m]\cap\varepsilon\mathbb{Z}$ such that

$$\gamma_1(r),\gamma_2(r)\in(x-\varepsilon,x+\varepsilon),\qquad \gamma_1(r+\delta),\gamma_2(r+\delta)\in(y-\varepsilon,y+\varepsilon),$$
$$\gamma_1(r+2\delta),\gamma_2(r+2\delta)\in(z-\varepsilon,z+\varepsilon),\qquad \gamma_1(r+3\delta),\gamma_2(r+3\delta)\in(w-\varepsilon,w+\varepsilon).$$

Since subpaths of geodesics are again geodesics by Lemma 2.1(b), the disjointness between γ_1 and γ_2 now implies

$$\mathrm{MaxDisjtGeo}^{\mathbb{R}}_{r,r+\delta}((x-\varepsilon,x+\varepsilon),(y-\varepsilon,y+\varepsilon))\ge 2,$$
$$\mathrm{MaxDisjtGeo}^{\mathbb{R}}_{r+\delta,r+2\delta}((y-\varepsilon,y+\varepsilon),(z-\varepsilon,z+\varepsilon))\ge 2,$$
$$\mathrm{MaxDisjtGeo}^{\mathbb{R}}_{r+2\delta,r+3\delta}((z-\varepsilon,z+\varepsilon),(w-\varepsilon,w+\varepsilon))\ge 2.$$

See Figure 6 for an illustration. Notice that the three random variables appearing above are i.i.d., since the time intervals $(r,r+\delta)$, $(r+\delta,r+2\delta)$, and $(r+2\delta,r+3\delta)$ are disjoint. Consequently, if we define $\mathcal{B}_{\varepsilon,x,y,z,w,r}$ to be the intersection of the three events in the above display, then

$$\mathbb{P}(\mathcal{B}_{\varepsilon,x,y,z,w,r})=\mathbb{P}\Big(\mathrm{MaxDisjtGeo}^{\mathbb{R}}_{r,r+\delta}((x-\varepsilon,x+\varepsilon),(y-\varepsilon,y+\varepsilon))\ge 2\Big)^3,\tag{3.17}$$

and our discussion has shown

$$\{R\le m\}\cap\bigcap_{\varepsilon>0}\mathcal{B}_\varepsilon\subset\bigcap_{k=1}^{\infty}\ \bigcup_{r\in[-m,m]\cap3\delta\mathbb{Z}}\ \bigcup_{x,y,z,w\in[-m,m]\cap k^{-1}\mathbb{Z}}\mathcal{B}_{k^{-1},x,y,z,w,r}.\tag{3.18}$$

Now recall the constant G from Theorem 1.15. Since we will soon take $\varepsilon\searrow 0$, we may assume ε is sufficiently small that

$$\frac{\varepsilon}{\delta^{2/3}}\le G^{-16},\quad \frac{2m}{\delta^{2/3}}<\frac{\varepsilon^{-1/2}}{\delta^{-1/3}}\Big(\log\frac{\delta^{2/3}}{\varepsilon}\Big)^{-2/3}G^{-2},\quad G^8\exp\Big\{G^2\Big(\log\frac{\delta^{2/3}}{2\varepsilon}\Big)^{5/6}\Big\}\frac{(2\varepsilon)^{3/2}}{\delta}\le\varepsilon^{11/8}.$$

We can then apply Theorem 1.16 to obtain

$$\mathbb{P}\Big(\mathrm{MaxDisjtGeo}^{\mathbb{R}}_{r,r+\delta}((x-\varepsilon,x+\varepsilon),(y-\varepsilon,y+\varepsilon))\ge 2\Big)\le\varepsilon^{11/8}.\tag{3.19}$$

Putting together (3.17)–(3.19) now yields the desired result:

$$\begin{aligned}\mathbb{P}\Big(\{R\le m\}\cap\bigcap_{\varepsilon>0}\mathcal{B}_\varepsilon\Big)&\le\mathbb{P}\Big(\bigcap_{k=1}^{\infty}\ \bigcup_{r\in[-m,m]\cap3\delta\mathbb{Z}}\ \bigcup_{x,y,z,w\in[-m,m]\cap k^{-1}\mathbb{Z}}\mathcal{B}_{k^{-1},x,y,z,w,r}\Big)\\&\le\limsup_{k\to\infty}\mathbb{P}\Big(\bigcup_{r\in[-m,m]\cap3\delta\mathbb{Z}}\ \bigcup_{x,y,z,w\in[-m,m]\cap k^{-1}\mathbb{Z}}\mathcal{B}_{k^{-1},x,y,z,w,r}\Big)\\&\le\limsup_{k\to\infty}\sum_{r\in[-m,m]\cap3\delta\mathbb{Z}}\ \sum_{x,y,z,w\in[-m,m]\cap k^{-1}\mathbb{Z}}k^{-33/8}\\&\le\limsup_{k\to\infty}\,(2m(3\delta)^{-1}+2)(2mk+1)^4k^{-33/8}=0.\end{aligned}$$

□

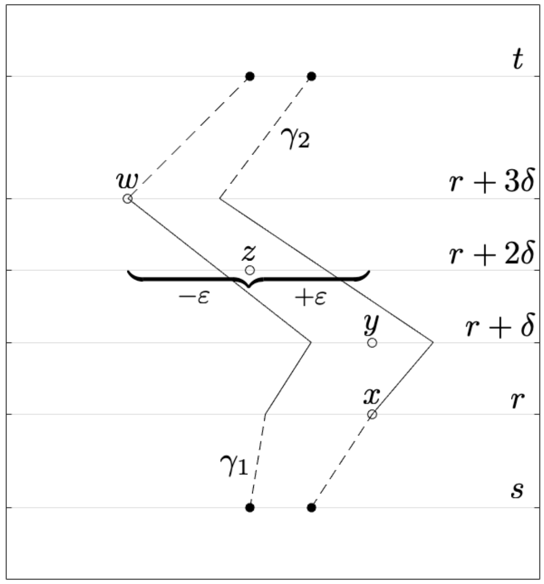


Figure 6. Scenario implied by $\mathcal{B}_\varepsilon$. Because $s, t \in [-m, m]$ satisfy $t - s \geq 6\delta$, there is some $r \in [-m, m] \cap 3\delta\mathbb{Z}$ for which $[r, r + 3\delta]$ is a subinterval of $[s, t]$. This subinterval is divided into three further subintervals, each of which admits the two disjoint subgeodesics arising from γ_1 and γ_2. (Here, three is the minimum number needed for our analysis to ensure probability 0, but it could be replaced by any larger integer.) The points $x, y, z, w \in [-m, m] \cap \varepsilon\mathbb{Z}$ are chosen so that intervals of radius ε about these points contain both γ_1 and γ_2 at the associated times.

4. Proofs of Theorems 1.9(a) and 1.10(a)

In this section, we fix $x_1 < x_2$, $s < t$, and prove Theorems 1.9(a) and 1.10(a), which are restated together here.

Proposition 4.1. *For any $x_1 < x_2$ and $s < t$, we almost surely have*

$$\mathrm{Supp}(\mu_{(x_1,x_2,s;t)}) \subset \mathcal{D}_{(x_1,x_2,s;t)} \quad \textit{and} \quad \mathrm{Supp}(\mu_{s,t}) \subset \mathcal{D}_{s,t}.$$

In conjunction with the next two inputs, the above containments imply the dimension lower bounds in Theorems 1.9(b) and 1.10(b).

Theorem 4.2. [10, Theorem 1.1] *For any $x_1 < x_2$ and $s < t$, the Hausdorff dimension of* $\mathrm{Supp}(\mu_{(x_1,x_2,s;t)})$ *is equal to $\frac{1}{2}$ almost surely.*

Theorem 4.3. *For any $s < t$, the Hausdorff dimension of* $\mathrm{Supp}(\mu_{s,t})$ *is equal to $\frac{1}{2}$ almost surely.*

For completeness, we note that Theorem 4.2 was proved in [10] for the case $x_1 = -1$, $x_2 = 1$, $s = 0$, and $t = 1$. Of course, by rescaling via (1.10), one infers the result for general $x_1 < x_2$ and $s < t$.

In the next section, we prove Proposition 4.1. In Section 4.2, we establish the lower bound for Theorem 4.3. The upper bound will be implied by Propositions 4.1 and 5.1.

4.1. Proof of Proposition 4.1: containment of supports by the exceptional sets.

We begin with the following property sequences of geodesics whose endpoints are monotonically converging. Recall Definition 2.5 of leftmost and rightmost geodesics, whose existence is given by the almost sure event $\mathcal{E}$ from (2.3). Recall also that $\mathcal{E}$ is implied by $\mathcal{P}$, the event from Definition 2.3.

Lemma 4.4. *On the event $\mathcal{P}$, the following statements hold for any $u = (x, s; y, t) \in \mathbb{R}^4_\uparrow$. Let γ^{L} and γ^{R} be the leftmost and rightmost geodesics in G_u.*

(a) If $x_j \nearrow x$ and $y_j \nearrow y$, then there is a sequence of $\gamma_j \in G_{(x_j,s;y_j,t)}$ so that $\gamma_j \nearrow \gamma^{\mathrm{L}}$ uniformly.

(b) If $x_j \searrow x$ and $y_j \searrow y$, then there is a sequence of $\gamma_j \in G_{(x_j,s;y_j,t)}$ so that $\gamma_j \nearrow \gamma^{\mathrm{R}}$ uniformly.

Proof. The statements (a) and (b) are symmetric to one another, and so we will prove only (a).

Take any sequences $x_j \nearrow x$ and $y_j \nearrow y$. By the assumed occurrence of $\mathcal{E} \supset \mathcal{P}$, there exists a leftmost geodesic $\gamma^{\mathrm{L}} \in G_{(x,s;y,t)}$, and $G_{(x_j,s;y_j,t)}$ is nonempty for every j. By successive applications of Lemma 2.6, we may assume that $\gamma_j(r) \le \gamma_{j+1}(r) \le \gamma^{\mathrm{L}}(r)$ for all $r \in [s,t]$. Therefore, $\gamma_j(r)$ must converge as $j \to \infty$ to some value we call $\gamma(r)$, which is at most $\gamma^{\mathrm{L}}(r)$. We claim that $\gamma \in G_{(x,s;y,t)}$, and so γ is necessarily equal to γ^{L}.

First we show that $\gamma : [s,t] \to \mathbb{R}$ is continuous. In particular, the convergence $\gamma_j \nearrow \gamma$ is uniform by Dini's theorem. Suppose toward a contradiction that γ is discontinuous at some $r \in [s,t]$. We will assume γ is right-discontinuous at r (in particular, $r < t$); the case of left-discontinuity is handled in a symmetric fashion. That is, there exists some $\varepsilon > 0$ such that for every $\delta > 0$, we have

$$\sup_{r' \in (r, r+\delta] \cap [s,t]} |\gamma(r') - \gamma(r)| \ge 4\varepsilon.$$

In particular, there is a sequence $r_\ell \searrow r$ such that

$$|\gamma(r_\ell) - \gamma(r)| \ge 3\varepsilon \quad \text{for every } \ell.$$

From the pointwise convergence $\gamma_j \to \gamma$, we can select indices $j_\ell \nearrow \infty$ such that

$$|\gamma_{j_\ell}(r_\ell) - \gamma(r_\ell)| \le \varepsilon \quad \text{and} \quad |\gamma_{j_\ell}(r) - \gamma(r)| \le \varepsilon \quad \text{for every } \ell.$$

These choices yield

$$|\gamma_{j_\ell}(r_\ell) - \gamma_{j_\ell}(r)| \ge \varepsilon \quad \text{for every } \ell.$$

In light of Remark 2.4(b) and the fact that $r_\ell \searrow r$, this last display implies

$$\lim_{\ell \to \infty} \mathcal{L}(\gamma_{j_\ell}(r), r; \gamma_{j_\ell}(r_\ell), r_\ell) = -\infty. \tag{4.1}$$

On the other hand, Remark 2.4(a) guarantees

$$\limsup_{\ell \to \infty} |\mathcal{L}(\gamma_{j_\ell}(r_\ell), r_\ell; y_{j_\ell}, t)| < \infty, \tag{4.2a}$$

as well as

$$\limsup_{\ell \to \infty} |\mathcal{L}(\gamma_{j_\ell}(r), r; y_{j_\ell}, t)| < \infty. \tag{4.2b}$$

Given that each γ_{j_ℓ} is a geodesic, we have

$$\mathcal{L}(\gamma_{j_\ell}(r), r; y_{j_\ell}, t) = \mathcal{L}(\gamma_{j_\ell}(r), r; \gamma_{j_\ell}(r_\ell), r_\ell) + \mathcal{L}(\gamma_{j_\ell}(r_\ell), r_\ell; y_{j_\ell}, t),$$

and so (4.1) is in contradiction with (4.2). Consequently, γ must be continuous on all of $[s,t]$.

To complete the proof, we need to show $\mathcal{L}(\gamma) = \mathcal{L}(x,s;y,t)$. For any partition $s = t_0 < t_1 < \cdots < t_k = t$, we have

$$\begin{aligned}\sum_{i=1}^k \mathcal{L}(\gamma(t_{i-1}), t_{i-1}; \gamma(t_i), t_i) &= \sum_{i=1}^k \lim_{j\to\infty} \mathcal{L}(\gamma_j(t_{i-1}), t_{i-1}; \gamma_j(t_i), t_i) \\ &= \lim_{j\to\infty} \mathcal{L}(x_j, s; y_j, t) = \mathcal{L}(x,s;y,t),\end{aligned}$$

where the first and last equalities hold by continuity of $\mathcal{L}$, and the middle equality is valid because each γ_j is a geodesic. Taking an infimum over all partitions, we conclude that $\mathcal{L}(\gamma) = \mathcal{L}(x,s;y,t)$. ∎

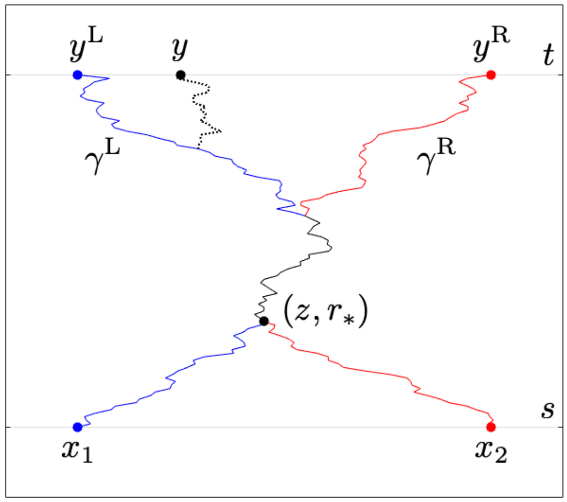


Figure 7. Proof sketch of Lemma 4.5. In the above diagram, $\gamma^{\mathrm{L}} \in G_{(x_1,s;y^{\mathrm{L}},t)}$ and $\gamma^{\mathrm{R}} \in G_{(x_2,s;y^{\mathrm{R}},t)}$. It is assumed that these two geodesics intersect so that $\mathrm{NonInt}_{s,t}(x_1, x_2, y^{\mathrm{L}}, y^{\mathrm{R}})$ does not occur, and (z, r_*) is their lowest point of intersection. If $y \in (y^{\mathrm{L}}, y^{\mathrm{R}})$, then a combination of Lemmas 2.2 and 2.6(c) yields geodesics $\gamma_1 \in G_{(x_1,s;y,t)}$ and $\gamma_2 \in G_{(x_2,s;y,t)}$ which agree at all points above and including (z, r_*), and then follow γ^{L} and γ^{R} respectively below time r_*. As this can be done for every $y \in (y^{\mathrm{L}}, y^{\mathrm{R}})$, we conclude that $\mathcal{Z}_{(x_1,x_2,s;t)}$ is constant on this interval (recall from Figure 5 how the value of $\mathcal{Z}_{(x_1,x_2,s;t)}(y)$ can be determined), the precise constant being $\mathcal{L}(x_2, s; z, r_*) - \mathcal{L}(x_1, s; z, r_*)$. From definitions (1.13) and (1.14), this implies $\mu_{(x_1,x_2,s;t)}([y_1, y_2]) = \mu_{s,t}([x_1, x_2] \times [y_1, y_2]) = 0$, meaning $y \notin \mathrm{Supp}(\mu_{(x_1,x_2,s;t)})$ and $(x, y) \notin \mathrm{Supp}(\mu_{s,t})$ for any $x \in (x_1, x_2)$.

We will need one more fact from [10] which is stated as the next lemma. It links the disjointness of geodesics to the measures $\mu_{(x_1,x_2,s;t)}$ and $\mu_{s,t}$. Although originally proved for Brownian LPP, it needs no revision in its extension to the directed landscape. Nevertheless, given the conceptual importance, we recall the ideas of the proof in Figure 7. Since only part (a) was explicitly stated in [10], we also point out in Figure 7 the equivalence of part (b).

Lemma 4.5. [10, Lemma 3.4] *The following statements hold on the event $\mathcal{P}$.*

(a) If $\mathrm{NonInt}_{s,t}(x_1, x_2; y^{\mathrm{L}}, y^{\mathrm{R}})$ does not occur for some $y^{\mathrm{L}} < y < y^{\mathrm{R}}$, then $y \notin \mathrm{Supp}(\mu_{(x_1,x_2,s;t)})$.

(b) If $\mathrm{NonInt}_{s,t}(x^{\mathrm{L}}, x^{\mathrm{R}}; y^{\mathrm{L}}, y^{\mathrm{R}})$ does not occur for some $x^{\mathrm{L}} < x < x^{\mathrm{R}}$ and $y^{\mathrm{L}} < y < y^{\mathrm{R}}$, then $(x, y) \notin \mathrm{Supp}(\mu_{s,t})$.

Proof of Proposition 4.1. We first prove that $\mathrm{Supp}(\mu_{(x_1,x_2,s;t)}) \subset \mathcal{D}_{(x_1,x_2,s;t)}$ by establishing the reverse containment for their complements. As usual, we assume for simplicity the occurrence of $\mathcal{P}$. Consider any $y \in \mathbb{R} \setminus \mathcal{D}_{(x_1,x_2,s;t)}$. Let γ^{L} and γ^{R} be the leftmost and rightmost geodesics in $G_{(x_1,s;y,t)}$ and $G_{(x_2,s;y,t)}$, respectively. As $y \notin \mathcal{D}_{(x_1,x_2,s;t)}$, there must be some $r_* \in (s, t)$ such that $\gamma^{\mathrm{L}}(r_*) = \gamma^{\mathrm{R}}(r_*)$.

Next take any sequences $y_j^{\mathrm{L}} \nearrow y$ and $y_j^{\mathrm{R}} \searrow y$, along with geodesics $\gamma_j^{\mathrm{L}} \in G_{(x_1,s;y_j^{\mathrm{L}},t)}$ and $\gamma_j^{\mathrm{R}} \in G_{(x_2,s;y_j^{\mathrm{R}},t)}$ guaranteed by Lemma 4.4; see Figure 8a. That is, $\gamma_j^{\mathrm{L}} \nearrow \gamma^{\mathrm{L}}$ and $\gamma_j^{\mathrm{R}} \searrow \gamma^{\mathrm{R}}$, uniformly as $j \to \infty$. So for any $\varepsilon > 0$, we have the following for all j sufficiently large:

$$\|\gamma_j^{\mathrm{L}} - \gamma^{\mathrm{L}}\|_\infty < \varepsilon \quad \text{and} \quad \|\gamma_j^{\mathrm{R}} - \gamma^{\mathrm{R}}\|_\infty < \varepsilon.$$

In particular, we can choose some j for which

$$|\gamma_j^{\mathrm{L}}(r) - \gamma^{\mathrm{L}}(r)| < \varepsilon \quad \text{and} \quad |\gamma_j^{\mathrm{R}}(r) - \gamma^{\mathrm{R}}(r)| < \varepsilon \quad \text{for all } r \in [r_*, t].$$

Given Theorem 1.17—applied to a *random* compact set $K \subset \mathbb{R}^4_\uparrow$ which accommodates the value of r_* relative to t—we can take ε to be so small that the above display implies

$$\gamma_j^{\mathrm{L}}(r^{\mathrm{L}}) = \gamma^{\mathrm{L}}(r^{\mathrm{L}}) \quad \text{and} \quad \gamma_j^{\mathrm{R}}(r^{\mathrm{R}}) = \gamma^{\mathrm{R}}(r^{\mathrm{R}}) \quad \text{for some } r^{\mathrm{L}}, r^{\mathrm{R}} \in [r_*, t]. \tag{4.3}$$

We now claim that

$$\gamma_j^{\mathrm{L}}(r') = \gamma^{\mathrm{L}}(r') \quad \text{for some } r' \in (s,t) \quad \Rightarrow \quad \gamma_j^{\mathrm{L}}(r) = \gamma^{\mathrm{L}}(r) \quad \text{for all } r \in [s, r']. \tag{4.4}$$

If this implication were false, then we would have $\gamma_j^{\mathrm{L}}(r) < \gamma^{\mathrm{L}}(r)$ for some $r \in (s, r')$, since we already know $\gamma_j^{\mathrm{L}}(r) \le \gamma^{\mathrm{L}}(r)$ and $\gamma_j^{\mathrm{L}}(s) = x_1 = \gamma^{\mathrm{L}}(s)$. But Lemma 2.2(iii) shows that $\gamma_j^{\mathrm{L}}\big|_{[s,r']} \oplus \gamma^{\mathrm{L}}\big|_{[r',t]}$ belongs to $G_{(x_1,s;y,t)}$, which contradicts the choice of γ^{L} as the leftmost geodesic. Therefore, our claim (4.4) is true; by the same logic, (4.4) holds also for γ_j^{R} and γ^{R}. Hence (4.3) together with the assumption $\gamma^{\mathrm{L}}(r_*) = \gamma^{\mathrm{R}}(r_*)$ implies

$$\gamma_j^{\mathrm{L}}(r_*) = \gamma^{\mathrm{L}}(r_*) = \gamma^{\mathrm{R}}(r_*) = \gamma_j^{\mathrm{R}}(r_*),$$

as shown in Figure 8b. In particular, $\mathrm{NonInt}_{s,t}(x_1, x_2, y_j^{\mathrm{L}}, y_j^{\mathrm{R}})$ has not occurred, and so $y \notin \mathrm{Supp}(\mu_{(x_1,x_2,s;t)})$ by Lemma 4.5(a).

The argument to show $\mathrm{Supp}(\mu_{s,t}) \subset \mathcal{D}_{s,t}$ will be similar, again establishing the contrapositive. Consider any $(x,y) \in \mathbb{R}^2 \setminus \mathcal{D}_{s,t}$. Let γ^{L} and γ^{R} be the leftmost and rightmost geodesics between (x,s) and (y,t). As before, there must be some $r_* \in (s,t)$ such that $\gamma^{\mathrm{L}}(r_*) = \gamma^{\mathrm{R}}(r_*)$. Using Lemma 4.4 once more, we take sequences $x_j^{\mathrm{L}} \nearrow x$, $x_j^{\mathrm{R}} \searrow x$, $y_j^{\mathrm{L}} \nearrow y$, $y_j^{\mathrm{R}} \searrow y$, and consider geodesics $\gamma_j^{\mathrm{L}} \in G_{(x_j^{\mathrm{L}},s;y_j^{\mathrm{L}},t)}$ and $\gamma_j^{\mathrm{R}} \in G_{(x_j^{\mathrm{R}},s;y_j^{\mathrm{R}},t)}$ such that $\gamma_j^{\mathrm{L}} \nearrow \gamma^{\mathrm{L}}$ uniformly and $\gamma_j^{\mathrm{R}} \searrow \gamma^{\mathrm{R}}$ uniformly. An illustration is provided in Figure 8c.

The same argument leading to (4.4)—but now using Lemma 2.2(i)—tells us that if γ_j^{L} intersects γ^{L} at two distinct times, then the two geodesics must agree at all intermediate times. That is,

$$\gamma_j^{\mathrm{L}}(r') = \gamma^{\mathrm{L}}(r'),\ \gamma_j^{\mathrm{L}}(r'') = \gamma^{\mathrm{L}}(r''),\ s < r' < r'' < t \quad \Rightarrow \quad \gamma_j^{\mathrm{L}}(r) = \gamma^{\mathrm{L}}(r) \quad \text{for all } r \in [r', r''], \tag{4.5}$$

and similarly for γ_j^{R} and γ^{R}. Meanwhile, by invoking Theorem 1.17 twice—once for each of the two intervals $[s, r_*]$ and $[r_*, t]$—we can conclude that for all j sufficiently large,

$$\gamma_j^{\mathrm{L}}(r') = \gamma^{\mathrm{L}}(r') \quad \text{and} \quad \gamma_j^{\mathrm{L}}(r'') = \gamma^{\mathrm{L}}(r'') \quad \text{for some } r' \in [s, r_*],\ r'' \in [r_*, t],$$

and similarly for γ_j^{R} and γ^{R}. By (4.5), it follows that $\gamma_j^{\mathrm{L}}(r_*) = \gamma^{\mathrm{L}}(r_*) = \gamma^{\mathrm{R}}(r_*) = \gamma_j^{\mathrm{R}}(r_*)$; see Figure 8d. Now Lemma 4.5(b) gives the desired conclusion: $(x,y) \notin \mathrm{Supp}(\mu_{s,t})$. $\square$

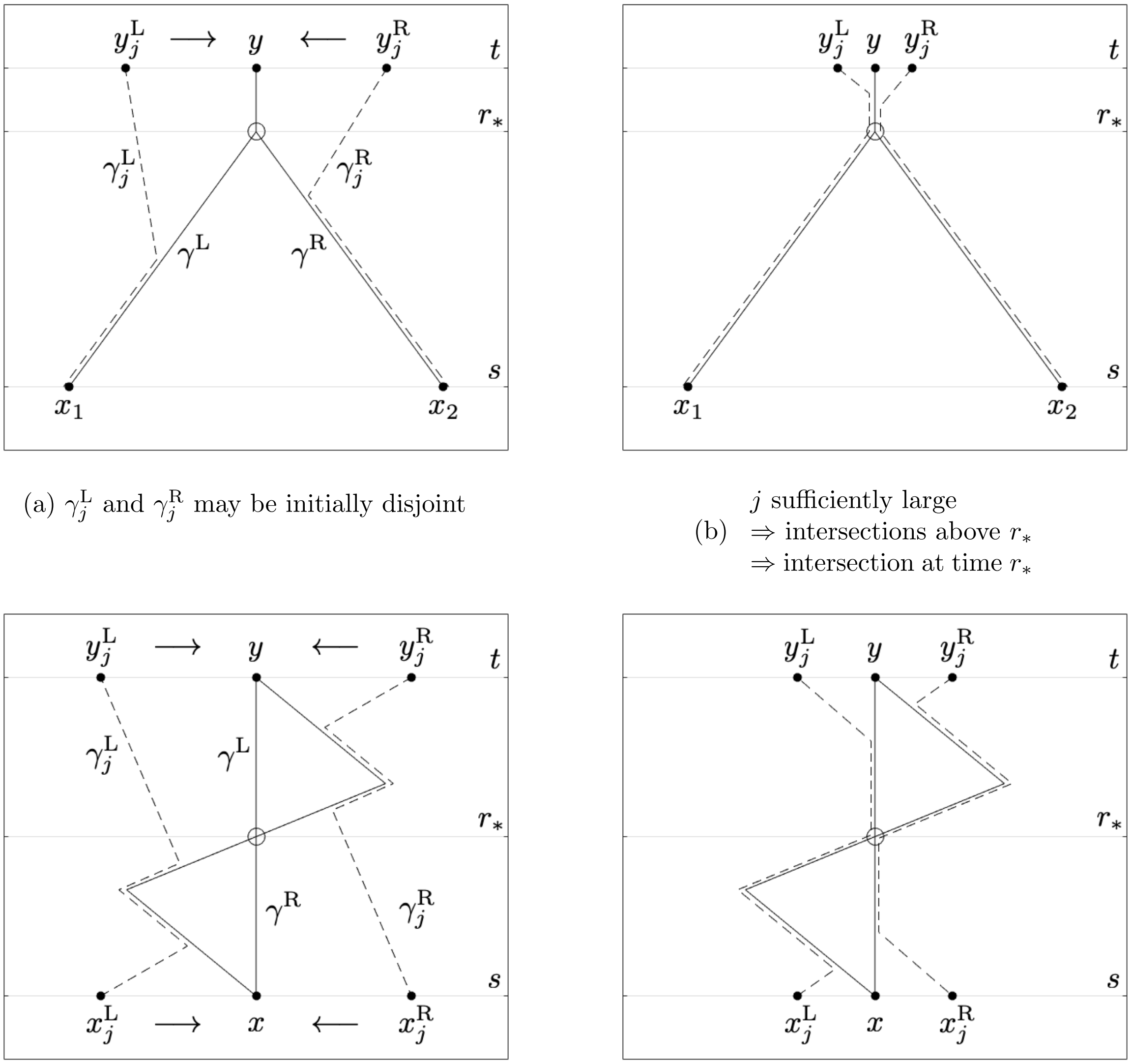
4.2. **The lower bound in Theorem 4.3.** Recall that for $d \in [0, \infty)$, the d-dimensional *Hausdorff content* of a metric space $\mathcal{X}$ is

$$H^d(\mathcal{X}) \coloneqq \inf\bigg\{\sum_i \mathrm{diam}(U_i)^d : \{U_i\} \text{ is a countable cover of } \mathcal{X}\bigg\}. \tag{4.6}$$

The *Hausdorff dimension* of $\mathcal{X}$ is

$$d_H(\mathcal{X}) \coloneqq \inf\{d \ge 0 : H^d(\mathcal{X}) = 0\}. \tag{4.7}$$

We now prove that $d_H(\mathrm{Supp}(\mu_{s,t})) \ge \frac{1}{2}$ almost surely.



(a) γ_j^{L} and γ_j^{R} may be initially disjoint

(b) j sufficiently large
$\Rightarrow$ intersections above r_*
$\Rightarrow$ intersection at time r_*

(c) γ_j^{L} and γ_j^{R} may be initially disjoint

(d) j sufficiently large
$\Rightarrow$ intersections above/below r_*
$\Rightarrow$ intersection at time r_*

Figure 8. Geodesics considered in the proof of Proposition 4.1. Diagrams (a) and (b) illustrate the argument for $\mathrm{Supp}(\mu_{(x_1,x_2,s;t)}) \subset \mathcal{D}_{(x_1,x_2,s;t)}$, while (c) and (d) illustrate the argument for $\mathrm{Supp}(\mu_{s,t}) \subset \mathcal{D}_{s,t}$. In each case, there is assumed to be some time $r_* \in (s,t)$ at which γ^{L} and γ^{R} intersect; the point of intersection is marked with an open circle. If γ_j^{L} intersects γ^{L} in (a), then the two geodesics coincide at all lower times; similarly for γ_j^{R} with γ^{R}. When $j \to \infty$, both pairs must experience intersections above time r_*, thereby forcing an intersection between γ_j^{L} and γ_j^{R} at time r_*, where γ^{L} and γ^{R} agree. A slightly different argument is needed for the scenario in (c). If γ_j^{L} intersects γ^{L} at two distinct times, then the two geodesics coincide at all intermediate times; similarly for γ_j^{R} with γ^{R}. For both pairs, sending $j \to \infty$ forces at least one intersection before time r_* and one after time r_*. Hence a common intersection eventually appears at r_*.

Proof of the lower bound in Theorem 4.3. By Theorem 4.2, it suffices to show that for any $x_1 < x_2$ and $s < t$, we have almost surely have

$$d_H(\mathrm{Supp}(\mu_{(x_1,x_2,s;t)})) \le d_H(\mathrm{Supp}(\mu_{s,t})). \tag{4.8}$$

To this end, we first prove that

$$y \in \mathrm{Supp}(\mu_{(x_1,x_2,s;t)}) \quad \Rightarrow \quad ([x_1,x_2] \times \{y\}) \cap \mathrm{Supp}(\mu_{s,t}) \neq \varnothing. \tag{4.9}$$

Indeed, let us check the contrapositive.

If $(x,y) \notin \mathrm{Supp}(\mu_{s,t})$, then there exists an open ball $B_r(x,y) \subset \mathbb{R}^2$ of radius $r > 0$ and centered at (x,y), such that $\mu_{s,t}(B_r(x,y)) = 0$. For each x, define

$$r_x \coloneqq \sup\{r > 0 : \mu_{s,t}(B_r(x,y)) = 0\} \vee 0.$$

The map $x \mapsto r_x$ is continuous, in fact with Lipschitz constant 1, as seen from the following chain of implications (if $r \le 0$, then $B_r(\cdot,\cdot)$ is taken to be the empty set):

$$\begin{aligned}
&\mu_{s,t}(B_{r_x-\varepsilon}(x,y)) = 0 < \mu_{s,t}(B_{r_x+\varepsilon}(x,y)) \quad \forall\, \varepsilon > 0\\
\Rightarrow\quad &\mu_{s,t}(B_{r_x-\delta-\varepsilon}(x',y)) = 0 < \mu_{s,t}(B_{r_x+\delta+\varepsilon}(x',y)) \quad \forall\, x' \in [x-\delta, x+\delta],\, \delta > 0,\, \varepsilon > 0\\
\Rightarrow\quad &r_{x'} \in [r_x - \delta - \varepsilon, r_x + \delta + \varepsilon] \quad \forall\, x' \in [x-\delta, x+\delta],\, \delta > 0,\, \varepsilon > 0\\
\Rightarrow\quad &r_{x'} \in [r_x - \delta, r_x + \delta] \quad \forall\, x' \in [x-\delta, x+\delta],\, \delta > 0.
\end{aligned}$$

Now, if $([x_1,x_2] \times \{y\}) \cap \mathrm{Supp}(\mu_{s,t}) = \varnothing$, then $r_x > 0$ for every $x \in [x_1,x_2]$. By the continuity just observed, there is then some $r > 0$ such that $r_x \ge r$ for all $x \in [x_1,x_2]$. Consequently, $\mu_{s,t}([x_1,x_2] \times (y-r, y+r)) = 0$, which means $y \notin \mathrm{Supp}(\mu_{(x_1,x_2,s;t)})$. We have now proved (4.9).

Now suppose $\{U_i\}$ is a countable cover of $\mathrm{Supp}(\mu_{s,t})$. For each i, let

$$\widetilde{U}_i \coloneqq \{y \in \mathbb{R} : ([x_1,x_2] \times \{y\}) \cap U_i \neq \varnothing\}.$$

By (4.9), $\{\widetilde{U}_i\}$ is a cover of $\mathrm{Supp}(\mu_{(x_1,x_2,s;t)})$. Furthermore, it is trivial that $\mathrm{diam}(\widetilde{U}_i) \le \mathrm{diam}(U_i)$. From the definitions (4.6) and (4.7) of Hausdorff content and Hausdorff dimension, the inequality (4.8) immediately follows. ☐

5. Proofs of Theorems 1.9(b) and 1.10(b)

Recall from the previous section that Proposition 4.1 and Theorems 4.2 and 4.3 (or rather, just the lower bounds from these theorems) combine to give the following almost sure statements:

$$d_H(\mathcal{D}_{(x_1,x_2,s;t)}) \ge \frac{1}{2} \quad \text{and} \quad d_H(\mathcal{D}_{s,t}) \ge \frac{1}{2}.$$

In this section, we prove the matching upper bounds.

Proposition 5.1. *For any $x_1 < x_2$ and $s < t$, we almost surely have*

$$d_H(\mathcal{D}_{(x_1,x_2,s;t)}) \le \frac{1}{2} \quad \textit{and} \quad d_H(\mathcal{D}_{s,t}) \le \frac{1}{2}.$$

For Sections 5.1–5.3, let us fix $x_1 < x_2$ and $s < t$.

5.1. Step 1: Reduce to bounded sets.

Suppose we can show the following for any $R > 0$.

Claim 5.2. *We almost surely have*

$$d_H(\mathcal{D}_{(x_1,x_2,s;t)} \cap [-R,R]) \le \frac{1}{2} \quad \textit{and} \quad d_H(\mathcal{D}_{s,t} \cap [-R,R]^2) \le \frac{1}{2}. \tag{5.1}$$

Proposition 5.1 then immediately follows by taking a countable sequence $R_j \nearrow \infty$ and using the fact that if $\mathcal{X} \subset \mathbb{R}^m$ satisfies $d_H(\mathcal{X} \cap [-R,R]^m) \le d$ for every R, then $d_H(\mathcal{X}) \le d$. So let us fix the value of R and aim simply to prove Claim 5.2. We will assume R is at least large enough that $x_1, x_2 \in [-R,R]$.

5.2. **Step 2: Relate the exceptional sets to pairs of disjoint geodesics.** Recall the events $\mathcal{E}$ and $\mathcal{O}_{(x,s;t)}$ from (2.3) and (2.5), guaranteeing geodesic existence and geodesic ordering. We next quote a useful lemma from [10] concerning the event $\mathrm{NonInt}_{s,t}(x_1,x_2;y_1,y_2)$ that every $\gamma_1 \in G_{(x_1,s;y_1;t)}$ is disjoint from every $\gamma_2 \in G_{(x_2,s;y_2,t)}$. It was originally stated for Brownian LPP, but the proof carries over without modification given that we have established Lemma 2.2.

Lemma 5.3. [10, Proposition 3.5] *Let $\varepsilon \in (0, x_2 - x_1)$. On the event $\mathcal{E}$, if $\mathrm{NonInt}_{s,t}(x_1,x_2;y,y+\varepsilon)$ occurs, then there exists an interval $I \subset [x_1,x_2]$ of length ε for which*

$$\mathrm{MaxDisjtGeo}^{\mathbb{R}}_{s,t}(I,[y,y+\varepsilon]) \geq 2.$$

For brevity, let us henceforth write

$$\begin{aligned}\mathcal{W}^{\varepsilon}_{(z,s;w,t)} &\coloneqq \{\mathrm{MaxDisjtGeo}^{\mathbb{R}}_{s,t}((z-\varepsilon,z+\varepsilon),(w-\varepsilon,w+\varepsilon)) \geq 2\},\\ \mathcal{V}^{\varepsilon}_{(s;w,t)} &\coloneqq \bigcup_{z\in\frac{\varepsilon}{2}\mathbb{Z}\cap[-R,R]} \mathcal{W}^{\varepsilon}_{(z,s;w,t)}.\end{aligned} \tag{5.2}$$

Claim 5.4. *We have the following containments.*

(a) On the event $\mathcal{E}$,

$$\big\{\mathcal{D}_{s,t} \cap \big([z,z+\varepsilon)\times[w,w+\varepsilon)\big) \neq \varnothing\big\} \subset \mathcal{W}^{\varepsilon}_{(z,s;w,t)} \quad \forall\ z,w\in\mathbb{R},\ \varepsilon>0, \tag{5.3}$$

(b) On the event $\mathcal{E}\cap\mathcal{O}_{(x_1,s;t)}\cap\mathcal{O}_{(x_2,s;t)}$,

$$\{\mathcal{D}_{(x_1,x_2,s;t)} \cap [w,w+\varepsilon) \neq \varnothing\} \subset \mathcal{V}^{\varepsilon}_{(s;w,t)} \quad \forall\ w\in[-R,R],\ \varepsilon\in(0,x_2-x_1). \tag{5.4}$$

Proof. First we prove (5.3), which is illustrated in Figure 9a. Suppose $(x,y) \in \mathcal{D}_{s,t} \cap \big([z,z+\varepsilon)\times[w,w+\varepsilon)\big)$. That is, there are $\gamma_1^*,\gamma_2^* \in G_{(x,s;y,t)}$ such that $\gamma_1(r) < \gamma_2(r)$ for all $r\in(s,t)$. Take any $x_1 \in (x-\varepsilon,z)$, $y_1\in(y-\varepsilon,w)$, and set $x_2 = x_1+\varepsilon$, $y_2 = y_1+\varepsilon$. We then have

$$z-\varepsilon < x_1 < z < x_2 < z+\varepsilon \quad \text{and} \quad w-\varepsilon < y_1 < y < y_2 < w+\varepsilon.$$

By Lemma 2.6 and the assumed occurrence of $\mathcal{E}$, there are $\gamma_1 \in G_{(x_1,s;y_1,t)}$ and $\gamma_2 \in G_{(x_2,s;y_2,t)}$ such that

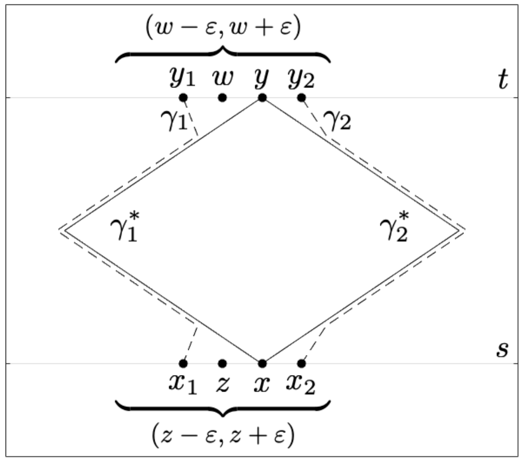
$$\gamma_1(r) \leq \gamma_1^*(r) < \gamma_2^*(r) \leq \gamma_2(r) \quad \text{for all } r\in(s,t). \tag{5.5}$$

Of course, we also know $\gamma_1(s) = x_1 < x_2 = \gamma_2(s)$ and $\gamma_1(t) = y_1 < y_2 = \gamma_2(t)$, and so γ_1 and γ_2 are disjoint. By our choice of endpoints, $\mathcal{W}^{\varepsilon}_{(z,s;w,t)}$ has occurred.

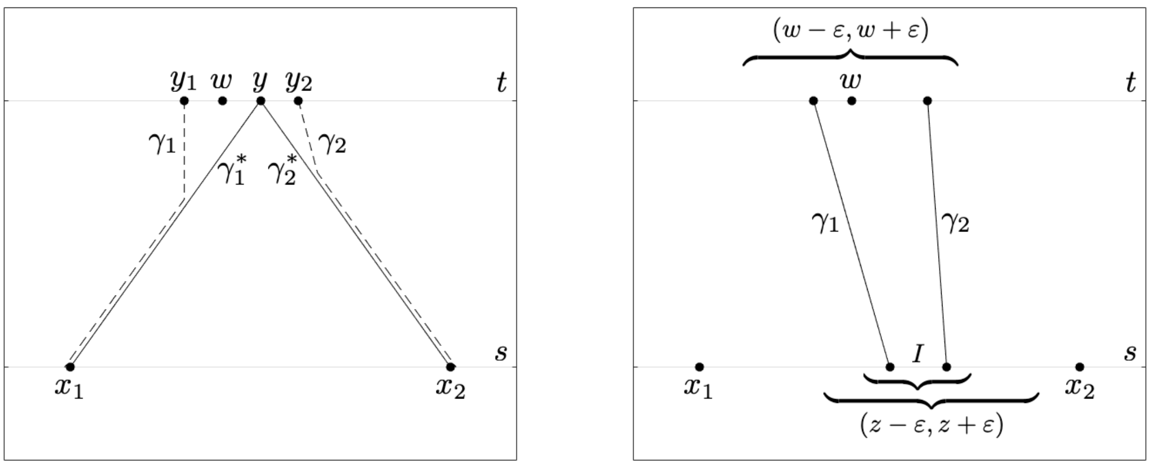
Next we argue (5.4), which requires just one additional step. Supposing $y \in \mathcal{D}_{(x_1,x_2,s;t)}\cap[w,w+\varepsilon)$, we have $\gamma_1^* \in G_{(x_1,s;y,t)}$ and $\gamma_2^* \in G_{(x_2,s;y,t)}$ such that $\gamma_1^*(r) < \gamma_2^*(r)$ for all $r\in[s,t)$. As before, we choose $y_1 \in (y-\varepsilon,w)$ and set $y_2 = y_1+\varepsilon$, but now we consider *any* $\gamma_1 \in G_{(x_1,s;y_1,t)}$, $\gamma_2 \in G_{(x_2,s;y_2,t)}$. By the assumed occurrence of $\mathcal{O}_{(x_1,s;t)}\cap\mathcal{O}_{(x_2,s;t)}$, we necessarily have the disjointness condition (5.5), as portrayed in Figure 9b. Therefore, so long as $\varepsilon\in(0,x_2-x_1)$, Lemma 5.3 gives the existence of some interval $I\subset[x_1,x_2]$ of length ε such that

$$\mathrm{MaxDisjtGeo}^{\mathbb{R}}_{s,t}(I,(w-\varepsilon,w+\varepsilon)) \geq \mathrm{MaxDisjtGeo}^{\mathbb{R}}_{s,t}(I,[y_1,y_1+\varepsilon]) \geq 2.$$

This scenario is depicted in Figure 9c. Finally, notice that I is contained in some interval of the form $(z-\varepsilon,z+\varepsilon)$ with $z\in(\varepsilon/2)\mathbb{Z}\cap[-R,R]$. Hence $\mathcal{V}^{\varepsilon}_{(s;w,t)}$ has occurred, thus proving (5.4). ☐



(a) $(x, y) \in \mathcal{D}_{s,t} \cap \big([z, z+\varepsilon) \times [w, w+\varepsilon)\big)$



(b) $y \in \mathcal{D}_{(x_1,x_2,s;t)} \cap [w, w+\varepsilon)$

(c) Implied scenario $\mathcal{V}^{\varepsilon}_{(s;w,t)}$

Figure 9. Geodesics considered in the proof of Claim 5.4. In (a) and (b), the disjointness of the solid geodesics implies the disjointness of the dashed geodesics. The case of (a) is simpler, since the occurrence of $\mathcal{W}^{\varepsilon}_{(z,s;w,t)}$ is immediate. Meanwhile, deducing from (b) the occurrence of $\mathcal{V}^{\varepsilon}_{(s;w,t)}$ requires Lemma 5.3: the disjointness of γ_1 and γ_2 in (b) implies the existence of γ_1 and γ_2 in (c).

5.3. **Step 3: Use tail estimate to deduce dimension upper bound.** Suppose we can show the following for any $d > \frac{1}{2}$.

Claim 5.5. *We almost surely have* $H^d(\mathcal{D}_{(x_1,x_2,s;t)} \cap [-R, R]) = 0 = H^d(\mathcal{D}_{s,t} \cap [-R, R]^2)$.

Claim 5.2 immediately follows by taking a countable sequence $d_j \searrow \frac{1}{2}$. Therefore, let us fix $d > \frac{1}{2}$ and complete the proof of Proposition 5.1 by verifying Claim 5.5. Let $\eta \coloneqq (2d-1)/6 > 0$.

Proof of Claim 5.5. Let G be the constant from Theorem 1.15, and choose $\varepsilon' \in (0,1]$ sufficiently small that the following inequalities hold for all $\varepsilon \in (0,\varepsilon']$:

$$\frac{\varepsilon}{(t-s)^{2/3}} \leq G^{-16}, \quad \frac{2R}{(t-s)^{2/3}} \leq \Big(\frac{\varepsilon}{(t-s)^{2/3}}\Big)^{-1/2}\Big(\log\frac{(t-s)^{2/3}}{\varepsilon}\Big)^{-2/3} G^{-2}, \tag{5.6a}$$

$$G^8 \exp\Big\{G^2\Big(\log\frac{(t-s)^{2/3}}{\varepsilon}\Big)^{5/6}\Big\}\Big(\frac{\varepsilon}{(t-s)^{2/3}}\Big)^{3/2} \leq \varepsilon^{3/2-\eta}, \tag{5.6b}$$

$$\varepsilon^{\eta} \leq \frac{1}{2(2R+1)^2}. \tag{5.6c}$$

The assumption (5.6a) allows us to apply Theorem 1.16 whenever the relevant spatial coordinates belong to $[-R,R]$, and (5.6b) makes the resulting estimate easier to write:

$$\mathbb{P}(\mathcal{W}^{\varepsilon}_{(z,s;w,t)}) \leq \varepsilon^{3/2-\eta} \quad \text{for all } z,w \in [-R,R],\ \varepsilon \in (0,\varepsilon']. \tag{5.7}$$

Now take any summable sequence $\delta_j \searrow 0$. Because $d - 1/2 - 2\eta = \eta > 0$, we can subsequently choose a sequence $\varepsilon_j \searrow 0$ such that

$$\lim_{j\to\infty} \varepsilon_j^{d-1/2-2\eta}\delta_j^{-1} = 0. \tag{5.8}$$

For convenience, let us always choose ε_j so that $R/\varepsilon_j \in \mathbb{Z}$. As soon as $\varepsilon_j \leq \varepsilon'$, the estimate (5.7) leads to

$$\mathbb{E}\Big[\sum_{z,w\in\varepsilon_j\mathbb{Z}\cap[-R,R]} \mathbb{1}_{\mathcal{W}^{\varepsilon_j}_{(z,s;w,t)}}\Big] \leq \Big(\frac{2R}{\varepsilon_j}+1\Big)^2 \varepsilon_j^{3/2-\eta} \leq (2R+1)^2\varepsilon_j^{-1/2-\eta} \overset{(5.6c)}{\leq} \varepsilon_j^{-1/2-2\eta}. \tag{5.9a}$$

Similarly, when $\varepsilon_j \leq \varepsilon' \wedge (x_2 - x_1)$, we have

$$\begin{aligned} \mathbb{E}\Big[\sum_{w\in\varepsilon_j\mathbb{Z}\cap[-R,R]} \mathbb{1}_{\mathcal{V}^{\varepsilon_j}_{(s;w,t)}}\Big] \leq \mathbb{E}\Big[\sum_{\substack{w\in\varepsilon_j\mathbb{Z}\cap[-R,R]\\ z\in(\varepsilon_j/2)\mathbb{Z}\cap[-R,R]}} \mathbb{1}_{\mathcal{W}^{\varepsilon_j}_{(z,s;w,t)}}\Big] &\leq \Big(\frac{2R}{\varepsilon_j}+1\Big)\Big(\frac{2R}{\varepsilon_j/2}+1\Big)\varepsilon_j^{3/2-\eta} \\ &\leq 2(2R+1)^2\varepsilon_j^{-1/2-\eta} \\ &\overset{(5.6c)}{\leq} \varepsilon_j^{-1/2-2\eta}. \end{aligned} \tag{5.9b}$$

Applying Markov's inequality to (5.9) results in

$$\mathbb{P}\Big(\sum_{z,w\in\varepsilon_j\mathbb{Z}\cap[-R,R]} \mathbb{1}_{\mathcal{W}^{\varepsilon_j}_{(z,s;w,t)}} \geq \varepsilon_j^{-1/2-2\eta}\delta_j^{-1}\Big) \leq \delta_j, \tag{5.10a}$$

$$\mathbb{P}\Big(\sum_{w\in\varepsilon_j\mathbb{Z}\cap[-R,R]} \mathbb{1}_{\mathcal{V}^{\varepsilon_j}_{(s;w,t)}} \geq \varepsilon_j^{-1/2-2\eta}\delta_j^{-1}\Big) \leq \delta_j. \tag{5.10b}$$

Our final step will be to use these inequalities to deduce that the d-dimensional Hausdorff content of $\mathcal{D}_{s,t} \cap [-R,R]^2$ and of $\mathcal{D}_{(x_1,x_2,s;t)} \cap [-R,R]$ is zero.

Let us first consider $\mathcal{D}_{s,t}$. If the event appearing in (5.10a) does not occur, then Claim 5.4 implies that $\mathcal{D}_{s,t} \cap \big([z,z+\varepsilon_j)\times[w,w+\varepsilon_j)\big)$ is nonempty for at most $\varepsilon_j^{-1/2-2\eta}\delta_j^{-1}$ values of $(z,w) \in (\varepsilon_j\mathbb{Z}\cap[-R,R])^2$. In this case, $\mathcal{D}_{s,t}\cap[-R,R]^2$ can be covered by $\varepsilon_j^{-1/2-2\eta}\delta_j^{-1}$ rectangles of diameter less than $2\varepsilon_j$, meaning that

$$H^d(\mathcal{D}_{s,t}\cap[-R,R]^2) \leq 2^d\varepsilon_j^{d-1/2-2\eta}\delta_j^{-1}.$$

Since $\sum_j \delta_j < \infty$, it follows from (5.10a) and Borel–Cantelli that with probability one, the above display is true for all large j. Because of (5.8), this implies $H^d(\mathcal{D}_{s,t} \cap [-R,R]^2) = 0$ with probability one. The case of $\mathcal{D}_{(x_1,x_2,s;t)}$ is argued in exactly the same fashion from (5.10b). □

5.4. **Almost sure statement regarding fixed initial locations.** This final section verifies a claim made in Remark 1.11 which is not needed elsewhere in the paper. Before stating the result, we note that the definition (1.12) of $\mathcal{D}_{(x_1,x_2,s;t)}$ makes sense for $x_1 = x_2$, even though we have only considered $x_1 < x_2$ up to this point. More precisely, $y \in \mathcal{D}_{(x,x,s;t)}$ if and only if $(x,y) \in \mathcal{D}_{s,t}$.

Theorem 5.6. *For any fixed $y \in \mathbb{R}$ and $s < t$, it is almost surely the case that $y \notin \mathcal{D}_{(x_1,x_2,s;t)}$ for every $x_1 \le x_2$.*

Proof. Let us write $\mathcal{D}^{s,t} := \bigcup_{x_1 \le x_2} \mathcal{D}_{(x_1,x_2,s;t)}$. We wish to prove that for fixed $y \in \mathbb{R}$, we have $\mathbb{P}(y \in \mathcal{D}^{s,t}) = 0$. It clearly suffices to show that for any $R > 0$, we have

$$\mathbb{P}\Big(y \in \bigcup_{-R \le x_1 \le x_2 \le R} \mathcal{D}_{(x_1,x_2,s;t)}\Big) = 0. \tag{5.11}$$

So let us fix $R > 0$ and define the events $\mathcal{W}^{\varepsilon}_{(z,s;w,t)}$ and $\mathcal{V}^{\varepsilon}_{(s;w,t)}$ as in (5.2). If $y \in \mathcal{D}_{(x_1,x_2,s;t)}$ for some $x_1 \le x_2$ in $[-R,R]$, then Claim 5.4, specifically part (a) if $x_1 = x_2$ and part (b) if $x_1 < x_2$, shows that $\mathcal{V}^{\varepsilon}_{(s;y,t)}$ occurs for every $\varepsilon > 0$. That is,

$$\Big\{y \in \bigcup_{-R \le x_1 \le x_2 \le R} \mathcal{D}_{(x_1,x_2,s;t)}\Big\} \subset \bigcap_{\varepsilon > 0} \mathcal{V}^{\varepsilon}_{(s;y,t)}. \tag{5.12}$$

Now let G be the constant from Theorem 1.15, and choose $\varepsilon' \in (0,1]$ sufficiently small that the (5.6) holds for all $\varepsilon \in (0,\varepsilon']$, say with $\eta = \frac{1}{6}$. As in the proof of Claim 5.5, we can appeal to Theorem 1.16 and obtain

$$\mathbb{P}(\mathcal{V}^{\varepsilon}_{(s;y,t)}) \le \sum_{z \in \frac{\varepsilon}{2}\mathbb{Z} \cap [-R,R]} \mathbb{P}(\mathcal{W}^{\varepsilon}_{(z,s;y,t)}) \overset{(5.7)}{\le} \Big(\frac{2R}{\varepsilon/2} + 2\Big)\varepsilon^{4/3} \quad \text{for all } z \in [-R,R],\ \varepsilon \in (0,\varepsilon'].$$

By allowing ε to tend to zero, we see that $\mathbb{P}(\bigcap_{\varepsilon > 0} \mathcal{V}^{\varepsilon}_{(s;y,t)}) = 0$. Hence (5.11) follows from (5.12). □

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