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**BROWNIAN STRUCTURE IN THE
KPZ FIXED POINT**

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Abstract. — Many models of one-dimensional local random growth are expected to lie in the Kardar-Parisi-Zhang (KPZ) universality class. For such a model, the interface profile at advanced time may be viewed in scaled coordinates specified via characteristic KPZ scaling exponents of one-third and two-thirds. When the long time limit of this scaled interface is taken, it is expected—and proved for a few integrable models—that, up to a parabolic shift, the Airy_2 process $\mathcal{A} : \mathbb{R} \rightarrow \mathbb{R}$ is obtained. This process may be embedded via the Robinson-Schensted-Knuth correspondence as the uppermost curve in an \mathbb{N} -indexed system of random continuous curves, the Airy line ensemble.

Among our principal results is the assertion that the Airy_2 process enjoys a very strong similarity to Brownian motion B (of rate two) on unit-order intervals. This result yields bounds on the Airy_2 probabilities of a large class of events from the counterpart bounds on Brownian motion probabilities. The result has the consequence that the Radon-Nikodym derivative of the law of \mathcal{A} on say $[-1, 1]$ after a suitable vertical shift, with respect to the law of B on the same interval, lies in every L^p space for $p \in (1, \infty)$. In fact, the quantitative comparison of probability bounds we prove also holds for the scaled energy profile with Dirac delta initial condition of the model of Brownian last passage percolation, a model that lies in the KPZ universality class and in which the energy of paths in a random Brownian environment is maximised.

Our technique of proof harnesses a probabilistic resampling or *Brownian Gibbs* property satisfied by the Airy line ensemble after parabolic shift, and this article develops Brownian Gibbs analysis of this ensemble begun in work of Corwin and Hammond (2014) and pursued by Hammond (2019). Our Brownian comparison for scaled interface profiles is an element in the ongoing programme of studying KPZ universality via probabilistic and geometric methods of proof, aided by limited but essential use of integrable inputs. Indeed, the comparison result is a useful tool for studying this universality class. We present and prove several applications, concerning for example the structure of near ground states in Brownian last passage percolation, or Brownian structure in scaled interface profiles that arise from the evolution from any element in a very general class of initial data.

Résumé (Le structure brownien dans le point fixe de KPZ). — On s’attend à ce que de nombreux modèles de croissance aléatoire uni-dimensionnelle appartiennent à la classe d’universalité de Kardar-Parisi-Zhang (KPZ). En rescalant convenablement le temps via les exposants caractéristiques de KPZ d’un tiers et de deux tiers, on considère le profil déterminé par un tel modèle. Dans la limite de temps infini, ce profil devrait converger vers le processus Airy₂ $\mathcal{A} : \mathbb{R} \rightarrow \mathbb{R}$ à l’addition d’une parabole près—constatation dont la preuve est connue dans les cas de certains modèles intégrables. Ce processus peut-être intégré par le correspondance de Robinson-Schensted-Knuth comme la courbe la plus haute d’un système indexé par \mathbb{N} de courbes aléatoires continues, l’ensemble de lignes d’Airy.

Parmi nos résultats principaux est l’énoncé que le processus Airy₂ présente une grande similarité avec le mouvement brownien (de taux deux) sur un intervalle de longueur de l’ordre de l’unité. Grâce à ce résultat, nous obtenons des bornes sur les probabilités Airy₂ d’une large classe d’événements des bornes homologues browniennes. Ce résultat a le corollaire que la dérivée Radon-Nikodym de la loi $\mathcal{A} : \mathbb{R} \rightarrow \mathbb{R}$ sur l’intervalle $[-1, 1]$ par rapport à la loi brownienne sur cet intervalle appartient à tous les espaces L^p où $p \in (1, \infty)$. En fait, la comparaison quantitative de bornes probabilistiques est également vraie pour le profil d’énergie après changement d’échelle (avec condition initiale delta Dirac) du modèle de percolation de passage dernier brownien—un modèle qui appartient à la classe d’universalité de KPZ et dans lequel l’énergie de chemins est maximisée dans un environnement aléatoire brownien.

Notre technique de preuve utilise une propriété de rééchantillonnage probabiliste—la propriété de Brownian-Gibbs—qui est satisfaite par l’ensemble parabolique de lignes d’Airy; cet article développe l’analyse Brownian-Gibbs de cet ensemble commencée par Corwin et Hammond (2014) et poursuivie par Hammond (2019). Notre comparaison brownienne de profils après changement d’échelle est un élément du programme plus vaste de l’étude de l’universalité de KPZ par les moyens géométriques et probabilistiques, facilité par l’usage, limité mais critique, des énoncés intégrables. En effet, le résultat de comparaison est un outil puissant dans l’étude de cette classe d’universalité. Nous prouvons quelques corollaires au sujet de la structure d’états quasi-minimaux dans la percolation de dernier passage brownien, ou de la structure brownienne trouvée dans les profils d’interface après changement d’échelle donnée par l’évolution d’un élément arbitraire dans une classe très large de conditions initiales.

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CHAPTER 1

INTRODUCTION

1.1. KPZ universality

The field of Kardar-Parisi-Zhang (KPZ) universality concerns one-dimensional interfaces that evolve randomly in time, and universal random structures that describe, independently of the microscopic details that specify the local evolution of such random models, the geometry and fluctuation of the interface when time is advanced. The KPZ universality class, whose members are random processes that are expected to evince these late-time characteristics, is very broad. The basic features of a random growth model that indicate that it may be a member of the KPZ universality class are that growth occurs in a direction normal to the present local slope of the interface at a rate influenced by the slope, alongside two competing forces: a smoothing effect generated by surface tension, and a roughening effect caused by forces in the environment that are local and random.

The principal results of this monograph offer a very strong assertion concerning the geometry of the canonical random object—the *Airy₂ process with parabolic curvature*—that describes the scaled attributes of what are known as the narrow wedge solutions to models in the KPZ universality class. Our results further offer a counterpart description that holds in a rather uniform sense in the prelimit for a model in the KPZ class known as *Brownian last passage percolation* (LPP). The geometric inference that we make is a powerful assertion of the Brownian nature of the scaled interface profile and it is the engine for a wide array of applications about scaled random growth models and their KPZ universality limiting structure.

The parabolically shifted Airy₂ process is the first in a family of limiting processes, indexed by the initial condition, which are expected to be universal objects in the KPZ universality class, and which may be referred to collectively as the *KPZ fixed point*. The name arises from the physical view that such objects are expected to be fixed points of suitable renormalization operators. (Indeed, an important related object has been constructed by [MQR21] in a paper bearing the name “The KPZ fixed point”. Our usage of this term is a corruption of theirs, but only mildly so. We will comment further on [MQR21] in Section 1.3.2.) One application of our results will be a statement about a certain form of Brownian regularity for these KPZ fixed point profiles, i.e., limiting scaled interface profiles from general initial data in Brownian LPP.

We defer the definition of Brownian last passage percolation and the presentation of our main theorem, expressed in prelimiting terms that capture Brownian LPP, to Section 3.2. What we are able to indicate in the ensuing paragraphs is the form of our principal assertion in the limiting case of scaled KPZ structure. Indeed, we will next introduce the parabolic Airy₂ process; and then we state our principal conclusion as it applies to this process.

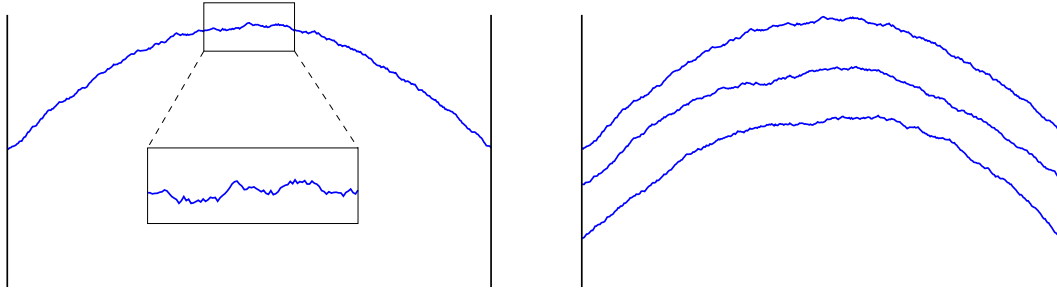


FIGURE 1.1. On the left panel is an illustration of the locally Brownian, globally parabolic nature of the limiting process \mathcal{L} . On the right, we depict the top three curves of the parabolic Airy line ensemble, a collection of continuous, non-intersecting curves whose top line is \mathcal{L} .

The Airy₂ process $\mathcal{A} : \mathbb{R} \rightarrow \mathbb{R}$ is a stationary process first introduced by Prähofer and Spohn [PS02] in a scaled description of the polynuclear growth (PNG) model; or equivalently, of another famous last passage percolation model, Poissonian LPP. It was defined in [PS02] in terms of its finite-dimensional distributions—written via determinantal formulas involving the Airy kernel—and shown to have a continuous version. Its basic role in KPZ universality may be expressed via the scaled “narrow wedge” interface $L_n : \mathbb{R} \rightarrow \mathbb{R}$ for various KPZ models, where n is an abstract parameter measuring the size of the system in the model. It is a major open problem to rigorously show in a broad class of models that the distributional high- n limit, with respect to the topology of locally uniform convergence on the space of continuous functions with domain and co-domain the real line, of the process $x \mapsto L_n(x)$ (after adopting model-dependent constants in the scalings) is $\mathcal{A}(x) - x^2$, a parabolically shifted Airy₂ process.

This inference has recently been proven for the KPZ equation [QS20; Vir20], and had been validated for several stochastic growth models with narrow wedge initial data beyond PNG. These models include the totally asymmetric simple exclusion process (TASEP) [BF08] as well as the model that will be the principal object of rigorous attention in this monograph, namely Brownian LPP. In the latter case, the convergence is proved via a distributional relation with Dyson Brownian motion that will be reviewed in Section 3.2.

1.1.1. Locally Brownian nature of the limiting process. — In the case of Brownian LPP, as mentioned, the limiting process is $\mathcal{A}(x) - x^2$. We define

$$\mathcal{L}(x) := 2^{-1/2} (\mathcal{A}(x) - x^2),$$

and call it the *parabolic Airy₂ process*, in spite of the factor of $2^{-1/2}$. The factor $2^{-1/2}$ is included to make comparisons with Brownian motion more convenient and will be made clearer momentarily.

The limiting process globally adopts a parabolic form, but it is locally Brownian—see Figure 1.1. The term “locally Brownian” may be interpreted in several ways, with a progression to stronger forms of interpretation, reflecting recent progress in understanding this limiting scaled profile. “Locally Brownian” could mean that, for any given $x \in \mathbb{R}$, the distributional process limit of $y \mapsto \varepsilon^{-1/2} (\mathcal{L}(x + \varepsilon y) - \mathcal{L}(x))$ as $\varepsilon \searrow 0$ is a standard Brownian motion (where it is the presence of the factor $2^{-1/2}$ in the definition of \mathcal{L} that permits the diffusion rate to equal one, as \mathcal{A} itself is locally of rate two). Hägg proved such a convergence for the finite-dimensional distributions [Häg08] by analysing the determinantal structure possessed by the Airy₂ process.

“Locally Brownian” could mean the stronger assertion that the process $\mathcal{L} - \mathcal{L}(a)$, when restricted to any given compact interval $[a, b]$, is absolutely continuous with respect to standard Brownian motion on this same interval. This probabilistic assertion does not seem amenable to approaches that utilise directly the determinantal structure of the Airy₂ process. It was proved in [CH14] by a technique that lies at the heart

of the investigation of the present monograph. This Brownian Gibbs technique involves embedding the process $\mathcal{L} : \mathbb{R} \rightarrow \mathbb{R}$ as the uppermost curve in an infinite system of random continuous non-intersecting curves, called the parabolic Airy line ensemble (see the right panel of Figure 1.1). The ensemble of curves may be viewed as a growing system of mutually avoiding rate one Brownian bridges viewed at their edge (i.e., any fixed number of the uppermost curves as the number of curves in the system grows). This implies that the uppermost curves have an attractive Gibbs resampling property involving the rejection sampling of independent Brownian bridges on a condition of avoidance of the lower curves.

However, to say merely that one measure ν is absolutely continuous with respect to another μ is to leave unquantified the relation between ε and δ in the assertion that $\mu(A) < \varepsilon$ implies that $\nu(A) < \delta$.

A quantitative notion of “locally Brownian” would be that the Radon-Nikodym derivative of $\mathcal{L} - \mathcal{L}(a)$ on $[a, b]$ with respect to standard Brownian motion on the same interval lies in $L^{\infty-}$, i.e., in every L^p space, for $p \geq 1$; if so, it can be seen that an event whose Brownian motion probability is a low value $\varepsilon \in (0, 1)$ would have probability at most $\varepsilon^{1-o(1)}$ for $\mathcal{L} - \mathcal{L}(a) : [a, b] \rightarrow \mathbb{R}$. Our principal result, as it applies to limiting structure, establishes that this is the case.

Theorem 1.1. — *For fixed $d \geq 1$, let \mathcal{C} be the space of continuous functions on the interval $[-d, d]$ that vanish at $-d$. Let $A \subseteq \mathcal{C}$ be Borel, and let $\varepsilon = \mathcal{B}(A)$, where \mathcal{B} is the law of rate one Brownian motion on $[-d, d]$. There exists $\varepsilon_0 = \varepsilon_0(d) > 0$ and an absolute finite constant G , such that, if $\varepsilon \in (0, \varepsilon_0)$, then*

$$\mathbb{P}\left(\mathcal{L}(\cdot) - \mathcal{L}(-d) \in A\right) \leq \varepsilon \cdot \exp\left(Gd(\log \varepsilon^{-1})^{5/6}\right).$$

Theorem 1.1 is the culmination of a probabilistic study of narrow wedge KPZ structure across several papers, including [CH14] and [Ham19b]. The form of the Brownian comparison made by this result and by its upcoming prelimiting counterpart Theorem 3.10 is strong enough to open up an exciting array of applications concerning KPZ universality and last passage percolation models. There are six applications that we discuss in this monograph—two or three are simple and direct, while our treatment of another will take the form of a summary of upcoming work in which the Brownian comparison proved here will play a foundational role; and the final two concern not narrow-wedge but *general* initial conditions, which we will discuss in Section 1.4.

We want to indicate promptly several of these applications. In the next section, we introduce the Brownian LPP model for which our main result, Theorem 3.10, proves a similar prelimiting and uniform-in- n version of Theorem 1.1. This model will also aid in our presentation of applications in Section 1.2.

The road to results such as Theorem 1.1 via the probabilistic Brownian Gibbs technique begins with the absolute continuity comparison made by [CH14]. An important intermediate step was achieved in [Ham19b], in which a comparison formally very similar to that made by Theorem 1.1 was made, but with the compared processes being affinely shifted so that their interval endpoint values vanish. The relation of Theorem 1.1 to this counterpart result in [Ham19b] is important both for its formal similarity and its striking differences; and for the technique of proof. Indeed, our proof of Theorem 1.1 will harness a substantial part of the technical apparatus of [Ham19b], but employ it in a substantially new manner. We turn to a more detailed overview of previous work in Section 1.3 and of the relations of the present work to that of [Ham19b] in the final section of the introduction, Section 1.5.

1.1.2. A further overview, via Brownian LPP. — In Brownian LPP we are given a \mathbb{Z} -indexed collection of independent and identically distributed two-sided rate one Brownian motions $B : \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}$, which we regard as a random environment. We then consider *directed paths* in this random environment. For $i, j \in \mathbb{Z}$ with $i \leq j$ and $x, y \in \mathbb{R}$ with $x \leq y$, a directed path γ from (x, i) to (y, j) is parametrized by its *jump times* $t_i < \dots < t_{j-1}$, where t_k is the location at which the path jumps from line k to line $k + 1$. See Figure 1.2.

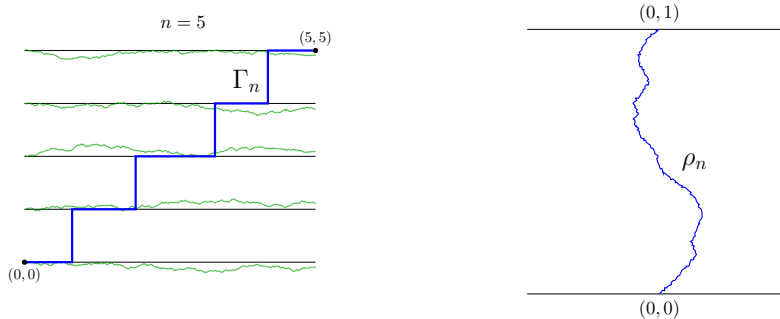


FIGURE 1.2. On the left panel is a depiction of the Brownian environment in green, with a blue geodesic path Γ_n from $(0,0)$ to (n,n) indicated; here $n = 5$. On the right panel is the polymer ρ_n after the scaling map S_n has been applied to the geodesic (not the same as in the first panel); here n is unspecified but large. The right panel is simply a depiction and not a simulation.

A given directed path γ from (x, i) to (y, j) is assigned a weight based on the values of the random environment that it passes through, given by

$$B[\gamma] = \sum_{k=i}^j (B(k, t_k) - B(k, t_{k-1})),$$

where for notational convenience we set $t_{i-1} = x$ and $t_j = y$. In words, we sum the increments of the Brownian motions along the path, which can be regarded as a path integral through the noise.

As in all LPP models, we now fix a starting and ending point and *maximize* the path weight over all directed paths between these two points. More precisely, the LPP value between (x, i) and (y, j) is denoted $M[(x, i) \rightarrow (y, j)]$ and is defined by

$$(1) \quad M[(x, i) \rightarrow (y, j)] = \sup_{\gamma: (x, i) \rightarrow (y, j)} B[\gamma].$$

We also adopt the shorthand $M_n = M[(0,0) \rightarrow (n,n)]$. The maximum energy M_n is attained by a *geodesic* Γ_n from $(0,0)$ to (n,n) . This is the directed path between these endpoints whose energy equals M_n . It is easy to see that, by the non-atomic nature of the noise, the geodesic between two fixed points is almost surely unique.

The Brownian LPP model is known to lie in the KPZ universality class. Indeed, if we set $a_n = n^{-1} \mathbb{E} M_n$, a simple subadditivity argument yields the existence of the leading order growth coefficient $a = \lim a_n$. It has been proven that $a = 2$ (via a relation to Dyson Brownian motion, see Section 3.2). If we further write

$$(2) \quad M_n = 2n + 2^{-1/2} n^{1/3} \text{Wgt}_n,$$

the random *weight* Wgt_n is a measure of the scaled fluctuation of the maximum energy for the route $(0,0) \rightarrow (n,n)$; this is the analogue of the observable L_n mentioned in the context of general KPZ models above. Indeed, the system of random variables $\{\text{Wgt}_n : n \in \mathbb{N}\}$ is tight, and, after scaling by a further constant, converges to the GUE Tracy-Widom distribution (the constant is $2^{1/2}$, which cancels the $2^{-1/2}$ in the definition of Wgt_n that we have introduced to match the same factor in the definition of \mathcal{L} above); see Section 3.2 for a discussion. Similar statements are expected to hold for a broad class of LPP models.

Just as the weight Wgt_n offers a scaled description of energy, it is natural to represent the geodesic Γ_n in scaled coordinates, as a path that traverses a unit-order distance while making random fluctuation also of unit order. Since the endpoints are separated by n on the horizontal axis, we will also need to make an affine transformation before scaling to ensure that the scaled path moves at most unit order in both directions.

To set up such a description, consider the scaling map $S_n : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that scales horizontally by $n^{2/3}$ and vertically by n after doing an affine shearing, i.e., the unique affine map that is such that $(0, 0)$ is mapped to itself; (n, n) is mapped to $(0, 1)$; and $(2n^{2/3}, 0)$ is mapped to $(1, 0)$. If we treat any directed path in $\mathbb{Z} \times \mathbb{R}$ as a subset of \mathbb{R}^2 by viewing it as the union of the nearest-neighbour edges that it crosses, then we may set ρ_n equal to the image under S_n of the geodesic Γ_n . This object ρ_n is then the image of a piecewise affine curve in \mathbb{R}^2 that connects $(0, 0)$ and $(0, 1)$. We call it a *polymer*, and regard it as a scaled version of Γ_n . (The factor of 2 in the horizontal scaling is merely to ensure that we obtain the parabolic Airy_2 process as a suitable limit of an object we will shortly introduce.)

(This use of the terms weight and polymer is hardly standard, but agrees with the terminology in [Ham19b]. Indeed, in the literature the term polymer often refers to realizations of the path measure in last passage percolation models at *positive* temperature, unlike the zero temperature case discussed here.)

We may further set $\text{Wgt}_n[(0, 0) \rightarrow (0, 1)] = \text{Wgt}_n$ and $\rho_n[(0, 0) \rightarrow (0, 1)] = \rho_n$ with a view to generalising these objects into a broader scaled description of geodesics and their energies. Indeed, for $x, y \in \mathbb{R}$, we may specify $\rho_n[(x, 0) \rightarrow (y, 1)]$ to be the image under S_n of the geodesic that runs between $S_n^{-1}(x, 0) = (2n^{2/3}x, 0)$ and $S_n^{-1}(y, 1) = (n + 2n^{2/3}y, n)$. The polymer $\rho_n[(x, 0) \rightarrow (y, 1)]$ has weight $\text{Wgt}_n[(x, 0) \rightarrow (y, 1)]$ given by the scaled expression for the energy of the mentioned geodesic. That is, we set

$$(3) \quad M[(n^{2/3}x, 0) \rightarrow (n + n^{2/3}y, n)] = 2n + 2(y - x)n^{2/3} + 2^{-1/2}n^{1/3}\text{Wgt}_n[(x, 0) \rightarrow (y, 1)],$$

which generalizes the definition of Wgt_n given in (2).

We will often be interested in the situation when the starting point is held fixed at the origin, i.e., $x = 0$. For future reference, let us highlight a supremum formula for $\text{Wgt}_n[(0, 0) \rightarrow (y, 1)]$. For $w \in (0, 1)$,

$$(4) \quad \text{Wgt}_n[(0, 0) \rightarrow (y, 1)] = \sup_{x \in \mathbb{R}} \left(\text{Wgt}_n[(0, 0) \rightarrow (x, w)] + \text{Wgt}_n[(x, w) \rightarrow (y, 1)] \right).$$

This formula is easy to understand: the expression in the supremum is the weight of the best scaled path forced to pass through (x, w) , and, fixing w , the polymer $\rho_n[(0, 0) \rightarrow (y, 1)]$ will clearly pass through (x, w) for the best such x . We note also that the two terms inside the supremum are independent by the independence of increments of Brownian motion, and that, if we have also fixed y , both terms can be considered as weight profiles. In fact, as in this decomposition, the sum of two independent weight profiles often arises in LPP studies, and one of the applications we will discuss later, though not proved in this paper, extends Theorem 1.1 and its prelimiting version Theorem 3.10 to such objects.

The parabolic Airy_2 process $\mathcal{L} : \mathbb{R} \rightarrow \mathbb{R}$ is the description expected to arise in the limit of high n of the scaled energy of scaled LPP paths that, in accordance with narrow wedge initial data, emanate from the origin. Simply put, $\mathcal{L}(y)$ equals $\text{Wgt}_\infty[(0, 0) \rightarrow (y, 1)]$, where we set Wgt_∞ to be the putatively existing high n limit of the weight system Wgt_n .

With the example of Brownian LPP in mind, we may move to describing the applications of our main results.

1.2. Applications of Theorem 1.1

We have postponed the full statement of our main theorem as it requires the introduction of more general objects than just the parabolic Airy_2 process, one of which is the Brownian LPP weight profile $y \mapsto \text{Wgt}_n[(0, 0) \rightarrow (y, 1)]$. The full class of general objects arises naturally as associated weight profiles in Brownian last passage percolation, and we will define them in Section 3.1. It is because of this example of Brownian LPP that we have chosen to prove our results in a more general framework than Theorem 1.1, and we anticipate a number of future applications that make use of this general form.

Remark 1.2. — While the applications proved in this paper are stated mainly for the parabolic Airy₂ process, we point out that essentially the same statements can be obtained for the prelimiting Brownian LPP narrow-wedge weight profile in each case by replacing the usage of Theorem 1.1 in the proofs with our upcoming main result Theorem 3.10, which is applicable to Brownian LPP. This is simply because the underlying Brownian motion estimates which serve as input to Theorem 1.1 in the each of the application's proofs can equally well be used as input for Theorem 3.10.

1.2.1. Movement of Airy₂ in an interval. — An immediate application of Theorem 1.1 which illustrates its utility is the following corollary, which gives a tail bound on the amount the Airy process or its parabolic version moves in a unit-order interval.

Corollary 1.3. — *There exist $C < \infty$, $C' < \infty$ such that, for all $d \geq 1$, there exists $x_0 = x_0(d) > 0$ such that, for $x > x_0$,*

$$\mathbb{P} \left(\sup_{s \in [-d, d]} |\mathcal{L}(s) - \mathcal{L}(-d)| \geq x \right) \leq e^{-x^2/4d + Cd^{1/6}x^{5/3}}$$

and

$$\mathbb{P} \left(\sup_{s \in [-d, d]} |\mathcal{A}(s) - \mathcal{A}(-d)| \geq x \right) \leq e^{-x^2/8d + C'dx^{5/3}}.$$

Recall that \mathcal{L} and \mathcal{A} are respectively, in an idealised sense, rate one and rate two processes, and that we are considering the tail probability of an increment over an interval of length $2d$. For Brownian motions of rates one and two, these probabilities can be respectively understood as being roughly $\exp(-x^2/4d)$ and $\exp(-x^2/8d)$, and this accounts for the dominant terms in the exponents in the bounds in Corollary 1.3; the remaining terms in the exponents of the form $d^{1/6}x^{5/3}$ or $dx^{5/3}$ are sub-dominant corrections arising from Theorem 1.1.

We remark that estimates on similar quantities have previously appeared in the literature with a weaker tail bound exponent of $3/2$, instead of 2 as obtained here, such as in [DV21, Proposition 1.6] and [Ham19b, Theorem 2.14]; additionally, we obtain an explicit coefficient for the $-x^2$ term in the exponent, as well as a quantified sub-dominant correction. The two just cited estimates are for the prelimiting Brownian LPP weight profiles, denoted in this paper by $\text{Wgt}_n[(0, 0) \rightarrow (\cdot, 1)]$ but, as mentioned in Remark 1.2 above, the statement of Corollary 1.3 can be easily obtained for Wgt_n as well.

PROOF OF COROLLARY 1.3. Let B be standard Brownian motion on $[-d, d]$ started at zero. Writing $N(0, 2d)$ for a normal random variable with mean zero and variance $2d$, we see that, by the reflection principle for Brownian motion,

$$\mathbb{P} \left(\sup_{s \in [-d, d]} |B(s)| \geq x \right) \leq 2 \cdot \mathbb{P} \left(\sup_{s \in [-d, d]} B(s) \geq x \right) = 4 \cdot \mathbb{P} \left(N(0, 2d) \geq x \right) \leq 4 \cdot e^{-x^2/4d}.$$

The last inequality is due to the Chernoff bound. Now we apply Theorem 1.1 after letting x be large enough that the last quantity is less than the ε_0 from Theorem 1.1, and raise the value of G obtained from Theorem 1.1 further to absorb the multiplicative constant of 4, to get the first bound in Corollary 1.3. The second follows from the first by noting that $\sup_{s \in [-d, d]} |\mathcal{L}(s) - \mathcal{L}(-d)|$ differs from $2^{-1/2} \sup_{s \in [-d, d]} |\mathcal{A}(s) - \mathcal{A}(-d)|$ by at most $2^{-1/2}d^2$, and by bounding $xd/4 + Cd^{1/6}x^{5/3}$ by $C'dx^{5/3}$, where C' is defined by say $C' = C + 1$. \square

The proof of Corollary 1.3 illustrates that the usefulness of Theorem 1.1 lies in allowing us to use all of the many powerful probabilistic tools and symmetries available for Brownian motion and the normal distribution

in the problem of estimating the probabilities of very naturally arising events for the parabolic Airy₂ process. The remaining applications make more sophisticated use of Brownian motion.

1.2.2. A quantified local version of Johansson’s conjecture. — As we saw in the discussion on Brownian LPP in Section 1.1.2, the weight profile $\text{Wgt}_n[(0, 0) \rightarrow (y, 1)]$ captures the weight of the polymer path as the endpoint $(y, 1)$ varies. When the starting point $(0, 0)$ is held fixed, the parabolic Airy₂ process is the limiting weight profile process as n tends to infinity in a number of LPP models, and so it is of interest to understand its maximiser—this corresponds to maximum weight scaled paths with fixed starting points. Johansson conjectured in [Joh03] that the process \mathcal{L} defined above almost surely has a unique global maximiser, a fact that now has a number of proofs: Corwin and Hammond [CH14] establish this via comparison to Brownian motion; Moreno Flores, Quastel, and Remenik [FQR13] prove an explicit formula for the maximiser; and an argument of Pimentel [Pim14] shows that any stationary process minus a parabola has a unique maximiser.

In many LPP situations, it is of interest not only to look at the energy-maximising path, but also at paths which are *nearly* energy-maximising, as these would perhaps become the maximising path under a small perturbation of the random environment. So, it is useful to know that paths far away from the maximising path are not very close in energy to the maximum.

In terms of the parabolic Airy₂ process, this corresponds to understanding the occurrence of *near maxima* at some distance from the unique maximiser, and this is our next focus of study. In order to respect the Brownian scaling we expect to see on unit order intervals, the closeness of the near maximum energy we consider has a square root relationship with the magnitude of the separation we impose from the unique actual maximiser. We prove a result bounding the probability of an event of this type, which we refer to as a quantified local Johansson result.

Let $X : [-d, d] \rightarrow \mathbb{R}$ be a stochastic process with an almost surely unique maximiser in $[-d, d]$, and let this unique maximiser be x_{\max} . Let $M = X(x_{\max})$. For $a, \eta \in (0, 1)$, define the *near touch* event $\text{NT}(X, \eta, a)$ by

$$\text{NT}(X, \eta, a) := \left\{ \sup_{|z| \geq 1} X(x_{\max} + z\eta) \geq M - a\eta^{1/2} \right\},$$

where for $x \notin [-d, d]$ we take $X(x) = -\infty$ by convention. We omit the d -dependence of the event in the notation.

Theorem 1.4 (A quantified local version of Johansson’s conjecture)

Let $d \geq 1$. There exist $a_0 = a_0(d)$ and a universal $G < \infty$ such that, for $a \in (0, a_0)$ and $\eta \in (0, 1)$,

$$\mathbb{P}(\text{NT}(\mathcal{L}, \eta, a)) \leq a \cdot \exp\left(Gd(\log a^{-1})^{5/6}\right).$$

Note that when we consider the event $\text{NT}(\mathcal{L}, \eta, a)$, we do not say that the global maximiser lies in $[-d, d]$; we are considering only near touches with the maximiser when restricted to $[-d, d]$. We note the estimate [CH14, Corollary 4.6], which bounds the probability that the global maximiser lies outside a given interval centred at 0. By taking large d , applying this estimate, and using the parabolic curvature of \mathcal{L} , along with a union bound, it is plausible that Theorem 1.4 can be extended to a statement about the global maximiser. However, we do not pursue this line of reasoning further here.

Theorem 1.4 is proved in Section 2. The idea of the proof is again to make strong use of information about Brownian motion. We will use the classical decomposition of Brownian motion around its maximiser in terms of independent Brownian meanders and the explicit transition probability formulas for the latter process.

1.2.3. Many near touch. — In the previous result we bounded the probability of occurrence of a near maximum as a function of the degree of nearness. Our next application provides a bound on a related quantity, the *number* of near maxima.

In order to have a reasonable notion of the number of near maxima, we define for $d \geq 1$ the *number of near touches* random variable $\text{NumNT}(X)$ for a random process $X : [-d, d] \rightarrow \mathbb{R}$ as follows, where $M = \sup_{t \in [-d, d]} X(t)$:

$$\text{NumNT}(X, \eta) = \max \left\{ |S| : \begin{array}{l} S \subseteq [-d, d], \text{ s.t. } s \in S \implies X(s) \geq M - \eta^{1/2} \\ \text{and } s, t \in S, s \neq t \implies |s - t| \geq \eta. \end{array} \right\}.$$

In words, this quantity is the size of the maximum collection of η -separated times at which X comes within $\eta^{1/2}$ of its global maximum on $[-d, d]$.

The following result says that the number of near maxima has exponential tails.

Theorem 1.5 (Many near touch). — *Let $d \geq 1$ and $0 < \eta < d$. Then there exists $\ell_0 = \ell_0(d)$ and a universal $c > 0$ such that, for $\ell > \ell_0$,*

$$\mathbb{P}(\text{NumNT}(\mathcal{L}, \eta) \geq \ell) \leq e^{-c\ell}.$$

We again note that the stated result studies near maxima with respect to the maximiser on $[-d, d]$. However, unlike Theorem 1.4, the result proved implies the same bound for the number of near touches in $[-d, d]$ of the *global* maximiser, i.e., the random variable with the same definition as $\text{NumNT}(\mathcal{L}, \eta)$ but with M being the global maximum value. This is simply because if $x \in [-d, d]$ is a point of near touch with the global maximum, it must also be a point of near touch with the maximum on $[-d, d]$. Bounding the number of near touches of the global maximiser which occur anywhere, not necessarily in $[-d, d]$, is again likely to be tractable using [CH14, Corollary 4.6] and the parabolic curvature of \mathcal{L} .

Bounding the number of near maxima in the model of exponential LPP proved to be an important ingredient in [SSZ21]. This paper proves a quantitative version of the sensitivity to microscopic perturbations in the “slow bond” problem, which was established in the paper [BSS14]. A result qualitatively similar to Theorem 1.5, but with a weaker tail decay and in the model of exponential LPP, was established in [SSZ21] after the first posting of the present manuscript.

As with Theorem 1.4, the proof of Theorem 1.5 is given in Chapter 2 and relies on bounding the probability for the same event under Brownian motion using information about Brownian meander.

The next application we discuss involves Brownian LPP and is not proved in this paper; as such, we aim at giving only a heuristic description underscoring the points of interest.

1.2.4. Brownianity of the routed weight profile. — Recall the discussion around the supremum formula (4) for Wgt_n . Let $w \in (0, 1)$ be a height in the scaled copy of the plane through which the polymer $\rho_n = \rho_n[(0, 0) \rightarrow (0, 1)]$ passes. Define the *routed weight profile* $Z_w : \mathbb{R} \rightarrow \mathbb{R}$ indexed by height w to be the function that reports the highest weight of a scaled path that is forced to pass through a given location at height w . That is, we set

$$(5) \quad Z_w(x) = \text{Wgt}_n[(0, 0) \rightarrow (x, w)] + \text{Wgt}_n[(x, w) \rightarrow (0, 1)],$$

the right-hand side being the maximum scaled energy of a scaled path that begins at $(0, 0)$; ends at $(0, 1)$; and visits (x, w) . Thus, $Z_w(x)$ is analogous to (what may be called) the *supremand* in (4). Note that by (4), any point in ρ_n of the form (x, w) is such that $Z_w(x)$ achieves the maximum value of Z_w . A natural question in the study of near polymers is: “how probable is it that a near maximiser of Z_w exists at a significant

distance from the maximiser?” The two right-hand terms in (5) locally resemble standard Brownian motion in the sense of Theorem 1.1 (though as we are discussing a weight profile in Brownian LPP, this requires the upcoming main result, Theorem 3.10); so Z_w plausibly resembles Brownian motion of rate two.

For example, this intuition suggests that the probability of occurrence of a near maximiser of Z_w should obey a similar bound as in Theorem 1.4. That is, if x is a maximiser of Z_w , the probability that $Z_w(y)$ attains a value that exceeds $Z_w(x) - \alpha h^{1/2}$ at a location $y \in \mathbb{R}$ for which $|y - x| \geq h$ should be at most α , up to a multiplicative error that rises subpolynomially in the limit $\alpha \searrow 0$. This is in fact true, and will follow from the next result in the same way that Theorem 1.4 follows from Theorem 1.1.

Theorem 1.6. — *Let w lie in a compact interval in $(0, 1)$, $d \geq 1$, and \mathcal{C} be as in Theorem 1.1. The routed weight profile $Z_w : \mathbb{R} \rightarrow \mathbb{R}$ is strongly comparable to rate-two Brownian motion B on $[-d, d]$. That is, there exists $G < \infty$ and $g > 0$ such that if A is a Borel measurable subset of \mathcal{C} for which $\mathbb{P}(B \in A) = \varepsilon \in (0, 1)$, and if ε is both sufficiently small and greater than $\exp(-gn^{1/12})$, then $\mathbb{P}(Z_w - Z_w(-d) \in A) \leq \varepsilon \exp\{Gd(\log \varepsilon^{-1})^{5/6}\}$.*

This theorem will appear in [GH20], where it will be derived from our main result, Theorem 3.10.

The stretched-exponential-in- n lower bound condition on ε expressed in Theorem 1.6 is an artifact of Theorem 3.10, which is the prelimiting counterpart of Theorem 1.1 that is valid for Brownian LPP. In most applications the condition is irrelevant, as events of interest do not usually have probabilities which decay faster than polynomially in n , (and sometimes do not decay with n at all). The condition does not appear in Theorem 1.1 because there n takes the value ∞ , in which case the lower bound condition is vacuously true.

Recall that we have till now focused on the narrow wedge initial condition, which leads to the parabolic Airy_2 process in the limit. The last two applications will be discussed in Section 1.4, after we have described a form of Brownian regularity for scaled energy profiles begun from *general* initial conditions. But we first turn to reviewing work in whose purview this article falls.

1.3. Pertinent recent work

We start by giving a brief account of the general background of the Airy_2 process. The interested reader is referred to the survey [QR14] for a more detailed review, though from the slightly different viewpoint of integrable probability.

The one-point distribution of the stationary Airy_2 process is the GUE Tracy-Widom distribution, first discovered in random matrix theory as the distribution of the limiting scaled fluctuations of the largest eigenvalue of the Gaussian Unitary Ensemble [TW94]. A breakthrough in the field of KPZ was Baik, Deift, and Johansson [BDJ99] proving that the same GUE Tracy-Widom distribution arises as the distribution of the limiting scaled fluctuations of the point-to-point energy in Poissonian LPP, through an equivalent description in terms of the longest increasing subsequence of a uniform random permutation.

The jump from the GUE Tracy-Widom distribution to the full Airy_2 process was made in [PS02], where it was shown that the weight profile in Poissonian LPP (which has a bijection with the PNG model) converges weakly to the Airy_2 process minus a parabola, in the sense of finite dimensional distributions. The convergence was strengthened to hold on the space of continuous functions in a closely related model in [Joh03].

As we mentioned above, the locally Brownian nature of the Airy_2 process has been previously established in a number of different formulations. One is that of *local limits* of the Airy_2 process; i.e., that $\varepsilon^{-1/2}(\mathcal{A}(x + \varepsilon) - \mathcal{A}(x))$ for a given $x \in \mathbb{R}$ as $\varepsilon \searrow 0$ is Gaussian. The appearance of Brownian motion in this limit was proven in [Häg08; CP15; QR13]. Additionally, [QR13] establishes Hölder $\frac{1}{2}$ -continuity of the

Airy_2 (as well as Airy_1) process, which is extended to limiting weight profiles arising from a very general class of initial conditions in [MQR21, Theorem 4.13]. A stronger notion of the locally Brownian nature of the Airy_2 process is absolute continuity of \mathcal{A} with respect to Brownian motion on a unit order compact interval. This was first proved in [CH14], where it was used to prove Johansson’s conjecture mentioned above.

Another line of work has established various Brownian features in the *pre-limiting* weight profiles. For instance, [BG21] establishes local Brownian fluctuations (in the sense of sub-Gaussian tails) in the weight profile of point-to-point exponential LPP, while [Ham19c] establishes a (sharp) version of the Holder $\frac{1}{2}$ –continuity mentioned above for the pre-limiting weight profiles in Brownian LPP (which also applies with quite general initial conditions).

However, none of these results addresses the question of bounding probabilities involving the Airy_2 process in terms of Brownian probabilities, or, equivalently, providing growth bounds on the Radon-Nikodym derivative with respect to some Brownian process.

A result in this direction was proved in [Ham19b]. There the comparison was between a modification of \mathcal{L} , denoted $\mathcal{L}^{[-d,d]}$, that is defined by affinely shifting \mathcal{L} to be zero at both endpoints of $[-d,d]$, and Brownian bridge, instead of between a vertically shifted version of \mathcal{L} and Brownian motion as in Theorem 1.1. The form of the result, however, is otherwise much the same:

Theorem 1.7 (Theorem 1.10 of [Ham19b]). — *Let $d \geq 1$ and let $\mathcal{C}_{0,0}$ be the space of continuous functions which vanish at both endpoints of $[-d,d]$. Let $A \subseteq \mathcal{C}_{0,0}$ be Borel, and let $\varepsilon = \mathcal{B}^{[-d,d]}(A)$, where $\mathcal{B}^{[-d,d]}$ is the law of Brownian bridge of rate one on $[-d,d]$ (i.e., with vanishing endpoints). There exists $\varepsilon_0 = \varepsilon_0(d) > 0$ and an absolute finite constant G such that, if $\varepsilon \in (0, \varepsilon_0)$, then*

$$\mathbb{P}\left(\mathcal{L}^{[-d,d]}(\cdot) \in A\right) \leq \varepsilon \cdot \exp\left(Gd^2(\log \varepsilon^{-1})^{5/6}\right).$$

This also follows immediately from Theorem 1.1 and the fact that performing the affine shift described on Brownian motion results in Brownian bridge.

Theorem 1.7 and our new Theorem 1.1 are formally very similar, the latter obtained merely by substituting Brownian motion for Brownian bridge. However, it is found in many contexts that Theorem 1.7 is unable to provide the kind of information that is desired. This is because, though the process $\mathcal{L}(\cdot) - \mathcal{L}(-d)$ can be obtained from the bridge $\mathcal{L}^{[-d,d]}(\cdot)$ and the endpoint $\mathcal{L}(d)$, the desired information escapes our grasp due to potentially pathological correlations between these two random objects. Controlling this correlation is especially required to understand the slope or maximum of \mathcal{L} on an interval; the slope or maximum are often of relevance in LPP problems, as can be seen in the applications discussed in Section 1.2.

The proof of Theorem 1.1 is significantly more involved and subtle than that of Theorem 1.7 in [Ham19b] because of the need to handle these correlations. We comment more and contrast the proofs in Section 1.5.

Theorem 1.7 was a crucial tool in the four-part study of Brownian LPP undertaken in [Ham19b; Ham19c; Ham20; Ham19a]. In the final paper [Ham19a], a form of Brownian regularity was proved for pre-limiting weight profiles for general initial conditions, to which we return shortly. But we first turn to discussing the Brownian Gibbs property, a crucial idea in the proofs of Theorem 1.7 as well as our own main result.

1.3.1. The Brownian Gibbs property. — A central player in our approach is the *Brownian Gibbs property*. The Brownian Gibbs property was first employed in [CH14], to study the *Airy line ensemble*. The Airy line ensemble is an \mathbb{N} -indexed collection of continuous, non-intersecting curves, whose uppermost curve is the Airy_2 process. The Brownian Gibbs property is an explicit spatial Markov property enjoyed by the Airy line ensemble after a parabolic shift and multiplication by a factor $2^{-1/2}$, resulting in the *parabolic Airy line ensemble*. In short, the Brownian Gibbs property says that the conditional distribution of any k

consecutive curves on an interval $[a, b]$, conditionally on all the other curves on all of \mathbb{R} and the k curves themselves on $(a, b)^c$, is given by k independent rate one Brownian bridges between appropriate endpoints and conditioned to intersect neither each other nor the preceding and succeeding curves.

The Brownian Gibbs property and various softening of it have proved to be a versatile tool in probabilistic investigations of KPZ. Beyond the already mentioned [CH14], there have been numerous works on line ensembles enjoying this or an analogous property, which we briefly discuss.

The Brownian Gibbs property was a central theme in the previously mentioned four-part study [Ham19b; Ham19c; Ham20; Ham19a] of Brownian LPP. While [CH14] established that the Brownian Gibbs property is enjoyed by the Airy line ensemble, and hence by the limiting weight profiles in a number of LPP models, Brownian LPP is special in that its weight profile satisfies the Brownian Gibbs property even in the *pre-limit*. This is a crucial integrable input first observed by [OY02] (who related the energy profiles in Brownian LPP to Dyson Brownian motion), and is the reason why Brownian LPP is the setting of the mentioned four-part study, as well as why our main results will apply to it. Apart from this four-part study, we mention some other works in this vein. The work [CS14] establishes the ergodicity of the Airy line ensemble using the Brownian Gibbs property. The fractal nature of a certain limiting weight difference profile in Brownian LPP is investigated in [BGH21], using inputs from the four-part study mentioned earlier. The Brownian Gibbs property is used in [CIW19; CIW18] to analyse tightness of families of non-intersecting Brownian bridges above a hard wall, subject to a tilting of measure via the area that the curves capture below them; these papers also show that an area-tilted form of the Brownian Gibbs property holds for the limiting ensemble.

A softened version of Brownian Gibbs, in which intersection is not prohibited but suffers an energetic penalty, was used in an investigation of the *scaled* solution to the KPZ equation with narrow-wedge initial condition [CH16], establishing absolute continuity of that process with respect to Brownian motion on compact intervals. This form of Brownian Gibbs was also used in [CG20] to obtain bounds on the one-point upper and lower tails for the solution to the KPZ equation from quite general initial data, and in [CGH21] to establish the rate of correlation decay with time of the narrow wedge solution at the origin. A discrete Gibbs property was used in [CD18] to study the transversal fluctuation exponent and tightness of the appropriately scaled height function in the asymmetric simple exclusion process and stochastic six vertex model, started with step initial conditions. A sequence of discrete line ensembles associated to the inverse gamma directed polymer, which obeys a softened discrete version of the Brownian Gibbs property, was shown to be tight in [Wu19].

Finally, we mention the valuable contribution [DOV22], aided by [DV21], which establishes the existence of the space-time Airy sheet using Brownian LPP and the Brownian Gibbs property. We shall say more on this in the next subsection on limiting weight profiles from general initial conditions.

1.3.2. The KPZ fixed point and the directed landscape. — As mentioned, the Airy_2 process arises as a limiting process under very particular initial conditions, often called narrow-wedge. Given our knowledge of the strong comparison to Brownian motion that the Airy_2 process enjoys via Theorem 1.1, a natural question is whether such a comparison extends to the limiting processes arising from general initial conditions.

Of course, before discussing the Brownian regularity of a limiting profile under general initial conditions, we must show that such a limiting process exists. Two recent results are pertinent to this basic question.

The first is the construction in [MQR21] of a scale invariant Markov process, the *KPZ fixed point*⁽¹⁾, in the context of the totally asymmetric simple exclusion process (TASEP) model. In this work, the authors establish formulas for the finite dimensional distributions of the height function under very general initial data, which, on taking the scaling limit according to KPZ exponents, results in a scale invariant Markov process of central importance in the KPZ class. Simply put, the time-one evolution of the Markov process of [MQR21] on given initial data coincides with the limiting weight profile begun from the same initial condition. (A technical point is that [MQR21] works in TASEP and not Brownian LPP, and so the objects are not a priori the same; however, [NQR20] has now established the equivalence by performing the construction via Brownian LPP.) For example, the time-one evolution of this limiting Markov process begun at the narrow-wedge initial condition (i.e., step initial condition for TASEP) results in the parabolic Airy₂ process. With the results of [NQR20], our results can say something about the time-one evolutions of this Markov process, and we will discuss this soon.

But first we discuss the recent advance, made by Dauvergne, Ortmann, and Virág in [DOV22] and assisted by [DV21], which proves the existence of the *space-time Airy sheet*. The space-time Airy sheet is a previously conjectured [CQR15] universal object in the KPZ universality class and is a process with two temporal and two spatial arguments; these arguments should be thought of as the coordinates of a pair of planar points, with each point's coordinates consisting of one spatial and one temporal argument. For fixed values of the temporal arguments, i.e., restricting the mentioned points to lie on two fixed lines, the marginal of the space-time Airy sheet in either of its two spatial arguments is the Airy₂ process. The marginal spatial process when the temporal arguments take fixed values is sometimes referred to as simply the Airy sheet.

The pre-limiting model used in the construction of the Airy sheet in [DOV22] is Brownian last passage percolation, and an analysis of the bulk behaviour of curves deep in the parabolic Airy line ensemble is undertaken in [DV21] in a manner that assists the construction of the scaling limit of Brownian LPP, i.e., the process limit of $\text{Wgt}_n[(x, s) \rightarrow (y, t)]$ as a function of all four arguments $x, s, y,$ and t ; this scaling limit is called the *directed landscape* in [DOV22] and is a parabolically shifted space-time Airy sheet. The proof of the construction of this scaling limit in [DOV22] proceeds via equating last passage percolation values in the original environment to an LPP problem in a new environment, which is defined by last passage values in the original environment—a novel extension of the Robinson-Schensted-Knuth correspondence. Using this result, it follows that limiting weight profiles from general initial conditions exist, and we may further say something about its Brownian regularity using our results. We move in this direction next.

1.4. A form of Brownian regularity for weight profiles with general initial conditions

In this section we describe the two remaining applications of Theorem 1.1. As was just mentioned, a fairly direct consequence of the results of [DOV22] is the existence of the limiting weight profile from general initial conditions in Brownian LPP (Proposition 6.6). A natural next question is whether these general limiting weight profiles enjoy a similar comparison to Brownian motion as provided by Theorem 1.1 for the limiting narrow wedge weight profile, i.e., the parabolic Airy₂ process. In fact, convergence of the *local* limit to Brownian motion was already known in different senses for various classes of initial conditions; for example, this Brownian local limit was shown for the Airy₁ process in [QR13] in the sense of finite dimensional distributions; for a class of Airy processes arising from quite general initial conditions as constructed in

⁽¹⁾The usage of this term in our title mildly corrupts that in [MQR21]. In speaking of Brownian structure in the KPZ fixed point, we refer to Brownian structure in the random function obtained by evolving the Markov process in question from given initial data, whether narrow wedge or more general. Properly, and as used in [MQR21], the KPZ fixed point is the Markov process, rather than the resulting profile.

[MQR21, Theorem 3.13], again in the sense of finite dimensional distributions, in [MQR21, Theorem 4.14]; while the same local limit in the space of continuous functions has been proven in [Pim18] under an assumption, which is verified for certain specific initial conditions (such as the mixed profile flat \rightarrow stationary).

General initial conditions do not enjoy the algebraic properties present in the narrow wedge and the other previously studied cases mentioned, and so results are much weaker or absent for those profiles. In particular, though it is believed that even with general initial condition the limiting weight profile should be locally Brownian in a strongly quantifiable sense, it seems difficult to establish something similar to Theorem 1.1.

Nevertheless, it is possible to use the general form of Theorem 1.1 to prove a form of Brownian regularity similar to one introduced in [Ham19a]. But before addressing the Brownian regularity of the limiting profile, we look at the pre-limiting profile, as this has a direct interpretation as the weight associated to polymers.

1.4.1. Discovering a polymer forest and a patchwork quilt. — We return to the model of Brownian LPP to illustrate what is meant by general initial conditions. The initial condition is described by a function $f : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$, and for such a fixed f , we consider scaled (via the scaling function S_n) paths which may begin anywhere on the real line at time zero, but must end at $(y, 1)$. The f -rewarded weight of the path is the sum of f evaluated at the starting point—a reward—and the weight collected by the path on its journey. The maximum f -rewarded weight over all such paths is denoted $\text{Wgt}_n^f[(*, 0) \rightarrow (y, 1)]$. More precisely,

$$\text{Wgt}_n^f[(*, 0) \rightarrow (y, 1)] := \sup \{ \text{Wgt}_n[(x, 0) \rightarrow (y, 1)] + f(x) : x \in \mathbb{R} \};$$

the $*$ in the notation is to indicate that the starting point is free. A mild condition that f grows at most linearly is needed to make this object suitable for study.

The narrow-wedge initial condition we have been focusing on thus far corresponds to the case that $f(x)$ is zero when x is zero, and $-\infty$ otherwise. In that case, we saw that the weight profile has a globally parabolic shape, and that, in the $n \rightarrow \infty$ limit, it converges to the parabolic Airy₂ process. What will the weight and limiting weight profiles look like when f is more general?

For example, suppose that we set $f(x)$ to be zero for $x = 0$ and $x = 1$ and $-\infty$ everywhere else; in other words, growth must be from either of the starting locations $x = 0$ or $x = 1$. For each ending point $(y, 1)$, we may trace back the polymer ending there and observe at which starting point it originated. In fact, the polymers originating from either of the points 0 and 1 will form a tree structure; the energy-maximising objective means that the trees can be viewed as *competing*. Within the canopy of a single tree, where its weight is dominant, we can expect the profile to look like that of growth from a single point (see Figure 1.3).

Thus we may surmise that in this situation of growth from two points, the full weight profile can be seen as a piecewise function, where each piece (called a *patch*) has the distribution of growth from a single point; each patch can be expected to enjoy a certain Brownian regularity similar to that in Theorem 1.1. Under fully general initial conditions, where growth may be from any point on the lower line with a certain reward associated to each point that is added to the energy of the path, essentially the same picture holds: the weight profile can be broken up into a number of patches, corresponding to the canopies of the surviving polymer trees. The only difference is that the number of patches will be random. (Actually, this description is slightly simplified: for technical reasons, the final patches will be sub-patches of the patches that we have described. The reader is referred to [Ham19a] for a fuller discussion.)

This leads us to the following notion of regularity for general weight profiles. The (pre-limiting) weight profile on a unit interval is divided into a random number of subintervals, the patches, with random boundary points in such a way that the restriction of the profile (called a *fabric piece*) to each patch enjoys a comparison to Brownian motion similar to the one described in Theorem 1.1, though perhaps in the weaker L^{p^-} form

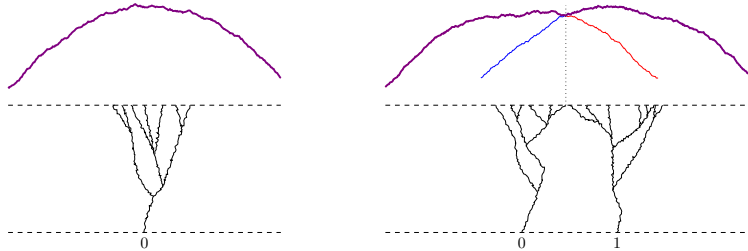


FIGURE 1.3. An illustration of polymer forest and the patchwork quilt. Growth is from a single point in the left figure, and from two points in the right one. In both figures, the actual weight profile is represented as a thicker curve in purple, and is the upper envelope. In the right figure, the two polymer trees compete, and the canopies of each are adjacent to each other; the boundaries of the canopies correspond to where the weight profile is divided into two patches. The fabric piece on each patch is the weight profile of growth from a single point, and are represented by the red and blue parabolic profiles, which become thicker and purple in the patch where they agree with the actual weight profile.

rather than $L^{\infty-}$. In this way, we may say that the weight profile is a *patchwork quilt of Brownian fabrics*. The strength of the regularity of the patchwork quilt depends largely on the control available for the number of patches, but also on the Brownian motion regularity guaranteed for each fabric piece as specified by the value of p . A precise definition is provided in Definition 6.3. Proving an analogue to Theorem 1.1 would correspond to establishing this regularity with a *single* patch and with $p = \infty$. (We mention that since the original posting of this article, [SV21], building on [DOV22], proves absolute continuity to Brownian motion of the KPZ fixed point on compact intervals; in our language, a single patch, but only L^1 regularity.)

Using our main theorem, we are able to show that this notion of Brownian motion regularity holds for the pre-limiting weight profiles from a general class of initial conditions, with a certain decay on the number of patches. We will state this precisely in Chapter 6 as Theorem 6.4 after the relevant definitions are made. This result is a refinement of a result in [Ham19a], where the comparison of each fabric piece was made to Brownian bridge instead of Brownian motion, using a generalised form of Theorem 1.7 that was proved there.

It is natural to expect that this form of Brownian regularity for the pre-limiting weight profiles should pass to the limiting profile if it exists, as was remarked in [Ham19a]. With the existence of the limiting profile established (in Proposition 6.6) using the result of [DOV22], we also show that this notion of regularity passes to the limit (denoted Wgt_{∞}^f), which we state informally now and formally as Theorem 6.11. This theorem may be seen as the most general form of Brownian regularity in the KPZ fixed point proved here.

Theorem 1.8 (Informal version of Theorem 6.11). — *Let $f : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$ be such that there exists $\Psi < \infty$ such that $f \not\equiv -\infty$ and $f(x) \leq \Psi(1 + |x|)$. Then we have that $y \mapsto \text{Wgt}_{\infty}^f[(*, 0) \rightarrow (y, 1)]$ is Brownian motion patchwork quiltable; the comparison with Brownian motion may be made in L^{3-} , and the random number of patches has a polynomial tail with exponent $2 - \varepsilon$ for any $\varepsilon > 0$.*

With the fact that the limiting weight profile enjoys the Brownian motion patchwork quilt description, we may state our final application as our final theorem, which gives a uniform Brownian-motion-like bound on the $2 - \eta$ moment of the limiting weight profile increment, for any $\eta > 0$ and a broad class of initial conditions.

Theorem 1.9. — *Let $f : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$ be such that there exists $\Psi < \infty$ such that $f \not\equiv -\infty$ and $f(x) \leq \Psi(1 + |x|)$, and let $0 < \eta < \frac{1}{2}$. Then there exist constants $G < \infty$ and $y_0 > 0$ such that, for $|y| < y_0$,*

$$\mathbb{E} \left[\left| \text{Wgt}_{\infty}^f[(*, 0) \rightarrow (y, 1)] - \text{Wgt}_{\infty}^f[(*, 0) \rightarrow (0, 1)] \right|^{2-\eta} \right] \leq G|y|^{1-\eta/2}.$$

In fact, G can be taken uniformly over all f in a certain class of initial conditions that we will define in Section 3.1. We will state and prove that result as Corollary 6.12, which immediately implies Theorem 1.9.

Theorem 1.9 applies, for example, to flat initial conditions $f \equiv 0$. The limiting weight profile is then the Airy_1 process. It is widely expected that the second moment of the increment of Airy_1 grows linearly in the size of the increment when the size is small [Fer08, page 324], but a formal result does not appear to be known.

A slight improvement to Theorem 1.9 will lead to the bound for Airy_1 increments posited in [Fer08]. The improvement required concerns control over the decay of the number of patches in the patchwork quilt. Currently, the probability there are at least ℓ patches intersecting a unit interval is only known to decay as $\ell^{-(2-\varepsilon)}$ for any $\varepsilon > 0$, as stated in Theorem 1.8; the improvement to Theorem 1.9 would require the bound to be at most $\ell^{-(2+\varepsilon)}$ for some $\varepsilon > 0$. Obtaining this additional control may be technically challenging and would require a furthering of the methods of [Ham19a], which we do not pursue here.

[QR13, Lemma 3.3] states a bound related to Theorem 1.9 in the case $f \equiv 0$, giving a bound on even moments of a truncated version of the increment. However, the truncation is significant enough that the behaviour of the bound is no longer Brownian, as it is in Theorem 1.9, and so it does not seem possible to recover the expected Brownian nature of the second moment (i.e., that it is linear in y for small $|y|$) raised in [Fer08].

1.5. Method of proof

In this final section of Chapter 1 we compare, on a high level, the method of proof of the main theorem with that of Theorem 1.7 as given in [Ham19b]. Chapter 4 is devoted to describing the general framework for the proof of Theorem 3.10, with Section 4.2 addressing the conceptual framework specific to its proof.

At the highest level, the method of proof of Theorem 1.7 in [Ham19b] relies on embedding the parabolic Airy_2 curve as the uppermost curve in the parabolic Airy line ensemble and employing the Brownian Gibbs property. In [Ham19b], a significant amount of additional technical apparatus, known as the jump ensemble, was developed to further this technique, which culminated in the proof of Theorem 1.7.

The proof of our main theorem is based squarely on the Brownian Gibbs property via the jump ensemble as well, but the details of the proof differ quite substantially from that of Theorem 1.7 because of the difficulties that arise from possibly pathological correlations between the bridge $\mathcal{L}^{[-d,d]}$ and the endpoints $\mathcal{L}(\pm d)$.

A flavour of this difficulty can be seen even in a purely Brownian toy example quite easily, and this example will be fairly representative because of the Brownian Gibbs property. Suppose we are trying to bound the probability that a Brownian process lies in a particular measurable subset of continuous functions. We are contrasting the situation when the Brownian process is Brownian motion with when it is Brownian bridge; we note that applying the affine shifting procedure described before Theorem 1.7, which defines $\mathcal{L}^{[-d,d]}$ from \mathcal{L} , to Brownian motion results in Brownian bridge. Let B be a standard rate one Brownian motion on $[-d, d]$ started at zero, and let $B^{[-d,d]}$ be the Brownian bridge on $[-d, d]$ resulting from the affine shifting procedure.

A standard fact is that $B^{[-d,d]}$ is *independent of the original endpoint value* $B(d)$ of the Brownian motion. Thus, when evaluating the probability that $B^{[-d,d]}$ lies in some subset of continuous functions, one simply has to integrate over $B(d)$; the conditional probability given $B(d)$ is the same for all of them.

In contrast, consider the probability that B lies in a subset A of continuous functions. If we here try to decompose the process by conditioning on its endpoint value $B(d)$, the conditional probability of A depends on $B(d)$. More importantly, the nature of the dependence is not the same for all A , and so there is no clear way to decouple the conditional probability of A from the endpoint values in an event-agnostic way.

The Brownian Gibbs property in some sense relates the statement to be proved here, regarding the process \mathcal{L} in the form of Theorem 1.1, to considerations similar to this toy example. Recall that, in a loose sense, the Brownian Gibbs property says that the conditional distribution of \mathcal{L} on an interval is that of a Brownian bridge with appropriate endpoints conditioned on being above a lower curve over the whole interval; the lower curve is the second curve of the parabolic Airy line ensemble. Let us call the event of non-intersection with the second curve NonInt . We are considering the probability that $\mathcal{L}^{[-d,d]}$ belongs to an event A . On a heuristic level, applying the Brownian Gibbs property and the independence from endpoints enjoyed by Brownian bridge, we see, with \tilde{B} a Brownian bridge on $[-d, d]$ with endpoint values given by $\mathcal{L}(\pm d)$,

$$\mathbb{P}\left(\mathcal{L}^{[-d,d]} \in A \mid \mathcal{L}(\pm d)\right) = \frac{\mathbb{P}(\tilde{B}^{[-d,d]} \in A, \text{NonInt} \mid \mathcal{L}(\pm d))}{\mathbb{P}(\text{NonInt} \mid \mathcal{L}(\pm d))} \leq \frac{\mathbb{P}(B^{[-d,d]} \in A)}{\mathbb{P}(\text{NonInt} \mid \mathcal{L}(\pm d))}.$$

In particular, we have reduced the problem to lower bounding the probability of non-intersection given $\mathcal{L}(-d)$ and $\mathcal{L}(d)$; the probability of A under Brownian bridge has no interaction with the endpoint values. So on taking expectations to get the unconditional probability, we only need to lower bound $\mathbb{P}(\text{NonInt} \mid \mathcal{L}(\pm d))$.

A simplifying feature of $\mathbb{P}(\text{NonInt} \mid \mathcal{L}(\pm d))$ is that it enjoys an intuitive monotonicity in the endpoint values: when they are higher, avoiding the lower curve is more probable. Using this monotonicity, it is sufficient for the proof of Theorem 1.7 to lower bound the non-intersection probability (rather, upper bound its reciprocal) by obtaining an upper bound on the density of $(\mathcal{L}(-d), \mathcal{L}(d))$ in only the case when the endpoints are very low. This is a crucial technical result in [Ham19b], stated as Lemma 5.17. (This description is not completely accurate as in the proof of Theorem 1.7 the technical apparatus of the jump ensemble allows the non-intersection condition to be with only a certain subset of the lower curve and instead of all of it.)

However, for $\mathcal{L}(\cdot) - \mathcal{L}(-d)$, analogous to the Brownian motion discussion, the conditional probability of A given $\mathcal{L}(\pm d)$ also depends on $\mathcal{L}(\pm d)$ and so cannot be decoupled from $\mathbb{P}(\text{NonInt} \mid \mathcal{L}(\pm d))$. The combination of the two probabilities does not enjoy a monotonicity property in the endpoint values for all events. Thus, while in the proof of Theorem 1.7 it was sufficient to have an endpoint value density bound valid only when the endpoint values are very low, here we will need corresponding density bounds for all ranges of endpoint values. The case of low endpoint values is handled by using the same statement of [Ham19b], (here stated as Proposition 5.1) but the other ranges of endpoint values lead to additional cases of greater technical difficulty.

A fuller discussion of the ideas and approach of the proof is provided in Chapter 4, with Section 4.1 discussing the jump ensemble and Section 4.2 discussing the framework specific to the proof of the main result.

1.6. Organization of the article

In Chapter 2 we prove some of the consequences of Theorem 1.1 discussed in Section 1.2. In Chapter 3, we introduce the Brownian Gibbs property and the more general objects to which our results apply, and then state the main result in its general form as Theorem 3.10. Chapter 4 sets up the framework in which our proof operates: in Section 4.1 we introduce the jump ensemble, and in Section 4.2 we provide a conceptual framework for the proof of the principal result. Finally, the main theorem is proved in Chapter 5 across four sections, each covering a different case. Chapter 6 is devoted to introducing the definitions of the patchwork quilt and stating and proving precise versions of the statements qualitatively described in Section 1.4.

We have attempted to state lemmas as they are required and give proofs immediately, but with a few exceptions. One that we highlight here is in Chapter 2, where some straightforward calculations involving Brownian meander required for the proofs of Theorems 1.4 and 1.5 have been deferred to Appendix A.

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CHAPTER 2

PROOFS OF APPLICATIONS

In this chapter we use Theorem 1.1 to provide the proofs of two of the applications described in Section 1.2, namely Theorems 1.4 and 1.5. The strategy of proof, of course, is to prove a similar bound as desired for Brownian motion and then translate it to one for the parabolic Airy₂ process \mathcal{L} .

2.1. Local Johansson

We start with the proof of Theorem 1.4, the quantified and local version of Johansson's conjecture. In fact, we will prove a stronger result which immediately implies Theorem 1.4. For an interval $I \subseteq [-d, d]$, and a stochastic process X on $[-d, d]$, we define the *maximiser location event* $\text{MaxLoc}(X, I)$ that $x_{\max} \in I$. The arcsine distribution on an interval $[a, b]$ is given by its density $\pi^{-1}((b-x)(x-a))^{-1/2}$; let $\text{ArcSin}_d(I)$ be the measure of I under the arcsine distribution on $[-d, d]$.

Proposition 2.1. — *Let $d \geq 1$, $I \subseteq [-d, d]$ an interval, $a \in (0, 1)$, $\eta > 0$, and $\varepsilon = \text{ArcSin}_d(I)$. Then there exist $\varepsilon_0 = \varepsilon_0(d)$ and an absolute constant $G < \infty$ such that, for $0 < a\varepsilon < \varepsilon_0$,*

$$\mathbb{P}(\text{NT}(\mathcal{L}, \eta, a) \cap \text{MaxLoc}(\mathcal{L}, I)) \leq a\varepsilon \cdot \exp\left(Gd(\log(a\varepsilon)^{-1})^{5/6}\right).$$

By taking $I = [-d, d]$, we obtain Theorem 1.4. To prove Proposition 2.1, we isolate the result for Brownian motion as a separate proposition.

Proposition 2.2. — *Let $d \geq 1$, $I \subseteq [-d, d]$ an interval, $a \in (0, 1)$, $\eta > 0$, and B be rate one Brownian motion on $[-d, d]$. Then*

$$\mathbb{P}(\text{NT}(B, \eta, a) \cap \text{MaxLoc}(B, I)) \leq 4a \cdot \text{ArcSin}_d(I).$$

We first apply Theorem 1.1 via Proposition 2.2 to obtain Proposition 2.1.

PROOF OF PROPOSITION 2.1. This follows immediately from Proposition 2.2 and Theorem 1.1 after noting that both events are unchanged by a vertical shift, i.e.,

$$\text{NT}(\mathcal{L}, \eta, a) = \text{NT}(\mathcal{L}(\cdot) - \mathcal{L}(-d), \eta, a) \quad \text{and} \quad \text{MaxLoc}(\mathcal{L}, I) = \text{MaxLoc}(\mathcal{L}(\cdot) - \mathcal{L}(-d), I). \quad \square$$

To prove Proposition 2.2 (as well as other statements in this chapter), we will make use of the classic connection between Brownian motion around its unique maximiser and *Brownian meander*. Recall that standard Brownian meander on $[0, 1]$ is a (non-homogeneous) Markov process which can be intuitively understood as a Brownian motion conditioned to remain positive (though this singular conditioning requires rigorous treatment). See [BY88; Imh84; Pit99; RY13] for references on Brownian meander.

We will need a number of calculations involving the Brownian meander; to maintain this article's focus, in this chapter we will state the results of the calculations as needed, but we will defer the proofs to Appendix A in cases where we think the proof is not illuminating.

We now record the classical decomposition of Brownian motion around its maximiser to which we alluded above; a proof can be found, for example, in [Den84].

Proposition 2.3. — *Let x_{\max} be the a.s. unique maximiser of a standard Brownian motion B on $[0, 1]$, and let $B(x_{\max}) = M$. Then the processes*

$$B_{\text{me}}^{\rightarrow} := (1 - x_{\max})^{-1/2} \left(M - B(x_{\max} + u(1 - x_{\max})) \right)_{u \in [0,1]}, \quad \text{and}$$

$$B_{\text{me}}^{\leftarrow} := x_{\max}^{-1/2} \left(M - B(x_{\max}(1 - u)) \right)_{u \in [0,1]}$$

are standard Brownian meanders, independent of each other and of x_{\max} .

This proposition suggests that we should look at the event that a Brownian meander comes near zero. Let us define, for a Brownian meander B_{me} , the event $\text{NZ}(B_{\text{me}}, \eta, a)$ by

$$\text{NZ}(B_{\text{me}}, \eta, a) = \left\{ \inf_{u \in [\eta, 1]} B_{\text{me}}(u) < a\eta^{1/2} \right\},$$

where the event is empty if $\eta > 1$. The following lemma is the main input for Proposition 2.2.

Lemma 2.4. — *Let B_{me} be a standard Brownian meander on $[0, 1]$. Let $a \in (0, 1)$ and $\eta > 0$. Then*

$$\mathbb{P}(\text{NZ}(B_{\text{me}}, \eta, a)) \leq 2a.$$

The proof of Lemma 2.4 proceeds by conditioning on $B_{\text{me}}(\eta)$ and using transition probability formulas for Brownian meander to obtain the bound; as the calculations are straightforward, we have deferred the proof to Appendix A. We now show how we may use Proposition 2.3 and Lemma 2.4 to obtain Proposition 2.2, which completes the proof of Theorem 1.4.

PROOF OF PROPOSITION 2.2. By Brownian scaling and translation, it is sufficient to prove Proposition 2.2 on the interval $[0, 1]$, which corresponds to $d = 1/2$. Observe that

$$(6) \quad \mathbb{P}(\text{NT}(B, \eta, a) \cap \text{MaxLoc}(B, I) \mid x_{\max}) = \mathbb{P}(\text{NT}(B, \eta, a) \mid x_{\max}) \cdot \mathbb{1}_{x_{\max} \in I};$$

and that, from Proposition 2.3,

$$(7) \quad \mathbb{P}(\text{NT}(B, \eta, a) \mid x_{\max}) \leq \mathbb{P}(\text{NZ}(B_{\text{me}}^{\rightarrow}, \tilde{\eta}_1, a) \mid x_{\max}) + \mathbb{P}(\text{NZ}(B_{\text{me}}^{\leftarrow}, \tilde{\eta}_2, a) \mid x_{\max}),$$

where $\tilde{\eta}_1$ and $\tilde{\eta}_2$ are defined as $\tilde{\eta}_1 = (1 - x_{\max})^{-1}\eta$ and $\tilde{\eta}_2 = x_{\max}^{-1}\eta$.

From Lemma 2.4 and the independence of $B_{\text{me}}^{\rightarrow}$, $B_{\text{me}}^{\leftarrow}$, and x_{\max} , each term on the right-hand side of (7) is bounded by $2a$, giving an overall bound of $4a$ for $\mathbb{P}(\text{NT}(B, \eta, a) \mid x_{\max})$. Finally, the statement of Proposition 2.2 follows from (6) and the well-known arcsine law of the maximiser of Brownian motion (see, for example, [MP10, Chapter 5.4]). \square

2.2. Many near touch

Now we turn to proving Theorem 1.5. We will of course first establish a similar result for Brownian motion.

Proposition 2.5. — *Let $d \geq 1$, $B : [-d, d] \rightarrow \mathbb{R}$ be a rate one Brownian motion, and let $0 < \eta < d$. Then there exist universal constants ℓ_0 and $c > 0$ such that, for $\ell > \ell_0$,*

$$\mathbb{P}(\text{NumNT}(B, \eta) \geq \ell) \leq e^{-c\ell}.$$

Similarly to before, combining this with Theorem 1.1 readily yields Theorem 1.5.

PROOF OF THEOREM 1.5. Again observing that $\text{NumNT}(\mathcal{L}, \eta) = \text{NumNT}(\mathcal{L}(\cdot) - \mathcal{L}(-d), \eta)$, we may apply Theorem 1.1 to Proposition 2.5 to get that, for large enough ℓ and some $G < \infty$,

$$\mathbb{P}(\text{NumNT}(\mathcal{L}, \eta) \geq \ell) \leq e^{-c\ell + Gd\ell^{5/6}}.$$

We may reduce c to $c' = c/2$ and enforce a sufficiently high lower bound on ℓ to obtain that this probability is bounded by $e^{-c'\ell}$, which is the statement of Theorem 1.5 with c' replacing c in the exponent. \square

To prove Proposition 2.5 we will again rely on Proposition 2.3. To do so, we must bound the number of times that standard Brownian meander comes within (a particular multiple of) $\eta^{1/2}$ of 0. Let us define the random variable recording the *number of near zeroes* $\text{NumNZ}(X)$ of a process $X : [0, 1] \rightarrow \mathbb{R}$ with $X(0) = 0$ by

$$\text{NumNZ}(X, \eta) = \max \left\{ |S| \mid \begin{array}{l} S \subseteq [0, 1] \text{ finite, } s \in S \implies |X(s)| \leq \eta^{1/2}, \\ \text{and } s, t \in S \text{ with } s \neq t \implies |s - t| \geq \eta, \end{array} \right\}.$$

Let B_{me} denote the standard Brownian meander. The following estimate on $\text{NumNZ}(B_{\text{me}}, \eta)$ is the main estimate needed for Proposition 2.5.

Proposition 2.6. — *Let $0 < \eta \leq 1/5$. Then there exist universal ℓ_0 and $c > 0$ such that, for $\ell > \ell_0$,*

$$\mathbb{P}(\text{NumNZ}(B_{\text{me}}, \eta) \geq \ell) \leq e^{-c\ell}.$$

We show how Proposition 2.5 follows from Proposition 2.6, and then turn to proving the latter.

PROOF OF PROPOSITION 2.5. Again by Brownian scaling and translation of the domain, it is enough to prove Proposition 2.5 on the interval $[0, 1]$, which corresponds to $d = 1/2$. (Notice that $\eta < d$ before scaling becomes $\eta < 1/2$ after.)

As in Proposition 2.3, conditionally on x_{max} , let $B_{\text{me}}^{\rightarrow}$ and $B_{\text{me}}^{\leftarrow}$ denote the two independent Brownian meanders on $[0, 1]$, with $B_{\text{me}}^{\rightarrow}$ corresponding to the right of the maximiser and $B_{\text{me}}^{\leftarrow}$ to the left. It will be helpful in the proof to consider separately the number of near touches that occur on the right and on the left. Accordingly, we define, with $M = X(x_{\text{max}})$,

$$\begin{aligned} \text{NumNT}^{\rightarrow}(X, \eta) &= \max \left\{ |S| \mid \begin{array}{l} S \subseteq [0, 1] \text{ finite, } s \in S \implies M - X(s) \leq \eta^{1/2}, \text{ and} \\ s, t \in S \text{ with } s \neq t \implies |s - t| \geq \eta, \text{ min } S > x_{\text{max}} \end{array} \right\} \quad \text{and} \\ \text{NumNT}^{\leftarrow}(X, \eta) &= \max \left\{ |S| \mid \begin{array}{l} S \subseteq [0, 1] \text{ finite, } s \in S \implies M - X(s) \leq \eta^{1/2}, \text{ and} \\ s, t \in S \text{ with } s \neq t \implies |s - t| \geq \eta, \text{ max } S < x_{\text{max}} \end{array} \right\}, \end{aligned}$$

the superscript arrows indicating in which direction from x_{max} we are considering the process. Now we have

$$\mathbb{P}(\text{NumNT}(B, \eta) \geq \ell) \leq \mathbb{P}(\text{NumNT}^{\rightarrow}(B, \eta) \geq \ell/2) + \mathbb{P}(\text{NumNT}^{\leftarrow}(B, \eta) \geq \ell/2).$$

We give only the argument to bound the first term in this break-up, as the argument for the second is identical. Observe that, if $\ell \geq 11$, we have

$$\mathbb{P}(\text{NumNT}^{\rightarrow}(B, \eta) \geq \ell/2 \mid x_{\text{max}}) = \mathbb{P}(\text{NumNT}^{\rightarrow}(B, \eta) \geq \ell/2 \mid x_{\text{max}}) \cdot \mathbb{1}_{x_{\text{max}} \leq 1 - 5\eta},$$

since the number of near touches that can occur in an interval of length 5η is at most five. Now for the given $\eta \in (0, 1/5)$, define $\tilde{\eta} = (1 - x_{\text{max}})^{-1}\eta$. Observe that, when $x_{\text{max}} \leq 1 - 5\eta$, we have that $\tilde{\eta} \leq 1/5$, allowing us to apply Proposition 2.6. Applying Proposition 2.3 and then Proposition 2.6, we have

$$\mathbb{P}(\text{NumNT}^{\rightarrow}(B, \eta) \geq \ell/2 \mid x_{\text{max}}) \cdot \mathbb{1}_{x_{\text{max}} \leq 1 - 5\eta} = \mathbb{P}(\text{NumNZ}(B_{\text{me}}^{\rightarrow}, \tilde{\eta}) \geq \ell/2 \mid x_{\text{max}}) \cdot \mathbb{1}_{x_{\text{max}} \leq 1 - 5\eta} \leq e^{-c\ell},$$

so that the proof is complete. \square

So it remains to prove Proposition 2.6. The strategy is to consider a random variable N which dominates $\text{NumNZ}(B_{\text{me}}, \eta)$. Define a sequence of stopping times τ_i inductively. We set $\tau_0 = 0$, and, for $k \geq 1$,

$$\tau_k = \begin{cases} \tau_{k-1} + \eta & \text{if } B_{\text{me}}(\tau_{k-1} + \eta) \leq 1.1\eta^{1/2}, \tau_{k-1} < 1 - 9\eta \\ \inf \{t \geq \tau_{k-1} + \eta : B_{\text{me}}(t) \leq \eta^{1/2}, t \leq 1 - 10\eta\} & \text{if } B_{\text{me}}(\tau_{k-1} + \eta) > 1.1\eta^{1/2}, \tau_{k-1} < 1 - 9\eta \\ \infty & \text{if } \tau_{k-1} \geq 1 - 9\eta. \end{cases}$$

We adopt the convention that the infimum of the empty set is $+\infty$. Now we define N to be

$$N = 5 + \max \{j : \tau_j < \infty\}.$$

In essence, N counts the consecutive η -separated instances of $B_{\text{me}}(x) < 1.1\eta^{1/2}$; but, when this condition is violated, no count is made until B_{me} returns to $\eta^{1/2}$. As such, N is an overcount of $\text{NumNZ}(B_{\text{me}}, \eta)$. Because some of our upcoming estimates on Brownian meander only hold on $[0, 1 - 10\eta]$, we simply assume that the maximum possible number of η -separated near zeroes occur in the interval $[1 - 5\eta, 1]$, which is 5. From here the claimed stochastic domination of N over $\text{NumNZ}(B_{\text{me}}, \eta)$ is apparent.

The idea of the proof is that the probability that τ_1, \dots, τ_ℓ are all finite can be bound essentially by a product of the probabilities of the random variables being individually finite using the fact that the τ_i are stopping times and the strong Markov property of B_{me} . To do this, we will need to know bounds on probabilities like that of B_{me} being within $1.1\eta^{1/2}$ of zero at a given time conditional on its behaviour at the previous τ_i , which we record in the following estimates. The proofs of these are straightforward but somewhat tedious calculations which have been deferred to Appendix A.

The first two lemmas bound the probability that the Brownian meander is within $1.1\eta^{1/2}$ of 0 after an interval of time η , given that it is so at the current time. The second lemma treats the case that the current time is not 0, while the first does so when the time is 0, as the transition densities are slightly different in the two cases.

Lemma 2.7. — *Let B_{me} be a standard Brownian meander on $[0, 1]$. For $\eta \in (0, 1/2)$,*

$$\mathbb{P}\left(B_{\text{me}}(\eta) < 1.1\eta^{1/2}\right) \leq \frac{1}{2}.$$

Lemma 2.8. — *Let B_{me} be a standard Brownian meander on $[0, 1]$. Fix $\eta \in (0, 1/2)$ and let $t \in (\eta, 1 - \eta]$. For $\eta > 0$ and $x \in (0, 1.1\eta^{1/2}]$,*

$$\mathbb{P}\left(B_{\text{me}}(t) < 1.1\eta^{1/2} \mid B_{\text{me}}(t - \eta) = x\right) \leq \frac{3}{4}.$$

The final lemma bounds the probability that the Brownian meander returns to level $\eta^{1/2}$ before time one if it is currently above $1.1\eta^{1/2}$.

Lemma 2.9. — *Let B_{me} be a standard Brownian meander on $[0, 1]$. Fix $\eta \in (0, 1/5)$ and $t \in (\eta, 1 - 4\eta]$. There exists an absolute constant $\delta > 0$ (independent of η and t) such that, for $\eta > 0$ and $x > 1.1\eta^{1/2}$,*

$$\mathbb{P}\left(\inf_{s \in [t-\eta, 1]} B_{\text{me}}(s) < \eta^{1/2} \mid B_{\text{me}}(t - \eta) = x\right) \leq 1 - \delta.$$

Given these estimates, we may turn to the proof of Proposition 2.6.

PROOF OF PROPOSITION 2.6. As noted, we have $\mathbb{P}(\text{NumNZ}(B_{\text{me}}, \eta) \geq \ell) \leq \mathbb{P}(N \geq \ell)$. We observe that, for $\ell \geq 10$ so that $\ell/2 \leq \ell - 5$,

$$\mathbb{P}(N \geq \ell) \leq p_{\lfloor \ell/2 \rfloor} := P(\tau_1 < \infty, \dots, \tau_{\lfloor \ell/2 \rfloor} < \infty).$$

We will show that the strong Markov property implies that p_ℓ is bounded by $(1 - \delta/4)^\ell$ for any $\ell \in \mathbb{N}$, with δ as in Lemma 2.9. We prove this by induction. The base case that $p_1 \leq (1 - \delta/4)$ is immediate from Lemmas 2.7–2.9, which are applicable since $\eta < \frac{1}{5}$:

$$\begin{aligned} \mathbb{P}(\tau_1 < \infty) &= 1 - \mathbb{P}(\tau_1 = \infty) \\ &= 1 - \mathbb{P}\left(B_{\text{me}}(\eta) > 1.1\eta^{1/2}, \inf_{\eta \leq t \leq 1-10\eta} B_{\text{me}}(t) > \eta^{1/2}\right) \\ &= 1 - \mathbb{P}\left(B_{\text{me}}(\eta) > 1.1\eta^{1/2}\right) \cdot \mathbb{P}\left(\inf_{\eta \leq t \leq 1-10\eta} B_{\text{me}}(t) > \eta^{1/2} \mid B_{\text{me}}(\eta) > 1.1\eta^{1/2}\right) \\ &\geq 1 - \frac{\delta}{2}. \end{aligned}$$

Now we establish the induction step. We know that $p_{\ell-1} < (1 - \delta/4)^{\ell-1}$. Then

$$\begin{aligned} (8) \quad p_\ell &= \mathbb{P}(\tau_1 < \infty, \dots, \tau_{\ell-1} < \infty, \tau_\ell < \infty) \\ &= \mathbb{E}\left[\mathbb{1}_{\tau_1 < \infty, \dots, \tau_{\ell-1} < \infty} \cdot \mathbb{P}(\tau_\ell < \infty \mid \mathcal{F}_{\tau_{\ell-1}})\right] \\ &= \mathbb{E}\left[\mathbb{1}_{\tau_1 < \infty, \dots, \tau_{\ell-1} < \infty} \cdot \mathbb{P}(\tau_\ell < \infty \mid B_{\text{me}}(\tau_{\ell-1}))\right], \end{aligned}$$

using the strong Markov property for the final equality, and where $\mathcal{F}_{\tau_{\ell-1}}$ is the usual σ -algebra generated by B_{me} before $\tau_{\ell-1}$, i.e., $\mathcal{F}_{\tau_{\ell-1}} = \{A \in \mathcal{F} : A \cap \{\tau_{\ell-1} \leq t\} \in \mathcal{F}_t \text{ for all } t \in [0, 1]\}$ with \mathcal{F}_t the canonical filtration of B_{me} and \mathcal{F} the σ -algebra of the probability space.

Observe that, on the event $\{\tau_{\ell-1} < \infty\}$, it holds that $B_{\text{me}}(\tau_{\ell-1}) \leq 1.1\eta^{1/2}$. Thus we have that, on the event $\{\tau_1 < \infty, \dots, \tau_{\ell-1} < \infty\}$,

$$\mathbb{P}(\tau_\ell < \infty \mid B_{\text{me}}(\tau_{\ell-1})) \leq \text{ess sup } \mathbb{P}(\tau_\ell < \infty \mid B_{\text{me}}(\tau_{\ell-1})) \mathbb{1}_{B_{\text{me}}(\tau_{\ell-1}) \leq 1.1\eta^{1/2}}.$$

It can be shown using Lemmas 2.7–2.9 in the same way as in the base case above that the latter quantity is at most $1 - \delta/4$ (with the detail that here we need the uniformity in the bounds in those lemmas over $x = B_{\text{me}}(t - \eta)$ in their respective ranges). Putting this back into (8) gives that $p_\ell \leq (1 - \delta/4)^\ell$, as required. This completes the proof of Proposition 2.6 \square

CHAPTER 3

NOTATION AND SETUP

In this chapter we introduce some notation we will be using throughout the article; give the definitions of the main objects of study; and then state the main result, Theorem 3.10.

3.1. Notation, Brownian Gibbs, and regular ensembles

3.1.1. General notation. — We take the set of natural numbers \mathbb{N} to be $\{1, 2, \dots\}$. For $k \in \mathbb{N}$, we use an overbar to denote a k -vector, i.e., $\bar{x} \in \mathbb{R}^k$. We denote the integer interval $\{i, i+1, \dots, j\}$ by $\llbracket i, j \rrbracket$. For a function $f : \llbracket 1, k \rrbracket \times \mathbb{R} \rightarrow \mathbb{R}$, we write $\bar{f}(x)$ for $(f(1, x), \dots, f(k, x))$. A k -vector $\bar{x} = (x_1, \dots, x_k) \in \mathbb{R}^k$ is called a k -decreasing list if $x_1 > x_2 > \dots > x_k$. For a set $I \subseteq \mathbb{R}$, let $I_{\geq}^k \subseteq I^k$ be the set of k -decreasing lists of elements of I , and I_{\leq}^k be the analogous set of k -non-increasing lists.

For a real valued function f whose domain of definition contains an interval $[a, b]$, we define $f^{[a,b]} : [a, b] \rightarrow \mathbb{R}$ to be the affinely shifted bridge version of f that is zero at both endpoints, i.e., for $x \in [a, b]$,

$$f^{[a,b]}(x) := f(x) - \frac{x-a}{b-a} \cdot f(b) - \frac{b-x}{b-a} \cdot f(a).$$

For an interval $[a, b] \subseteq \mathbb{R}$, we denote the space of continuous functions with domain $[a, b]$ which vanish at a by $\mathcal{C}_{0,*}([a, b], \mathbb{R})$, and the space of continuous functions which may take any value at the endpoints by $\mathcal{C}_{*,*}([a, b], \mathbb{R})$. The asterisk should be thought of as a wildcard indicating that any value may be taken.

3.1.2. Line ensembles and the Brownian Gibbs property. —

Definition 3.1 (Line ensembles). — Let Σ be an (possibly infinite) interval of \mathbb{Z} , and let Λ be a (possibly unbounded) interval of \mathbb{R} . Let \mathcal{X} be the set of continuous functions $f : \Sigma \times \Lambda \rightarrow \mathbb{R}$ endowed with the topology of uniform convergence on compact subsets of $\Sigma \times \Lambda$, and let \mathcal{C} denote the Borel σ -algebra of \mathcal{X} .

A Σ -indexed line ensemble \mathcal{L} is a random variable defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, taking values in \mathcal{X} such that \mathcal{L} is a $(\mathcal{B}, \mathcal{C})$ -measurable function. We regard \mathcal{L} as a Σ -indexed collection of random continuous curves (despite the usage of the word “line”), each of which maps Λ into \mathbb{R} . We will slightly abuse notation and write $\mathcal{L} : \Sigma \times \Lambda \rightarrow \mathbb{R}$, even though it is not \mathcal{L} which is such a function, but rather $\mathcal{L}(\omega)$ for each $\omega \in \Omega$. A line ensemble is *ordered* if, for all $i, j \in \Sigma$ with $i < j$, it holds that $\mathcal{L}(i, x) > \mathcal{L}(j, x)$ for all $x \in \Lambda$. Statements such as this are understood as being asserted almost surely with respect to \mathbb{P} .

Definition 3.2 (Normal, Brownian bridge, and Brownian motion laws)

We will use $N(m, \sigma^2)$ to denote the normal distribution with mean m and variance σ^2 , and sometimes, with abuse of notation, a random variable with this distribution.

Let $k \in \mathbb{N}$, $a, b \in \mathbb{R}$ with $a < b$, and $\bar{x}, \bar{y} \in \mathbb{R}_{>}^k$. We write $\mathcal{B}_{k;\bar{x},\bar{y}}^{[a,b]}$ for the law of k independent Brownian bridges (B_1, \dots, B_k) of rate one, with $B_i : [a, b] \rightarrow \mathbb{R}$ and $B_i(a) = x_i$ and $B_i(b) = y_i$, for $i = 1, \dots, k$. We remind the reader that this law is the same as the law of standard Brownian bridge (i.e., on $[0, 1]$ with vanishing endpoints) after performing a scaling and an affine transformation to have the desired endpoint values.

We will also need the law of standard Brownian motion started at 0 on the interval $[a, b]$, which we will denote by $\mathcal{B}_{0,*}^{[a,b]}$; i.e., $\mathcal{B}_{0,*}^{[a,b]}$ is the law of a rate one Brownian motion B with $B(a) = 0$.

Now let $f : [a, b] \rightarrow \mathbb{R} \cup \{-\infty\}$ be a continuous function such that $x_k > f(a)$ and $y_k > f(b)$. Define the non-intersection event on a closed set $A \subseteq [a, b]$ with lower boundary curve f by

$$(9) \quad \text{NonInt}_f^A = \left\{ \text{for all } x \in A, B(i, x) > B(i+1, x) \text{ for each } 1 \leq i \leq k-1, \text{ and } B(k, x) > f(x) \right\}.$$

(Note that NonInt_f^A is only an event if it is a measurable subset of \mathcal{X} (see Definition 3.1), which is not guaranteed for general A and f ; here, however, we have assumed that A is closed and f is continuous, and it is easy to verify that this is sufficient for NonInt_f^A to be measurable, as we do in Lemma B.8. In our applications, f and A will always satisfy these conditions.)

When $A = [a, b]$, we omit its mention in the notation, i.e., we write NonInt_f .

With this definition, we can move to defining the Brownian Gibbs property.

Definition 3.3 (Brownian Gibbs property). — Let $n \in \mathbb{N} \cup \{\infty\}$, $I \subseteq \mathbb{R}$ be an interval, $k \in \llbracket 1, n \rrbracket$, and $a, b \in I$ with $a < b$. Let $D_{k;a,b} = \llbracket 1, k \rrbracket \times (a, b)$ and $D_{k;a,b}^c = (\llbracket 1, n \rrbracket \times I) \setminus D_{k;a,b}$. Let $\mathcal{L} : \llbracket 1, n \rrbracket \times I \rightarrow \mathbb{R}$ be an ordered line ensemble, and let $\mathcal{F}_{\text{ext}}(k, a, b)$ be the σ -algebra generated by $\mathcal{L}|_{D_{k;a,b}^c}$. We say that \mathcal{L} has the Brownian Gibbs property if the following holds for all such choices of k, a , and b :

$$\text{Law} \left(\mathcal{L}|_{D_{k;a,b}} \text{ conditionally on } \mathcal{F}_{\text{ext}}(k, a, b) \right) = \mathcal{B}_{k;\bar{x},\bar{y}}^{[a,b]} (\cdot \mid \text{NonInt}_f),$$

where $\bar{x} = \bar{\mathcal{L}}(a)$, $\bar{y} = \bar{\mathcal{L}}(b)$, and $f(\cdot) = \mathcal{L}(k+1, \cdot)$ on $[a, b]$; if $k = n$, then $\mathcal{L}(k+1, \cdot)$ is defined as $-\infty$. (Observe that, in these situations, NonInt_f has positive probability, and so the conditioning is not singular.)

Equivalently, for any Borel measurable function $F : \mathcal{C}(\llbracket 1, k \rrbracket \times [a, b]) \rightarrow \mathbb{R}$, it should hold that, almost surely,

$$\mathbb{E} [F(\mathcal{L}(1, \cdot), \dots, \mathcal{L}(k, \cdot)) \mid \mathcal{F}_{\text{ext}}(k, a, b)] = \mathbb{E}[F(B_1, \dots, B_k)],$$

where (B_1, \dots, B_k) are k independent Brownian bridges on $[a, b]$ from \bar{x} to \bar{y} , conditioned on not intersecting each other or $\mathcal{L}(k+1, \cdot)$ on $[a, b]$.

In words, the conditional distribution of the top k curves of \mathcal{L} on $[a, b]$, given the form on \mathcal{L} on $D_{k;a,b}^c$, is the law of k independent Brownian bridges, the i^{th} from $\mathcal{L}(i, a)$ to $\mathcal{L}(i, b)$, which are conditioned to intersect neither each other nor the lower curve $\mathcal{L}(k+1, \cdot)$ on $[a, b]$.

In the next definition we define *regular ensembles*, which are the general objects to which our main result will apply. The definition is the same as [Ham19b, Definition 2.4], with the parameter $\bar{\varphi}$ in that definition taking the value $(1/3, 1/9, \infty)$; the value of ∞ for the third parameter is a formal device to indicate that the range of s in point (2) below is $[1, \infty)$ instead of $[1, n^{\varphi_3}]$ for a finite value of φ_3 .

Definition 3.4 (Regular Brownian Gibbs ensemble). — Consider a Brownian Gibbs ensemble that has the form

$$\mathcal{L} : \llbracket 1, n \rrbracket \times [-z_n, \infty) \rightarrow \mathbb{R},$$

and which is defined on a probability space under the law \mathbb{P} . The number $n = n(\mathcal{L})$ of ensemble curves and the absolute value z_n of the finite endpoint may take any values in $\mathbb{N} \cup \{\infty\}$ and $[0, \infty)$, respectively. (In fact, we may also take $z_n = \infty$, in which case we would take the domain of \mathcal{L} to be $\llbracket 1, n \rrbracket \times \mathbb{R}$.) Let C and c be two positive constants. The ensemble \mathcal{L} is said to be (c, C) -regular if the following conditions are satisfied.

1. **Endpoint escape.** $z_n \geq cn^{1/3}$.
2. **One-point lower tail.** If $z \geq -z_n$ satisfies $|z| \leq cn^{1/9}$, then

$$\mathbb{P}\left(\mathcal{L}(1, z) + 2^{-1/2}z^2 \leq -s\right) \leq C \exp\{-cs^{3/2}\}$$

for all $s \in [1, \infty)$.

3. **One-point upper tail.** If $z \geq -z_n$ satisfies $|z| \leq cn^{1/9}$, then

$$\mathbb{P}\left(\mathcal{L}(1, z) + 2^{-1/2}z^2 \geq s\right) \leq C \exp\{-cs^{3/2}\}$$

for all $s \in [1, \infty)$.

We reserve the symbols c and C for this usage in the remainder of this paper.

The symbol n will be reserved in the rest of the paper for the number of curves in the regular ensemble under consideration, which we will denote by \mathcal{L}_n .

Though the definition of regular ensembles only includes one-point tail information for the top curve, this actually extends to the lower curves as well [Ham19b, Proposition 2.7]. Though we do not state this result, we will have need of two associated sequences of constants for the statement of our main results. For a (c, C) -regular ensemble, define $C_1 = 140C$, $c_1 = 2^{-5/2}c \wedge 1/8$; and, for each $k \geq 2$,

$$C_k = \max\left\{10 \cdot 20^{k-1}5^{k/2}\left(\frac{10}{3-2^{3/2}}\right)^{k(k-1)/2}C, e^{c/2}\right\}$$

and

$$(10) \quad c_k = \left((3 - 2^{3/2})^{3/2}2^{-1}5^{-3/2}\right)^{k-1}c_1.$$

These symbols will retain these meanings throughout the article.

One example of a regular Brownian Gibbs line ensemble is the *parabolic Airy line ensemble*, given by

$$\mathcal{L}(i, x) = 2^{-1/2}(\mathcal{A}(i, x) - x^2),$$

for $(i, x) \in \mathbb{N} \times \mathbb{R}$, where $\mathcal{A} : \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R}$ is the Airy line ensemble. (We again include a factor of $2^{-1/2}$ to allow comparisons to be made with rate *one* Brownian objects.) The Airy line ensemble was constructed as an ensemble of continuous non-intersecting curves in [CH14, Theorem 3.1], and tightness estimates furnished by each of [DV21], [DNV19], and [Ham19b] lead to simplified constructions. It is defined as follows.

Definition 3.5 (Airy line ensemble). — The Airy line ensemble $\mathcal{A} : \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R}$ is a collection of random continuous curves $\mathcal{A}(j, \cdot)$ for $j \in \mathbb{N}$. For any finite set $I \subset \mathbb{R}$, define the random object $\mathcal{A}[I]$ to be the point process on $I \times \mathbb{R}$ given by $\{(s, \mathcal{A}(j, s)) \mid j \in \mathbb{N}, s \in I\}$. The law of \mathcal{A} is defined as the unique distribution supported on such collections of continuous curves such that, for each finite $I = \{t_1, \dots, t_m\}$, $\mathcal{A}[I]$ is a determinantal point process whose kernel is the extended Airy₂ kernel K_2^{ext} , specified by

$$K_2^{\text{ext}}(s_1, x_1; s_2, x_2) = \begin{cases} \int_0^\infty e^{-\lambda(s_1-s_2)} \text{Ai}(x_1 + \lambda) \text{Ai}(x_2 + \lambda) d\lambda & \text{if } s_1 \geq s_2, \\ -\int_{-\infty}^0 e^{-\lambda(s_1-s_2)} \text{Ai}(x_1 + \lambda) \text{Ai}(x_2 + \lambda) d\lambda & \text{if } s_1 < s_2, \end{cases}$$

where $\text{Ai} : \mathbb{R} \rightarrow \mathbb{R}$ is the Airy function. The Airy line ensemble's curves are ordered, with $\mathcal{A}(1, \cdot)$ uppermost.

3.2. An important example of regular ensembles: Brownian LPP weight profiles

Here we recall the Brownian last passage percolation model from Chapter 1, which will generate an important example of regular ensembles via the RSK correspondence, and weight profiles from general initial conditions. These definitions are not logically required for the proof of our main theorem, but do motivate our decision to prove the result in the more general context of regular ensembles. Additionally, these objects will be used in the applications involving the patchwork quilt.

3.2.1. The model's definition. — We recall the definitions of the quantities $M[(x, i) \rightarrow (y, j)]$ and $\text{Wgt}_n[(x, 0) \rightarrow (y, 1)]$ from (1) and (3).

The process $M[(0, 1) \rightarrow (\cdot, n)] : [0, \infty) \rightarrow \mathbb{R}$ may have been considered first by [GW91]; it was studied further in [OY02].

As mentioned in Chapter 1, the weight profile $\text{Wgt}_n[(x, 0) \rightarrow (y, 1)]$ may be expected to be, for given real choices of x and y , a unit-order random quantity, whose law is tight in the scaling parameter $n \in \mathbb{N}$. In fact, more is true: for $x = 0$, the function $y \mapsto \text{Wgt}_n[(0, 0) \rightarrow (y, 1)]$ is a tight sequence of random functions which converges to \mathcal{L} , the parabolic Airy₂ process mentioned in Section 1.1, which is the top curve in the parabolic Airy line ensemble mentioned above. These inferences follow from the relation between $M[(0, 1) \rightarrow (\cdot, n)]$ and Dyson Brownian motion proved in [OY02] and the fact that the scaling limit of Dyson Brownian motion is the Airy₂ process [AVM05] in the sense of finite-dimensional distributions, upgraded to the space of continuous functions by [CH14]. The equality in distribution with Dyson Brownian motion for the top line alone was proved earlier in [GTW01] and also [Bar01].

We may regard this function $y \mapsto \text{Wgt}_n[(0, 0) \rightarrow (y, 1)]$ as the top line in an ensemble of n continuous curves which we denote $\mathcal{L}_n^{\text{sc}} : \llbracket 1, n \rrbracket \times [-\frac{1}{2}n^{1/3}, \infty) \rightarrow \mathbb{R}$, i.e., $\mathcal{L}_n^{\text{sc}}(1, y) = \text{Wgt}_n[(0, 0) \rightarrow (y, 1)]$. We will not formally define the remaining curves in this ensemble as doing so is not required for this article, but merely say that their definition goes via the Robinson-Schensted-Knuth correspondence and the weights of multi-polymer watermelons, which are maximal energy collections of disjoint (except for endpoints) point-to-point paths. We also have that $\mathcal{L}_n^{\text{sc}}$ converges to the parabolic Airy line ensemble under the standard notion of weak convergence given the locally uniform topology on curves. This is proved by the same references mentioned in the previous paragraph for the top line of the ensemble.

Our reason for considering this ensemble of curves is that it enjoys the Brownian Gibbs property and is in fact regular.

Proposition 3.6. — *There exist choices of the positive constants c and C such that each of the scaled Brownian LPP line ensembles $\mathcal{L}_n^{\text{sc}} : \llbracket 1, n \rrbracket \times [-\frac{1}{2}n^{1/3}, \infty) \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, is (c, C) -regular.*

In fact, this is almost [Ham19b, Proposition 2.5], which proves that $\mathcal{L}_n^{\text{sc}}$ is (c, C) regular in a slightly weaker sense, namely with point (2) in Definition 3.4 holding for $s \in [1, n^{1/3}]$ only. Below we point out how to modify that proof to obtain the complete claimed range of s .

PROOF. [Ham19b, Proposition 2.5] proves that $\mathcal{L}_n^{\text{sc}}$ satisfies points (1) and (3) of Definition 3.4, and so we only need to prove point (2) for all of $s \in [1, \infty)$. To do this, we simply replace the use of [Ham19b, Lemma A.1(1)] in the proof of [Ham19b, Proposition 2.5] with [DV21, Theorem 3.1]; this latter theorem is an improved moderate deviation bound for the k^{th} line of Dyson Brownian motion (equivalently, the k^{th} eigenvalue of the Gaussian Unitary Ensemble), which we need for only $k = 1$. \square

We have stated this slightly improved regularity of $\mathcal{L}_n^{\text{sc}}$ in comparison to the statement of [Ham19b, Proposition 2.5] for completeness. If Proposition 3.6 were used in place of [Ham19b, Proposition 2.5] in the arguments

of [Ham19b], minor improvements to certain statements quoted from [Ham19b] that we use later in this paper, in Section 4.1, could be made; additionally, an improvement in the regularity of the Brownian motion patchwork quilt result proved in Chapter 6 as Theorem 6.4 (in particular, $\beta_4 = 1/12$ stated in that theorem would improve to $1/6$) would be available. However, in view of the minor and technical nature of these improvements, we do not formally claim, state, or use them in our arguments, and therefore we will not carry through these improved effects of Proposition 3.6 further.

3.2.2. Basic parabolic symmetry of regular ensembles. — Here we record a straightforward proposition that allows us to translate the interval of consideration and still retain a regular ensemble (with an extra linear term).

Let $Q : \mathbb{R} \rightarrow \mathbb{R}$ denote the parabola $Q(x) = 2^{-1/2}x^2$, and let $l : \mathbb{R}^2 \rightarrow \mathbb{R}$ be given by $l(x, y) = -2^{-1/2}y^2 - 2^{1/2}y(x - y)$. Note that $x \mapsto l(x, y)$ is the tangent line of the parabola $x \mapsto -Q(x)$ at the point $(y, -Q(y))$. Note also that, for any $x, y \in \mathbb{R}$,

$$(11) \quad Q(x) = -l(x, y) + Q(x - y).$$

For $z_n \geq 0$, consider a regular ensemble $\mathcal{L}_n : \llbracket 1, n \rrbracket \times [-z_n, \infty) \rightarrow \mathbb{R}$. For any $y_n > -z_n$, define $\mathcal{L}_{n, y_n}^{\text{shift}} : \llbracket 1, n \rrbracket \times [-z_n - y_n, \infty) \rightarrow \mathbb{R}$ to be the shifted ensemble given by

$$\mathcal{L}_{n, y_n}^{\text{shift}}(i, x) = \mathcal{L}_n(i, x + y_n) - l(x + y_n, y_n)$$

By (11), $\mathcal{L}_{n, y_n}^{\text{shift}} = \mathcal{L}_n(i, x + y_n) + Q(x + y_n) - Q(x)$.

Lemma 3.7 (Lemma 2.26 of [Ham19b]). — *Let $c, C > 0$ and $n \in \mathbb{N}$. Suppose that $\mathcal{L}_n : \llbracket 1, n \rrbracket \times [-z_n, \infty) \rightarrow \mathbb{R}$ is a (c, C) -regular ensemble. Whenever $y_n \in \mathbb{R}$ satisfies $|y_n| \leq c/2 \cdot n^{1/9}$, the ensemble $\mathcal{L}_{n, y_n}^{\text{shift}}$ is $(c/2, C)$ -regular.*

This lemma will allow our main result to apply to an interval $[K - d, K + d]$ not necessarily centred at the origin.

3.2.3. Other initial conditions. — The initial condition of the weight profile $\text{Wgt}_n[(0, 0) \rightarrow (y, 1)]$, since it is started from the single point $(0, 0)$, is the *narrow-wedge* initial condition, and we would like to define energy profiles from general initial conditions. We introduced the notation $\text{Wgt}_\infty^f[(*, 0) \rightarrow (y, 1)]$ in Section 1.4 for the limiting weight profile from general initial conditions, and we now formally define the pre-limiting version, $\text{Wgt}_n^f[(*, 0) \rightarrow (y, 1)]$, whose limit will be $\text{Wgt}_\infty^f[(*, 0) \rightarrow (y, 1)]$.

Given a general initial condition f , the setting should be understood as taking the highest energy path ending at $(y, 1)$, where the initial point is allowed to be $(x, 0)$ for any $x \in \mathbb{R}$, but with a reward $f(x)$ associated to each x which is added to the weight of any path starting at $(x, 0)$. We define spaces of admissible reward functions and the general initial condition weight profile associated to a reward function from one of these spaces in the next two definitions.

Definition 3.8. — Writing $\bar{\Psi} = (\Psi_1, \Psi_2, \Psi_3) \in (0, \infty)^3$ for a triple of positive reals, we let $\mathcal{I}_{\bar{\Psi}}$ denote the set of measurable functions $f : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$ such that $f(x) \leq \Psi_1(1 + |x|)$ and $\sup_{x \in [-\Psi_2, \Psi_2]} f(x) > -\Psi_3$.

Definition 3.9. — For f lying in one of the function spaces $\mathcal{I}_{\bar{\Psi}}$, we now define the f -rewarded line-to-point polymer weight $\text{Wgt}_n^f[(*, 0) \rightarrow (y, 1)]$ according to

$$\text{Wgt}_n^f[(*, 0) \rightarrow (y, 1)] = \sup \left\{ \text{Wgt}_n[(x, 0) \rightarrow (y, 1)] + f(x) : x \in (-\infty, 2^{-1}n^{1/3} + y] \right\}.$$

Unlike in the narrow-wedge case, for general initial conditions we do not define a corresponding ensemble of non-intersecting curves, as this ensemble does not enjoy the Brownian Gibbs property. However, as previously discussed in Section 1.4, we may still consider a form of Brownian regularity possessed by the f -rewarded

weight profile using the Brownian motion regularity of the narrow-wedge weight profile, which we discuss more fully in Chapter 6.

3.3. Main result

For $k \in \mathbb{N}$, let D_k be a sequence of constants depending only on k , given by

$$(12) \quad D_k = \max \left\{ k^{1/3} c_k^{-1/3} \left(2^{-9/2} - 2^{-5} \right)^{-1/3}, 36(k^2 - 1), 2 \right\}$$

for $k \geq 2$, and set $D_1 = D_2$; here c_k is as given in (10). This will be the value of D_k for the rest of the article.

Our main result will concern an interval $[K - d, K + d]$ for $K \in \mathbb{R}$ and $d \geq 1$. For such K , define the linear function $\ell_{K,d} : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\ell_{K,d}(x) = 2^{1/2} K(x - K + d).$$

Our main result is a generalization of Theorem 1.1 that applies to the k^{th} curve (for fixed $k \in \mathbb{N}$) of any regular ensemble with sufficiently many curves.

Theorem 3.10. — *Suppose that \mathcal{L}_n is an n -curve (c, C) regular ensemble for some $(c, C) \in (0, \infty)^2$. Let $d \geq 1$ denote a parameter. Let $K \in \mathbb{R}$ satisfy $[K - d, K + d] \subset c/2 \cdot [-n^{1/9}, n^{1/9}]$, and let $k \in \mathbb{N}$.*

There exist constants $n_0 = n_0(c)$, $\varepsilon_0 = \varepsilon_0(k) > 0$, and $\tilde{c} > 0$ such that the following holds. Suppose that $n \geq (k + 1) \vee n_0$. For any Borel measurable $A \subset \mathcal{C}_{0,}([K - d, K + d])$, write $\varepsilon = \mathcal{B}_{0,*}^{[K-d, K+d]}(A)$. Suppose that ε satisfies the (k, d) -dependent upper bound $\varepsilon < \varepsilon_0 \wedge \exp(-24)^6 d^6 / D_k^3$; as well as the n -dependent lower bound*

$$(13) \quad \varepsilon \geq \exp \left\{ -\tilde{c} D_k^{-1} n^{1/12} \right\}.$$

Then there exist $G_1, G_2 < \infty$ such that

$$\mathbb{P} \left(\mathcal{L}_n(k, \cdot) - \mathcal{L}_n(k, K - d) + \ell_{K,d}(\cdot) \in A \right) \leq \varepsilon \cdot G_1 \cdot \exp \left\{ G_2 \cdot d \cdot D_k^{5/2} (\log \varepsilon^{-1})^{5/6} \right\}.$$

Specifically, this probability is $\varepsilon \cdot \exp \{ d (\log \varepsilon^{-1})^{5/6} O_k(1) \}$, where $O_k(1)$ denotes a k -dependent term that is independent of ε and d .

Remark 3.11. — The upper bound on ε is only a technical one and is of no real consequence. The rapid decay in n of the lower bound (13) means that no difficulty is created in applications, since, roughly put, events whose probabilities have decay that is superpolynomial in n are in practice irrelevant. In the case that $n = \infty$, such as for the parabolic Airy line ensemble, this lower bound becomes the vacuous $\varepsilon > 0$.

Remark 3.12. — The linear term $\ell_{K,d}$ introduced in the event in the general result is necessary. It arises from the parabolic curvature of regular ensembles, which cannot be ignored when the interval $[K - d, K + d]$ is far from the origin. In fact, we will prove the theorem for $K = 0$, and then use the parabolic invariance introduced in Section 3.2.2 to get the general statement, as we have $\mathcal{L}_{n,K}^{\text{shift}}(k, \cdot) - \mathcal{L}_{n,K}^{\text{shift}}(k, -d) = \mathcal{L}_n(k, \cdot + K) - \mathcal{L}_n(k, K - d) + \ell_{K,d}(\cdot + K)$ on $[-d, d]$.

Next, we present a result that permits us to dispense with Theorem 3.10's inconsequential but practically irksome Brownian probability hypothesis of a d - and k -dependent upper bound on ε .

Corollary 3.13. — *Under the circumstances of Theorem 3.10, suppose on the parameter ε we assume only the lower bound $\varepsilon \geq \exp(-gn^{1/12})$ holds, where $g = g(k) = \tilde{c} D_k^{-1}$. Then there exists $H = H(k)$ such that*

$$\mathbb{P} \left(\mathcal{L}_n(k, \cdot) - \mathcal{L}_n(k, K - d) + \ell_{K,d}(\cdot) \in A \right) \leq \varepsilon \cdot \exp \left\{ H d^6 \right\} \exp \left\{ H d (\log \varepsilon^{-1})^{5/6} \right\}.$$

PROOF. Let $h = \varepsilon_0 \wedge \exp(-(24)^6 d^6 / D_k^3)$ be the expression upper bounding ε in Theorem 3.10.

When ε satisfies the upper bound $\varepsilon \leq h$ of Theorem 3.10 in addition to the assumed lower bound $\varepsilon \geq \exp(-gn^{1/12})$, the claim follows from the same theorem by taking $H = H(k) \geq G_2 D_k^{5/2}$ large enough that $\exp(Hd^6) \geq G_1$, where G_1 and G_2 are as in that theorem; this can be done such that H does not depend on d since $d \geq 1$.

When we instead have $\varepsilon \geq h$, note that $\varepsilon \cdot h^{-1} \geq 1$, and so

$$\mathbb{P}\left(\mathcal{L}_n(k, \cdot) - \mathcal{L}_n(k, K-d) + \ell_{K,d}(\cdot) \in A\right) \leq 1 \leq \varepsilon \cdot h^{-1}.$$

Now let $H = H(k)$ be such that $h^{-1} \leq \exp(Hd^6)$, which is possible by the definition of h and since $d \geq 1$. This yields our claim since $\exp(Hd(\log \varepsilon^{-1})^{5/6}) \geq 1$. \square

CHAPTER 4

PROOF FRAMEWORK

In this chapter we introduce the two frameworks required for our proof: the *jump ensemble*, a general technique introduced in [Ham19b] which allows one to analyse regular Brownian Gibbs line ensembles using a more explicitly Brownian proxy; and a conceptual framework of *costs* which is specific to our proof of Theorem 3.10. We will also reduce the proof of Theorem 3.10 to a statement, Theorem 4.10, about the jump ensemble, and Chapter 5 will be devoted to providing a major part of the proof of this statement using the introduced framework of costs.

4.1. The jump ensemble

We start with a working description of the technical framework in which our proof approach operates, known as the jump ensemble. This ensemble should be thought of as a sort of “half-way house” between Brownian motion and the line ensemble \mathcal{L}_n that we wish to study. Roughly speaking, what we mean by this is that the jump ensemble conditioned on a certain manageable event has the same distribution as \mathcal{L}_n ; but since the jump ensemble can be described in terms of Brownian objects, we can estimate probabilities involving the jump ensemble using knowledge about Brownian motion.

The construction we describe is the same as that in [Ham19b, Chapter 4]. The reader is referred to that article for a fuller discussion; here we restrict ourselves to providing a complete, though perhaps sometimes not fully motivated, description of the jump ensemble that allows the reader to understand the proofs of the paper. The notation used in this section is largely the same as in [Ham19b] for the convenience of the reader. We stress that some of the proofs underlying the correctness and usefulness of the jump ensemble as given in [Ham19b] are technically involved, and so we choose to not reproduce them here, instead focusing only on illustrating the ideas and statements of the jump ensemble.

We use only three statements from [Ham19b], reproduced here as Lemma 4.2, Proposition 4.8, and Lemma 4.9. We call these three statements *the side interval test*; *the jump ensemble candidate proficiency*; and *the high probability of the favourable event*. The reason for the use of these names will become clearer over the next few subsections.

4.1.1. Motivation and main themes. — Before turning to the details of the jump ensemble, let us bring to focus some of the main themes. Recall that we aim to study the k^{th} curve of \mathcal{L}_n on an interval $[\ell, r]$. To do this, we initially consider the top k curves together. The basic tool we have at our disposal in studying regular line ensembles is the Brownian Gibbs property. To recall it, let $\mathcal{F}_{\text{BB}}^{[\ell, r]}$ be the σ -algebra generated by the following collection of random variables (where the BB subscript stands for “Brownian bridge”):

- all the lower curves $\mathcal{L}_n : \llbracket k+1, n \rrbracket \times [-z_n, \infty) \rightarrow \mathbb{R}$;
- and the top k curves $\mathcal{L}_n : \llbracket 1, k \rrbracket \times ([-z_n, \ell] \cup [r, \infty)) \rightarrow \mathbb{R}$ outside (ℓ, r) .

(Though this σ -algebra's definition clearly depends on k , we suppress this dependence in the notation $\mathcal{F}_{\text{BB}}^{[\ell, r]}(\cdot)$.)

The statement of the Brownian Gibbs property is then that, conditionally on $\mathcal{F}_{\text{BB}}^{[\ell, r]}$, the top k curves of \mathcal{L}_n on $[\ell, r]$ have the same distribution as a collection of k independent Brownian bridges, the i^{th} from $\mathcal{L}_n(i, \ell)$ to $\mathcal{L}_n(i, r)$, with the curves in the collection conditioned on intersecting neither $\mathcal{L}_n(k+1, \cdot)$ nor each other on all of $[\ell, r]$.

4.1.1.1. Candidate ensembles. — We interpret this description as a *resampling property*, which is to say that, given the data in $\mathcal{F}_{\text{BB}}^{[\ell, r]}$, the top k curves of \mathcal{L}_n on $[\ell, r]$ are obtained by rejection sampling collections of k independent Brownian bridges with the given endpoints until they fully avoid $\mathcal{L}_n(k+1, \cdot)$ and each other on $[\ell, r]$. We call the curves' avoidance of each other on $[\ell, r]$ *internal non-intersection*.

This resampling interpretation suggests a slightly different viewpoint on the Brownian Gibbs property. Let us call the collection of k independent Brownian bridges with the given endpoints a *candidate ensemble*; it ignores all information about the lower curve, as its definition only involves \mathcal{L}_n on $\llbracket 1, k \rrbracket \times \{\ell, r\}$. Our desire is for the candidate ensemble to gain the correct $\mathcal{F}_{\text{BB}}^{[\ell, r]}$ -conditional distribution of the top k curves of \mathcal{L}_n on $[\ell, r]$. In order for this to happen, the candidate ensemble must *reconstruct* the effect of the ignored data as well as satisfy the other constraints that \mathcal{L}_n does.

The basic relation between the $(k+1)^{\text{st}}$ curve of \mathcal{L}_n and the top k curves of \mathcal{L}_n is that the top k curves must not intersect the $(k+1)^{\text{st}}$; beyond this, the top k curves of \mathcal{L}_n must not intersect internally, though the curves in the Brownian bridge candidate may. The mentioned reconstruction is done by passing a *test of non-intersection*, both with the lower curve and internally. The reinterpretation of the Brownian Gibbs property is that the candidate ensemble, on passing the non-intersection test, gains the target line ensemble's $\mathcal{F}_{\text{BB}}^{[\ell, r]}$ -conditional distribution. The probability of a successful given step in rejection sampling is thus exactly the probability that the candidate ensemble passes this test.

Here, the role of $\mathcal{F}_{\text{BB}}^{[\ell, r]}$ is to specify the data conditioned on, and so available to the candidate ensemble; some of this data is 'used' by the candidate, in the sense that it appears in the definition of the candidate; while the rest is in effect ignored by the candidate. This idea that the candidate ensemble ignores some part of the available data is an important one, which we will develop over the next few pages. In particular, we will consider the effects of using and ignoring different aspects of data; as we shall see shortly, the example here, of ignoring the entire bottom curve, is too extreme and will not be useful for our purposes.

Broadly then we work with a two-step process. First, we condition on a certain selection of data, here represented by $\mathcal{F}_{\text{BB}}^{[\ell, r]}$; and second, we consider candidate ensembles which use some subset of this data and ignore the rest. The candidate ensemble recovers the correct conditional distribution, specified by the data first conditioned on, in spite of the ignored data, by resampling till the appropriate constraint is met, which is that of non-intersection.

4.1.1.2. Features of a useful candidate. — Our final aim is to estimate probabilities for the k^{th} curve of the line ensemble \mathcal{L}_n . For this purpose, a candidate ensemble is useful if we can

- (i) estimate probabilities of interest for the candidate ensemble; and
- (ii) translate them to estimates on probabilities for the line ensemble.

To successfully estimate probabilities for the candidate ensemble, it must be amenable to the tools at our disposal, which in practice means it must be sufficiently Brownian (a property that is also imposed by our intention to use the Brownian Gibbs property); while to successfully translate estimates to the line ensemble,

it should be intuitively clear that we need the probability of passing the non-intersection test to be not too low.

4.1.1.3. Stepping out to a larger interval. — So let us consider how the Brownian bridge candidate fares in meeting the aims (i) and (ii). Since the candidate is an ensemble of independent Brownian bridges, the point (i) from the previous paragraph is clearly easily met. However, in regard to (ii), the weak control that we have over the intricate random function $\mathcal{L}_n(k+1, \cdot)$ makes it difficult to obtain useful bounds on the probability of passing the test of non-intersection with the lower curve.

In these terms, we do not have good control over the test-passing probability of pure Brownian bridge, and so this candidate is not directly useful. To understand how a better candidate should be designed, let us consider what made the Brownian bridge candidate have a low test passing probability; we will focus on the $k=1$ case for simplicity. There are in essence two problems that the candidate ensemble must successfully overcome to achieve non-intersection, and these are depicted in Figure 4.1: (i) the boundary points of $\mathcal{L}_n(1, \cdot)$ at $\{\ell, r\}$ may be very close to those of $\mathcal{L}_n(2, \cdot)$, which makes non-intersection near the boundaries difficult; and (ii) $\mathcal{L}_n(2, \cdot)$ may have a very high peak inside $[\ell, r]$ that the candidate ensemble will struggle to surmount.

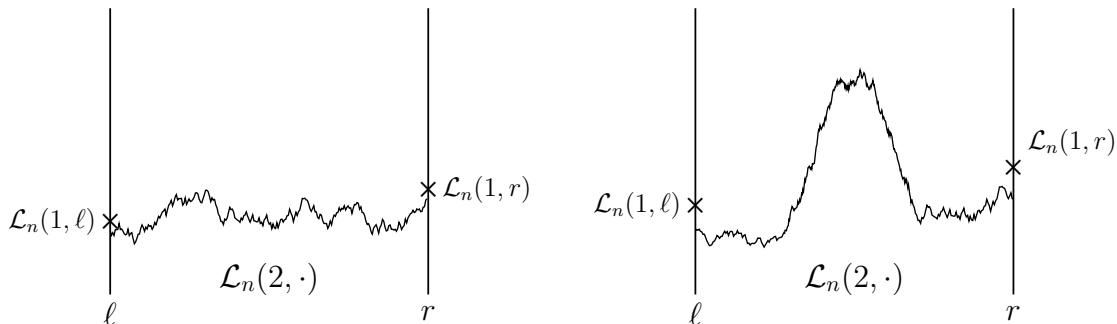


FIGURE 4.1. Two illustrations in the $k=1$ case of instances of data from \mathcal{F}_{BB} which are difficult for the Brownian bridge candidate to handle. The black crosses indicate the values of $\mathcal{L}_n(1, \cdot)$ at the endpoints, which are the points between which the Brownian bridge candidate must move. In the left panel, the boundary points are very close to those of $\mathcal{L}_n(2, \cdot)$, thus making it hard to avoid this second curve locally. In the right panel, there is a high peak which presents a global difficulty.

A first step to resolve both these difficulties will be to step out from our interval of interest, $[\ell, r]$, to a larger interval $[-2T, 2T]$. What this means on a technical level is that the candidate will be defined on $[-2T, 2T]$ and, in its definition, will have access to all the data in $\mathcal{F}_{\text{BB}}^{[-2T, 2T]}$ (instead of $\mathcal{F}_{\text{BB}}^{[\ell, r]}$), each portion of which it may use or ignore; however, this is not to say that our ultimate candidate will *only* use data in $\mathcal{F}_{\text{BB}}^{[-2T, 2T]}$. Indeed, recall that we have the freedom to augment the information in $\mathcal{F}_{\text{BB}}^{[-2T, 2T]}$ and consider a larger σ -algebra from which the candidate can then use data. We will make use of this option shortly.

Now, it is easy to see that having a larger interval makes it easier for a Brownian-type candidate to jump over large peaks inside $[\ell, r]$ (though, as we will see soon, a larger interval alone will not suffice to obtain a high enough probability of non-intersection for our purposes). But it is not clear why a larger interval helps with the issue of close boundary points; indeed, all that has happened is that we have the problem of close points at $\pm 2T$ now instead of $\{\ell, r\}$. It is to address this point that we will need to use the mentioned freedom to consider an expanded σ -algebra.

A second step will be needed. By selective release of conditional information to the candidate, this candidate's overall geometry will be made to adhere to a coarse-grained description of its shape in the event that it does

not intersect the lower curves. In this way, the candidate will be much more likely to pass the non-intersection test to which it is subject.

4.1.1.4. The high jump difficulty. — We have said that, while stepping out from $[\ell, r]$ to a larger interval $[-2T, 2T]$ makes it easier for the pure Brownian candidate to jump over large peaks, the resulting probabilities of non-intersection are not high enough for us to obtain quantitative Brownian comparison estimates of the strength we desire; we call this the *high jump difficulty*. We expand on this point briefly here; a fuller discussion may be found in [Ham19b, Section 4.1.1].

We are faced with the task of choosing T appropriately. Making T larger aids us in one way, but introduces extra difficulties in another. It aids us by increasing the probability of the Brownian candidate jumping over peaks of fixed height. However, as T increases, the boundary values which serve as the starting and ending values of the pure Brownian candidate *decrease* in a parabolic manner, as this is the behavior of $\mathcal{L}_n(1, \cdot)$. Thus, in essence, the peak which has to be jumped over effectively becomes higher because of the parabolic decay of the endpoint values, and this reduces the probability of jumping over the peaks.

With these two opposing forces in mind, optimizing T so as to maximize the non-intersection probability turns out not to yield a high enough value of the latter in order to obtain the L^∞ - bounds on the Radon-Nikodym derivative that we seek—this is the high jump difficulty. (The exact value of T that we will adopt for the jump ensemble, which is essentially the result of this optimization, will be introduced in Section 4.1.6.)

The solution proposed in [Ham19b] is to permit the candidate to make use of a little more of the available data about the lower curve; unlike the Brownian bridge ensemble, which did not use lower curve information at all, the jump ensemble will use a *coarsened* version of the lower curve in order to adopt a geometry that allows it to make the high jump with the requisite high probability, thereby addressing the high jump difficulty. More precisely, the candidate will be conditioned on avoiding this coarsened version of $\mathcal{L}_n(2, \cdot)$, which can be thought of as a test that is preliminary to the full non-intersection test; the candidate, on passing the preliminary test, will naturally have a more suitable overall geometry to pass the final test. The exact form of this coarsening, which we will describe in Section 4.1.5, is at the heart of the jump ensemble method.

4.1.1.5. Using side bridges. — Now we return to the question of the difficulty of non-intersection when boundary points are close. As we saw, stepping out to $[-2T, 2T]$ does not avoid this problem as we just face it at $\pm 2T$ instead of at ℓ and r . We introduce some notation: we label as *side intervals* the intervals $[-2T, \ell]$ and $[r, 2T]$, and the remaining interval $[\ell, r]$ as the *middle interval*.

Focus on the left side interval $[-2T, \ell]$, and consider $\mathcal{L}_n(1, \cdot)$ on this interval. This process starts at the same boundary points and manages non-intersection with $\mathcal{L}_n(2, \cdot)$, as does our Brownian bridge candidate. If our candidate ensemble can adopt some of the geometry of $\mathcal{L}_n(1, \cdot)$ on $[-2T, \ell]$, it would naturally be inclined towards non-intersection with the second curve and achieve it with higher probability.

This motivates the solution we propose next, which also makes precise what we mean by the geometry of $\mathcal{L}_n(1, \cdot)$ on the side intervals. First note that no aspect of $\mathcal{L}_n(1, \cdot)$ on $[-2T, \ell]$ is present in $\mathcal{F}_{\text{BB}}^{[-2T, 2T]}$ and so is not available to any candidate that can only access $\mathcal{F}_{\text{BB}}^{[-2T, 2T]}$. So we first need to consider an augmented σ -algebra. Define \mathcal{F} to be the σ -algebra generated by the following collection of random variables:

- all the lower curves $\mathcal{L}_n : \llbracket k+1, n \rrbracket \times [-z_n, \infty) \rightarrow \mathbb{R}$;
- the top k curves $\mathcal{L}_n : \llbracket 1, k \rrbracket \times ([-z_n, -2T] \cup [2T, \infty)) \rightarrow \mathbb{R}$ outside $(-2T, 2T)$;
- and the $2k$ standard bridges $\mathcal{L}_n^{[-2T, \ell]}(i, \cdot)$ and $\mathcal{L}_n^{[r, 2T]}(i, \cdot)$ for $i = 1, \dots, k$.

(The notation $f^{[a, b]}$ for the affinely shifted bridge version of a function f has been introduced in Section 3.1.1, though it is being mildly abused here, to refer to the bridge version of the i^{th} curve of the ensemble.) We again suppress the k dependence of the σ -algebra in the notation \mathcal{F} .

In words, we include data of the entirety of all the lower curves; the top k curves outside $(-2T, 2T)$; and, on $[-2T, \ell]$ and $[r, 2T]$, the standard bridge paths of the top k curves on these intervals, which we will call the *side bridges*; these capture an aspect of the geometry of $\mathcal{L}_n(1, \cdot)$ on the side intervals. Nothing is retained on $[\ell, r]$, and, in particular, the values of the candidate at ℓ and r are not determined. That nothing is known at these endpoints will help to resolve the problem of close boundary points at $\{\ell, r\}$ and nearby local peaks, because the candidate has the opportunity to rise globally at ℓ and r and thus to sail well above these local obstacles.

Remark 4.1. — The side bridges may appear to be complicated objects to condition upon; in fact, they are easy to handle because of the Brownian Gibbs property and an independence present in the corresponding side bridge decomposition of Brownian bridges. See Lemma 4.3 ahead.

The σ -algebra \mathcal{F} and the selection of data included in it is of great importance for the jump ensemble method, and will be used throughout the arguments of Theorem 3.10. As such, the conditional law $\mathbb{P}(\cdot \mid \mathcal{F})$ will be used extensively, and so we use the notation

$$\mathbb{P}_{\mathcal{F}}(\cdot) := \mathbb{P}(\cdot \mid \mathcal{F})$$

to denote it. Our aim is to understand the law of the top k curves of \mathcal{L}_n on $[-2T, 2T]$ under $\mathbb{P}_{\mathcal{F}}$. (That such a conditional distribution exists can be a subtle point, and to ensure its existence we will assume that the probability space on which we work is a Polish space with the associated Borel σ -algebra; the precise justification for the existence of $\mathbb{P}_{\mathcal{F}}$ is provided in Lemma B.9.)

How do we make the candidate use this data, in particular the side bridges? We will combine the candidate ensemble on $[\ell, r]$ with the data from \mathcal{F} to create a new ensemble on $[-2T, 2T]$. The form of this combination is dictated by the Brownian Gibbs property and the linear operation involved in the definition of $f^{[a,b]}$.

Let the candidate ensemble be denoted by $X : \llbracket 1, k \rrbracket \times [\ell, r] \rightarrow \mathbb{R}$; the new ensemble created using X and data from \mathcal{F} will be called the *resampled* ensemble $\mathcal{L}^{\text{re}, X} : \llbracket 1, k \rrbracket \times [-2T, 2T] \rightarrow \mathbb{R}$. Intuitively, the values of the candidate ensemble at ℓ and r are used to affinely shift the side bridges; the affinely shifted bridges define the resampled ensemble on $[-2T, \ell] \cup [r, 2T]$, while the candidate ensemble determines the resampled ensemble on $[\ell, r]$. This is illustrated in Figure 4.2, and the formal definition of $\mathcal{L}^{\text{re}, X}$ is given by the following, for $i = 1, \dots, k$:

$$(14) \quad \mathcal{L}^{\text{re}, X}(i, x) = \begin{cases} \mathcal{L}_n^{[-2T, \ell]}(i, x) + \frac{x+2T}{\ell+2T} \cdot X(i, \ell) + \frac{\ell-x}{\ell+2T} \cdot \mathcal{L}_n(i, -2T) & x \in [-2T, \ell] \\ \mathcal{L}_n^{[r, 2T]}(i, x) + \frac{2T-x}{2T-r} \cdot X(i, r) + \frac{x-r}{2T-r} \cdot \mathcal{L}_n(i, 2T) & x \in [r, 2T] \\ X(i, x) & x \in [\ell, r]. \end{cases}$$

(Implicit in the above discussion is the promise that the resampled ensemble $\mathcal{L}^{\text{re}, X}$ will be able to pass the non-intersection test on the side intervals sufficiently well due to the use of data in \mathcal{F} . We discuss and make good on this promise in Section 4.1.2.)

As with the earlier discussion of the Brownian bridge ensemble and \mathcal{F}_{BB} , the Brownian Gibbs property says that for certain candidate ensembles X , the distribution of $\mathcal{L}^{\text{re}, X}$, conditioned on passing the non-intersection tests on $[-2T, 2T]$, will be the \mathcal{F} -conditional law of \mathcal{L}_n on $\llbracket 1, k \rrbracket \times [-2T, 2T]$. The candidate ensembles X for which this is true are Brownian bridge ensembles conditioned on avoiding the lower curve on some subset of $[\ell, r]$; the jump ensemble, which will be conditioned on avoiding a coarsened version of the lower curve, will fit this description. Section 4.1.3 is devoted to setting up a precise version of this statement, which is recorded in Lemma 4.4.

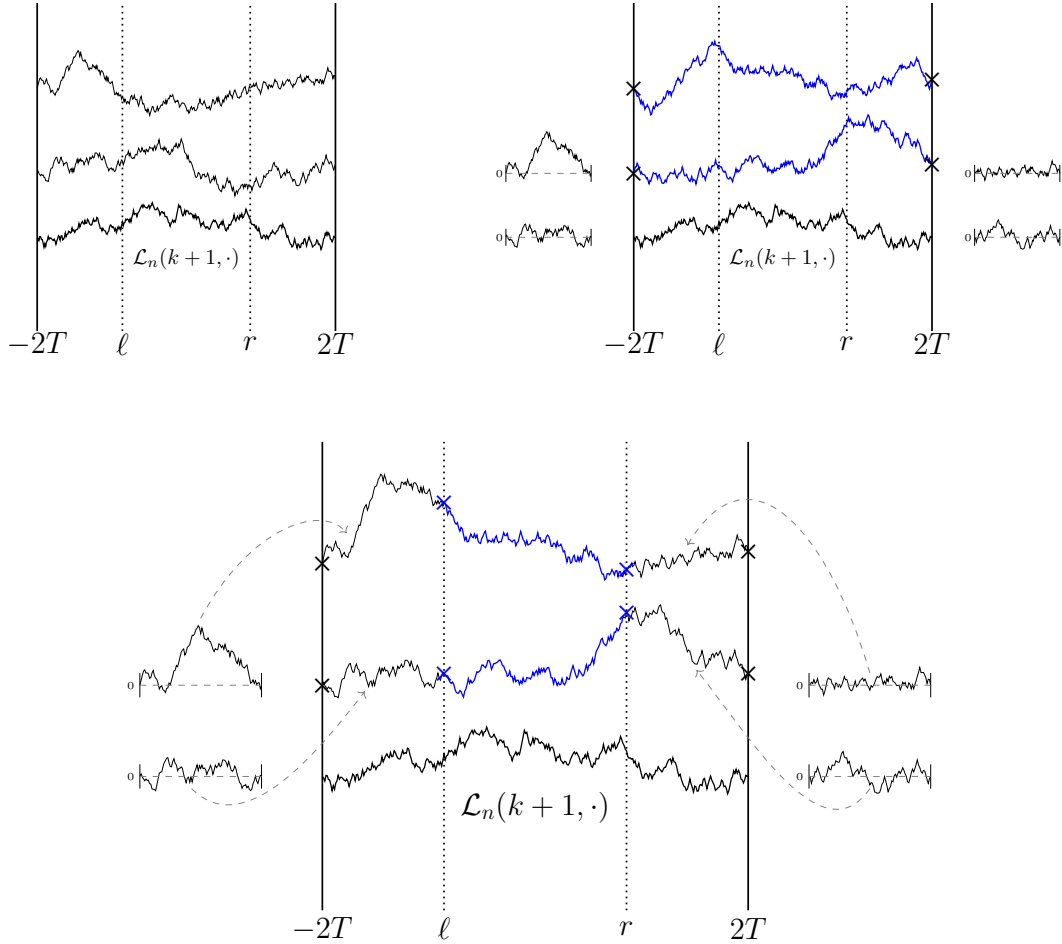


FIGURE 4.2. Constructing $\mathcal{L}^{\text{re},X}$ from the candidate process X (the curves in blue) when $k = 2$. In the first figure we have the original line ensemble \mathcal{L}_n . In the second figure, the black elements are the data available in \mathcal{F} , namely the entirety of the $(k+1)^{\text{st}}$ curve, the positions at $\pm 2T$ of the first k curves (denoted by crosses), and the bridges obtained from the side intervals by affine shift. The blue curves comprise the candidate process X (though technically X is restricted to $[\ell, r]$). In the final figure we complete the reconstruction by pasting the side interval bridges according to the positions dictated by X (blue crosses at ℓ and r) on $[\ell, r]$. Note that in this figure, $\mathcal{L}^{\text{re},X}$ passes both the side interval tests and the middle interval test.

In summary, we are looking to define a candidate process which has estimable probabilities by virtue of being in some sense Brownian, and which has a not-too-low probability of passing the non-intersection test. To accomplish this, we saw in this and the preceding subsection that the candidate process will make use of a coarsened version of the lower curve profile in its definition; and will use side bridge information to aid it in non-intersection on side intervals, for which we work with a more sophisticated selection of data captured by \mathcal{F} . This data will be combined with the candidate ensemble to give the resampled ensemble.

In the next subsection, we expand on the idea that including the data of the side bridges in \mathcal{F} makes it easy for the candidate ensemble to pass the non-intersection test on the side intervals.

4.1.2. The side intervals test. — We formulate the non-intersection test on the side intervals as *the side intervals test*. The side intervals test has two parts: that $\mathcal{L}^{\text{re},X}(k, \cdot)$ does not intersect $\mathcal{L}_n(k+1, \cdot)$; and

that $\mathcal{L}^{\text{re},X}(i, \cdot)$ does not intersect $\mathcal{L}^{\text{re},X}(i+1, \cdot)$ for $i = 1, \dots, k-1$ —both of these on $[-2T, \ell] \cup [r, 2T]$. In symbols, for all $i = 1, \dots, k-1$ and $x \in [-2T, \ell] \cup [r, 2T]$,

$$(15) \quad \begin{aligned} \mathcal{L}^{\text{re},X}(i, x) &> \mathcal{L}^{\text{re},X}(i+1, x) \\ \mathcal{L}^{\text{re},X}(k, x) &> \mathcal{L}_n(k+1, x). \end{aligned}$$

A look at the first two cases in the definition (14) of $\mathcal{L}^{\text{re},X}$ suggests that whether this test is passed is simply a question of whether $X(i, \ell)$ and $X(i, r)$ are high enough in value, as the remaining quantities are \mathcal{F} -measurable and thus not affected by the candidate X . This intuition is roughly correct, and a precise version is the content of the next lemma, which we refer to as *the side intervals test criterion*.

Lemma 4.2 (Side intervals test criterion, Lemma 3.8 of [Ham19b])

There exist \mathcal{F} -measurable random vectors $\overline{\text{Corner}}^{\ell, \mathcal{F}}, \overline{\text{Corner}}^{r, \mathcal{F}} \in \mathbb{R}_{\geq}^k$ such that $\mathcal{L}^{\text{re},X}$ passes the side intervals tests (15) if and only if $\bar{X}(x) - \overline{\text{Corner}}^{x, \mathcal{F}} \in (0, \infty)_{\geq}^k$ for $x = \ell, r$.

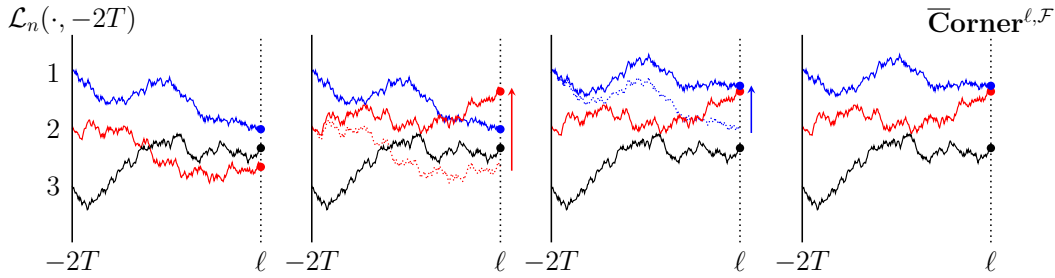


FIGURE 4.3. Building $\overline{\text{Corner}}^{\ell, \mathcal{F}}$ for $k = 3$. $\overline{\text{Corner}}^{\ell, \mathcal{F}}$ is a vector, the $(i-1)^{\text{st}}$ entry of which is the unique smallest value which $\mathcal{L}_n(i-1, \ell)$ can adopt via affine translation before crossing $\mathcal{L}_n(i, \cdot)$ on $[-2T, \ell]$. The line $\mathcal{L}_n(2, \cdot)$ (red) intersects $\mathcal{L}_n(3, \cdot)$ (black) on $[-2T, \ell]$ (first panel), so $\mathcal{L}_n(2, \cdot)$ (dotted red) is affinely translated until just touching, but not crossing $\mathcal{L}_n(3, \cdot)$ (second panel); the translated value of $\mathcal{L}_n(2, \ell)$ is $\text{Corner}_2^{\ell, \mathcal{F}}$. This process is then repeated for the line $\mathcal{L}_n(1, \cdot)$ (blue). Then $\mathcal{L}_n(1, \cdot)$ (dotted blue) intersects the translated curve $\mathcal{L}_n(2, \cdot)$ (red), so $\mathcal{L}_n(1, \cdot)$ is affinely translated until just touching, but not crossing $\mathcal{L}_n(2, \cdot)$ (third panel); the translated value of $\mathcal{L}_n(1, \ell)$ is $\text{Corner}_1^{\ell, \mathcal{F}}$. The result of these translations is shown in the fourth panel. This procedure depends on the collection of bridges on $[-2T, \ell]$, justifying the dependence of $\overline{\text{Corner}}^{\ell, \mathcal{F}}$ on \mathcal{F} .

The proof is given in [Ham19b], but we include here Figure 4.3 which captures the essential argument.

The conclusion we draw from Lemma 4.2 is that analysing the passing of the side interval tests by the candidate X is very simple in practice: we merely need to consider the event that $\bar{X}(\ell) - \overline{\text{Corner}}^{\ell, \mathcal{F}} \in (0, \infty)_{\geq}^k$ and $\bar{X}(r) - \overline{\text{Corner}}^{r, \mathcal{F}} \in (0, \infty)_{\geq}^k$.

Next we give a precise description of candidate ensembles X which are such that $\mathcal{L}^{\text{re},X}$, conditioned on the non-intersection tests, has the \mathcal{F} -conditional distribution of the top k curves of \mathcal{L}_n . In our ultimate definition of the jump ensemble, we will need to work with random sets, and our discussion of candidate ensembles in the next section will allow various sets to be random. The necessary definitions and facts about measurability will be collected in Appendix B and we will reference statements from that appendix when needed.

4.1.3. Applying the Brownian Gibbs property to $\mathcal{L}^{\text{re},X}$. — The resampled ensemble $\mathcal{L}^{\text{re},X}$ passing the non-intersection tests on $[-2T, 2T]$ requires

- $\mathcal{L}^{\text{re},X}(i, x) > \mathcal{L}^{\text{re},X}(i+1, x)$ for $i = 1, \dots, k-1$ and $x \in [-2T, 2T]$; and
- $\mathcal{L}^{\text{re},X}(k, x) > \mathcal{L}_n(k+1, x)$ for $x \in [-2T, 2T]$.

We denote by $\text{Pass}(X)$ the indicator for the event described by these two bullet points. (In [Ham19b], an analogous indicator obtained by restricting these two bullet points to $x \in [\ell, r]$ is denoted $T_3(X)$, where 3 represents the test of non-intersection on the middle interval being the third in a sequence of tests.)

Now we may describe a class of candidate ensembles which, conditioned on $\{\text{Pass}(X) = 1\}$, have the desired \mathcal{F} -conditional distribution. Let $[\ell, r]$ be a subset of $[-2T, 2T]$; we will also allow it to be a *random* interval as long as it is \mathcal{F} -measurable. Let $A \subseteq [\ell, r]$ be an \mathcal{F} -measurable random closed set (a random element of the space of closed subsets of $[\ell, r]$ endowed with a certain σ -algebra, whose definition is given in Definition B.1). If $[\ell, r]$ is random, we assume the inclusion $A \subseteq [\ell, r]$ holds almost surely.

Conditionally on \mathcal{F} if $[\ell, r]$ is random, define the candidate ensemble $X : \llbracket 1, k \rrbracket \times [\ell, r] \rightarrow \mathbb{R}$ as the restriction to $[\ell, r]$ of a collection of k independent Brownian bridges, the i^{th} one from $(-2T, \mathcal{L}_n(i, -2T))$ to $(2T, \mathcal{L}_n(i, 2T))$, conditioned on $\text{NonIntLow}_{\mathcal{L}_n(k+1, \cdot)}^A$, which we define to be the event that the curves $X(1, \cdot), \dots, X(k, \cdot)$ are each larger than the lower curve $\mathcal{L}_n(k+1, \cdot)$ on A .

More precisely, we assume that there exists a probability space which supports both \mathcal{L}_n and a collection of k independent Brownian bridges conditioned on $\text{NonIntLow}_{\mathcal{L}_n(k+1, \cdot)}^A$. Conditionally on \mathcal{F} , the latter is almost surely a positive probability event, as the event that a collection of k independent Brownian bridges with endpoints above $\mathcal{L}_n(k+1, \cdot)$ satisfies $\text{NonIntLow}_{\mathcal{L}_n(k+1, \cdot)}^{[-2T, 2T]}$ has positive probability. Thus the existence of these conditioned Brownian bridges is clear (for example, by a standard procedure of sampling independent Brownian bridges on $[-2T, 2T]$ till the avoidance condition is met, which will almost surely eventually happen if the probability of avoidance is positive), as long as $\text{NonIntLow}_{\mathcal{L}_n(k+1, \cdot)}^A$ is indeed an event, i.e., is measurable. This needs to be verified as A itself is a random closed set, and we perform this straightforward task using results on random closed sets from [Mol05] in Lemma B.8.

Also define $X' : \llbracket 1, k \rrbracket \times [\ell, r] \rightarrow \mathbb{R}$ in the same way, with the additional conditioning that $\bar{X}'(x) - \bar{\text{Corner}}^{x, \mathcal{F}} \in (0, \infty)_{\geq}^k$ for $x \in \{\ell, r\}$; equivalently, X' may be described in the same way as X with A replaced by $A \cup [-2T, \ell] \cup [2T, r]$. We introduce this variant candidate ensemble as it is the form that the jump ensemble will take.

Both X and X' have the desired \mathcal{F} -conditional distribution on passing the non-intersection tests, as we prove in Lemma 4.4. To prove this, we first need a fact about decompositions of Brownian bridges. This property of Brownian bridges also explains why there is no difficulty in conditioning on the potentially complicated objects, the side bridges of \mathcal{L}_n , as mentioned in Remark 4.1. The proof of this fact is a straightforward checking of covariances and is omitted.

Lemma 4.3. — *Let $T > 0$ and $x_1, \dots, x_m \in [-2T, 2T]$ with $x_1 < \dots < x_m$, for some $m \in \mathbb{N}$. Let $x_0 = -2T$ and $x_{m+1} = 2T$. Let B be a Brownian bridge (with arbitrary fixed starting and ending point values) on $[-2T, 2T]$. Then, the distribution of $(B^{[x_i, x_{i+1}]})_{i=0}^m$ is that of $m+1$ independent Brownian bridges, with the i^{th} one of duration $x_i - x_{i-1}$.*

Lemma 4.4. — *For $[\ell, r]$ and X as defined above, conditionally on \mathcal{F} , the following two laws on $\mathcal{C}_{*,*}(\llbracket 1, k \rrbracket \times [-2T, 2T], \mathbb{R})$ are equal:*

$$\mathbb{P}_{\mathcal{F}}(\mathcal{L}^{\text{re}, X} \in \cdot \mid \text{Pass}(X) = 1) \quad \text{and} \quad \mathbb{P}_{\mathcal{F}}(\mathcal{L}_n \in \cdot)$$

(where \mathcal{L}_n is implicitly restricted to $\llbracket 1, k \rrbracket \times [-2T, 2T]$). The same holds with X' in place of X .

PROOF OF LEMMA 4.4. We will prove the statement for X , as the argument works verbatim for X' .

We need to show that almost surely, for any bounded measurable function $g : \mathcal{C}_{*,*}([-2T, 2T], \mathbb{R}) \rightarrow \mathbb{R}$,

$$\mathbb{E}[g(\mathcal{L}_n) \mid \mathcal{F}] = \mathbb{E}[g(\mathcal{L}^{\text{re}, X}) \mid \text{Pass}(X) = 1].$$

We can couple together all the conditional expectations on the LHS (as g varies over all bounded measurable functions) using the regular conditional distribution $\mathbb{P}_{\mathcal{F}}$. This exists by Lemma B.9. Thus it is sufficient to show the above equality for each fixed bounded measurable g almost surely.

Let $E(\mathcal{L}_n)$ denote an \mathcal{F} -measurable event of the process on $[-2T, 2T]$. Showing the previous display means we have to show that, for any such $E(\mathcal{L}_n)$,

$$\mathbb{E}[g(\mathcal{L}_n)\mathbb{1}_{E(\mathcal{L}_n)}] = \mathbb{E}[\mathbb{E}[g(\mathcal{L}^{\text{re},X}) \mid \text{Pass}(X) = 1]\mathbb{1}_{E(\mathcal{L}_n)}].$$

We start with the LHS. We adopt the shorthand $\text{NonInt} = \text{NonInt}_{\mathcal{L}_n(k+1,\cdot)}$. Let $B = (B_1, \dots, B_k)$ be a collection of k independent Brownian bridges with B_i from $(-2T, \mathcal{L}_n(i, -2T))$ to $(2T, \mathcal{L}_n(i, 2T))$. We adopt the following convention: symbols like B , X , and \mathcal{L}_n will all refer to the same mapping on the probability space, and the change in symbol will indicate a change in the underlying probability measure (which thus changes the stochastic process). For example, by the Brownian Gibbs property, conditionally on \mathcal{F}_{BB} , \mathcal{L}_n will be replaced by B under the suitable non-intersection conditioning to indicate that the *measure* is given by that of suitably non-intersecting Brownian bridges. With this convention and by the Brownian Gibbs property,

$$(16) \quad \mathbb{E}[g(\mathcal{L}_n)\mathbb{1}_{E(\mathcal{L}_n)}] = \mathbb{E}[\mathbb{E}_{\mathcal{F}_{\text{BB}}}[g(B)\mathbb{1}_{E(B)} \mid \text{NonInt}]] = \mathbb{E}\left[\frac{\mathbb{E}_{\mathcal{F}_{\text{BB}}}[g(B)\mathbb{1}_{E(B)} \cdot \mathbb{1}_{\text{NonInt}}]}{\mathbb{P}_{\mathcal{F}_{\text{BB}}}(\text{NonInt})}\right].$$

Let $\{B \in \text{NonIntLow}^A\}$, where $\text{NonIntLow}^A = \text{NonIntLow}_{\mathcal{L}_n(k+1,\cdot)}^A$, denote the event that B meets the requirements in the definition of X , so that the law of B conditioned on $\{B \in \text{NonIntLow}^A\}$ is the law of X ; more precisely, conditionally on \mathcal{F} , the Radon-Nikodym derivative of the law of X with respect to the law of B is $\mathbb{1}_{B \in \text{NonIntLow}^A} / \mathbb{P}_{\mathcal{F}}(B \in \text{NonIntLow}^A)$. (This fact implicitly makes use of the independence property of Brownian bridges recorded in Lemma 4.3, as the definition of X is given on $[\ell, r]$ while the Radon-Nikodym derivative here is for processes on $[-2T, 2T]$). We observe that the event NonIntLow^A may (and indeed, does) involve \mathcal{F} -measurable parameters, such as the \mathcal{F} -measurable set A on which non-intersection is imposed, or the Corner random variables involved in the definition of X' , without affecting our argument.

Let us now simplify the numerator of the inner expression of (16):

$$\mathbb{E}_{\mathcal{F}_{\text{BB}}}[g(B)\mathbb{1}_{E(B)} \cdot \mathbb{1}_{\text{NonInt}}] = \mathbb{E}_{\mathcal{F}_{\text{BB}}}\left[\mathbb{E}_{\mathcal{F}}[g(B)\mathbb{1}_{\text{NonInt}}]\mathbb{1}_{E(B)}\right] = \mathbb{E}_{\mathcal{F}_{\text{BB}}}\left[\mathbb{E}_{\mathcal{F}}[g(\mathcal{L}^{\text{re},B})\mathbb{1}_{\text{NonInt}}]\mathbb{1}_{E(B)}\right],$$

using the definition (14) of $\mathcal{L}^{\text{re},B}$ for the second equality. From this equality, since $\mathbb{1}_{\text{NonInt}} = \mathbb{1}_{B \in \text{NonIntLow}^A} \cdot \mathbb{1}_{\text{Pass}(B)}$, and using the definition of the Radon-Nikodym derivative of the law of X with respect to the law of B , we see

$$\begin{aligned} & \mathbb{E}_{\mathcal{F}_{\text{BB}}}[g(B)\mathbb{1}_{E(B)} \cdot \mathbb{1}_{\text{NonInt}}] \\ &= \mathbb{E}_{\mathcal{F}_{\text{BB}}}\left[\mathbb{E}_{\mathcal{F}}[g(\mathcal{L}^{\text{re},X})\mathbb{1}_{\text{Pass}(X)=1}] \cdot \mathbb{P}_{\mathcal{F}}(B \in \text{NonIntLow}^A) \cdot \mathbb{1}_{E(B)}\right] \\ &= \mathbb{E}_{\mathcal{F}_{\text{BB}}}\left[\mathbb{E}_{\mathcal{F}}[g(\mathcal{L}^{\text{re},X}) \mid \text{Pass}(X) = 1] \cdot \mathbb{P}_{\mathcal{F}}(B \in \text{NonIntLow}^A) \cdot \mathbb{P}_{\mathcal{F}}(\text{Pass}(X) = 1) \cdot \mathbb{1}_{E(B)}\right]. \end{aligned}$$

Now, since $\mathbb{P}_{\mathcal{F}}(\text{Pass}(X) = 1) = \mathbb{P}_{\mathcal{F}}(\text{Pass}(B) = 1, B \in \text{NonIntLow}^A) / \mathbb{P}_{\mathcal{F}}(B \in \text{NonIntLow}^A)$, and since $\mathbb{1}_{\text{Pass}(B)=1, B \in \text{NonIntLow}^A} = \mathbb{1}_{\text{NonInt}}$, the previous display is equal to

$$\mathbb{E}_{\mathcal{F}_{\text{BB}}}\left[\mathbb{E}_{\mathcal{F}}[g(\mathcal{L}^{\text{re},X}) \mid \text{Pass}(X) = 1] \cdot \mathbb{P}_{\mathcal{F}}(\text{NonInt}) \cdot \mathbb{1}_{E(B)}\right]$$

By the tower property of conditional expectation, the last expression equals

$$\mathbb{E}_{\mathcal{F}_{\text{BB}}}\left[\mathbb{E}_{\mathcal{F}}[g(\mathcal{L}^{\text{re},X}) \mid \text{Pass}(X) = 1] \cdot \mathbb{1}_{\text{NonInt}} \cdot \mathbb{1}_{A(B)}\right].$$

Putting the last display into (16) and again using the tower property, we see that

$$\begin{aligned} \mathbb{E}[g(\mathcal{L}_n)\mathbb{1}_{E(\mathcal{L}_n)}] &= \mathbb{E}\left[\frac{\mathbb{E}_{\mathcal{F}}[g(\mathcal{L}^{\text{re},X}) \mid \text{Pass}(X) = 1] \cdot \mathbb{1}_{\text{NonInt}} \cdot \mathbb{1}_{E(B)}}{\mathbb{P}_{\mathcal{F}_{\text{BB}}}(\text{NonInt})}\right] \\ &= \mathbb{E}\left[\mathbb{E}_{\mathcal{F}}[g(\mathcal{L}^{\text{re},X}) \mid \text{Pass}(X) = 1] \cdot \mathbb{1}_{E(\mathcal{L}_n)}\right], \end{aligned}$$

using the Brownian Gibbs property for the last equality. This is what was to be shown. \square

Having completed the general set up and groundwork of candidate ensembles, we may now turn to describing the jump ensemble itself.

4.1.4. Parameters of the jump ensemble. — We start with two parameters, $k \in \mathbb{N}$ and $\varepsilon > 0$. The first is simply the number of curves of \mathcal{L}_n that we are studying, which will also be the number of curves in the jump ensemble. The second is to be understood as the Brownian probability of the event that we wish to analyse under the law of \mathcal{L}_n , but is formally simply a positive parameter. The logic of the jump ensemble is to set the parameters according to the event we wish to study.

Note that the statements made in Section 4.1 allowed the interval $[\ell, r]$ to be random as long as the endpoints are \mathcal{F} -measurable. We will be working with a particular such choice of ℓ and r , which we will denote \mathfrak{l} and \mathfrak{r} , that we will introduce shortly. The important property of these variables is that they are defined purely in terms of the lower curve $\mathcal{L}_n(k+1, \cdot)$ and so are \mathcal{F} -measurable.

In Section 4.1.1 we introduced a parameter T . For the jump ensemble, the value of T is determined by both parameters k and ε , and is given by

$$(17) \quad T := D_k(\log \varepsilon^{-1})^{1/3},$$

where D_k is given by (12). The value of T given in (17) will be its fixed value for the remainder of this article.

Remark 4.5. — Though not needed for our arguments, here is a heuristic idea of why this is the form of T we select; a fuller discussion is available in the beginning of [Ham19b, Chapter 4]. We mentioned in Section 4.1.1.3 the high jump difficulty, and pointed out that a Brownian bridge can more easily make a large jump if there is more space, i.e., T is increased. However, as T gets larger we must also grapple with the globally parabolic curvature of \mathcal{L}_n , which means that the starting and ending points of the candidate ensemble will fall quadratically. Simply put, there is more space for the run-up before the jump, but the required jump is higher as the starting point is lower. This selection of T —in particular the $1/3$ exponent of $\log \varepsilon^{-1}$ —balances these opposing forces and in some sense maximises the non-intersection test probability of the to-be-defined jump ensemble.

Remark 4.6. — This same reasoning of balancing these opposing forces of curvature and run-up advantage is what shows that a run-up is not sufficient to handle the second instance of lower curve data discussed in Subsection 4.1.1.3, as explained in the beginning of [Ham19b, Chapter 4]. Indeed, what that discussion essentially shows is that even with the well-tuned choice of T made above, there exists data in \mathcal{F} with sufficiently high probability for which the Brownian bridge candidate ensemble (which has ignored all data about the lower curve) is unable to pass the non-intersection test with sufficiently high probability.

We now record a certain upper and lower bound that the parameter ε is required to meet for technical reasons; these constraints also previously appeared in the statement of Theorem 3.10.

$$(18) \quad \begin{aligned} \varepsilon &< e^{-1} \wedge (17)^{-1/k} C_k^{-1/k} D_k^{-1} \wedge \exp\left\{- (24)^6 d^6 / D_k^3\right\} \quad \text{and} \\ \varepsilon &> \exp\left\{- (c/2 \wedge 2^{1/2}) D_k^{-1} n^{1/12}\right\}. \end{aligned}$$

As we noted in Remark 3.11, both these bounds do not cause any difficulties in practice. And, in the case that $n = \infty$, the lower bound becomes simply $\varepsilon > 0$.

With these definitions, we may start making precise the notion introduced earlier of a coarsened version of the underlying curve.

4.1.5. Coarsening the lower curve. — Let $\mathbf{c}_+ : [-T, T] \rightarrow \mathbb{R}$ be the least concave majorant of $\mathcal{L}_n(k+1, \cdot) : [-T, T] \rightarrow \mathbb{R}$, and define a random interval $[\mathfrak{l}, \mathfrak{r}]$ by

$$\begin{aligned} \mathfrak{l} &= \inf \{x \in [-T, T] : \mathbf{c}'_+(x) \leq 4T\} \quad \text{and} \\ \mathfrak{r} &= \sup \{x \in [-T, T] : \mathbf{c}'_+(x) \geq -4T\}; \end{aligned}$$

for concreteness, in these definitions we take $\inf \emptyset = -T$ and $\sup \emptyset = T$, though in all our arguments we will be working on favourable events where this extreme case does not arise.

We can think of \mathbf{c}_+ as a first coarsening of the lower curve $\mathcal{L}_n(k+1, \cdot)$. As indicated earlier, the interval $[\mathfrak{l}, \mathfrak{r}]$ will play the role of $[\ell, r]$ in Section 4.1.1. Note that though random, \mathfrak{l} and \mathfrak{r} are functions of the curve $\mathcal{L}_n(k+1, \cdot)$ and not of the k curves we are attempting to study, and so meets the hypotheses of Lemma 4.4. The purpose of defining this random interval is that on it we are guaranteed some control over the coarsened lower curve profile (which we will be further coarsening before the definition of the jump ensemble); we will not use any data about $\mathcal{L}_n(k+1, \cdot)$ outside of $[\mathfrak{l}, \mathfrak{r}]$ in defining the jump ensemble, though it is available.

By the concavity of \mathbf{c}_+ , it follows that $\mathfrak{l} \leq \mathfrak{r}$, and in fact on a high probability favourable event we will discuss in Section 4.1.8, $[\mathfrak{l}, \mathfrak{r}]$ will be an interval of length at least T . We use the σ -algebra \mathcal{F} defined in Section 4.1.1 on page 39, except with \mathfrak{l} and \mathfrak{r} in the place of ℓ and r .

As we saw, we need to define a coarsening that the candidate ensemble will be defined to jump over. So far we have defined a preliminary coarsening, the least concave majorant \mathbf{c}_+ , which was used to define the interval $[\mathfrak{l}, \mathfrak{r}]$. Now we define a further (and final) coarsening, which we will then use in the next subsection to finally define the jump ensemble J .

To precisely describe the final coarsening of $\mathcal{L}_n(k+1, \cdot)$ on $[\mathfrak{l}, \mathfrak{r}]$, we first define a subset of extreme points of \mathbf{c}_+ . Let $\text{xExt}(\mathbf{c}_+)$ be the x -coordinates of the set of extreme (or corner) points of the convex set $\{(x, y) : \mathfrak{l} \leq x \leq \mathfrak{r}, y \leq \mathbf{c}_+(x)\}$. Note that necessarily $\mathfrak{l}, \mathfrak{r} \in \text{xExt}(\mathbf{c}_+)$. Then define the *pole set* P to be a subset of $\text{xExt}(\mathbf{c}_+)$ such that

- $\mathfrak{l}, \mathfrak{r} \in P$,
- $p_1, p_2 \in P, p_1 \neq p_2 \implies |p_1 - p_2| \geq d_{\text{ip}}$, and
- if $x \in \text{xExt}(\mathbf{c}_+)$, then some element $p \in P$ satisfies $|p - x| \leq d_{\text{ip}}$.

Here $d_{\text{ip}} \in [1, \mathfrak{r} - \mathfrak{l}]$ is a parameter called the *inter-pole distance*; typically it is set to a unit order quantity independent of ε and k , and usually it is comparable to the interval of interest under study. For example, in the proof of our results, we will set it to be a multiple of d . The parameter d_{ip} defines the minimum separation between consecutive elements of P . These elements of the pole set P will be called *poles*. These above three properties do not necessarily define P uniquely, and to address this we take P to be the subset satisfying these conditions obtained as follows.⁽¹⁾ More explicitly, we can define P in an iterative manner. First it includes \mathfrak{l} . Then we add the first element of $\text{xExt}(\mathbf{c}_+)$ that is greater than or equal to $\mathfrak{l} + d_{\text{ip}}$, unless that is within d_{ip} of \mathfrak{r} , in which case we include \mathfrak{r} . We repeat the previous sentence with the largest current

⁽¹⁾In [Ham19b], P was defined to be the subset of $\text{xExt}(\mathbf{c}_+)$ satisfying the conditions of being maximal in cardinality and then maximal in lexicographically. However, the maximality properties did not play any role in the arguments and were chosen purely to uniquely define P . We have adopted a more explicit unique definition to make measurability properties easier to verify.

element of the set in place of \mathfrak{l} as long as \mathfrak{r} is not yet in the set. It is easy to see the set finally obtained from this procedure satisfies the above three bullet points.

(Observe that $\text{xExt}(\mathfrak{c}_+)$ and P are random sets, so we must specify the measure space in which they take values in order to ensure that our constructions and the events involved are all measurable. As mentioned earlier, we collect the definition of the measure space and facts about the measurability of standard set operations in Appendix B. In particular, we verify in Proposition B.7 that the mapping from $\mathcal{L}_n(k+1, \cdot)$ to P is measurable, so that P is a genuine random closed set, and is also \mathcal{F} -measurable.)

Remark 4.7. — By the definition of d_{ip} , it is clear that the size of the pole set, $|P|$, is at most $2T/d_{\text{ip}}$. So for constant order values of d_{ip} , $|P|$ is potentially rather large. However, in arguments the only poles which must be considered are essentially the ones within or adjacent to the interval under study. In the proof of our main result our arguments will be focused on the single pole contained in the interval $[-2d, 2d]$ (if it is present), and will in one instance make use of the preceding and succeeding poles. We guarantee ourselves this control on the number of poles in $[-d, d]$ by making an appropriate choice of d_{ip} .

The coarsened profile of $\mathcal{L}_n(k+1, \cdot)$ that will be used in defining the jump ensemble J is exactly the set $\{(p, \mathcal{L}_n(k+1, p)) \mid p \in P\}$, which J will be conditioned to *jump* over. We next make precise what we mean by J jumping over the poles, and also give the definition of J .

4.1.6. Defining the jump ensemble. — Conditional on \mathcal{F} , let $B : \llbracket 1, k \rrbracket \times [-2T, 2T] \rightarrow \mathbb{R}$, with $\{B(i, \cdot)\}_{i=1}^k$ a collection of k independent Brownian bridges on $[-2T, 2T]$ and $B(i, \cdot)$ having endpoints $(-2T, \mathcal{L}_n(i, -2T))$ and $(2T, \mathcal{L}_n(i, 2T))$ for $i = 1, \dots, k$. Note that the required information about the endpoint values of \mathcal{L}_n is present in \mathcal{F} . The jump ensemble $J : \llbracket 1, k \rrbracket \times [\mathfrak{l}, \mathfrak{r}] \rightarrow \mathbb{R}$ is the restriction to $[\mathfrak{l}, \mathfrak{r}]$ of B conditioned on

- (i) $\overline{B}(x) - \overline{\text{Corner}}^{x, \mathcal{F}} \in (0, \infty)_{>}^k$ for $x \in \{\mathfrak{l}, \mathfrak{r}\}$; and
- (ii) $B(i, p) \geq \mathcal{L}_n(k+1, p)$ for all $p \in P$ and $i = 1, \dots, k$.

As we saw in Lemma 4.2, the conditioning present in point (i) ensures that J passes the side interval tests (15). We will refer to the event in point (ii), namely $J(p) \geq \mathcal{L}_n(k+1, p)$ for $p \in P$, as *jumping over the pole* p .

As we noted when we stated its definition, the pole set P represents the coarsened version of the lower curve $\mathcal{L}_n(k+1, \cdot)$ that J has access to. By conditioning J to avoid this coarsened version of $\mathcal{L}_n(k+1, \cdot)$, we increase the probability of J successfully avoiding all of $\mathcal{L}_n(k+1, \cdot)$ on $[\mathfrak{l}, \mathfrak{r}]$, compared (heuristically) to a candidate process with no information about the underlying curve. Indeed we will see ahead in Proposition 4.8 that this increased probability is high enough to be useful for our intended application.

It will be necessary in our arguments to consider how much J deviates from the shape defined by the poles. To do this, define the \mathcal{F} -measurable random piecewise affine function $\text{Tent} : [\mathfrak{l}, \mathfrak{r}] \rightarrow \mathbb{R}$ which linearly interpolates between the points $(p, \mathcal{L}_n(k+1, p))$ for $p \in P$. Note that from the definition of \mathfrak{c}_+ we have that Tent is concave, and from the definition of \mathfrak{l} and \mathfrak{r} we have that the slope of every linear segment of Tent lies in $[-4T, 4T]$, which for future reference we will express (with abuse of notation, as Tent is only piecewise linear and not linear) as

$$(19) \quad \text{slope}(\text{Tent}) \in [-4T, 4T].$$

See Figure 4.4 for an illustration of the jump ensemble and the Tent map.

Note also that Lemma 4.4 implies that $\mathcal{L}^{\text{re}, J}$, conditionally on \mathcal{F} and $\{\text{Pass}(J) = 1\}$, has the \mathcal{F} -conditional distribution of the top k curves of \mathcal{L}_n on $[-2T, 2T]$. Since $\mathcal{L}^{\text{re}, J} = J$ on $\llbracket 1, k \rrbracket \times [\mathfrak{l}, \mathfrak{r}]$, this implies that the distribution of J , conditionally on \mathcal{F} and $\{\text{Pass}(J) = 1\}$, is that of the top k curves of \mathcal{L}_n on $[\mathfrak{l}, \mathfrak{r}]$.

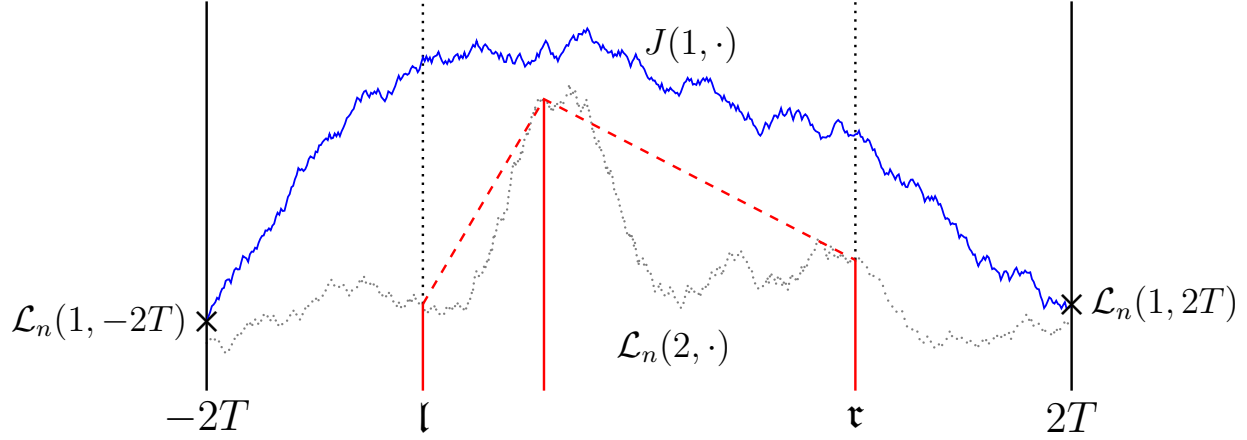


FIGURE 4.4. The jump ensemble candidate J (in blue) when $k = 1$. The lines in red are the *poles*, i.e., the elements of the pole set P , which J is conditioned to jump over (recall that necessarily $l, r \in P$). The dashed red piecewise linear function is the Tent map. The lower curve is drawn dotted and in light gray to indicate that J does not have access to this data, only the heights of the solid poles. Making this jump forces the candidate to avoid a coarsened version of the lower curve profile, which makes it more likely that it also avoids the full lower curve, as in this figure. Moving back to $[-2T, 2T]$ from $[l, r]$ gives the candidate space to make the jump. Though we have shown the blue curve on the entirety of $[-2T, 2T]$, J is only the restriction to $[l, r]$, and so does *not* need to avoid the lower curve outside $[l, r]$ (though it does in the illustrated instance). The avoidance on the full interval $[-2T, 2T]$ is a requirement imposed only on $\mathcal{L}^{\text{re}, J}$, which is made by combining J with the data in \mathcal{F} as in Figure 4.2.

4.1.7. The probability that J passes the non-intersection test. — Now we shall address whether the jump ensemble is in fact able to pass the non-intersection test on the whole interval $[-2T, 2T]$ with sufficiently high probability. Recall from Section 4.1.2 that we have an easy criterion for J passing the non-intersection test on the side intervals of $[-2T, l]$ and $[r, 2T]$, which we called the side intervals test (15). This criterion is that $(J(i, x) - \text{Corner}_i^{x, \mathcal{F}})_{i=1}^k \in (0, \infty)_{>}^k$ for $x \in \{l, r\}$, which J is in fact conditioned to satisfy above in point (i) of its definition. Thus all that remains is for J to pass the non-intersection test on the middle interval $[l, r]$, i.e., for J to satisfy

$$J(1, x) > J(2, x) > \dots > J(k, x) > \mathcal{L}_n(k+1, x) \quad \forall x \in [l, r].$$

In other words, the indicator of this last event is the same as the indicator $\text{Pass}(J)$. It is important for our approach that the event $\{\text{Pass}(J) = 1\}$ being conditioned on does not have too low a probability. Unlike the side intervals test, there is no simple criterion for the middle interval test. In fact, an analysis was undertaken in [Ham19b] to obtain a sufficiently strong lower bound on this probability, which holds on a high probability \mathcal{F} -measurable favourable event Fav (that we will define shortly). As that argument does not serve our expository purpose, we do not present it here; instead we reproduce the statement from [Ham19b]. This is the statement we previously referenced as *jump ensemble candidate proficiency*.

Proposition 4.8 (Jump ensemble candidate proficiency, Proposition 4.2 of [Ham19b])

We have that

$$\mathbb{P}_{\mathcal{F}}(\text{Pass}(J) = 1) \geq \exp\left\{-3973k^{7/2}d_{\text{ip}}^2D_k^2(\log \varepsilon^{-1})^{2/3}\right\} \cdot \mathbb{1}_{\text{Fav}}.$$

We will now define Fav , before returning to discuss the important role of Proposition 4.8 in our approach.

4.1.8. The definition of Fav & the role of Proposition 4.8. — The favourable event Fav is defined as the intersection

$$\text{Fav} = F_1 \cap F_2 \cap F_3,$$

where

$$\begin{aligned} F_1 &= \left\{ \mathcal{L}_n(i, x) \in T^2[-2\sqrt{2} - 1, -2\sqrt{2} + 1] \text{ for } (i, x) \in \llbracket 1, k \rrbracket \times \{-2T, 2T\} \right\} \\ F_2 &= \left\{ -T^2 \leq \mathcal{L}_n(k+1, x) \leq T^2 \text{ for } x \in [-T, T] \right\}, \\ F_3 &= \bigcap_{i \in \llbracket 1, k \rrbracket} \left\{ \text{Corner}_i^{\mathfrak{l}, \mathcal{F}} \in [-T^2, T^2] \right\} \cap \left\{ \text{Corner}_i^{\mathfrak{r}, \mathcal{F}} \in [-T^2, T^2] \right\}. \end{aligned}$$

Note that Fav is an \mathcal{F} -measurable event. As its name suggests, this event fixes good data in \mathcal{F} on which we have strong enough control to make our arguments. The reader should view this data as being fixed in the arguments involving the jump ensemble, as we will be working only on this event; the bound on $\mathbb{P}(\text{Fav}^c)$ just ahead allows us to take this liberty.

The form of the favourable event respects the parabolic curvature possessed by \mathcal{L}_n . In particular, since we are working on the interval $[-2T, 2T]$, we expect that at the endpoints the location of \mathcal{L}_n will be $O(-T^2)$, which dictates the form of the three subevents F_1, F_2 , and F_3 above.

It is a simple calculation based on the definition of \mathfrak{l} and \mathfrak{r} that, on F_2 ,

$$\mathfrak{l} \leq -T/2 \quad \text{and} \quad \mathfrak{r} \geq T/2.$$

We need the knowledge that the favourable event occurs with sufficiently high probability; this is provided to us from [Ham19b]:

Lemma 4.9 (High probability of favourable event, Lemma 4.1 of [Ham19b])

$$\mathbb{P}(\text{Fav}^c) \leq \varepsilon^{2^{-5}c_k D_k^3}.$$

In fact for our purposes it will be sufficient to note that $2^{-5}c_k D_k^3 \geq 1$ for all k , and so the upper bound above is further bounded by ε .

Now we may discuss the central role of Proposition 4.8 in our argument. We will use it to reduce the problem of understanding the probability of an event under the \mathcal{F} -conditional law of \mathcal{L}_n to understanding the same under the law of J . For concreteness, let us illustrate this by attempting to bound the probability that the vertically shifted k^{th} curve $\mathcal{L}_n(k, \cdot) - \mathcal{L}_n(k, -d)$ lies in a measurable subset $A \subseteq \mathcal{C}_{0,*}([-d, d], \mathbb{R})$ of continuous functions vanishing at $-d$, where $d > 0$. Recall again from Lemma 4.4 that the \mathcal{F} -conditional distribution of \mathcal{L}_n on $[-2T, 2T]$ is the same as the distribution of $\mathcal{L}^{\text{re}, J}$ on $[-2T, 2T]$ conditioned on the event that $\text{Pass}(J) = 1$, and also that (by definition) $\mathcal{L}^{\text{re}, J}(i, \cdot) = J(i, \cdot)$ on $[\mathfrak{l}, \mathfrak{r}] \subseteq [-2T, 2T]$. We assume that ε is small enough that $[-d, d] \subseteq [-T/2, T/2] \subseteq [\mathfrak{l}, \mathfrak{r}]$, the last inclusion on the event Fav. (This assumption on ε is implied by the condition that $\varepsilon < \exp(-(24)^6 d^6 / D_k^3)$ that is imposed in Theorem 3.10.) We also have to set the last parameter of the jump ensemble, the inter-pole distance d_{ip} , which we set as

$$(20) \quad d_{\text{ip}} = 5d,$$

which will be its value in our application of the jump ensemble. Then we see that

$$\begin{aligned}
(21) \quad \mathbb{P}(\mathcal{L}_n(k, \cdot) - \mathcal{L}_n(k, -d) \in A) &= \mathbb{E} [\mathbb{P}_{\mathcal{F}}(\mathcal{L}_n(k, \cdot) - \mathcal{L}_n(k, -d) \in A) \cdot \mathbb{1}_{\text{Fav}}] + \mathbb{P}(\text{Fav}^c) \\
&= \mathbb{E} [\mathbb{P}_{\mathcal{F}}(J(k, \cdot) - J(k, -d) \in A \mid \text{Pass}(J) = 1) \cdot \mathbb{1}_{\text{Fav}}] + \mathbb{P}(\text{Fav}^c) \\
&\leq \mathbb{E} \left[\frac{\mathbb{P}_{\mathcal{F}}(J(k, \cdot) - J(k, -d) \in A)}{\mathbb{P}_{\mathcal{F}}(\text{Pass}(J) = 1)} \cdot \mathbb{1}_{\text{Fav}} \right] + \mathbb{P}(\text{Fav}^c) \\
&\leq \mathbb{E} [\mathbb{P}_{\mathcal{F}}(J(k, \cdot) - J(k, -d) \in A) \cdot \mathbb{1}_{\text{Fav}}] \cdot \exp \left\{ O_k(1) (\log \varepsilon^{-1})^{2/3} \right\} \\
&\quad + \mathbb{P}(\text{Fav}^c),
\end{aligned}$$

using Proposition 4.8 in the last inequality.

Theorem 3.10 asserts a bound of the form $\varepsilon \cdot \exp(O_k(1)(\log \varepsilon^{-1})^{5/6})$ on the left-hand side of the first line of the above display, where ε is the probability of A under the law of Brownian motion, and so the main step to prove the theorem is to bound the first term in the final expression by a quantity of the same form. The notational equivalence we have just made between the parameter ε of the jump ensemble and the Brownian motion probability of the event of interest is one we will adopt formally: in the remainder of the proof of Theorem 3.10, the parameter ε of the jump ensemble will have the value

$$(22) \quad \varepsilon := \mathcal{B}_{0,*}^{[-d,d]}(A).$$

(As we will record formally soon, the parabolic invariance Lemma 3.7 allows us to reduce Theorem 3.10 to the case where $K = 0$.) With this choice of ε , the importance of Proposition 4.8 in achieving the goal mentioned in the last paragraph is now clear, in particular that the exponent of the $\log \varepsilon^{-1}$ in the exponent in its statement is $2/3 < 5/6$. Looking back at (21), to actually achieve this goal we need two bounds: that $\mathbb{P}_{\mathcal{F}}(J(k, \cdot) - J(k, -d) \in A) \cdot \mathbb{1}_{\text{Fav}}$ and $\mathbb{P}(\text{Fav}^c)$ are both bounded by $\varepsilon \exp(O_k(1)(\log \varepsilon^{-1})^{5/6})$.

The second bound is implied by Lemma 4.9. Finally, in the following Theorem 4.10, we have the first bound, proving which will be the work of the next chapter. Recall that $\mathcal{B}_{0,*}^{[-d,d]}$ is the law on $\mathcal{C}_{0,*}([-d,d], \mathbb{R})$ of a Brownian motion started at coordinates $(-d, 0)$.

Theorem 4.10. — *Let $d \geq 1$ and $A \subseteq \mathcal{C}_{0,*}([-d,d], \mathbb{R})$. Then there exist $\varepsilon_0 = \varepsilon_0(d, k) = \exp(-(24)^6 d^6 / D_k^3)$ and absolute constants $G_1, G_2 < \infty$ such that, if $\mathcal{B}_{0,*}^{[-d,d]}(A) = \varepsilon < \varepsilon_0$,*

$$\mathbb{P}_{\mathcal{F}}(J(k, \cdot) - J(k, -d) \in A) \cdot \mathbb{1}_{\text{Fav}} \leq \varepsilon \cdot G_1 d^{\frac{1}{2}} \cdot D_k^4 (\log \varepsilon^{-1})^{\frac{4}{3}} \cdot \exp \left(G_2 \cdot d \cdot D_k^{5/2} \cdot (\log \varepsilon^{-1})^{5/6} \right).$$

Given this theorem we may prove the main Theorem 3.10:

PROOF OF THEOREM 3.10. By applying the parabolic invariance Lemma 3.7 with $y_n = -K$, proving Theorem 3.10 reduces to the case $K = 0$. The condition that $[K - d, K + d] \subset c/2 \cdot [-n^{1/9}, n^{1/9}]$ is exactly the one required to apply Lemma 3.7.

Using Proposition 4.8, Lemma 4.9 (with our choice of D_k and since we have assumed (13)), and Theorem 4.10 in (21) gives that $\mathbb{P}(\mathcal{L}_n(k, \cdot) - \mathcal{L}_n(k, -d) \in A)$ is bounded above, up to a constant factor, by

$$\begin{aligned}
&\varepsilon \cdot d^{\frac{1}{2}} \cdot D_k^4 (\log \varepsilon^{-1})^{\frac{4}{3}} \cdot \exp \left(G_2 \cdot d \cdot D_k^{5/2} \cdot (\log \varepsilon^{-1})^{5/6} + 3973 k^{7/2} d_{\text{ip}}^2 D_k^2 (\log \varepsilon^{-1})^{2/3} \right) + \varepsilon \\
&\leq 2\varepsilon \cdot d^{\frac{1}{2}} \cdot D_k^4 (\log \varepsilon^{-1})^{\frac{4}{3}} \cdot \exp \left(G_2 \cdot d \cdot k^{7/2} \cdot D_k^{5/2} \cdot (\log \varepsilon^{-1})^{5/6} \right);
\end{aligned}$$

we have used that $d_{\text{ip}} = 5d$ from (20), $d \geq 1$, $D_k^{1/2} (\log \varepsilon^{-1})^{1/6} \geq 24d$ from the assumed upper bound on ε , and allowed G_2 to change its value. Since $x^{11/5} \leq e^x$ for all $x \geq 1$, we may absorb the factor $d^{\frac{3}{2}} D_k^{\frac{11}{2}} (\log \varepsilon^{-1})^{\frac{11}{6}}$ by increasing the coefficient of the exponent by 1. This proves Theorem 3.10. \square

Hence the remaining task is to prove Theorem 4.10. This is accomplished in Chapter 5. Before proceeding to this, we give in the next subsection an important statement about stochastic domination relations between the jump ensemble and certain Brownian bridges, and finally a concluding subsection giving a brief summary of the jump ensemble which may act as a quick reference for the reader.

4.1.9. A stochastic domination property of the jump ensemble. — In our proof of Theorem 4.10, at various points we will need to stochastically dominate $J(k, \cdot)$ by, or have $J(k, \cdot)$ stochastically dominate, certain Brownian bridges. For this we will make use of the following statement, which will be proven essentially using Lemma 4.3 to reduce stochastic domination of processes to obvious stochastic dominations of point-values. Statements like the law of $J(k, \cdot)$ on $[l, \tau]$ stochastically dominates that of B on $[l, \tau]$, conditionally on \mathcal{F} and on the event Fav , means that there exists a probability space supporting J and B such that

$$\mathbb{P}(J(k, x) \geq B(x) \text{ for all } x \in [l, \tau] \mid \mathcal{F}) \cdot \mathbb{1}_{\text{Fav}} = 1.$$

Lemma 4.11. — Fix $n \in \mathbb{N}$ and $k \leq n$. Let $\phi_{\text{start}} \geq \mathcal{L}_n(k, -2T)$ and $\phi_{\text{end}} \geq \mathcal{L}_n(k, 2T)$, and let $x \in [l, \tau]$. Then, on Fav and conditionally on \mathcal{F} ,

- (i) The law of $J(k, \cdot)$ (as a law on $[l, \tau]$) stochastically dominates the law of a Brownian bridge from $(l, -T^2)$ to $(\tau, -T^2)$.
- (ii) Conditionally on $J(k, x)$, the law of $J(k, \cdot)$ restricted to $[l, x]$ stochastically dominates the law of a Brownian bridge from $(l, -T^2)$ to $(x, J(k, x))$. Under the same conditioning, the law of $J(k, \cdot)$ restricted to $[x, \tau]$ stochastically dominates the law of a Brownian bridge from $(x, J(k, x))$ to $(\tau, -T^2)$.
- (iii) The law of $J(k, \cdot)$ is stochastically dominated by the law of the restriction to $[l, \tau]$ of a Brownian bridge from $(-2T, \phi_{\text{start}})$ to $(2T, \phi_{\text{end}})$ which is conditioned to be above all poles, above $(l, \text{Corner}_k^{l, \mathcal{F}})$, and above $(\tau, \text{Corner}_k^{\tau, \mathcal{F}})$.
- (iv) Conditionally on $J(k, x)$, the law of $J(k, \cdot)$, restricted to $[x, \tau]$, is stochastically dominated by the law of the restriction to $[x, \tau]$ of a Brownian bridge from $(x, J(k, x))$ to $(2T, \phi_{\text{end}})$ which is conditioned to be above all poles in $[x, \tau]$ and above $(\tau, \text{Corner}_k^{\tau, \mathcal{F}})$.

PROOF. Lemma 4.11(i) follows from (ii) by taking $x = \tau$ and averaging, and noting that, on Fav , $J(k, \tau) \geq \text{Corner}_k^{\tau, \mathcal{F}} \geq -T^2$. So we prove (ii); in fact, we prove the first part of (ii) as the second part is analogous.

Let B be a Brownian bridge from $(l, J(k, l))$ to $(x, J(k, x))$. Letting $m = |P|$ and x_1, \dots, x_m be the elements of P , we may apply Lemma 4.3 to decompose B at the elements of P . Recall that, conditionally on $J(k, l)$ and $J(k, x)$, the distribution of $J(k, \cdot)$ is that of B conditioned on the values of B at elements of P being sufficiently high. Also note that the decomposition provided by Lemma 4.3 says that the bridges in between consecutive elements of P , once affinely shifted to vanish at the endpoints, are independent. These two facts yield that the law of $J(k, \cdot)$, conditionally on its values at l and x , stochastically dominates that of B . Since, on Fav , $J(k, l) \geq \text{Corner}_k^{l, \mathcal{F}} \geq -T^2$, it is clear that B stochastically dominates the Brownian bridge described in Lemma 4.11(i), yielding the claim.

Now we turn to (iii). Given two intervals I_x for $x \in \{l, \tau\}$, consider the restriction to $[l, \tau]$ of a Brownian bridge B with starting point $(-2T, \mathcal{L}_n(k, -2T))$ and ending point $(2T, \mathcal{L}_n(k, 2T))$, conditioned on $B(x) \geq \mathcal{L}_n(k+1, x)$ for all $x \in P$, and on $B(x) \in I_x$ for $x \in \{l, \tau\}$. Call this law $\mathcal{B}(I_l, I_\tau)$.

Conditionally on $J(k-1, \cdot)$ as a process on $[l, \tau]$, the law of $J(k, \cdot)$ is $\mathcal{B}(I_l, I_\tau)$ with I_x a *finite* interval determined by $J(k-1, \cdot)$, in such a way that $\inf I_x = \text{Corner}_k^{x, \mathcal{F}}$, for $x \in \{l, \tau\}$. Using the same decomposition from Lemma 4.3, it is clear that $\mathcal{B}(I_l, I_\tau)$ is stochastically dominated by $\mathcal{B}(I'_l, I'_\tau)$, where $I'_x = [\text{Corner}_k^{x, \mathcal{F}}, \infty)$ for $x \in \{l, \tau\}$.

Averaging over $J(k-1, \cdot)$, we find that the law of $J(k, \cdot)$ is stochastically dominated by $\mathcal{B}(I'_l, I'_r)$. On Fav , the law of the process described in Lemma 4.11(iii) is that of the Brownian bridge B in the definition of $\mathcal{B}(I'_l, I'_r)$ with the endpoints shifted vertically upwards, which clearly maintains the described stochastic domination.

The proof of (iv) is along the same lines as (iii). \square

4.1.10. Summary of the jump ensemble. — The definition of the jump ensemble was rather involved, and here we provide a quick summary of the main aspects of its definition which the reader should keep in mind in order to understand the arguments leading to the proof of Theorem 4.10.

The jump ensemble has three parameters, $\varepsilon > 0$, $k \in \mathbb{N}$, and $d_{\text{ip}} > 0$; and is defined on an interval $[-2T, 2T]$, where $T = D_k(\log \varepsilon^{-1})^{1/3}$, with D_k given by (12). The interval $[-2T, 2T]$ contains a random subinterval $[l, r]$. The jump ensemble J is the restriction to $[l, r]$ of a collection of k independent Brownian bridges, the i^{th} from $(-2T, \mathcal{L}_n(i, -2T))$ to $(2T, \mathcal{L}_n(i, 2T))$, conditioned on $\bar{J}(x) - \bar{\text{Corner}}^{l, \mathcal{F}} \in (0, \infty)_{\geq}^k$ for $x \in \{l, r\}$ and on $J(p) > \mathcal{L}_n(k+1, p)$ for all p in the pole set P . The elements of the pole set P have a minimum separation of d_{ip} , and are a subset of $[l, r]$. This interval is defined in terms of the underlying curve $\mathcal{L}_n(k+1, \cdot)$, and the relevant consequence of its definition is that the slope (of each linear segment) of the Tent map (which linearly interpolates the points $(p, \mathcal{L}_n(k+1, p))$) lies in $[-4T, 4T]$. In our application, ε is set according to the Brownian motion probability of the event under consideration, and $d_{\text{ip}} = 5d$.

We gain control over several of the random objects present in the above paragraph on a high probability favourable event Fav . On this event, $[-T/2, T/2] \subseteq [l, r]$; that $\bar{\text{Corner}}^{l, \mathcal{F}}, \bar{\text{Corner}}^{r, \mathcal{F}} \in [-T^2, T^2]^k$; that $\mathcal{L}_n(k+1, x) \in [-T^2, T^2]$ for $x \in [-T, T]$; and that $\mathcal{L}_n(i, x) \in T^2[-2\sqrt{2}-1, -2\sqrt{2}+1]$ for $(i, x) \in \llbracket 1, k \rrbracket \times \{-2T, 2T\}$.

In the next section we discuss the conceptual framework underlying the proof of Theorem 4.10.

4.2. A conceptual framework in terms of costs

Notation. — We will use the notation $A \lesssim B$ to indicate that there exists a constant $G < \infty$ which is independent of ε , k , and d such that $A \leq GB$. The value of G may however vary from line to line.

We now begin discussing the ideas underlying the approach of the proof of Theorem 4.10. We fix the curve index k that we are studying, and for the proof of Theorem 4.10 in this section and in Chapter 5 adopt the abuse of notation

$$J(\cdot) = J(k, \cdot),$$

which we will refer to as the *jump curve*; none of our arguments will involve the other curves of the full jump ensemble.

The overarching conceptual framework is one of *costs* that a certain joint density of the jump curve J must satisfy. This section will set up the quantities the proof will be working with, introduce the costs these quantities must interact with, and conclude by showing that the proof of Theorem 4.10 can be reduced to showing that these costs are met.

We would like to study the probability that the jump curve J lies in some $A \subseteq \mathcal{C}_{0,*}([-d, d], \mathbb{R})$, where the probability of A under the law of Brownian motion is ε . As mentioned at the end of the last section, the parameters of the jump ensemble are ε and k , with $d_{\text{ip}} = 5d$. Our aim is to get a bound of ε on the jump curve probability of A , up to a constant multiplicative factor which is subpolynomial in ε^{-1} .

Recall that the jump ensemble involves the notion of a pole set. We have control over the minimum separation of consecutive poles, so by our choice of $d_{\text{ip}} = 5d$ we ensure that there is at most one pole in the larger interval $[-2d, 2d]$. Thus there are two possibilities: either the interval $[-2d, 2d]$ contains a single pole, or no poles.

In the case where there are no poles in $[-2d, 2d]$, there can of course be poles which are arbitrarily close to $[-2d, 2d]$, and we will see in our analysis that our bounds become too weak if a pole is too close to a point under consideration. This is why we consider the presence of poles in $[-2d, 2d]$ even though the interval of interest is $[-d, d]$: when there is no pole in $[-2d, 2d]$ we can focus the analysis at the points $\pm d$, which are then ensured a distance of at least d from the nearest pole. When there *is* a pole, we will adopt a trick that will be described later to allow us to step back from it, again giving us an order d distance from it.

In the scenario where there is no pole in $[-2d, 2d]$, the jump curve is essentially just a Brownian bridge on $[-2d, 2d]$, and the argument for the probability comparison we claim is much more straightforward. So for the purposes of the exposition in this section we discuss the more difficult case where there is a pole in $[-2d, 2d]$.

By the Brownian Gibbs property, we have a direct way to write the probability of A for the jump curve, given its endpoint values $J(-2d)$ and $J(2d)$, in terms of the Brownian bridge probability of A between those endpoint values. In some sense, our task is to show that this Brownian bridge probability becomes a Brownian motion probability when we take expectations over $J(-2d)$ and $J(2d)$.

4.2.1. Translating between Brownian motion and Brownian bridge. — It is instructive to look at how Brownian motion probabilities translate to Brownian bridge probabilities.

Now suppose we are again working on $[-2d, 2d]$, and suppose B is a rate one Brownian motion started from x -coordinate $-4d$ according to some probability distribution μ , independently of μ . Right now it is not clear why we need the device of the measure μ , but we will retain it to give ourselves some freedom which, by proper choice at a later point, will make the requirements more lenient. At present it can be thought of as, and indeed will later be defined to be, the uniform measure on a very large interval centred at 0.

On a heuristic level, the probability that $B(-4d) \in [y, y + dy]$ and $B(4d) \in [z, z + dz]$ is given by $(16\pi d)^{-\frac{1}{2}} \exp(-(y-z)^2/16d) d\mu(y) dz$. By the Markov property, $B(\cdot) - B(-d)$ is standard Brownian motion started at coordinates $(-d, 0)$ when restricted to $[-d, \infty)$. Now, recall that $\mathcal{B}_{0,*}^{[-d,d]}$ is the law of standard Brownian motion on $[-d, d]$, started at coordinates $(-d, 0)$, and that $\mathcal{B}_{y,z}^{[-4d,4d]}$ is the law of rate one Brownian bridge from $(-4d, y)$ to $(4d, z)$. Then for $A \subseteq \mathcal{C}_{0,*}([-d, d], \mathbb{R})$, we have the calculation

$$\begin{aligned}
 \mathcal{B}_{0,*}^{[-d,d]}(A) &= \mathbb{P}(B(\cdot) - B(-d) \in A) \\
 &= \mathbb{E} \left[\mathbb{P}(B(\cdot) - B(-d) \in A \mid B(-4d), B(4d)) \right] \\
 (23) \quad &= \frac{1}{\sqrt{16\pi d}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{B}_{y,z}^{[-4d,4d]}(\tilde{A}) \cdot e^{-(y-z)^2/16d} d\mu(y) dz,
 \end{aligned}$$

where \tilde{A} is the set of functions f in $\mathcal{C}_{*,*}([-4d, 4d], \mathbb{R})$ such that $f(\cdot) - f(-d)$, regarded as a function with domain $[-d, d]$, lies in A . We also used that Brownian motion, conditioned on its endpoints, is a Brownian bridge between those endpoints. As we said before, $(16\pi d)^{-\frac{1}{2}} \exp(-(y-z)^2/16d)$ is exactly the conditional density of $B(4d)$ at z given that $B(-4d)$ is y . A very similar calculation holds for the jump curve, which points us to what we should try to prove. Let $f_J(y, z)$ be the joint density of

$$Y^* := J(-4d) \quad \text{and} \quad Z^* := J(4d)$$

at the point $(y, z) \in \mathbb{R}^2$; the $*$ is an adornment that will be removed in the final definition of Y and Z that will be used in our actual arguments.

Suppose now, for simplicity, that the pole p in $[-2d, 2d]$ is at zero and has height zero (i.e., $p = 0$ and $\text{Tent}(p) = 0$). Then, on $\text{Fav} \cap \{P \cap [-2d, 2d] \neq \emptyset\}$,

$$\begin{aligned}
(24) \quad \mathbb{P}_{\mathcal{F}}(J(\cdot) - J(-d) \in A) &= \mathbb{E}_{\mathcal{F}} \left[\mathbb{P}_{\mathcal{F}} \left(J(\cdot) - J(-d) \in A \mid J(-4d), J(4d) \right) \right] \\
&= \mathbb{E}_{\mathcal{F}} \left[\mathcal{B}_{J(-4d), J(4d)}^{[-4d, 4d]}(\tilde{A} \mid J(0) \geq 0) \right] \\
&\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mathcal{B}_{y,z}^{[-4d, 4d]}(\tilde{A})}{\mathcal{B}_{y,z}^{[-4d, 4d]}(J(0) \geq 0)} \cdot f_J(y, z) \, dy \, dz.
\end{aligned}$$

In essence, our aim is to run this calculation forward, and the previous one backwards, in order to get from the jump curve probability of A to the standard Brownian motion probability of A . Then by direct comparison of the integrands, in order for the last line of the second calculation to be roughly equal to the last line of the first calculation we would need

$$\begin{aligned}
f_J(y, z) \text{ “=” } &\mathcal{B}_{y,z}^{[-4d, 4d]}(J(0) \geq 0) \cdot d^{-\frac{1}{2}} \exp\left(-\frac{(y-z)^2}{16d}\right) \cdot \frac{d\mu}{dy}(y) \\
&= \mathbb{P}\left(N\left(\frac{y+z}{2}, 2d\right) \geq 0\right) \cdot d^{-\frac{1}{2}} \exp\left(-\frac{(y-z)^2}{16d}\right) \cdot \frac{d\mu}{dy}(y).
\end{aligned}$$

Of course, we will have to provide ourselves some error margins in order to succeed. This is hiding in the “=” symbol above, which means that the left side is bounded above by the right side, possibly multiplied by a constant of the form $\exp(G(\log \varepsilon^{-1})^{5/6})$, which we will often refer to as the *leeway factor*; the notation of “=”, however, we use only in this instance. Since $T = D_k(\log \varepsilon^{-1})^{1/3}$, we will also often refer to constants of the form $\exp(GT^{5/2})$ as leeway factors.

So equivalently, what we require is that

$$(25) \quad f_J(y, z) \cdot \left[\mathbb{P}\left(N\left(\frac{y+z}{2}, 2d\right) \geq 0\right) \right]^{-1} \cdot d^{\frac{1}{2}} \exp\left(\frac{(y-z)^2}{16d}\right) \leq \exp\left(G(\log \varepsilon^{-1})^{5/6}\right).$$

Notice that we have temporarily ignored the issue of choosing μ and the slightly stronger demand (than the above inequality) that will arise from its Radon-Nikodym derivative with respect to Lebesgue measure. We will return to this in a few paragraphs, but only promise here that the cost will be polynomial in $\log \varepsilon^{-1}$ instead of exponential, and so will not substantively affect the analysis.

Remark 4.12. — This correspondence between the random variables, Y and Z (temporarily with the * adornment added), and the arguments of their joint density, y and z , is one that will be maintained throughout the proof of Theorem 4.10. Sometimes we will also refer to the marginal densities of Y or Z . By an abuse of notation (as Y and Z do not necessarily have the same distribution), we will refer to both densities as f_J , distinguishing whether we mean that of Y or Z based on whether the argument is y or z . This will not cause any confusion as at no point will we refer to a marginal density at a specific value. Similarly, $f_J(z \mid y)$ will be the conditional density of Z given $Y = y$, evaluated at the point z , and so on.

4.2.2. The vault and slope costs. — With this simplification, let us focus on the two terms multiplying $f_J(y, z)$ on the left-hand side of (25). They arise from two conceptually distinct sources. The first of these two terms comes from the potential difficulty a Brownian bridge faces in order to jump or *vault* over the pole, a task which J is conditioned to accomplish. Accordingly we refer to this factor as the *vault* cost, and it will be denoted by V^* (the * being again an adornment that will be removed in the final corrected version of V which we will use). The second of the two terms, on the other hand, is the potential difficulty faced by a Brownian motion to attain the slope specified by y and z , which may be equivalently be thought of as an

increment, across an interval of length $4d$. We refer to this as the *slope* cost, and denote it by S^* . Thus,

$$(26) \quad \begin{aligned} V^* &:= \left[\mathbb{P} \left(N \left(\frac{y+z}{2}, 2d \right) \geq 0 \right) \right]^{-1} \\ S^* &:= d^{\frac{1}{2}} \cdot \exp \left(\frac{(y-z)^2}{16d} \right). \end{aligned}$$

Note that V^* behaves differently depending on the values of y and z ; for example, if $y+z$ is positive, the probability is bounded below by a constant, and so V^* is bounded above by a constant and is easily managed by the margin of error we have provided. On the other hand, if $y+z$ is negative, V^* can be seen to be roughly $d^{\frac{1}{2}} \exp((y+z)^2/16d)$, thus posing a much more serious demand. More precisely,

$$(27) \quad V^* \lesssim \begin{cases} 1 & y+z > 0 \\ d^{\frac{1}{2}} \cdot \exp \left(\frac{(y+z)^2}{16d} \right) & y+z \leq 0, \end{cases}$$

Our analysis will later break into cases based on this fact.

Thus, (25) says that roughly what we need to show, for some $G < \infty$, is that

$$f_J(y, z) \cdot V^* \cdot S^* \leq \exp \left(G(\log \varepsilon^{-1})^{5/6} \right).$$

The above heuristic description was idealised to highlight the main features of the approach, but is in essence correct. We now discuss which aspects of the description change in the actual approach. There were three simplifying assumptions: that the pole position p equals zero; that the height of the pole is zero; and the postponement of the choice of μ . In addressing each of these simplifications, we will come to the final quantities V and S , and we will see that V and S respectively equal V^* and S^* up to the leeway factor.

4.2.3. Addressing the simplifications. —

Pole position $p = 0$. — The first simplifying assumption we made in the heuristic description was that the pole in $[-2d, 2d]$ lies at 0. In general, of course, we have very limited control over the pole position as it is determined by the $(k+1)^{\text{st}}$ curve of the original line ensemble. This point will be addressed essentially by changing our frame of reference horizontally.

Call the pole position $p \in [-2d, 2d]$ and suppose $p \neq 0$. The coefficient of $1/2$ for each of y and z in V^* was due to the pole position of 0 being equidistant from $-4d$ and $4d$. One option would be to maintain the random variables Y^* and Z^* to be the values at $\pm 4d$, in which case the coefficients would not be $1/2$ but λ and $1-\lambda$ for some $\lambda \in [0, 1]$. While the subsequent analysis could possibly be adapted for this case, this would introduce an undesirable level of complication to the formulas. So to maintain the symmetric coefficient of $1/2$, we instead have Y^* and Z^* be the values at $p-4d$ and $p+4d$, i.e., $J(p-4d)$ and $J(p+4d)$ respectively. (We will introduce a further modification shortly which will be the final definition of Y and Z , without the $*$ adornment.)

While on some level this is merely a trick, it is one which leads to very useful simplifications. At a technical level, though p is a random variable, it is an \mathcal{F} -measurable one, and so this trick is sound: the relevant point is that, given the \mathcal{F} -data, there is no obstacle to writing the law of the jump curve on $[-2d, 2d]$ as the marginal of Brownian bridge on $[p-4d, p+4d]$, between correctly distributed endpoints, which is conditioned to jump over the pole at p .

(To see that p is a \mathcal{F} -measurable random variable, note first that P is an \mathcal{F} -measurable random closed set by Proposition B.7. Now $P \cap [-2d, 2d]$ is also \mathcal{F} -measurable by Lemma B.3, as is the infimum of this set. Thus $\{P \cap [-2d, 2d] \neq \emptyset\}$ is an \mathcal{F} -measurable event, and so defining $p = \inf P \cap [-2d, 2d]$ on that event gives an \mathcal{F} -measurable random variable.)

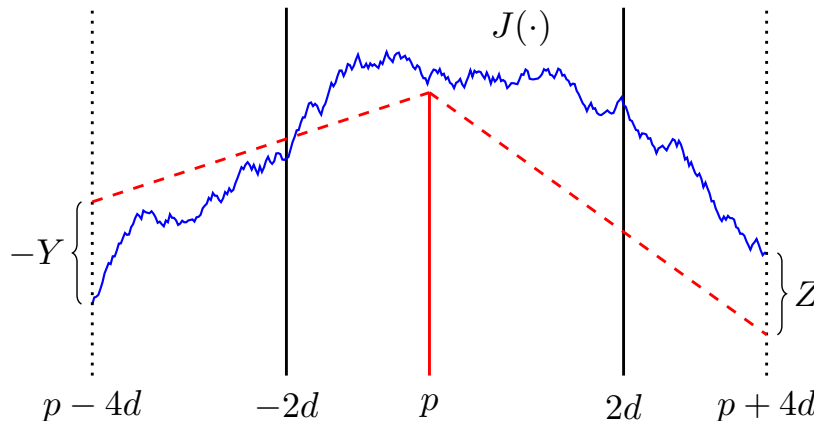


FIGURE 4.5. Illustrating the definition of Y and Z when $P \cap [-2d, 2d] \neq \emptyset$. The Tent map is depicted as the piecewise linear function drawn in dashed red. The single pole p in $[-2d, 2d]$ is depicted as a vertical red line. The jump curve is blue. In this figure Y is negative as J is below Tent at $p - 4d$, and thus the distance indicated is $-Y$. The quantity Z is positive as J is above Tent at $p + 4d$.

Pole height $Tent(p) = 0$. — Now we address the height of the pole. Here also we essentially employ a change in the frame of reference. At this point in our discussion, we have Y^* and Z^* being the values of $J(p - 4d)$ and $J(p + 4d)$ themselves, i.e., the deviation from $0 = Tent(p)$. When $Tent(p) \neq 0$, the obvious choice is to let Y and Z respectively represent the deviations of $J(p - 4d)$ and $J(p + 4d)$ from $Tent(p)$. But to make certain derivations slightly simpler in the sequel, we instead let the final definitions of Y and Z be the respective deviations of J from Tent at $p - 4d$ and $p + 4d$, i.e.,

$$(28) \quad \begin{aligned} Y &:= J(p - 4d) - Tent(p - 4d), \\ Z &:= J(p + 4d) - Tent(p + 4d). \end{aligned}$$

See Figure 4.5. This is the final definition of Y and Z which will be maintained for the rest of the argument, at least on the event that $P \cap [-2d, 2d] \neq \emptyset$. (On the other event, where there is no pole, they will have conceptually analogous but different definitions.)

The choice of μ . — First we provide some intuition as to the role of μ to motivate the choice we will make. Since our approach to move from jump curve probabilities (24) to Brownian motion probabilities (23) is somewhat crude, in the sense that we are trying to directly compare the integrands to conclude that the integrals are comparable, in order to succeed we cannot allow the integrand in (23) to be zero when the integrand in (24) is not small. This suggests that the support of μ should be the support of the law of Y , i.e., the support of $f_J(y)$ (recall from Remark 4.12 that this is the marginal density of Y). However, it is reasonable to assume that $f_J(y) > 0$ on the entire real line, and we run into a problem when we try to find a full support measure μ that also satisfies the constraints imposed by the costs V and S . This will be shown in the discussion in the following paragraphs. Heuristically, the solution will be to let μ have support only where $f_J(y)$ is not too small in some sense that we will specify.

From the above expressions for V^* and S^* , we see that, in the worst case, the total cost to be paid is $\exp\left(\frac{y^2}{8d} + \frac{z^2}{8d}\right)$. This tells us that we cannot afford for $\frac{d\mu}{dy}$ to be too small anywhere, as the additional cost corresponding to the choice of μ will be the reciprocal of this derivative. The logic behind this inference is as follows: in the worst case, we have (without loss of generality) $y < 0$, as well as the condition $J(0) \geq 0$, where for the sake of simplified discussion we have again assumed $p = 0$ and $Tent(p) = 0$. A Brownian bridge forced

to make a large jump (from y to 0) in unit order time will, because of the difficulty, make the required jump with very little extra margin. So we may think of the $J(0) \geq 0$ condition as being $J(0) = 0$. But the density of a Brownian motion which is 0 at 0 having value y at $-4d$ and z at $4d$ is exactly $(8\pi d)^{-1} \exp(-y^2/8d - z^2/8d)$. Thus this is the best density bound we can hope for J as well, which means any extra requirement imposed by $\frac{d\mu}{dy}$ must be absorbable in the leeway factor of $\exp(G(\log \varepsilon^{-1})^{5/6})$.

In other words, we require

$$(29) \quad \frac{d\mu}{dy}(y) \geq \exp\left(-G(\log \varepsilon^{-1})^{5/6}\right).$$

This bound cannot hold for all y for the density of a probability measure. But a closer look at (24) shows a modification we may make to that calculation: we can look for the density bound on $f_J(y, z)$ that we have been discussing for y and z in a *good* region, and find a separate argument for why the contribution of the integral from the bad regions is $O(\varepsilon)$. More precisely, call the good region $\mathcal{G}_R^{(1)} \subseteq \mathbb{R}^2$ (R is a parameter we will set later, and the 1 in the superscript refers to the fact that this is the first case, when a pole is present), and let $\tilde{y} = y + \text{Tent}(p - 4d)$, $\tilde{z} = z + \text{Tent}(p + 4d)$; this is so that $Y = y$ implies $J(p - 4d) = \tilde{y}$, and similarly $Z = z$ implies $J(p + 4d) = \tilde{z}$. Then, on $\text{Fav} \cap \{P \cap [-2d, 2d] \neq \emptyset\}$,

$$(30) \quad \begin{aligned} & \mathbb{P}_{\mathcal{F}}(J(\cdot) - J(-d) \in A) \\ &= \mathbb{E}_{\mathcal{F}} \left[\mathbb{P}_{\mathcal{F}}(J(\cdot) - J(-d) \in A \mid J(p - 4d), J(p + 4d)) \right] \\ &\leq \mathbb{E}_{\mathcal{F}} \left[\mathbb{P}_{\mathcal{F}}(J(\cdot) - J(-d) \in A \mid J(p - 4d), J(p + 4d)) \cdot \mathbb{1}_{(Y, Z) \in \mathcal{G}_R^{(1)}} \right] + \mathbb{P}_{\mathcal{F}}((Y, Z) \notin \mathcal{G}_R^{(1)}) \\ &= \mathbb{E}_{\mathcal{F}} \left[\mathcal{B}_{J(p-4d), J(p+4d)}^{[p-4d, p+4d]}(\tilde{A} \mid J(p) \geq \text{Tent}(p)) \cdot \mathbb{1}_{(Y, Z) \in \mathcal{G}_R^{(1)}} \right] + \mathbb{P}_{\mathcal{F}}((Y, Z) \notin \mathcal{G}_R^{(1)}) \\ &\leq \iint_{\mathcal{G}_R^{(1)}} \frac{\mathcal{B}_{\tilde{y}, \tilde{z}}^{[p-4d, p+4d]}(\tilde{A})}{\mathcal{B}_{\tilde{y}, \tilde{z}}^{[p-4d, p+4d]}(J(p) \geq \text{Tent}(p))} \cdot f_J(y, z) \, dy \, dz + \mathbb{P}_{\mathcal{F}}((Y, Z) \notin \mathcal{G}_R^{(1)}), \end{aligned}$$

where, recall, \tilde{A} is the set of functions f in $\mathcal{C}_{*,*}([-4d, 4d], \mathbb{R})$ such that $f(\cdot) - f(-d)$, regarded as a function with domain $[-d, d]$, lies in A . Thus we see that, if we can show the second term in the final displayed line is $O(\varepsilon)$, then we only require the density bounds associated with V and S for $(y, z) \in \mathcal{G}_R^{(1)}$, and the same holds true for the lower bound on $\frac{d\mu}{dy}(y)$. It will turn out that including the condition $(y, z) \in [-RT^{3/2}, RT^2]^2$ in the definition of $\mathcal{G}_R^{(1)}$ allows us to obtain the $O(\varepsilon)$ bound on the second term. (We will remark later in Section 4.2.4 on why we require the lower bound to be $-RT^{3/2}$ and not $-RT^2$.) So we only need to meet the condition (29) for $y, z \in [-RT^{3/2}, RT^2]$. An easy choice which meets this requirement is the uniform measure on $[-RT^2, RT^2]$, and so we let μ be this measure. Thus, we set

$$\frac{d\mu}{dy}(y) = (2RT^2)^{-1},$$

for $y \in [-RT^2, RT^2]$. We will treat separately the corresponding cost, which is equal to $2RT^2$, and not include it in either V or S .

The good regions. — We now give the definition of $\mathcal{G}_R^{(1)}$ as a subset of \mathbb{R}^2 .

$$(31) \quad \mathcal{G}_R^{(1)} = \left\{ (y, z) \in \mathbb{R}^2 : \begin{array}{l} y \in (-RT^{3/2}, RT^2), \\ z \in (-RT^{3/2}, RT^2), \\ |y - z| < 2RT^{3/2} \end{array} \right\}.$$

We have included an extra condition that $|y - z| \leq 2RT^{3/2}$. To have the $O(\varepsilon)$ upper bound on the probability of $(Y, Z) \in \mathcal{G}_R^{(1)}$ means that the increment of J across an interval of $8d$ must be bounded by $O(T^{3/2}) = O((\log \varepsilon^{-1})^{1/2})$ with probability at least $1 - \varepsilon$. This is plausible, on the basis that the probability

that a Brownian motion has an increment of size greater than $O((\log \varepsilon^{-1})^{1/2})$ over a unit order interval is polynomial in ε . In fact, we have the following bound on the event that $(Y, Z) \in \mathcal{G}_R^{(1)}$.

Lemma 4.13. — *We have for $R \geq 3$ and $d \geq 1$*

$$\mathbb{P}_{\mathcal{F}}\left((Y, Z) \notin \mathcal{G}_R^{(1)}\right) \cdot \mathbb{1}_{\text{Fav}, P \cap [-2d, 2d] \neq \emptyset} \lesssim \left(\varepsilon^{R^2 D_k^3 / 8d} + \varepsilon^{(R-3)^2 D_k^3 / 2}\right) \cdot \exp\left(13R^2 D_k^{5/2} (\log \varepsilon^{-1})^{5/6}\right).$$

To prove this, we will actually make use of a similar but weaker statement about a good region $\mathcal{G}_R^{(2)} \subseteq \mathbb{R}^2$, the difference being that the bound on the probability that $(Y, Z) \in \mathcal{G}_R^{(2)}$ will hold regardless of the presence or absence of a pole in $[-2d, 2d]$. We define

$$(32) \quad \mathcal{G}_R^{(2)} = \left\{ (y, z) \in \mathbb{R}^2 : \begin{array}{l} y \in (-RT^2, RT^2), \\ z \in (-RT^2, RT^2), \\ |y - z| < 2RT^{3/2} \end{array} \right\}.$$

Our earlier definitions of Y and Z were on the \mathcal{F} -measurable event that $P \cap [-2d, 2d] \neq \emptyset$ and do not make sense otherwise, as they are deviations of J from Tent at points defined relative to p . (Recall from before that $\{P \cap [-2d, 2d] \neq \emptyset\}$ is indeed an event which is \mathcal{F} -measurable.) So on the event that $P \cap [-2d, 2d] = \emptyset$, we define

$$(33) \quad \begin{aligned} Y &:= J(-d) - \text{Tent}(-d) \\ Z &:= J(d) - \text{Tent}(d). \end{aligned}$$

In the following lemma, Y and Z are defined in this case-specific manner.

Lemma 4.14. — *For $R \geq 3$,*

$$\mathbb{P}_{\mathcal{F}}\left((Y, Z) \notin \mathcal{G}_R^{(2)}\right) \cdot \mathbb{1}_{\text{Fav}} \lesssim \left(\varepsilon^{R^2 D_k^3 / 4d} + \varepsilon^{(R-3)^2 D_k^3 / 2}\right) \exp\left(13R^2 D_k^{5/2} (\log \varepsilon^{-1})^{5/6}\right).$$

While we defer the proofs of these lemmas to Section 5.1, let us say a few words about its approach. The statements about Y and Z being bounded above in absolute value by RT^2 are proved by stochastically dominating J by an appropriate Brownian bridge on Fav , and similarly with the roles of J and the Brownian bridge reversed for the lower bound. The slightly trickier issue is bounding the increment across the interval of length $8d$. We postpone discussing this point.

Based on the form of the statements of Lemmas 4.13 and 4.14, we set R (somewhat arbitrarily) as

$$(34) \quad R = 6\sqrt{d},$$

so that, for small enough ε , and since $d \geq 1$ and $D_k \geq 2$ (see (12)),

$$\mathbb{P}_{\mathcal{F}}\left((Y, Z) \notin \mathcal{G}_R^{(1)}\right) \cdot \mathbb{1}_{\text{Fav}, P \cap [-2d, 2d] \neq \emptyset} \leq \varepsilon \quad \text{and} \quad \mathbb{P}_{\mathcal{F}}\left((Y, Z) \notin \mathcal{G}_R^{(2)}\right) \cdot \mathbb{1}_{\text{Fav}} \leq \varepsilon.$$

4.2.4. The final costs. — Having addressed the three simplifications in the heuristic derivation, let us see how the costs have changed. Recall that the random variables Y and Z are respectively the quantities $J(p-4d) - \text{Tent}(p-4d)$ and $J(p+4d) - \text{Tent}(p+4d)$. As immediately preceding (30), let $\tilde{y} = y + \text{Tent}(p-4d)$ and $\tilde{z} = z + \text{Tent}(p+4d)$, so that $Y = y$ implies $J(p-4d) = \tilde{y}$, and similarly $Z = z$ implies $J(p+4d) = \tilde{z}$. The final definitions of V and S are

$$\begin{aligned} V &:= \mathcal{B}_{\tilde{y}, \tilde{z}}^{[p-4d, p+4d]} \left(J(p) \geq \text{Tent}(p) \right)^{-1} = \mathbb{P} \left(N \left(\frac{y+z}{2}, 2d \right) \geq \text{Tent}(p) - \frac{\text{Tent}(p-4d) + \text{Tent}(p+4d)}{2} \right)^{-1} \\ S &:= d^{\frac{1}{2}} \cdot \exp \left(\frac{1}{16d} (\tilde{y} - \tilde{z})^2 \right) = d^{\frac{1}{2}} \cdot \exp \left(\frac{1}{16d} (y - z + \text{Tent}(p-4d) - \text{Tent}(p+4d))^2 \right). \end{aligned}$$

For $(y, z) \in \mathcal{G}_R^{(1)}$, S and V differ from S^* and V^* by factors which can be absorbed within the leeway factor, which we show now. (We use the expressions for V^* and S^* from (26), which were defined in the special case

that $p = 0$ and $\text{Tent}(p) = 0$, for simplicity.) We will have, on expanding the second square in the exponent of S , some extra terms apart from the $(y - z)^2/16d$ present in S^* . The extra factor is

$$(35) \quad \exp \left(\frac{1}{16d} (\text{Tent}(p - 4d) - \text{Tent}(p + 4d))^2 + \frac{1}{8d} (y - z) (\text{Tent}(p - 4d) - \text{Tent}(p + 4d)) \right).$$

In this expression, we see immediately that the first term in the exponent does not cause a problem: since Tent has slope bounded in absolute value by $4T$ (recall (19)), it follows that $|\text{Tent}(p - 4d) - \text{Tent}(p + 4d)| \leq 32Td$, so that

$$\frac{1}{16d} (\text{Tent}(p - 4d) - \text{Tent}(p + 4d))^2 \leq 64T^2d,$$

which is well below $O((\log \varepsilon^{-1})^{5/6})$. We need the same bound to hold for the second term in the exponent in (35) as well. Again using that Tent has absolute value of slope bounded by $4T$, and that $|y - z| \leq 2RT^{3/2}$ on $\mathcal{G}_R^{(1)}$, what we find is

$$(y - z) (\text{Tent}(p - 4d) - \text{Tent}(p + 4d)) \leq 2RT^{3/2} \cdot 32Td = 64RT^{5/2}d.$$

This is why we included the bound on $|y - z|$ of order $T^{3/2}$ in the definition of $\mathcal{G}_R^{(1)}$. In other words,

$$(36) \quad S \lesssim S^* \cdot \exp \left(64T^2d + 8RT^{5/2} \right) \leq d^{\frac{1}{2}} \cdot \exp \left(\frac{1}{16d} (y - z)^2 + 10RT^{5/2} \right),$$

the last inequality since $d \leq \sqrt{T}/24$, $R \geq 6$ from (34), and $64/24 \leq 3$.

Remark 4.15. — We here used that $d \leq \sqrt{T}/24$, which is equivalent to the assumption $\varepsilon \leq \exp(-24)^6 d^6 / D_k^3$ made in Theorems 3.10 and 4.10; we had previously made use of this inequality in the proof of Theorem 3.10. We will be making use of this inequality many times in the sequel as well, as here to bound expressions of the form T^2d by $T^{5/2}/24$, and also to reduce coefficients when we have a margin to convert a small power of d to T .

Let us now finally analyse V in comparison with V^* , which will show us why we included in $\mathcal{G}_R^{(1)}$ that $y, z \geq -RT^{3/2}$.

We first focus on the right-hand quantity in the probability expression of V , namely

$$\text{Tent}(p) - \frac{\text{Tent}(p - 4d) + \text{Tent}(p + 4d)}{2}.$$

By the fact that the slope of Tent is bounded in absolute value by $4T$, this quantity is bounded above by

$$\frac{1}{2} \times 4d \times 4T + \frac{1}{2} \times 4d \times 4T = 16Td.$$

Thus V is bounded as

$$V \leq \mathbb{P} \left(N \left(\frac{y + z}{2}, 2d \right) \geq 16Td \right)^{-1}.$$

As in the analysis of V^* in (27), we see that this quantity behaves differently depending on the value of $y + z$:

$$(37) \quad V \lesssim \begin{cases} 1 & y + z > 32Td \\ d^{\frac{1}{2}} \cdot \exp \left(-\frac{1}{16d} (y + z - 32Td)^2 \right) & y + z \leq 32Td. \end{cases}$$

The bound in the worse case, on expanding the exponent and including only the factors which differ from those of V^* , is

$$\exp \left(64T^2d - 4(y + z)T \right).$$

Again the first term in the exponent, being $(\log \varepsilon^{-1})^{2/3}$, does not cause a problem, and so we turn to the second term. Firstly we see that we have the trivial upper bound of 0 on the term $-4(y + z)T$ when $y + z \geq 0$. When $y + z < 0$, we use that $y, z \geq -RT^{3/2}$ from the definition of $\mathcal{G}_R^{(1)}$ to see that

$$-(y + z) \cdot T \leq 2RT^{5/2}.$$

We note that any weaker lower bound on y, z would not have been sufficient to obtain an upper bound of order $T^{5/2}$ on the above quantity. From the previous few paragraphs we conclude that, when $y + z \leq 32Td$,

$$(38) \quad V \lesssim d^{\frac{1}{2}} \cdot \exp\left(\frac{1}{16d}(y+z)^2 + 11RT^{5/2}d\right),$$

since $d \leq \sqrt{T}/24$ and $R \geq 1$ imply $64T^2d + 8RT^{5/2} \leq (64/24)T^{5/2} + 8RT^{5/2} \leq 11RT^{5/2}$.

So overall what we have observed is that for $(y, z) \in \mathcal{G}_R^{(1)}$, it is true that

$$V \approx V^* \quad \text{and} \quad S \approx S^*,$$

in the sense that the left sides are bounded by the right sides up to multiplication by $\exp(G(\log \varepsilon^{-1})^{5/6})$. Thus, while we will work with V and S , the reader is advised to keep in mind the more convenient expressions from (36) and (38) after ignoring the leeway factor.

4.2.5. The bound to be proved. — With the quantities V and S defined, the following is what we will prove for (y, z) in the good region $\mathcal{G}_R^{(1)}$ and when $P \cap [-2d, 2d] \neq \emptyset$:

$$(39) \quad f_J(y, z) \cdot V \cdot S \cdot \left(\frac{d\mu}{dy}(y)\right)^{-1} \leq G_1 \cdot \exp\left(G_2(\log \varepsilon^{-1})^{5/6}\right),$$

for some finite constants $G_1 = G_1(\varepsilon, k, d)$ (with the ε -dependence polynomial in $\log \varepsilon^{-1}$) and $G_2 = G_2(k, d)$, and for $\varepsilon < \varepsilon_0$ for some $\varepsilon_0 > 0$. Since the Radon-Nikodym derivative term is $2RT^2$, it is sufficient to prove

$$(40) \quad f_J(y, z) \cdot V \cdot S \leq G'_1 \cdot T^2 \cdot \exp\left(G'_2(\log \varepsilon^{-1})^{5/6}\right),$$

for some $G'_1 = G'_1(k, d) < \infty$ and $G'_2 = G'_2(k, d) > 0$ (we include the factor of T^2 simply because this will be the form of what we will ultimately prove, and we wish to keep G'_1 independent of ε); then (40) implies (39) with $G_1 = G'_1 \cdot 2RT^2 \cdot T^2$ and $G_2 = G'_2$. We end this section by showing that (39) almost implies our main Theorem 4.10. More precisely, we have the following lemma, which, along with a proposition about the no-pole case, will allow us to prove Theorem 4.10 modulo proving these input statements.

Lemma 4.16. — *Suppose for all $(y, z) \in \mathcal{G}_R^{(1)}$ we have (39). Then with G_1 and G_2 as in (39) we have*

$$\mathbb{P}_{\mathcal{F}}(J(\cdot) - J(-d) \in A, (Y, Z) \in \mathcal{G}_R^{(1)}) \cdot \mathbb{1}_{\text{Fav}, P \cap [-2d, 2d] \neq \emptyset} \lesssim G_1 \cdot \varepsilon \cdot \exp\left(G_2(\log \varepsilon^{-1})^{5/6}\right).$$

PROOF. Let B be a Brownian motion begun at $p - 4d$ according to the distribution μ . We have on $\text{Fav} \cap \{P \cap [-2d, 2d] \neq \emptyset\}$,

$$\mathbb{P}_{\mathcal{F}}\left(J(\cdot) - J(-d) \in A, (Y, Z) \in \mathcal{G}_R^{(1)}\right) \leq \iint_{\mathcal{G}_R^{(1)}} \frac{\mathcal{B}_{\tilde{y}, \tilde{z}}^{[p-4d, p+4d]}(\tilde{A})}{\mathcal{B}_{\tilde{y}, \tilde{z}}^{[p-4d, p+4d]}(J(p) \geq \text{Tent}(p))} \cdot f_J(y, z) \, dy \, dz$$

Using (39), this integral is bounded by

$$\begin{aligned} G_1 d^{-\frac{1}{2}} \exp\left(G_2(\log \varepsilon^{-1})^{5/6}\right) & \iint_{\mathcal{G}_R^{(1)}} \mathcal{B}_{\tilde{y}, \tilde{z}}^{[p-4d, p+4d]}(\tilde{A}) e^{-\frac{1}{16d}(\tilde{y}-\tilde{z})^2} \, d\mu(y) \, dz \\ & \lesssim \mathbb{P}(B(\cdot) - B(-d) \in A) \cdot G_1 \cdot \exp\left(G_2(\log \varepsilon^{-1})^{5/6}\right) \\ & = \mathcal{B}_{0, * }^{[-d, d]}(A) \cdot G_1 \cdot \exp\left(G_2(\log \varepsilon^{-1})^{5/6}\right) = G_1 \cdot \varepsilon \cdot \exp\left(G_2(\log \varepsilon^{-1})^{5/6}\right). \end{aligned}$$

The second-to-last equality follows from the Markov property of Brownian motion. \square

In fact, we will establish (39) with $G_2 = 792 \cdot d \cdot D_k^{5/2}$ and $G_1 = G'RT^4$, where G' is an absolute constant independent of d, k, ε . This is the content of Lemmas 5.3, 5.6, and 5.14 ahead, as $792d \geq \max(22R^2, 41R)$ from R 's value set in (34), and the observation above that to go from (40) to (39) we must multiply G'_1 by $2RT^4$. We will also establish the following proposition in the no-pole case:

Proposition 4.17. — *There exists a positive constant $\varepsilon_0 = \varepsilon_0(d) > 0$ such that if $\varepsilon < \varepsilon_0$, then, on $\text{Fav} \cap \{P \cap [-2d, 2d] = \emptyset\}$,*

$$\mathbb{P}_{\mathcal{F}}\left(J(\cdot) - J(-d) \in A, (Y, Z) \in \mathcal{G}_R^{(2)}\right) \lesssim \varepsilon \cdot \exp\left(756 \cdot d \cdot D_k^{5/2} \cdot (\log \varepsilon^{-1})^{5/6}\right).$$

Admitting these statements for now, namely Lemmas 4.13, 4.14, Proposition 4.17, and that we have (39) with $G_1 = G'RT^4$ and $G_2 = 792 \cdot d \cdot D_k^{5/2}$, we may complete the proof of Theorem 4.10.

PROOF OF THEOREM 4.10. The quantity $\mathbb{P}(J(\cdot) - J(-d) \in A) \cdot \mathbb{1}_{\text{Fav}}$ satisfies the following upper bound:

$$(41) \quad \begin{aligned} & \mathbb{P}_{\mathcal{F}}(J(\cdot) - J(-d) \in A) \cdot \mathbb{1}_{\text{Fav}} \\ & \leq \left[\mathbb{P}_{\mathcal{F}}\left(J(\cdot) - J(-d) \in A, (Y, Z) \in \mathcal{G}_R^{(1)}\right) + \mathbb{P}_{\mathcal{F}}\left((Y, Z) \notin \mathcal{G}_R^{(1)}\right) \right] \mathbb{1}_{\text{Fav}, P \cap [-2d, 2d] \neq \emptyset} \\ & \quad + \left[\mathbb{P}_{\mathcal{F}}\left(J(\cdot) - J(-d) \in A, (Y, Z) \in \mathcal{G}_R^{(2)}\right) + \mathbb{P}_{\mathcal{F}}\left((Y, Z) \notin \mathcal{G}_R^{(2)}\right) \right] \mathbb{1}_{\text{Fav}, P \cap [-2d, 2d] = \emptyset}. \end{aligned}$$

Focus on the first term after the inequality of (41). By Lemma 4.13 and our choice of $R = 6\sqrt{d}$ from (34),

$$\mathbb{P}_{\mathcal{F}}\left((Y, Z) \notin \mathcal{G}_R^{(1)}\right) \cdot \mathbb{1}_{\text{Fav}, P \cap [-2d, 2d] \neq \emptyset} \lesssim \varepsilon \cdot \exp\left(468 \cdot d \cdot D_k^{5/2} (\log \varepsilon^{-1})^{5/6}\right).$$

From our assumption that we have (39) with $G_2 = 792 \cdot d \cdot D_k^{5/2}$ and $G_1 = G'RT^4$ for all $(y, z) \in \mathcal{G}_R^{(1)}$, we get from Lemma 4.16

$$(42) \quad \begin{aligned} & \mathbb{P}_{\mathcal{F}}\left(J(\cdot) - J(-d) \in A, (Y, Z) \in \mathcal{G}_R^{(1)}\right) \cdot \mathbb{1}_{\text{Fav}, P \cap [-2d, 2d] \neq \emptyset} \\ & \lesssim \varepsilon \cdot d^{\frac{1}{2}} \cdot D_k^4 (\log \varepsilon^{-1})^{\frac{4}{3}} \cdot \exp\left(792 \cdot d \cdot D_k^{5/2} (\log \varepsilon^{-1})^{5/6}\right) \end{aligned}$$

Now we turn to the second line of (41). From Lemma 4.14 and our choice of $R = 6\sqrt{d}$ in (34),

$$\mathbb{P}_{\mathcal{F}}\left((Y, Z) \notin \mathcal{G}_R^{(2)}\right) \cdot \mathbb{1}_{\text{Fav}, P \cap [-2d, 2d] = \emptyset} \lesssim \varepsilon \cdot \exp\left(468 \cdot d \cdot D_k^{5/2} (\log \varepsilon^{-1})^{5/6}\right).$$

Finally from Proposition 4.17 we have

$$\mathbb{P}_{\mathcal{F}}\left(J(\cdot) - J(-d) \in A, (Y, Z) \in \mathcal{G}_R^{(2)}\right) \lesssim \varepsilon \cdot \exp\left(756 \cdot d \cdot D_k^{5/2} \cdot (\log \varepsilon^{-1})^{5/6}\right).$$

Substituting these bounds into (41) gives

$$\mathbb{P}_{\mathcal{F}}\left(J(\cdot) - J(-d) \in A\right) \lesssim \varepsilon \cdot d^{\frac{1}{2}} \cdot D_k^4 (\log \varepsilon^{-1})^{\frac{4}{3}} \cdot \exp\left(792 \cdot d \cdot D_k^{5/2} \cdot (\log \varepsilon^{-1})^{5/6}\right),$$

completing the proof. \square

At this point we pause to review our progress. We have defined good regions $\mathcal{G}_R^{(1)}$ and $\mathcal{G}_R^{(2)}$ to which we can restrict our analysis. We have also defined the costs that need to be met on $\mathcal{G}_R^{(1)}$ in order to prove Theorem 4.10, and indeed we have given the proof of Theorem 4.10 modulo these costs being met and a few additional statements being proven. Concretely, our task is now to establish (40) with the claimed values of G'_1 and G'_2 ; to prove Lemmas 4.13 and 4.14; and to prove Proposition 4.17.

Establishing (40) will break into separate cases that will each require different arguments which will all be handled in the next Chapter 5. An easy and a moderate case, respectively when $y, z < 0$ or $y + z > 0$, will also supply us the bounds we need to prove Lemmas 4.13 and 4.14. Chapter 5 will also address the case where a pole is not present in $[-2d, 2d]$, i.e., Proposition 4.17.

CHAPTER 5

PROVING THE DENSITY BOUNDS

This chapter proves the statements of Chapter 4 that are needed in the proof of Theorem 4.10 as given in Section 4.2.5; in particular, we prove here Lemmas 4.13 and 4.14, Proposition 4.17, and that equation (40) holds with the claimed constants. In essence, these all follow from the last item, i.e., (40), which is a bound on the density $f_J(y, z)$.

The proof of this density bound is broken up into four sections. The first three are when a pole is present in $[-2d, 2d]$ and are distinguished by the values of $Y = J(p - 4d) - \text{Tent}(p - 4d)$ and $Z = J(p + 4d) - \text{Tent}(p + 4d)$, roughly corresponding to an easy case of being below the pole on both sides (Section 5.1); a moderate case of being above the pole on both sides (Section 5.2); and a difficult case of being above and below the pole on either side (Section 5.3). Lemmas 4.13 and 4.14 are proved at the end of the moderate case, Section 5.2. The last section, Section 5.4, addresses when there is no pole in $[-2d, 2d]$, i.e., Proposition 4.17.

5.1. The easy case: Below the pole on both sides

Having set up the problem and identified what bounds (40) we require on $f_J(y, z)$, we now prove such a bound in a simple case. This case is when $y < 0$ and $z < 0$, which implies, from (36) and (38), that

$$V \cdot S \lesssim d \cdot \exp\left(\frac{y^2}{8d} + \frac{z^2}{8d} + 21RT^{5/2}\right).$$

So, it is sufficient to prove

$$(43) \quad f_J(y, z) \lesssim d^{-1} \cdot \exp\left(-\frac{y^2}{8d} - \frac{z^2}{8d}\right).$$

In this case we are aided by the presence of the pole. Essentially, the desired density of J at (y, z) is bounded by the density of a particular pair of independent Brownian bridges at (y, z) . More precisely, on the event $\text{Fav} \cap \{P \cap [-2d, 2d] \neq \emptyset\}$, let p^- and p^+ be the elements of P immediately preceding and succeeding p ; and define

$$\begin{aligned} \sigma_{-4d}^2 &= 4d \cdot \frac{(p - p^- - 4d)}{p - p^-} \\ \sigma_{4d}^2 &= 4d \cdot \frac{(p^+ - p - 4d)}{p^+ - p}. \end{aligned}$$

The first quantity is the variance of a Brownian bridge defined on the interval $[p^-, p]$ at the point $p - 4d$, while the second is the same for a Brownian bridge defined on $[p, p^+]$ at the point $p + 4d$.

The following proposition is exactly [Ham19b, Lemma 5.17]. The proof is fairly straightforward and we will shortly reproduce it here for completeness and because it aids our exposition.

Proposition 5.1. — For $y, z < 0$,

1. *Joint bound*

$$f_J(y, z) \cdot \mathbb{1}_{\text{Fav}, P \cap [-2d, 2d] \neq \emptyset} \lesssim d^{-1} \cdot \exp\left(-\frac{1}{2\sigma_{-4d}^2} y^2 - \frac{1}{2\sigma_{4d}^2} z^2\right) \cdot \mathbb{1}_{\text{Fav}, P \cap [-2d, 2d] \neq \emptyset}.$$

2. *Marginal bounds*

$$f_J(y) \cdot \mathbb{1}_{\text{Fav}, P \cap [-2d, 2d] \neq \emptyset} \lesssim d^{-\frac{1}{2}} \cdot \exp\left(-\frac{1}{2\sigma_{-4d}^2} y^2\right) \cdot \mathbb{1}_{\text{Fav}, P \cap [-2d, 2d] \neq \emptyset}$$

$$f_J(z) \cdot \mathbb{1}_{\text{Fav}, P \cap [-2d, 2d] \neq \emptyset} \lesssim d^{-\frac{1}{2}} \cdot \exp\left(-\frac{1}{2\sigma_{4d}^2} z^2\right) \cdot \mathbb{1}_{\text{Fav}, P \cap [-2d, 2d] \neq \emptyset}.$$

We note the following simple bounds on the variances:

Lemma 5.2. — On $\text{Fav} \cap \{P \cap [-2d, 2d] \neq \emptyset\}$, we have $\sigma_{4d}^2, \sigma_{-4d}^2 \in [\frac{4}{5}d, 4d]$.

PROOF. The upper bound is obvious from the defining expressions. For the lower bound, we have

$$\sigma_{-4d}^2 = 4d \left(1 - \frac{4d}{p - p^-}\right) \geq 4d \left(1 - \frac{4d}{5d}\right) = \frac{4}{5}d,$$

since $p - p^- \geq d_{\text{ip}} = 5d$. A similar argument proves the corresponding bound for σ_{4d}^2 . \square

With these variance bounds and the density bounds of Proposition 5.1, the sufficient bound (40) is immediate and we simply record it:

Lemma 5.3. — When $y < 0$ and $z < 0$, we have (40) with $G'_2 = 21RD_k^{5/2}$ and with G'_1 independent of ε, k , and d .

PROOF OF PROPOSITION 5.1. The second statement follows from the first by integrating out one of the variables, so we prove only the first statement on the joint density bound. We are in the situation where $P \cap [-2d, 2d] \neq \emptyset$, and p is the unique element in this intersection. Let p^- and p^+ be the adjacent elements of P . Let $\mathcal{F}[p^-, p, p^+]$ denote the σ -algebra generated by \mathcal{F} and the random variables $J(x)$ for $x \in \{p^-, p, p^+\}$; more precisely, since the latter random variables are only defined when $P \cap [-2d, 2d] \neq \emptyset$, we define $\mathcal{F}[p^-, p, p^+]$ to be the σ -algebra generated by \mathcal{F} and the random vector $X \in \mathbb{R}^3$, which equals $(J(p^-), J(p), J(p^+))$ on $\{P \cap [-2d, 2d] \neq \emptyset\}$ and $(0, 0, 0)$ otherwise. (The presence of p^-, p, p^+ in the notation is merely for convenience and is not meant to imply that the σ -algebra is random; it is not.) Recall also our previous observation from page 55 that P and p are measurable and so are random objects; similar reasoning applies for why p^- and p^+ are measurable.

The density $f_J(y, z)$ has a counterpart $f_J^{\mathcal{F}[p^-, p, p^+]}$ under the augmented σ -algebra, and it is enough to show that

$$f_J^{\mathcal{F}[p^-, p, p^+]}(y, z) \cdot \mathbb{1}_{P \cap [-2d, 2d] \neq \emptyset} \lesssim \sigma_{-4d}^{-1} \cdot \sigma_{4d}^{-1} \cdot \exp\left(-\frac{y^2}{2\sigma_{-4d}^2} - \frac{z^2}{2\sigma_{4d}^2}\right),$$

since then Proposition 5.1(1) will arise by averaging.

Under the law $\mathbb{P}_{\mathcal{F}[p^-, p, p^+]}$, the processes $J(\cdot)$ on $[p^-, p]$ and $[p, p^+]$ are conditionally independent. Since we are working on the $\mathcal{F}[p^-, p, p^+]$ -measurable event that $\{P \cap [-2d, 2d] \neq \emptyset\}$ occurs, it is enough to argue that

- the conditional density of Y at $s \leq 0$ is at most a constant multiple of $\sigma_{-4d}^{-1} \exp(-s^2/2\sigma_{-4d}^2)$;
- and the conditional density of Z at $t \leq 0$ is at most a constant multiple of $\sigma_{4d}^{-1} \exp(-t^2/2\sigma_{4d}^2)$.

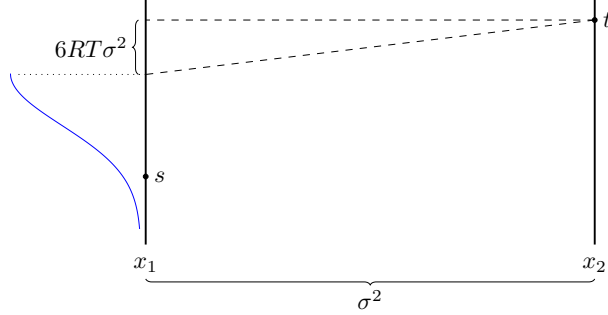


FIGURE 5.1. Illustrating Proposition 5.4 when $s < t$. The requirement that $|s - t| \geq 6RT\sigma^2$ comes from an error in an estimate of a certain slope; the blue curve on the left shows the Gaussian-like density centred at $t - 6RT\sigma^2$ that dominates the joint density $f_J^{x_1, x_2}(s, t)$. Note that the proposition does not require or use the presence of a pole in $[x_1, x_2]$.

These statements are straightforward to verify. Indeed, the conditional law under $\mathcal{F}[p^-, p, p^+]$ of $J(p - 4d)$ is normal with mean $\left(1 - \frac{4d}{p-p^-}\right)J(p) + \frac{4d}{p-p^-}J(p^-)$ and variance σ_{-4d}^2 . Note that $J(p^-) \geq \text{Tent}(p^-)$ and $J(p) \geq \text{Tent}(p)$ since $p^-, p \in P$, and that Tent is affine on the interval between consecutive pole set elements p^- and p ; thus, we see that this mean is at least $\text{Tent}(p - 4d)$. The first bullet point statement follows from the form of the normal density since we have shown that $\mathbb{E}_{\mathcal{F}[p^-, p, p^+]}[Y] \geq 0$, and we are concerned with the density only on $(-\infty, 0]$. The second bullet point is proved in the same fashion. This proves Proposition 5.1(1). \square

5.2. The moderate case: Above the pole on both sides

In this section, we address the case of bounding $f_J(y, z)$ when $y + z > 0$. Here is the main proposition to be proved.

Proposition 5.4 (Density bound on increment). — *Let $\iota/2 < x_1 < x_2 < \mathfrak{r}/2$ be \mathcal{F} -measurable, $\sigma^2 = x_2 - x_1 \geq d$, R be as in (34), and suppose that $\sigma \leq T^2 \cdot \sqrt{d}/2$. Suppose also that $[x_1 - d, x_1 + d] \cap P = \emptyset$ and $[x_2 - d, x_2 + d] \cap P = \emptyset$. Let $f_J^{x_1, x_2}(s, t)$ be the joint density of $(J(x_1) - \text{Tent}(x_1), J(x_2) - \text{Tent}(x_2))$ at (s, t) . If $s, t \in [-RT^2, RT^2]$ and $|s - t| > 6RT\sigma^2$, then*

$$f_J^{x_1, x_2}(s, t) \cdot \mathbb{1}_{\text{Fav}} \lesssim \sigma^{-1} \cdot d^{-\frac{1}{2}} T^2 \cdot \exp\left(-\frac{1}{2\sigma^2}(|s - t| - 6RT\sigma^2)^2 + \frac{4R^2 T^2}{\sigma^2} + 36R^2 \sigma^2\right).$$

We also have that $f_J^{x_1, x_2}(s, t) \lesssim d^{-1}$ for all $s, t \in \mathbb{R}$.

Remark 5.5. — The condition that $|s - t| > 6RT\sigma^2$ arises in the proof of Proposition 5.4 from the error of an estimate on certain slopes; see Figure 5.1. For our purposes it does not cause any difficulty, as when $|y - z| < 6RT\sigma^2$ the costs will be always absorbable in the leeway factor. For example, this is seen in the proof of Lemma 5.6 below.

It may not be immediately clear what is the relation of this proposition to the case where $y + z > 0$. In fact, this proposition has been carefully stated to apply to a more general situation than just the case of this section. For example, unlike Proposition 5.1, this proposition does not require a pole to be present, and we will make use of it in the no-pole case addressed in Section 5.4 as well. We will also use Proposition 5.4 to prove Lemmas 4.13 and 4.14 near the end of the section, as it provides a density bound in terms of the *increment* $|y - z|$, and so can be easily used to bound the increment of J across an interval, a requirement which was briefly discussed in Section 4.2.

Before proving Proposition 5.4, we apply it to show that it yields the sufficient bound (40) in the case that $y + z > 0$.

Lemma 5.6. — *When $y + z > 0$ and $(y, z) \in \mathcal{G}_R^{(1)}$, we have (40) with $G'_2 = 22R^2D_k^{5/2}$ and $G'_1 = G'T^2$, where G' is a constant independent of ε, k , and d .*

PROOF. In this case, we see that

$$V \lesssim \begin{cases} 1 & y + z > 32Td \\ d^{\frac{1}{2}} \cdot \exp(64T^2d) & 0 < y + z < 32Td, \end{cases}$$

i.e., V is bounded by the leeway factor $d^{\frac{1}{2}} \cdot \exp(G_2T^{5/2})$ with $G_2 = 8$, since $d \leq \sqrt{T}/24$ by the assumption $\varepsilon \leq \exp(-(24)^6d^6/D_k^3)$ (note that $8 < 22R^2$ since $R \geq 6$ from (34)). So we merely need to handle S , which from (36) is bounded above as

$$S \lesssim d^{\frac{1}{2}} \cdot \exp\left(\frac{1}{16d}(y-z)^2 + 10RT^{5/2}\right).$$

Note that if we set $x_1 = p - 4d$ and $x_2 = p + 4d$ in Proposition 5.4, then $\sigma^2 = 8d$. So with these parameters, we see from the second part of Proposition 5.4 that, when $|y - z| \leq 48RTd$,

$$\begin{aligned} f_J(y, z) \cdot V \cdot S &\lesssim \exp\left(\frac{1}{16d}|y-z|^2 + 64T^2d + 10RT^{5/2}\right) \leq \exp\left((48 \times 3R^2 + 64)T^2d + 10RT^{5/2}\right) \\ &\leq \exp\left(19R^2T^{5/2}\right), \end{aligned}$$

the last inequality since $R \geq 1$ and again using that $d \leq \sqrt{T}/24$; we have also used that $64/24 \leq 3$.

When $|y - z| > 48RTd$, we again make use of Proposition 5.4 with $x_2 = p + 4d$ and $x_1 = p - 4d$. The assumptions of Proposition 5.4 are satisfied since $d_{\text{ip}} = 5d$. So we obtain the bound

$$f_J(y, z) \cdot \mathbb{1}_{\text{Fav}} \lesssim d^{-1}T^2 \exp\left(-\frac{1}{16d}(|y-z| - 48RTd)^2 + \frac{4R^2T^2}{8d} + 36R^2 \times 8d\right)$$

when $|y - z| > 48RTd$. So, for $|y - z| > 48RTd$,

$$\begin{aligned} f_J(y, z) \cdot V \cdot S &\lesssim T^2 \cdot \exp\left(6RT|y-z| - 48 \times 3R^2T^2d + \frac{1}{2}R^2T^2 + 36R^2 \times 8d + 64T^2d + 10RT^{5/2}\right) \\ &\leq T^2 \cdot \exp\left(12R^2T^{5/2} + 10RT^{5/2}\right) = T^2 \cdot \exp\left(22R^2T^{5/2}\right), \end{aligned}$$

the last inequality since we have $|y - z| \leq 2RT^{3/2}$ on $\mathcal{G}_R^{(1)}$ and using that $1 \leq d \leq \sqrt{T}/24$ and $R \geq 1$ from (34) to see that $36R^2 \times 8d \leq 12R^2T^{1/2}$, allowing us to drop the middle four terms. This verifies (40) with the claimed values. \square

We may now turn to discussing the proof strategy of Proposition 5.4. In the proof of Proposition 5.1, we were greatly aided by the presence of the pole and the *difficulty* that a Brownian bridge faces in making large jumps while remaining negative on either side. An interesting feature of that argument is that no extra reasoning was needed to obtain a density bound from a comparison with a Brownian object. Typically, such comparisons easily yield bounds on tail probabilities, but these do not immediately imply a pointwise density bound.

In the proof of Proposition 5.4, as well as in the case addressed in Section 5.3, both of these features will be missing. Firstly, when J (still thought of as essentially a Brownian bridge) is allowed to be positive on one side, the pole *assists* its attainment of the values on either side, and so it is not clear why the Brownian density should bound that of the jump curve. This is especially true for the case analysed in the next Section 5.3, where essentially the same bound as that proved in Section 5.1 must hold. Secondly, we will not

be able to access density bounds directly, but will need to make further technical arguments to move from tail probability bounds to density bounds. To accomplish the latter, we will make use of a technique of *local randomization*, in both the case of this section as well as that of Section 5.3.

A short description of what we mean by local randomization is the following: Suppose that we wish to obtain a bound on the density of $J(x)$ for some x . We first obtain bounds on the tail probabilities of J at certain points, say $x - \eta$ and $x + \eta$ for some $\eta > 0$, with no pole contained in $[x - \eta, x + \eta]$. To convert this to a density bound at x , we use that J , conditionally on its values at $x - \eta$ and $x + \eta$, is a Brownian bridge on $[x - \eta, x + \eta]$. Then the distribution of J at x can be written as a convolution of the distributions at $x - \eta$ and $x + \eta$ with a normal random variable, which, when combined with the tail bounds, can be used to give a density bound. Heuristically, the tail bound is being propagated and smoothed by the Brownian bridge to a density bound.

At the level of this description, no importance is given to the exact value of η . To aid our discussion, let us say that a random variable X has a *pseudo-variance at most* σ^2 if we have a tail bound of the form $\mathbb{P}(X < t) \leq \exp(-t^2/2\sigma^2)$. As the discussion of V and S in Section 4.2 showed, we require the pseudo-variance we obtain in sub-Gaussian density bounds to be essentially optimal. And indeed, if we knew that the distributions of $J(x - \eta)$ and $J(x + \eta)$ were actually Gaussian, then the density bound arising from the convolution mentioned would be precisely the correct one. However, when all we have is a sub-Gaussian tail bound and not an actual Gaussian distribution, there is some extra gain in the pseudo-variance we obtain for the final density bound. This is captured in the following lemma, whose proof will be given at the end of the section.

Lemma 5.7. — *Let X be a random variable such that $\mathbb{P}(X < x) \leq A \exp(-\frac{1}{2\sigma_2^2}(x - x_0)^2)$ for $x < x_0$, and let N be a normal random variable with mean 0 and variance σ_1^2 which is independent of X . Then the density f of $X + N$ satisfies*

$$(44) \quad f(x) \leq \frac{A+1}{\sqrt{2\pi\sigma_1}} \cdot \exp\left(-\frac{(x-x_0)^2}{2(\sigma_1+\sigma_2)^2}\right)$$

for $x < x_0$, and is bounded by $1/\sqrt{2\pi}\sigma_1$ for all $x \in \mathbb{R}$. The same conclusion holds if X satisfies $\mathbb{P}(X > x) \leq A \exp(-\frac{1}{2\sigma_2^2}(x - x_0)^2)$ for $x > x_0$.

The point we were expressing is seen in this formula by the fact that the pseudo-variance guaranteed by this bound is $(\sigma_1 + \sigma_2)^2$, which is greater than $\sigma_1^2 + \sigma_2^2$ as it would be had we known, in the notation of the lemma, that X has Gaussian distribution with variance σ_2^2 .

For η a positive constant, this gain in pseudo-variance gives a density bound that is too weak for our purposes; indeed, the bound is weaker than that claimed in Proposition 5.4. The solution is, roughly, to take $\eta \rightarrow 0$. In the language of the lemma, if $\sigma_2 = \sigma_2(\eta) \rightarrow \sigma$ and $\sigma_1(\eta) \rightarrow 0$ (we will later take σ_2 to be the pseudo-variance of $\frac{1}{2}(J(x - \eta) + J(x + \eta))$), so it will depend on η as $\eta \rightarrow 0$, then

$$\lim_{\eta \rightarrow 0} (\sigma_1 + \sigma_2)^2 = \sigma^2 = \lim_{\eta \rightarrow 0} (\sigma_1^2 + \sigma_2^2);$$

i.e., there is no gain in pseudo-variance in the limit. However, taking $\eta \rightarrow 0$ leads to a blow up in the constant in front of the exponential in (44), and so we actually take η to be a small ε -dependent quantity, small enough that the gain in pseudo-variance is manageable. This is the argument of Proposition 5.4, which we turn to next. We will then give the pending proofs of Lemmas 4.13 and 4.14, and finish the section by proving the technical tool Lemma 5.7.

The proof of Proposition 5.4 will actually obtain the claimed bound on the conditional density $f_J^{x_1, x_2}(s | t)$, so we also need that the marginal density $f_J^{x_2}(t)$ is bounded.

Lemma 5.8 (Marginal density is bounded). — For any \mathcal{F} -measurable $x \in [-2T, 2T]$ such that $[x - d, x + d] \cap P = \emptyset$, let f_J^x be the density of $J(x) - \text{Tent}(x)$ conditionally on \mathcal{F} . Then we have on Fav that $f_J^x(s) \leq \pi^{-\frac{1}{2}} d^{-\frac{1}{2}}$ for all $s \in \mathbb{R}$.

PROOF. By assumption there is no pole in $[x - d, x + d]$. So $J(x)$, conditionally on $J(x - d)$ and $J(x + d)$, is given by

$$J(x) = \frac{1}{2}J(x - d) + \frac{1}{2}J(x + d) + N\left(0, \frac{1}{2}d\right).$$

Thus a formula for f_J^x is

$$f_J^x(s) = \frac{1}{\sqrt{\pi d}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{d}(t - s)^2\right) d\nu(t),$$

where ν is the law of $\frac{1}{2}J(x - d) + \frac{1}{2}J(x + d)$. From this formula the claim follows. \square

We next cite a standard bound on normal probabilities before turning to the proof of Proposition 5.4.

Lemma 5.9 (Normal bounds). — Let $\sigma^2 > 0$. If $t > \sigma$ for the first inequality and $t > 0$ for the second,

$$\frac{\sigma}{2\sqrt{2\pi}t} \exp\left(-\frac{t^2}{2\sigma^2}\right) \leq \mathbb{P}\left(N(0, \sigma^2) > t\right) \leq \exp\left(-\frac{t^2}{2\sigma^2}\right).$$

PROOF. Replacing t by σt , it suffices to take $\sigma = 1$. The standard lower bound

$$\mathbb{P}(N(0, 1) > t) \geq (2\pi)^{-1/2} \frac{t}{t^2 + 1} \exp(-t^2/2)$$

for $t \geq 0$ may be found in [Wil91, Section 14.8]. Note that $\frac{t}{t^2 + 1} \geq (2t)^{-1}$ for $t \geq 1$. The upper bound is simply the Chernoff bound. \square

The proof of Proposition 5.4 has two steps, as described in the earlier discussion. The first is a tail bound on a quantity close to $J(x_1)$, conditionally on $J(x_2)$ (and with the roles of x_1 and x_2 reversed); the second is to convert this tail bound into a density bound using Lemma 5.7. The first step is isolated in the next lemma, while the second step is performed in the immediately following proof of Proposition 5.4.

Lemma 5.10. — Let $\mathfrak{l}/2 < x_1 < x_2 < \mathfrak{r}/2$ be \mathcal{F} -measurable, $\sigma^2 = x_2 - x_1 \geq d$, and R be as in (34). Suppose also that $[x_1 - d, x_1 + d] \cap P = [x_2 - d, x_2 + d] \cap P = \emptyset$. Then on Fav , for $r < t + \text{Tent}(x_2) - 4\sigma^2(R + 2)T$ and any $\eta < d/2$,

$$\begin{aligned} \mathbb{P}_{\mathcal{F}}\left(\frac{1}{2}J(x_1 + 2\eta) + \frac{1}{2}J(x_1 - 2\eta) < r \mid J(x_2) = t + \text{Tent}(x_2)\right) \\ \leq \exp\left(-\frac{1}{2\sigma_{\eta}^2}(r - t - \text{Tent}(x_2) + 4\sigma^2(R + 2)T)^2\right), \end{aligned}$$

where $\sigma_{\eta}^2 = \sigma^2 \cdot \frac{x_1 - \mathfrak{l}}{x_2 - \mathfrak{l}} - \eta$.

Similarly, for $r < t + \text{Tent}(x_1) - 4\sigma^2(R + 2)T$ and on Fav ,

$$\begin{aligned} \mathbb{P}_{\mathcal{F}}\left(\frac{1}{2}J(x_2 + 2\eta) + \frac{1}{2}J(x_2 - 2\eta) < r \mid J(x_1) = t + \text{Tent}(x_1)\right) \\ \leq \exp\left(-\frac{1}{2\tilde{\sigma}_{\eta}^2}(r - t - \text{Tent}(x_1) + 4\sigma^2(R + 2)T)^2\right), \end{aligned}$$

where $\tilde{\sigma}_{\eta}^2 = \sigma^2 \cdot \frac{\mathfrak{r} - x_2}{\mathfrak{r} - x_1} - \eta$.

PROOF. We will only prove the first bound as the second bound is analogous, by repeating the below argument with the roles of x_1 and x_2 switched, and \mathfrak{r} in place of \mathfrak{l} .

By assumption, there is no pole in $[x_1 - 2\eta, x_1 + 2\eta]$ for all $\eta \leq d/2$. For every such η , the distribution of $J(x_1)$ given $J(x_2)$, $J(x_1 - 2\eta)$, and $J(x_1 + 2\eta)$ depends on only $J(x_1 - 2\eta)$ and $J(x_1 + 2\eta)$, and is given by

$$(45) \quad J(x_1) = \frac{1}{2}J(x_1 - 2\eta) + \frac{1}{2}J(x_1 + 2\eta) + N(0, \eta).$$

Given $J(x_2) = t + \text{Tent}(x_2)$ and on Fav , Lemma 4.11(ii) implies that J , restricted to $[\mathfrak{l}, x_2]$, stochastically dominates the Brownian bridge with endpoints $(\mathfrak{l}, -T^2)$ and $(x_2, t + \text{Tent}(x_2))$. We call this Brownian bridge B . Then the slope of the line connecting these two points is

$$(46) \quad m := \frac{t + \text{Tent}(x_2) + T^2}{x_2 - \mathfrak{l}},$$

and so $\mathbb{E}[B(x_1 + r)] = t + \text{Tent}(x_2) - (x_2 - x_1 - r)m$ for any r such that $x_1 + r \in [\mathfrak{l}, x_2]$. Now, conditionally on $J(x_2)$, we have a coupling such that

$$(47) \quad \frac{1}{2}J(x_1 + 2\eta) + \frac{1}{2}J(x_1 - 2\eta) \geq \frac{1}{2}B(x_1 + 2\eta) + \frac{1}{2}B(x_1 - 2\eta).$$

Since the covariance of B is given for $r_1 \leq r_2$ by

$$\text{Cov}(B(r_1), B(r_2)) = \frac{(r_1 - \mathfrak{l})(x_2 - r_2)}{x_2 - \mathfrak{l}},$$

it follows after some algebraic simplification that the variance of the right hand side of (47) is σ_η^2 . The mean of $\frac{1}{2}B(x_1 + 2\eta) + \frac{1}{2}B(x_1 - 2\eta)$ is

$$t + \text{Tent}(x_2) - \frac{1}{2}(x_2 - x_1 - 2\eta)m - \frac{1}{2}(x_2 - x_1 + 2\eta)m = t + \text{Tent}(x_2) - \sigma^2 m.$$

Thus, for $r < t + \text{Tent}(x_2) - \sigma^2 m$, on $\text{Fav} \cap \{[x_1 - d, x_1 + d] \cap P = \emptyset\}$,

$$(48) \quad \begin{aligned} & \mathbb{P}_{\mathcal{F}}\left(\frac{1}{2}J(x_1 + 2\eta) + \frac{1}{2}J(x_1 - 2\eta) < r \mid J(x_2) = t + \text{Tent}(x_2)\right) \\ & \leq \mathbb{P}_{\mathcal{F}}\left(N(t + \text{Tent}(x_2) - \sigma^2 m, \sigma_\eta^2) < r\right) \leq \exp\left(-\frac{1}{2\sigma_\eta^2}(r - t - \text{Tent}(x_2) + \sigma^2 m)^2\right), \end{aligned}$$

the last inequality obtained for $r < t + \text{Tent}(x_2) - \sigma^2 m$ via the upper bound from Lemma 5.9.

Now returning to the definition (46) of m , on Fav ,

$$(49) \quad m = \frac{t + \text{Tent}(x_2) + T^2}{x_2 - \mathfrak{l}} \leq \frac{(R+2)T^2}{T/4} = 4(R+2)T,$$

since we have assumed that $t \leq RT^2$; that $x_2 \geq \mathfrak{l}/2$; and since, on Fav , $\mathfrak{l} \leq -T/2$ and $\text{Tent}(x_2) \leq T^2$.

Using this bound on m in (48) completes the proof of Lemma 5.10. \square

PROOF OF PROPOSITION 5.4. We prove only the case of $s < t$; the other case is analogous, making use of the second inequality of Lemma 5.10 instead of the first as we do in the case of $s < t$.

We first note that

$$(50) \quad \sigma_\eta^2 = \frac{(x_2 - x_1)(x_1 - \mathfrak{l})}{x_2 - \mathfrak{l}} - \eta \leq \sigma^2 - \eta.$$

We will apply Lemma 5.7 to Lemma 5.10 using (45). The parameters of Lemma 5.7 are set as follows (the formal notational conflict between σ_1 or σ_2 and σ_η should not cause confusion): $X = \frac{1}{2}J(x_1 - 2\eta) + \frac{1}{2}J(x_1 + 2\eta) - \text{Tent}(x_1)$, $\sigma_1^2 = \eta$, $\sigma_2^2 = \sigma_\eta^2$, $x_0 = t + \text{Tent}(x_2) - \text{Tent}(x_1) - \sigma^2 m$ (where m is the quantity on the right hand side of (49)), and $A = 1$ as this is the multiplicative constant in the first inequality of Lemma 5.10.

This yields, on the event $\mathbf{Fav} \cap \{[x_1 - d, x_1 + d] \cap P = \emptyset\}$, for each $\eta < d/2$, the following bound on the conditional density of $J(x_1) - \text{Tent}(x_1)$, conditionally on $J(x_2) - \text{Tent}(x_2) = t$:

$$\begin{aligned} f_J^{x_1, x_2}(s | t) &\lesssim \eta^{-\frac{1}{2}} \cdot \exp\left(-\frac{1}{2(\sigma_\eta + \eta^{1/2})^2} (s - t + \text{Tent}(x_1) - \text{Tent}(x_2) + 4\sigma^2(R+2)T)^2\right) \\ &\leq \eta^{-\frac{1}{2}} \cdot \exp\left(-\frac{1}{2(\sigma^2 + 2\eta^{1/2}\sigma)} (s - t + \text{Tent}(x_1) - \text{Tent}(x_2) + 4\sigma^2(R+2)T)^2\right) \end{aligned}$$

for $s < t + \text{Tent}(x_2) - \text{Tent}(x_1) - 4\sigma^2(R+2)T$. We used (50) when expanding the square in the denominator of the exponent in the last inequality. Note that

$$|\text{Tent}(x_1) - \text{Tent}(x_2)| \leq 4T(x_2 - x_1) = 4T\sigma^2.$$

Using the previous equation, we obtain, on $\mathbf{Fav} \cap \{[x_1 - d, x_1 + d] \cap P = \emptyset\}$, that

$$f_J^{x_1, x_2}(s | t) \lesssim \eta^{-\frac{1}{2}} \cdot \exp\left(-\frac{1}{2(\sigma^2 + 2\eta^{1/2}\sigma)} (s - t + MT\sigma^2)^2\right),$$

for $s \leq t - MT\sigma^2$, where $M = 4 + 4(R+2) = 4(R+3)$. Now using the inequality $(1+x)^{-1} \geq 1-x$ for $x = 2\eta^{1/2}\sigma^{-1}$, we find

$$(51) \quad f_J^{x_1, x_2}(s | t) \lesssim \eta^{-\frac{1}{2}} \cdot \exp\left(-\frac{1}{2\sigma^2} (s - t + MT\sigma^2)^2 + \frac{\eta^{1/2}}{\sigma^3} (s - t + MT\sigma^2)^2\right).$$

Let us focus on bounding the second term in the exponent. We expand the square and drop the cross-term, since $s - t \leq 0$, to get that the second term is bounded above by

$$\frac{\eta^{1/2}}{\sigma^3} ((s - t)^2 + M^2T^2\sigma^4) \leq \frac{\eta^{1/2}}{\sigma^3} (4R^2T^4 + M^2T^2\sigma^4),$$

the last inequality since $s - t \in [-2RT^2, 0]$. We now use this bound in (51) and set $\eta^{1/2} = T^{-2}\sigma$ (which satisfies $\eta < d/2$ by assumption), to obtain

$$f_J^{x_1, x_2}(s | t) \lesssim \frac{T^2}{\sigma} \exp\left(-\frac{1}{2\sigma^2} (s - t + MT\sigma^2)^2 + \frac{4R^2T^2}{\sigma^2} + M^2\sigma^2\right).$$

The argument is complete by noting that $f_J^{x_2}(t) \lesssim d^{-\frac{1}{2}}$ by Lemma 5.8 and that $M \leq 6R$, since $M = 4R + 12 \leq 4R + 2 \times 6\sqrt{d} = 6R$ from (34) and $d \geq 1$.

The final statement in Proposition 5.4 of a constant bound on $f_J^{x_1, x_2}(s, t)$ for all values of s and t follows immediately from (45) and the latter assertion of Lemma 5.7 with the parameters $\sigma_1^2 = \eta = d/4$, and again using that the marginal density satisfies $f_J^{x_2}(t) \lesssim d^{-\frac{1}{2}}$ from Lemma 5.8. \square

We will now move towards the proofs of Lemmas 4.13 and 4.14, which use Propositions 5.1 and 5.4. Then we will conclude this section and this part of the argument by proving Lemma 5.7.

For the proofs of Lemmas 4.13 and 4.14 we will need two further statements, the first bounding certain Gaussian integrals, and the other a standard tail bound on the supremum of a Brownian bridge. These are the next two lemmas. We will make frequent use of Lemma 5.11 in the next section as well.

Lemma 5.11. — For $a > 0$,

$$\int_0^\infty \exp(-ax^2 + bx) \, dx \leq \begin{cases} \pi^{\frac{1}{2}} a^{-\frac{1}{2}} & b \leq 0 \\ \pi^{\frac{1}{2}} a^{-\frac{1}{2}} \exp\left(\frac{b^2}{4a}\right) & b \in \mathbb{R}. \end{cases}$$

PROOF. Completing the square, we find

$$\begin{aligned} \int_0^\infty \exp(-ax^2 + bx) \, dx &= \int_0^\infty \exp\left(-a\left(x^2 - \frac{bx}{a} + \frac{b^2}{4a^2}\right) + \frac{b^2}{4a}\right) \, dx \\ &= \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right) \mathbb{P}\left(N\left(\frac{b}{2a}, \frac{1}{2a}\right) > 0\right) \\ &= \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right) \mathbb{P}\left(N(0, 1) > -\frac{1}{\sqrt{2a}}b\right). \end{aligned}$$

For all $b \in \mathbb{R}$ this probability factor is bounded by 1, which yields Case 2 of the statement. If $b < 0$ we may use the upper bound from Lemma 5.9 to obtain Case 1:

$$\sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right) \mathbb{P}\left(N(0, 1) > -\frac{1}{\sqrt{2a}}b\right) \leq \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a} - \frac{1}{2} \cdot \frac{b^2}{2a}\right) = \sqrt{\frac{\pi}{a}}. \quad \square$$

Lemma 5.12. — *Let B be a Brownian bridge of length T from $(0, 0)$ to $(T, 0)$. Then we have*

$$\mathbb{P}\left(\sup_{[0, T]} B(x) \geq r\right) = \mathbb{P}\left(\inf_{[0, T]} B(x) \leq -r\right) = e^{-2r^2/T}.$$

PROOF. The equality of the two quantities follows from Brownian symmetry. By Brownian scaling the statement reduces to when $T = 1$, which is given by equation (3.40) in [KS98, Chapter 4]. \square

PROOF OF LEMMA 4.14. By Lemma 4.11(iii), J is stochastically dominated by the restriction to $[l, \mathfrak{r}]$ of a Brownian bridge from $(-2T, 2T^2)$ to $(2T, 2T^2)$ conditioned to jump over all the poles as well as $\text{Corner}_k^{l, \mathcal{F}}$ and $\text{Corner}_k^{\mathfrak{r}, \mathcal{F}}$. This event being conditioned on has a probability uniformly bounded below by a constant since, on Fav , the value of Tent at the poles, $\text{Corner}_k^{l, \mathcal{F}}$, and $\text{Corner}_k^{\mathfrak{r}, \mathcal{F}}$ are all below T^2 and $T \geq 1$. We also have on Fav that $-T^2 \leq \text{Tent} \leq T^2$. Thus, for both the pole and no-pole cases of the definition of (Y, Z) from (28) and (33), on Fav ,

$$\begin{aligned} \mathbb{P}_{\mathcal{F}}\left(\max\{Y, Z\} > RT^2\right) &\lesssim \mathcal{B}_{2T^2, 2T^2}^{[-2T, 2T]} \left(\sup_{t \in [-2T, 2T]} B(t) > (R-1)T^2\right) \\ &= \exp\left(-\frac{1}{2}(R-3)^2 T^3\right) = \varepsilon^{(R-3)^2 D_k^3/2}, \end{aligned}$$

using Lemma 5.12 in the second inequality. Similarly for the lower bound, we have by Lemma 4.11(i) that, on Fav , J stochastically dominates a Brownian bridge from $(l, -T^2)$ to $(\mathfrak{r}, -T^2)$. Thus again using Lemma 5.12 and that $-T^2 \leq \text{Tent} \leq T^2$, and for both cases of the definition of (Y, Z) ,

$$\begin{aligned} \mathbb{P}_{\mathcal{F}}\left(\min\{Y, Z\} < -RT^2\right) \cdot \mathbb{1}_{\text{Fav}} &\leq \mathcal{B}_{-T^2, -T^2}^{[l, \mathfrak{r}]} \left(\inf_{t \in [l, \mathfrak{r}]} B(t) < -(R-1)T^2\right) \\ &= \exp\left(-\frac{2}{(\mathfrak{r}-l)}(R-2)^2 T^4\right) \leq \varepsilon^{(R-2)^2 D_k^3}, \end{aligned}$$

since $|\mathfrak{r}|, |l| \leq T$. We note that for our range of R , $\varepsilon^{(R-2)^2 D_k^3} \leq \varepsilon^{(R-3)^2 D_k^3/2}$.

We are left with bounding

$$\mathbb{P}_{\mathcal{F}}\left(|Y - Z| > 2RT^{3/2}, -RT^2 < Y, Z < RT^2\right)$$

on Fav . Since the definition of Y and Z depends on whether $P \cap [-2d, 2d] = \emptyset$ or not (see (28) and (33)), the bound we can obtain on the above probability depends on the same as well. However, the bound in the case where a pole is present (which is the one claimed in the statement of Lemma 4.14) actually holds for both cases; this is because the distance between the points where Y and Z measure the deviation of J from Tent is $8d$ when a pole is present, larger than the $2d$ it is when the pole is absent in $[-2d, 2d]$. So we will

present the case where $P \cap [-2d, 2d] \neq \emptyset$, but exactly the same argument works in the other case as well, where it yields a slightly stronger bound corresponding to $2d$ in place of $8d$. Let us define

$$(52) \quad \mathcal{G} = \left\{ (y, z) : -RT^2 < y, z < RT^2, |y - z| > 2RT^{3/2} \right\},$$

so that

$$\mathbb{P}_{\mathcal{F}} \left(|Y - Z| > 2RT^{3/2}, -RT^2 < Y, Z < RT^2 \right) = \iint_{\mathcal{G}} f_J(y, z) dy dz \cdot \mathbb{1}_{\text{Fav}}.$$

To bound this integral, we will use Proposition 5.4 to bound the density and make the change of variables $(u, v) = (y - z, y)$. Note that the range of y and z satisfies the hypotheses of Proposition 5.4 with $x_1 = p - 4d, x_2 = p + 4d$, so that $\sigma^2 = 8d$. These parameter choices satisfy the hypotheses of Proposition 5.4 since $d_{\text{ip}} = 5d$; in particular, $|y - z| \geq 48RTd$ for all $(y, z) \in \mathcal{G}$ as $2RT^{3/2} \geq 48RTd$ since $d \leq \sqrt{T}/24$. Note that $36\sigma^2 = 36 \times 8d = 288d$. So, on Fav,

$$\begin{aligned} & \iint_{\mathcal{G}} f_J(y, z) dy dz \\ & \lesssim d^{-1}T^2 \iint_{\mathcal{G}} \exp \left(-\frac{1}{16d}(|y - z| - 48RTd)^2 + \frac{R^2T^2}{2d} + 288R^2d \right) dy dz \\ & \leq 2d^{-1}T^2 \int_{-RT^2}^{RT^2} \int_{2RT^{3/2}}^{\infty} \exp \left(-\frac{1}{16d}(u - 48RTd)^2 + \frac{R^2T^2}{2d} + 288R^2d \right) du dv \\ & = 2d^{-1}T^2 \int_{-RT^2}^{RT^2} \int_0^{\infty} \exp \left(-\frac{1}{16d}(u + 2RT^{3/2} - 48RTd)^2 + \frac{R^2T^2}{2d} + 288R^2d \right) du dv. \end{aligned}$$

In the second line we used the definition (52) of \mathcal{G} , and in the third we did a change of variables to make the inner integral's range's lower boundary be zero. Next, evaluating the outer integral and expanding the square in the exponent, we see

$$\begin{aligned} & \lesssim RT^4 d^{-1} \exp \left(\frac{R^2T^2}{2d} + 288R^2d \right) \\ & \quad \times \int_0^{\infty} \exp \left(-\frac{1}{16d}(u^2 + 2u(2RT^{3/2} - 48RTd) + 4R^2T^3 - 192R^2T^{5/2}d + 48^2R^2T^2d^2) \right) du \\ & \lesssim RT^4 d^{-\frac{1}{2}} \exp \left(-\frac{1}{4d}R^2T^3 + 12R^2T^{5/2} - 48 \times 3R^2T^2d + \frac{R^2T^2}{2d} + 288R^2d \right) \\ & \leq RT^4 d^{-\frac{1}{2}} \exp \left(12R^2T^{5/2} \right) \varepsilon^{R^2D_k^3/4d}. \end{aligned}$$

We have used Case 1 of Lemma 5.11 with $a = 1/(16d)$ and $b = -2(2RT^{3/2} - 48RTd)$ for the integral in the second-to-last line, since $d \leq \sqrt{T}/24$ implies that $2RT^{3/2} - 48RTd \geq 0$, and thus that $b \leq 0$. In the last line, since $1 \leq d \leq \sqrt{T}/24$, we see that $288R^2d \leq 12R^2T^{1/2}$, and thus the sum of the last three terms in the exponent of the penultimate line is negative and may be dropped. Finally, since $x \leq \exp(x^{5/8})$ for $x \geq 1$, and since $1 \leq d \leq \sqrt{T}/24$ and $R \geq 1$ from (34), we have that $RT^4 d^{-\frac{1}{2}} \leq \exp(R^2T^{5/2})$. This completes the proof of Lemma 4.14. \square

PROOF OF LEMMA 4.13. Using Lemma 4.14, it is enough to show that, on Fav and when $P \cap [-2d, 2d] \neq \emptyset$, $\mathbb{P}(Y < -T^{3/2}) + \mathbb{P}_{\mathcal{F}}(Z < -T^{3/2})$ is bounded by the right-hand side in the Lemma 4.13's statement.

Since we are considering the situation where Y and Z are negative, we may use Proposition 5.1. So from Proposition 5.1(2) and Lemma 5.2, on Fav $\cap \{P \cap [-2d, 2d] \neq \emptyset\}$,

$$\mathbb{P}_{\mathcal{F}}(Y < -RT^{3/2}) \lesssim d^{-\frac{1}{2}} \int_{RT^{3/2}}^{\infty} \exp \left(-\frac{y^2}{8d} \right) dy \lesssim \exp \left(-\frac{1}{8d}R^2T^3 \right) = \varepsilon^{R^2D_k^3/8d},$$

where we have performed the change of variables $y \mapsto y + RT^{3/2}$ and applied Case 1 of Lemma 5.11 with $a = 1/(8d)$ and $b = -RT^{3/2}/(4d)$ in the second inequality. Similarly, we have

$$\mathbb{P}_{\mathcal{F}}(J(p+4d) - \text{Tent}(p+4d) < -RT^{3/2}) \lesssim \varepsilon^{R^2 D_k^3 / 8d}. \quad \square$$

We conclude the section by providing the proof of the technical tool Lemma 5.7.

PROOF OF LEMMA 5.7. We prove the case where X satisfies $\mathbb{P}(X < x) \leq A \exp(-\frac{1}{2\sigma_2^2}(x - x_0)^2)$ for $x < x_0$; the other case is the same by considering $-X$.

Let ν be the law of X . For $x < x_0$ and $0 < \delta < 1$, let $\tilde{x} = \delta(x_0 - x)$. Since X and N are independent, we have

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma_1} \int_{-\infty}^{\infty} e^{-(x-y)^2/2\sigma_1^2} d\nu(y) = \frac{1}{\sqrt{2\pi}\sigma_1} \left[\int_{[x-\tilde{x}, x+\tilde{x}]} + \int_{[x-\tilde{x}, x+\tilde{x}]^c} e^{-(x-y)^2/2\sigma_1^2} d\nu(y) \right].$$

From the first equality we see that the density is bounded by $(\sqrt{2\pi}\sigma_1)^{-1}$ for all x . For the stronger bound for small enough x , note using the hypothesis on ν that the first integral in the right-hand side is bounded by

$$\int_{[x-\tilde{x}, x+\tilde{x}]} d\nu(y) \leq \nu((-\infty, x + \tilde{x})) \leq A \cdot \exp\left(-\frac{(x + \tilde{x} - x_0)^2}{2\sigma_2^2}\right) = A \cdot \exp\left(-\frac{(1-\delta)^2(x - x_0)^2}{2\sigma_2^2}\right),$$

where we have used that $x + \tilde{x}$ is less than x_0 ; this is due to $\delta < 1$ and $x < x_0$.

The second integral is bounded by $\exp\left(-\frac{\tilde{x}^2}{2\sigma_1^2}\right) = \exp\left(-\frac{\delta^2(x-x_0)^2}{2\sigma_1^2}\right)$. These inequalities hold for all $0 < \delta < 1$, and so if we set $\delta = \sigma_1/(\sigma_1 + \sigma_2)$, we obtain our result. \square

5.3. The difficult case: Above and below the pole on either side

At this stage we have proved the required bound on $f_J(y, z)$ in the two cases where $y, z < 0$ or $y + z > 0$. This leaves the case where $y < 0$, $z > 0$, and $y + z < 0$ (the case where $y > 0$ and $z < 0$ is clearly symmetric). Perhaps surprisingly, this turns out to be the most difficult case. However, the difficulty can be understood by heuristically noting that the density $f_J(y, z)$ can be expected to be largest in this case.

Recall that $f_J(y, z)$ is the density of (Y, Z) , which, from (28), are respectively the deviations of J from Tent at $p - 4d$ and $p + 4d$. So, the size of the density $f_J(y, z)$ essentially represents a comparison of the probability that J takes the values $y + \text{Tent}(p - 4d)$ and $z + \text{Tent}(p + 4d)$ respectively at $p - 4d$ and $p + 4d$ to the probability of the same for Brownian motion. A larger value of this density is obtained if J finds it easier to adopt the specified values than a Brownian motion does. This is precisely what happens when $y < 0$, $z > 0$, and $y + z < 0$, as J has the pole at p which pushes it up and helps it attain the value of z at $p + 4d$; a Brownian motion has no such assistance. Some thought should reveal that in Section 5.1 the pole actually makes J 's task more difficult than B 's, which has no pole, while in Section 5.2, the pole has essentially no effect.

In this section, since $y + z < 0$, the vault cost V cannot be ignored. Thus we need a stronger bound on $f_J(y, z)$ than was required in Section 5.2; in fact, we need a bound of the same basic form as that proved in Section 5.1. This is why the previous paragraph's conclusion that the density is highest in this case indicates that the required argument will be more delicate.

Let $f_J(z | y)$ be the conditional density of Z at z given $Y = y$. Our aim will be the following proposition.

Proposition 5.13. — *Let R be as in (34). If $40Td \leq z \leq 2RT^{3/2}$ and $y + z < 0$,*

$$f_J(z | y) \cdot \mathbb{1}_{\text{Fav}, P \cap [-2d, 2d] \neq \emptyset} \lesssim d^{-\frac{1}{2}} \cdot \exp\left(-\frac{z^2}{8d} + 20RT^{5/2}\right).$$

Further, $f_J(z | y) \lesssim d^{-\frac{1}{2}}$ for all $y, z \in \mathbb{R}$ with $y + z < 0$.

First we show that Proposition 5.13 implies the sufficient bound (40) when $y < 0, z > 0$, and $y + z < 0$ (and symmetrically when $y > 0$ and $z < 0$).

Lemma 5.14. — *When $yz < 0, y + z < 0$, and $(y, z) \in \mathcal{G}_R^{(1)}$, we have (40) with $G'_2 = 41RD_k^{5/2}$ and $G'_1 = G'$, where G' is a constant independent of ε, k , and d .*

PROOF. We give the proof for when $y < 0$ and $z > 0$. From (36) and (38), we see that in this case

$$(53) \quad V \cdot S \lesssim d \cdot \exp\left(\frac{y^2}{8d} + \frac{z^2}{8d} + 21RT^{5/2}\right).$$

So it suffices to prove, for some $G \leq 20R$,

$$f_J(y, z) \lesssim d^{-1} \cdot \exp\left(-\frac{y^2}{8d} - \frac{z^2}{8d} + GT^{5/2}\right).$$

There are two cases to consider. If $z \leq 40Td$, then, from (53), $V \cdot S$ is bounded by

$$d \cdot \exp\left(\frac{y^2}{8d} + \frac{40^2 T^2 d^2}{8d} + 21RT^{5/2}\right) \leq d \cdot \exp\left(\frac{y^2}{8d} + 30RT^{5/2}\right)$$

since $d \leq \sqrt{T}/24$ and $40^2/(8 \times 24) \leq 9$. Noting that $30 < 41$ (i.e., the constant factor in front of our claimed G'_2), it suffices to prove

$$f_J(y, z) \lesssim d^{-1} \cdot \exp\left(-\frac{y^2}{8d}\right).$$

This is provided by Proposition 5.1 and Lemma 5.2 since $y < 0$, and by Proposition 5.13's latter statement that $f_J(z | y) \lesssim d^{-\frac{1}{2}}$.

Now suppose $z > 40Td$. Note that we also have $z \leq 2RT^{3/2}$, since $y < 0$ and $(y, z) \in \mathcal{G}_R^{(1)}$ implies that $|y - z| \leq 2RT^{3/2}$. So from Proposition 5.1, Lemma 5.2, and Proposition 5.13, we obtain

$$f_J(y, z) \lesssim d^{-1} \cdot \exp\left(-\frac{y^2}{8d} - \frac{z^2}{8d}\right) \cdot \exp\left(20RT^{5/2}\right).$$

This completes the proof, after taking into account the extra factor of $\exp(21RT^{5/2})$ which arises from the expressions for V and S as in (53). \square

We next turn to discussing the proof ideas of Proposition 5.13. The claim of the proposition may be a surprising one at first glance, for, in a slight abuse of the language of pseudo-variance used in Section 5.2, Proposition 5.13 says that $J(p + 4d) - \text{Tent}(p + 4d)$, conditionally on $J(p - 4d) - \text{Tent}(p - 4d)$ being negative, has pseudo-variance at most $4d$; in contrast, a Brownian motion, conditionally on its value at $p - 4d$, would have a much higher variance of $8d$ at the position $p + 4d$. So we must crucially use both that $J(p - 4d) - \text{Tent}(p - 4d) < 0$ (i.e., $y < 0$) and that J must satisfy $J(p) \geq \text{Tent}(p)$.

Heuristically, because J is jumping over $\text{Tent}(p)$ from a negative value at $p - 4d$, it will make the jump with a very low margin. Thus the variance at p is not $4d$ as it would be for a Brownian motion, but essentially 0. This intuition is captured in Lemma 5.16, which says that we may safely restrict our analysis to the case where J jumps over the pole at p by at most 1. To prove Lemma 5.16, we need a technical lemma about the monotonicity of conditional probabilities of Gaussians. The result is identical to [Ham19b, Lemma 2.21], but we include its short proof here for completeness.

Lemma 5.15. — *Fix $r > 0, m \in \mathbb{R}$, and $\sigma^2 > 0$, and let X be distributed as $N(m, \sigma^2)$. Then the quantity $\mathbb{P}(X \geq r + s | X \geq s)$ is a strictly decreasing function of $s \in \mathbb{R}$.*

PROOF. Note that

$$\log \mathbb{P}(X \geq s + r \mid X \geq s) = \log \int_{s+r}^{\infty} \exp \left\{ -\frac{(x-m)^2}{2\sigma^2} \right\} dx - \log \int_s^{\infty} \exp \left\{ -\frac{(x-m)^2}{2\sigma^2} \right\} dx$$

has derivative in s given by

$$\frac{\exp \left\{ -\frac{(s-m)^2}{2\sigma^2} \right\} \int_{s+r}^{\infty} \exp \left\{ -\frac{(x-m)^2}{2\sigma^2} \right\} dx - \exp \left\{ -\frac{(s+r-m)^2}{2\sigma^2} \right\} \int_s^{\infty} \exp \left\{ -\frac{(x-m)^2}{2\sigma^2} \right\} dx}{\int_{s+r}^{\infty} \exp \left\{ -\frac{(x-m)^2}{2\sigma^2} \right\} dx \cdot \int_s^{\infty} \exp \left\{ -\frac{(x-m)^2}{2\sigma^2} \right\} dx}.$$

The denominator is clearly positive. Performing the change of variable $x \mapsto x + r$ in the first integral of the numerator and manipulating the exponents show that the numerator equals

$$\int_s^{\infty} \exp \left\{ -\frac{(x-m)^2 + (s-m)^2 + r^2}{2\sigma^2} \right\} \left(\exp \left\{ -\frac{(x-m)r}{\sigma^2} \right\} - \exp \left\{ -\frac{(s-m)r}{\sigma^2} \right\} \right) dx.$$

The proof is complete by noting that this integrand is strictly negative for all $x > s$. □

In order to state Lemma 5.16, we define the random variable U to be the deviation of the jump ensemble from the Tent map at the pole p . For later use, we also take this opportunity to define W_η to be the same at $p + 4d + \eta$ for $\eta < d$. So, we define

$$(54) \quad \begin{aligned} U &:= J(p) - \text{Tent}(p), \\ W_\eta &:= J(p + 4d + \eta) - \text{Tent}(p + 4d + \eta). \end{aligned}$$

The parameter η , as in the previous section, will be set to a specific small value in a local randomization argument later. Recall also from (28) that Y and Z are respectively the deviation of J from Tent at $p - 4d$ and $p + 4d$. See Figure 5.2. We now turn to our assertion that J typically makes a narrow jump over p .

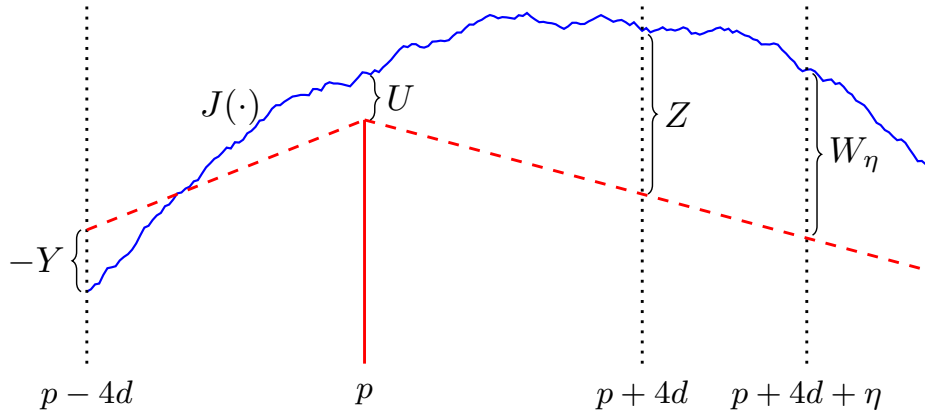


FIGURE 5.2. Illustrating the definitions of U , W , Y and Z in the subcase being addressed in this section; Y is negative, and so the length being shown is $-Y$. The blue curve is J , while the red dashed function is Tent. The red vertical line emphasises the height of the pole at p .

Lemma 5.16 (Narrow jump over p). — *Let $f_J(z, u \mid y)$ be the joint conditional density of Z and U given Y . For $y, z \in \mathbb{R}$ such that $y + z < 0$,*

$$f_J(z \mid y) \lesssim d^{\frac{1}{2}} \int_0^1 f_J(z, u \mid y) du.$$

PROOF. We have

$$(55) \quad \int_0^1 f_J(z, u | y) \, du = f_J(z | y) \int_0^1 f_J(u | y, z) \, du.$$

Let X be distributed as $N(0, 2d)$, which is the distribution of a Brownian bridge of duration $8d$ from 0 to 0 at its midpoint, and let $\Delta = \frac{1}{2}(\text{Tent}(p - 4d) + \text{Tent}(p + 4d)) - \text{Tent}(p)$. Note that $\Delta \leq 0$ by the concavity of the Tent map. Then the second factor on the right hand side is

$$(56) \quad \begin{aligned} \mathbb{P}_{\mathcal{F}}(U \in [0, 1] | Y = y, Z = z) &= \mathbb{P}\left(X + \frac{y+z}{2} + \Delta \in [0, 1] \mid X + \frac{y+z}{2} + \Delta \geq 0\right) \\ &= 1 - \mathbb{P}\left(X + \frac{y+z}{2} + \Delta \geq 1 \mid X + \frac{y+z}{2} + \Delta \geq 0\right). \end{aligned}$$

Now by Lemma 5.15 with $r = 1$ and $s = -(y+z)/2 - \Delta$,

$$\mathbb{P}\left(X \geq 1 - \frac{y+z}{2} - \Delta \mid X \geq -\frac{y+z}{2} - \Delta\right)$$

is an increasing function of $y+z$, i.e., $\mathbb{P}_{\mathcal{F}}(U \in [0, 1] | Y = y, Z = z)$ is a decreasing function of $y+z$. So for $y+z < 0$, we obtain that

$$\mathbb{P}_{\mathcal{F}}(U \in [0, 1] | Y = y, Z = z) \geq 1 - \mathbb{P}(X \geq 1 | X \geq 0) = \mathbb{P}(X \in [0, 1] | X \geq 0) = \mathbb{P}(|X| \leq 1);$$

the second quantity is the right hand side of (56) evaluated at $y+z = -2\Delta \geq 0$. An easy bound using the standard normal density gives that $\mathbb{P}(|X| \leq 1)^{-1} \lesssim d^{\frac{1}{2}}$ for $d \geq 1$; tracing back the relations and using this last bound in (55) completes the proof of Lemma 5.16. \square

The broad idea of the proof of Proposition 5.13 is to write the distribution of Z , conditionally on $U = J(p) - \text{Tent}(p)$ and $W_\eta = J(p + 4d + \eta) - \text{Tent}(p + 4d + \eta)$, in terms of a Brownian bridge using the definition of J , and then use this Brownian structure to obtain a bound on the conditional density of Z given Y via Lemma 5.7 and Lemma 5.16. This is the same strategy of local randomization that was used in Section 5.2 to prove Proposition 5.4.

So, as before, we will need tail probabilities on the distribution of W_η given U and Y . By the Markov property of Brownian bridges, W_η is conditionally independent of Y given U , and so in fact we need a tail probability for W_η given only U . Such a tail bound is the content of Lemma 5.17, whose argument is essentially again stochastically dominating J by a Brownian bridge.

Lemma 5.17. — *We have for $0 < \eta \leq d$, $u \geq 0$, $w > u + 9T(4d + \eta)$ and on the event $\text{Fav} \cap \{P \cap [-2d, 2d] \neq \emptyset\}$,*

$$\mathbb{P}_{\mathcal{F}}(W_\eta > w | U = u) \lesssim \exp\left(-\frac{(w - u - 9T(4d + \eta))^2}{2(4d + \eta)} + 7T\right).$$

To prove Lemma 5.17 we will need a lower bound on the probability of a Brownian bridge jumping over poles. This is recorded in the next lemma, whose straightforward proof is deferred to the end of the section to permit the flow of the overall argument.

Lemma 5.18. — *Let $N \geq 1$, $x_0 \in [l, \mathfrak{r}]$, and $x_N = \mathfrak{r}$. Let $x_1 < \dots < x_{N-1}$ be \mathcal{F} -measurable points in (x_0, x_N) , and B be a Brownian bridge with law $\mathcal{B}_{0,0}^{[x_0, 2T]}$. Then there exists a universal $G < \infty$ such that*

$$\mathbb{P}_{\mathcal{F}}(B(x_i) > 0, i = 1, \dots, N-1) > G^{-1}N^{-1/2} \exp(-3N).$$

PROOF OF LEMMA 5.17. Define the coordinate ϕ_{end} by

$$\phi_{\text{end}} := u + \text{Tent}(p) + 5T(2T - p),$$

which is the y -coordinate at $x = 2T$ of a line with slope $5T$ started at $(p, u + \text{Tent}(p))$. Since on Fav we have $|\text{slope}(\text{Tent})| \leq 4T$, it follows that $\phi_{\text{end}} > \mathcal{L}(2T)$. Lemma 4.11(iv) tells us that the Brownian bridge B from $(p, u + \text{Tent}(p))$ to $(2T, \phi_{\text{end}})$ conditioned to jump over all poles in $[p, 2T]$ and to be above $\text{Corner}_k^{\tau, \mathcal{F}}$ stochastically dominates J on $[p, 2T]$. Let A be the conditioning event just mentioned. Then we have, on $\text{Fav} \cap \{P \cap [-2d, 2d] \neq \emptyset\}$,

$$\mathbb{P}_{\mathcal{F}}\left(W_{\eta} > w \mid U = u\right) \leq \mathcal{B}_{u+\text{Tent}(p), \phi_{\text{end}}}^{[p, 2T]}\left(B(p+4d+\eta) > w + \text{Tent}(p+4d+\eta) \mid A\right).$$

We need to lower bound $\mathbb{P}_{\mathcal{F}}(A)$ on Fav . Note that on Fav we have $\text{Corner}_k^{\tau, \mathcal{F}} \leq T^2$, while $B(\tau)$ has mean bounded below by

$$J(p) + 5T \cdot (\tau - p) \geq -T^2 + 5T \cdot (T/2 - d) = \frac{3}{2}T^2 - 5Td;$$

here we used $J(p) \geq \text{Tent}(p) = \mathcal{L}_n(k+1, p) \geq -T^2$ on Fav . These bounds, along with the concavity of Tent and Lemma 5.18 (with $N = |P| \leq 2T$, as $d_{\text{ip}} \geq 1$ implies this bound), implies that $\mathbb{P}_{\mathcal{F}}(A) \cdot \mathbb{1}_{\text{Fav}}$ is bounded below, up to an absolute constant, by $T^{-\frac{1}{2}} \exp(-6T) \geq \exp(-7T)$ as $T \geq 1$. So, on $\text{Fav} \cap \{P \cap [-2d, 2d] \neq \emptyset\}$,

$$\mathbb{P}_{\mathcal{F}}\left(W_{\eta} > w \mid U = u\right) \lesssim \exp(7T) \cdot \mathcal{B}_{u+\text{Tent}(p), \phi_{\text{end}}}^{[p, 2T]}\left(B(p+4d+\eta) > w + \text{Tent}(p+4d+\eta)\right).$$

Let $\rho = \frac{4d+\eta}{2T-p}$. Then $\mathbb{E}_{\mathcal{F}}[B(p+4d+\eta)] = u + \text{Tent}(p) + 5T(4d+\eta)$ and $\text{Var}_{\mathcal{F}}(B(p+4d+\eta)) = (1-\rho)(4d+\eta)$. So, for $w > u + 9T(4d+\eta)$ and on Fav ,

$$\begin{aligned} & \mathcal{B}_{u+\text{Tent}(p), \phi_{\text{end}}}^{[p, 2T]}\left(B(p+4d+\eta) > w + \text{Tent}(p+4d+\eta)\right) \\ &= \mathbb{P}_{\mathcal{F}}\left(N(u + \text{Tent}(p) + 5T(4d+\eta), (1-\rho)(4d+\eta)) > w + \text{Tent}(p+4d+\eta)\right) \\ &\leq \mathbb{P}_{\mathcal{F}}\left(N(0, (1-\rho)(4d+\eta)) > w - u - 9T(4d+\eta)\right) \\ &\leq \exp\left(-\frac{(w - u - 9T(4d+\eta))^2}{2(4d+\eta)}\right) \end{aligned}$$

The third line uses that $\text{Tent}(p+4d+\eta) - \text{Tent}(p) \geq -4T(4d+\eta)$ from (19); the final inequality uses the upper bound of Lemma 5.9 with $\sigma^2 = (1-\rho)(4d+\eta)$ and $t = w - u - 9T(4d+\eta)$. \square

With this tail bound we may turn to the proof of Proposition 5.13.

PROOF OF PROPOSITION 5.13. Let λ be defined as

$$\lambda = \frac{\eta}{4d+\eta}$$

so that, conditionally on Y, U , and W_{η} , the distribution of Z is

$$(57) \quad Z = \lambda U + (1-\lambda)W_{\eta} + N(0, 4\lambda d).$$

This relation holds because, by the definition of J , the distribution of J on $[p, p+4d+\eta]$, conditionally on $J(p)$ and $J(p+4d+\eta)$, is a Brownian bridge with the prescribed endpoint values, and because Tent is affine on $[p, p+4d+\eta]$ for $\eta < d$ as $d_{\text{ip}} = 5d$. Note that the variance of this Brownian bridge at $p+4d$ is $\frac{4d\eta}{4d+\eta} = 4\lambda d$.

Let $\nu_p(\cdot | y)$ be the conditional law of U given that $Y = y$. From the narrow jump Lemma 5.16 we see

$$(58) \quad f_J(z | y) \lesssim d^{\frac{1}{2}} \cdot \int_0^1 f_J(z, u | y) du = d^{\frac{1}{2}} \cdot \int_0^1 f_J(z | u, y) d\nu_p(u | y),$$

where $f_J(z | u, y)$ is the conditional density of Z given U and Y . So our task is to bound $f_J(z | u, y)$ when $u \in [0, 1]$. By the Markov property, this object does not depend on y . We will obtain this bound by converting the conditional tail bound of Lemma 5.17 to a conditional density bound using Lemma 5.7.

We have from Lemma 5.17, by taking $w = \frac{t-\lambda u}{1-\lambda}$ and simplifying the resulting expression, that, for $t > u + 36Td$,

$$(59) \quad \mathbb{P}_{\mathcal{F}}\left(\lambda U + (1-\lambda)W_{\eta} > t \mid U = u, Y = y\right) \lesssim \exp\left(-\frac{1}{8d(1-\lambda)}(t-u-36Td)^2 + 7T\right).$$

While simplifying we used that $(4d+\eta)(1-\lambda) = 4d$. Now using (57) and (59), we apply Lemma 5.7. The parameters of this application are $\sigma_1^2 = 4\lambda d$, $\sigma_2^2 = 4(1-\lambda)d$, $x_0 = u + 36Td$, and A equal to the constant specified by \lesssim in (59) multiplied by $\exp(7T)$. Thus we obtain, for every $\eta < d$ and $z > u + 36Td$,

$$f_J(z \mid u, y) \lesssim \eta^{-\frac{1}{2}} \cdot \exp\left(-\frac{1}{2\sigma^2}(z-u-36Td)^2 + 7T\right),$$

where $\sigma^2 = 4d(\lambda^{1/2} + (1-\lambda)^{1/2})^2 \leq 4d(1+2\lambda^{1/2}) \leq 4d(1+\eta^{1/2})$ since $0 \leq \lambda \leq \eta/4$. Setting $\eta^{1/2} = 8T^{-2}d$ (which satisfies $\eta < d$ as $d \leq \sqrt{T}/24$ and $T \geq 1$) and using that $(1+x)^{-1} \geq 1-x$ for $x = \eta^{1/2}$ yields

$$f_J(z \mid u, y) \lesssim d^{-1}T^2 \cdot \exp\left(-\frac{1}{8d}(z-u-36Td)^2 + T^{-2}(z-u-36Td)^2 + 7T\right).$$

Using this in (58) and bounding the integral by a trivial bound on the integrand then shows

$$f_J(z \mid y) \lesssim d^{-\frac{1}{2}}T^2 \cdot \exp\left(-\frac{1}{8d}(z-1-36Td)^2 + T^{-2}z^2 + 7T\right)$$

for $z > 36Td + 1$. Since $4Td > 1$, and since $T^2 \leq e^T$ as $T \geq 1$, it also holds that

$$f_J(z \mid y) \lesssim d^{-\frac{1}{2}} \cdot \exp\left(-\frac{1}{8d}(z-40Td)^2 + T^{-2}z^2 + 8T\right)$$

for $z > 40Td$. Expanding the expression in the exponent gives the expression

$$-\frac{z^2}{8d} + \frac{z(40Td)}{4d} - \frac{40^2T^2d^2}{8d} + z^2T^{-2} + 8T \leq -\frac{z^2}{8d} + 20RT^{5/2} - 200T^2d + 4R^2T + 8T,$$

using that $z \leq 2RT^{3/2}$. Since $R = 6\sqrt{d}$ from (34) and since $T, d \geq 1$, the last three terms in the previous display are collectively negative and may be dropped. This proves the first part of Proposition 5.13.

Now we turn to the latter claim of Proposition 5.13. The tail bound (59), combined with the latter part of Lemma 5.7, says that $f_J(z \mid y, u) \lesssim d^{-\frac{1}{2}}$. Then (58) gives the latter claim of Proposition 5.13. This completes the proof of Proposition 5.13. \square

Finally, we provide the last piece of the proof of Proposition 5.13 by proving Lemma 5.18.

PROOF OF LEMMA 5.18. Let A denote the event that a Brownian motion B' with law $\mathcal{B}_{0,*}^{[x_0, 2T]}$ is negative at $2T$; write \tilde{B} for the process B' conditioned on A . Since Brownian motion on $[x_0, 2T]$ conditioned on its value at $2T$ is a Brownian bridge on $[x_0, 2T]$ with appropriate endpoints, Lemma 4.3 implies that the Brownian bridge B in the statement of Lemma 5.18 stochastically dominates \tilde{B} . Thus,

$$\mathbb{P}_{\mathcal{F}}\left(\bigcap_{i=1}^{N-1} \{B(x_i) > 0\}\right) \geq \mathbb{P}\left(\bigcap_{i=1}^{N-1} \{\tilde{B}(x_i) > 0\} \mid A\right) = \frac{\mathbb{P}\left(\bigcap_{i=1}^N \{\tilde{B}(x_i) > 0\}, \tilde{B}(2T) < 0\right)}{\mathbb{P}\left(\tilde{B}(2T) < 0\right)}.$$

The denominator is equal to $\frac{1}{2}$. We may lower bound the numerator by

$$(60) \quad \begin{aligned} & \mathbb{P}\left(\bigcap_{i=1}^N \{\tilde{B}(x_i) - \tilde{B}(x_{i-1}) \in [0, \sqrt{x_i - x_{i-1}}]\}, \tilde{B}(2T) - \tilde{B}(x_N) < -\sum_{i=1}^N \sqrt{x_i - x_{i-1}}\right) \\ & = p^N \cdot \mathbb{P}\left(N(0, 2T - x_N) < -\sum_{i=1}^N \sqrt{x_i - x_{i-1}}\right), \end{aligned}$$

using the independence of Brownian motion increments, where $p = \mathbb{P}(N(0, 1) \in [0, 1]) > (2\pi)^{-1/2}e^{-1/2} > e^{-2}$; the first inequality is by lower bounding the standard normal density on $[0, 1]$ by $(2\pi)^{-1/2}e^{-1/2}$, and the last inequality is by numerical calculation. Now,

$$\mathbb{P}\left(N(0, 2T - x_N) < -\sum_{i=1}^N \sqrt{x_i - x_{i-1}}\right) \geq \mathbb{P}\left(N(0, 1) > \frac{\sqrt{N}\sqrt{x_N}}{\sqrt{2T - x_N}}\right) \geq \mathbb{P}\left(N(0, 1) > \sqrt{N}\right) \gtrsim \frac{e^{-N/2}}{\sqrt{N}}.$$

We have used the Cauchy-Schwarz inequality in the first inequality; that $x_N = \tau \leq T$ in the second; and Lemma 5.9 in the last. Combining with (60) and that $\mathbb{P}(\tilde{B}(2T) < 0) = \frac{1}{2}$ completes the proof. \square

5.4. When no pole is present

The aim of this section is to perform the final task in the proof of Theorem 4.10, namely to provide the proof of Proposition 4.17. The analysis follows similar lines to the case where a pole was present, but here we are aided by the fact that there is no vault cost V . Because there is no pole, we cannot make the choice of stepping back to $p - 4d$ and $p + 4d$ in decomposing the jump curve probability. Since we know only that there is no pole in $[-2d, 2d]$, and we need to give ourselves some distance from the nearest pole, we make the choice to not step back and instead decompose at $-d$ and d . More precisely we recall, as mentioned before Lemma 4.14, that Y and Z are defined on the event $P \cap [-2d, 2d] = \emptyset$ as

$$\begin{aligned} Y &:= J(-d) - \text{Tent}(-d) \\ Z &:= J(d) - \text{Tent}(d). \end{aligned}$$

Correspondingly, on $P \cap [-2d, 2d] = \emptyset$, $f_J(y, z)$ is the joint density of (Y, Z) under this definition.

As there is no vault cost, we only need to consider the slope cost S . In this context, since the interval $[-d, d]$ has length $2d$, S is

$$S = d^{\frac{1}{2}} \cdot \exp\left(-\frac{1}{4d}(\tilde{y} - \tilde{z})^2\right);$$

here, $\tilde{y} = y + \text{Tent}(-d)$ and $\tilde{z} = z + \text{Tent}(d)$, so that $Y = y$ implies $J(-d) = \tilde{y}$ and similarly for $Z = z$ and $J(d) = \tilde{z}$; thus the tilde plays the same role as it did in Section 4.2 of going from the value of Y or Z to the value of J at the corresponding points.

The bound we will obtain from Proposition 5.4 is in terms of y and z , and here we see the exact form of the leeway factor needed to write S in terms of y and z , as was done in (36) in the pole case; this bound will hold for $(y, z) \in \mathcal{G}_R^{(2)}$, where we recall the definition of $\mathcal{G}_R^{(2)}$ from (32). Using from (19) that the slope of the Tent map is bounded in absolute value by $4T$, so that $|\text{Tent}(d) - \text{Tent}(-d)| \leq 8Td$, we see that, for $(y, z) \in \mathcal{G}_R^{(2)}$,

$$\begin{aligned} (61) \quad S &= d^{\frac{1}{2}} \cdot \exp\left(\frac{1}{4d}(\tilde{y} - \tilde{z})^2\right) \leq d^{\frac{1}{2}} \cdot \exp\left(\frac{1}{4d}(y - z)^2 + 16T^2d + \frac{1}{2d}(y - z)(\text{Tent}(d) - \text{Tent}(-d))\right) \\ &\leq d^{\frac{1}{2}} \cdot \exp\left(-\frac{1}{4d}(y - z)^2 + T^{5/2} + 8RT^{5/2}\right) \\ &\leq d^{\frac{1}{2}} \cdot \exp\left(-\frac{1}{4d}(y - z)^2 + 9RT^{5/2}\right); \end{aligned}$$

the first inequality using the mentioned bound on $|\text{Tent}(d) - \text{Tent}(-d)|$; the second since $(y, z) \in \mathcal{G}_R^{(2)}$ implies $|y - z| \leq 2RT^{3/2}$, and since $d \leq \sqrt{T}/24$ and $16/24 \leq 1$; and third since $R \geq 1$ from (34).

Apart from this, we will also need to make a suitable choice for the distribution μ , discussed in Section 4.2.3, which is here the distribution of the Brownian motion we start at $-d$ to which we compare J . We select μ to be uniform on $[-RT^2, RT^2]$ as this interval contains all values y may take when $(y, z) \in \mathcal{G}_R^{(2)}$. As before,

the cost for this choice of μ is only polynomial in T , and so does not affect the bound we need to prove on $f_J(y, z)$.

So overall our aim is to get a bound on the joint density of the form

$$f_J(y, z) \leq \exp\left(-\frac{1}{4d}(y-z)^2\right) \cdot \exp\left(GT^{5/2}\right)$$

for some $G < \infty$. This of course is essentially immediate from Proposition 5.4 when $|y-z|$ is sufficiently large; recall that Proposition 5.4 was stated carefully to not assume the presence of a pole in the interval of consideration.

PROOF OF PROPOSITION 4.17. On $\text{Fav} \cap \{P \cap [-2d, 2d] = \emptyset\}$,

$$\begin{aligned} \mathbb{P}_{\mathcal{F}}\left(J(\cdot) - J(-d) \in A, (Y, Z) \in \mathcal{G}_R^{(2)}\right) &= \mathbb{E}_{\mathcal{F}}\left[\mathbb{P}_{\mathcal{F}}\left(J(\cdot) - J(-d) \in A \mid J(d), J(-d)\right) \mathbb{1}_{(Y, Z) \in \mathcal{G}_R^{(2)}}\right] \\ &= \mathbb{E}_{\mathcal{F}}\left[\mathcal{B}_{J(-d), J(d)}^{[-d, d]}(\tilde{A}) \mathbb{1}_{(Y, Z) \in \mathcal{G}_R^{(2)}}\right], \end{aligned}$$

where \tilde{A} is the set of functions f in $\mathcal{C}_{*,*}([-d, d], \mathbb{R})$ such that $f(\cdot) - f(-d) \in A$. Recall the notation $\tilde{y} = y + \text{Tent}(-d)$ and $\tilde{z} = z + \text{Tent}(d)$. We may write the last displayed expression as an integral on the event $\text{Fav} \cap \{P \cap [-2d, 2d] = \emptyset\}$:

$$\begin{aligned} \mathbb{E}_{\mathcal{F}}\left[\mathcal{B}_{J(-d), J(d)}^{[-d, d]}(\tilde{A}) \mathbb{1}_{(Y, Z) \in \mathcal{G}_R^{(2)}}\right] &= \iint_{\mathcal{G}_R^{(2)}} \mathcal{B}_{\tilde{y}, \tilde{z}}^{[-d, d]}(\tilde{A}) f_J(y, z) \, dy \, dz \\ (62) \qquad \qquad \qquad &= d^{-\frac{1}{2}} \iint_{\mathcal{G}_R^{(2)}} \mathcal{B}_{\tilde{y}, \tilde{z}}^{[-d, d]}(\tilde{A}) e^{-\frac{1}{4d}(\tilde{y}-\tilde{z})^2} \cdot (f_J(y, z) \cdot S) \, dy \, dz. \end{aligned}$$

From Proposition 5.4 with parameters $x_1 = -d$ and $x_2 = d$, we know that on Fav and when $|y-z| > 12RTd$ and $(y, z) \in \mathcal{G}_R^{(2)}$,

$$\begin{aligned} f_J(y, z) &\lesssim d^{-1}T^2 \cdot \exp\left(-\frac{1}{4d}(|y-z| - 12RTd)^2 + \frac{4R^2T^2}{2d} + 72R^2d\right) \\ &= d^{-1}T^2 \cdot \exp\left(-\frac{1}{4d}((y-z)^2 - 24RTd|y-z| + 144R^2T^2d^2) + \frac{4R^2T^2}{2d} + 72R^2d\right) \\ &\leq d^{-1} \cdot \exp\left(-\frac{1}{4d}(y-z)^2 + 6RT|y-z| - 36R^2T^2d + \frac{4R^2T^2}{2d} + 72R^2d + T^2\right) \\ &\leq d^{-1} \cdot \exp\left(-\frac{1}{4d}(y-z)^2 + 12R^2T^{5/2}\right), \end{aligned}$$

the penultimate inequality follows from $T^2 \leq \exp(T^2)$ as $T \geq 1$; and the last inequality using that on $\mathcal{G}_R^{(2)}$ we have $|y-z| < 2RT^{3/2}$ and since $1 \leq d \leq \sqrt{T}/24$, which implies that the sum of the last four terms in the exponent in the penultimate line is negative and may be dropped.

For when $|y-z| < 12RTd$, we also use Proposition 5.4 to say that $f_J(y, z) \lesssim d^{-1}$; and so we see that

$$\begin{aligned} f_J(y, z) &\lesssim d^{-1} \cdot \exp\left(-\frac{(y-z)^2}{4d} + \frac{(y-z)^2}{4d}\right) \leq d^{-1} \cdot \exp\left(-\frac{1}{4d}(y-z)^2 + \frac{144R^2T^2d^2}{4d}\right) \\ &\leq d^{-1} \cdot \exp\left(-\frac{1}{4d}(y-z)^2 + 2R^2T^{5/2}\right), \end{aligned}$$

the last inequality since $d \leq \sqrt{T}/24$ and $144/(4 \times 24) \leq 2$.

So we see from (61) and the above two bounds on $f_J(y, z)$ depending on the size of $|y-z|$, that

$$f_J(y, z) \cdot S \leq \exp\left(21R^2T^{5/2}\right),$$

using that $R \leq R^2$ and $d \geq 1$.

Thus, we obtain that (62) is bounded up to an absolute multiplicative constant by

$$(63) \quad \exp\left(21R^2T^{5/2}\right) \cdot \frac{1}{\sqrt{4\pi d}} \iint_{\mathbb{R}^2} \mathcal{B}_{\tilde{y}, \tilde{z}}^{[-d, d]}(\tilde{A}) e^{-\frac{1}{4d}(\tilde{y}-\tilde{z})^2} d\mu(y) dz.$$

Let B be a Brownian motion started with distribution μ at $-d$. Focusing on the integral,

$$\frac{1}{\sqrt{4\pi d}} \iint_{\mathbb{R}^2} \mathcal{B}_{\tilde{y}, \tilde{z}}^{[-d, d]}(\tilde{A}) e^{-\frac{1}{4d}(\tilde{y}-\tilde{z})^2} d\mu(y) dz = \mathbb{P}\left(B(\cdot) - B(-d) \in A\right) = \mathcal{B}_{0, * }^{[-d, d]}(A) = \varepsilon,$$

using the Markov property of Brownian motion for the penultimate equality.

Combining this with the ignored factor in (63) and using that $R^2 = 36d$ from (34) gives that (62) is bounded above, up to an absolute multiplicative constant, by

$$\varepsilon \cdot \exp\left(756 \cdot D_k^{5/2} \cdot d \cdot (\log \varepsilon^{-1})^{5/6}\right),$$

since $21 \times 36 = 756$. □

CHAPTER 6

THE PATCHWORK QUILT RESEWN

In this chapter we make precise the notion of Brownian regularity for weight profiles from general initial conditions discussed in Section 1.4. Then we prove that the general pre-limiting weight profiles in Brownian LPP enjoy this regularity; and, using the results of [DOV22], that this inference passes over to the limiting profiles. Finally we state and prove in Corollary 6.12 the precise version of Theorem 1.9, giving a bound, for any $\eta > 0$, on the $(2 - \eta)$ moment of the increment of the limiting weight profile.

6.1. Brownian motion regularity of weight profiles with general initial conditions

In [Ham19a], a certain form of Brownian *bridge* regularity is proved for the weight profile started from a very general initial condition. With our Theorem 3.10, we are able to upgrade this result to be a form of Brownian *motion* regularity, and here we establish the notation to state this result.

We need to introduce the notion of a *quilt*, which is constructed from a *fabric sequence* and a set of *stitch points*. These are the same as the definitions from [Ham19a].

Definition 6.1. — Let $\bar{F} = \{F_i \in \mathcal{C}_{*,*}([a, b], \mathbb{R}) : i \in \mathbb{N}\}$ be a sequence of random continuous functions defined on $[a, b]$, which we will call the *fabric sequence*. Also let $S = \{s_1 < s_2 < \dots < s_N\} \subseteq [a, b]$ be an almost surely finite set, called the *stitch point set*.

The *quilt* constructed from \bar{F} and S is denoted $\text{Quilt}[\bar{F}, S]$, and is a random continuous function on $[a, b]$ defined by

$$\text{Quilt}[\bar{F}, S](x) = F_i(x) + v_i, \quad x \in [s_{i-1}, s_i],$$

for $i = 1, \dots, N + 1$, where $s_0 = a$ and $s_{N+1} = b$. We set $v_0 = 0$, and specify v_i for $i \geq 1$ so that $\text{Quilt}[\bar{F}, S]$ is continuous; more precisely, we identify S with the atomic measure $\sum_{i=1}^N \delta_{s_i}$, which takes values in the space of sigma-finite Borel measures on $[a, b]$ endowed with the weak topology. This identification will remain in place for all random sets considered in this chapter.

We modify the definition of a collection of stochastic processes $\{X_{n,\alpha}\}$ being Brownian patchwork quiltable from [Ham19a] so that the reference process is Brownian motion rather than Brownian bridge.

Definition 6.2. — Let $\beta \in (1, \infty)$. A random continuous function $X : [a, b] \rightarrow \mathbb{R}$ is said to *withstand $L^{\beta-}$ -comparison with Brownian motion* if, for any $\eta \in (0, 1 - \beta^{-1})$ and any measurable $A \subseteq \mathcal{C}_{0,*}([a, b], \mathbb{R})$,

$$\mathbb{P}(X(\cdot) - X(a) \in A) \leq C_0 \cdot \mathcal{B}_{0,*}^{[a,b]}(A)^{1-\beta^{-1}-\eta},$$

where C_0 is a finite constant which may depend upon η .

Suppose that \mathcal{I} is an arbitrary index set and that we have a collection of random continuous functions $\{X_{n,\alpha} : (n,\alpha) \in \mathbb{N} \times \mathcal{I}\}$ indexed by $\mathbb{N} \times \mathcal{I}$. For a function $h : \mathbb{N} \rightarrow (0, 1]$, the collection is said to *uniformly withstand L^{β^-} -comparison to Brownian motion above scale h* if

$$\mathbb{P}(X_{n,\alpha}(\cdot) - X_{n,\alpha}(a) \in A) \leq C_0 \cdot \mathcal{B}_{0,*}^{[a,b]}(A)^{1-\beta^{-1}-\eta},$$

for all $(n,\alpha) \in \mathbb{N} \times \mathcal{I}$ and all measurable $A \subseteq \mathcal{C}_{0,*}([a,b], \mathbb{R})$ with $\mathcal{B}_{0,*}^{[a,b]}(A) \geq h(n)$; here, C_0 is a constant that may depend upon η but not on n nor on α .

Definition 6.3. — Let \mathcal{I} be an arbitrary index set. Suppose we are given a collection of random continuous functions $X_{n,\alpha} : [a,b] \rightarrow \mathbb{R}$, indexed by $(n,\alpha) \in \mathbb{N} \times \mathcal{I}$ and defined on the same probability space $(\Omega, \mathcal{G}, \mathbb{P})$. (Here the $X_{n,\alpha}$ are taking values in $\mathcal{C}_{*,*}([a,b], \mathbb{R})$ endowed with the uniform topology and the corresponding Borel σ -algebra.) Let $\beta_1 > 0$, $\beta_2 \geq 1$, $\beta_3 > 0$ and $\beta_4 > 0$. This collection is said to be uniformly Brownian motion patchwork $(\beta_1, \beta_2, \beta_3, \beta_4)$ -quilttable if there exist

- sequences $p, q : \mathbb{N} \rightarrow [0, 1]$ verifying $p_j \leq j^{-\beta_1+\varepsilon}$ and $q_j \leq j^{-\beta_3+\varepsilon}$ for each $\varepsilon > 0$ and all j sufficiently high; and
- a constant $g > 0$

such that, for each $(n,\alpha) \in \mathbb{N} \times \mathcal{I}$, we may construct under the law \mathbb{P} ,

1. an error event $E_{n,\alpha} \in \mathcal{G}$ that satisfies $\mathbb{P}(E_{n,\alpha}) \leq q_n$;
2. a *fabric* sequence $\overline{F}_{n,\alpha} = \{F_{n,\alpha;i} : i \in \mathbb{N}\}$ (consisting of continuous random functions on $[a,b]$), where the collection $\{F_{n,\alpha;i} : (n, (\alpha, i)) \in \mathbb{N} \times \mathcal{K}\}$, with $\mathcal{K} = \mathcal{I} \times \mathbb{N}$, uniformly withstands $L^{\beta_2^-}$ -comparison to Brownian motion above scale $\exp\{-gn^{\beta_4}\}$;
3. a *stitch points* set $S_{n,\alpha} \subset [a,b]$ (interpreted as a counting measure as mentioned in Definition 6.2 above) whose cardinality verifies $\mathbb{P}(|S_{n,\alpha}| \geq \ell) \leq p_\ell$ for each $\ell \in \mathbb{N}$; and
4. all this in such a way that, for every $(n,\alpha) \in \mathbb{N} \times \mathcal{I}$, the random function $X_{n,\alpha}$ is equal to the patchwork quilt $\text{Quilt}[\overline{F}_{n,\alpha}, S_{n,\alpha}]$ throughout the interval $[a,b]$, whenever the error event $E_{n,\alpha}$ does *not* occur.

We may now state the theorem establishing this form of Brownian motion regularity for the pre-limiting weight profiles from general initial conditions.

Theorem 6.4. — Let $\bar{\Psi} \in (0, \infty)^3$ satisfy $\Psi_2 \geq 1$. The collection of random continuous functions

$$[-1, 1] \rightarrow \mathbb{R} : y \mapsto \text{Wgt}_n^f[(*, 0) \rightarrow (y, 1)]$$

indexed by $(n, f) \in \mathbb{N} \times \mathcal{I}_{\bar{\Psi}}$ (see Definition 3.8) is uniformly Brownian motion patchwork $(2, 3, 1/252, 1/12)$ -quilttable.

[Ham19a, Theorem 1.2] is exactly the same as Theorem 6.4, except with Brownian bridge comparison, with the obvious change in Definition 6.2.

PROOF OF THEOREM 6.4. Our aim is to replace the use of [Ham19a, Theorem 4.3] in the proof of [Ham19a, Theorem 1.2] with Theorem 3.10. To begin, we simply change A from being a subset of $\mathcal{C}_{0,0}([-1, 1], \mathbb{R})$ to be a subset of $\mathcal{C}_{0,*}([-1, 1], \mathbb{R})$, and replace all the occurrences of affinely shifted processes $Z_{n,j}^{[-1,1]}$ and $Y_{n,i}^{[-1,1]}$ with vertically shifted processes $Z_{n,j}(\cdot) - Z_{n,j}(-1)$ and $Y_{n,i}(\cdot) - Y_{n,i}(-1)$. Theorem 4.3 of [Ham19a] is used only on page 59 of [Ham19a] in the last display, and the proof works with our definition of uniformly Brownian motion patchwork quilttable by replacing this usage of [Ham19a, Theorem 4.3] with Theorem 3.10. \square

6.2. The limiting patchwork quilt

In the recent work of Dauvergne, Ortmann, and Virág [DOV22], the existence of the space-time Airy sheet is established. With some straightforward arguments that we will attend to shortly, this establishes the unambiguous existence of the random function $y \mapsto \text{Wgt}_\infty^f[(*, 0) \rightarrow (y, 1)]$, and it is easy to believe, as remarked in [Ham19a], that a theorem analogous to Theorem 6.4 should hold for the limiting process with a suitable analysis of weak convergence of the relevant objects. This is indeed the case, and in this subsection we carry out the needed task in Proposition 6.10.

The following is the result from [DOV22] establishing the joint convergence of the weight profiles, stated in our notation:

Theorem 6.5 (Theorem 1.3 of [DOV22]). — *The process $(x, y) \mapsto \text{Wgt}_n[(x, 0) \rightarrow (y, 1)]$ for $(x, y) \in \mathbb{R}^2$ has a unique limit in distribution as $n \rightarrow \infty$, denoted by $\text{Wgt}_\infty[(x, 0) \rightarrow (y, 1)]$. The convergence is on the space of continuous functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ endowed with the topology of uniform convergence on compact sets.*

This theorem establishes the existence of the limiting weight profile started from the narrow-wedge initial condition (i.e., $f(x) = 0$ and $f(z) = -\infty$ for $z \neq x$) jointly as x varies, and first we must perform the straightforward task of using this to obtain the existence of limiting weight profiles for general initial conditions. Define

$$\text{Wgt}_\infty^f[(*, 0) \rightarrow (y, 1)] := \sup \left\{ \text{Wgt}_\infty[(x, 0) \rightarrow (y, 1)] + f(x) : x \in \mathbb{R} \right\}.$$

Proposition 6.6. — *Let $\Psi \in (0, \infty)^3$ and $f \in \mathcal{I}_{\bar{\Psi}}$ (recall from Definition 3.8); assume further that f is upper semi-continuous. Then the process $y \mapsto \text{Wgt}_n^f[(*, 0) \rightarrow (y, 1)]$ for $y \in [-1, 1]$ has a unique limit in distribution, $\text{Wgt}_\infty^f[(*, 0) \rightarrow (y, 1)]$, on $\mathcal{C}_{*,*}([-1, 1], \mathbb{R})$ endowed with the topology of uniform convergence as $n \rightarrow \infty$.*

Before giving the proof, we record some statements from [Ham19a] which give us some control over the maximiser in the definition of $\text{Wgt}_n^f[(*, 0) \rightarrow (y, 1)]$. Let $x_y^n = \text{argmax} \{ \text{Wgt}_n[(x, 0) \rightarrow (y, 1)] + f(x) : x \in \mathbb{R} \}$ for $y \in [-1, 1]$ and for a fixed $f \in \mathcal{I}_{\bar{\Psi}}$. Note that x_y^n is well defined for every fixed y since this maximiser is unique a.s. for every fixed $y \in [-1, 1]$.

Lemma 6.7 (Lemma 4.10 of [Ham19a]). — *Fix $\Psi \in (0, \infty)^3$ and $f \in \mathcal{I}_{\bar{\Psi}}$, and let x_y^n be as above. Then $\{x_1^n, x_{-1}^n\}_{n \in \mathbb{N}}$ is a tight sequence of random variables.*

Lemma 6.8. — *In the notation of Lemma 6.7, $y \mapsto x_y^n$ is a.s. a non-decreasing function for each n .*

PROOF. This is an immediate consequence of a simple sandwiching property of polymers recorded in [Ham19a, Lemma 4.4]. \square

This sandwiching property just used is a common tool in arguments involving polymer geometry in other LPP models as well, and has also been called “polymer ordering”.

Lemma 6.9. — *Fix $\Psi \in (0, \infty)^3$ and $f \in \mathcal{I}_{\bar{\Psi}}$, and further assume that f is upper semi-continuous. Let $y \in \mathbb{R}$ and x_y^n be as above, and suppose x_y^∞ is a limit point of $\{x_y^n\}_{n \in \mathbb{N}}$. Then*

$$\text{Wgt}_\infty^f[(*, 0) \rightarrow (y, 1)] = \text{Wgt}_\infty[(x_y^\infty, 0) \rightarrow (y, 1)] + f(x_y^\infty).$$

In particular, $\text{Wgt}_\infty^f[(, 0) \rightarrow (y, 1)]$ is finite.*

PROOF. By the definition of x_y^n , for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$(64) \quad \text{Wgt}_n[(x_y^n, 0) \rightarrow (y, 1)] + f(x_y^n) \geq \text{Wgt}_n[(x, 0) \rightarrow (y, 1)] + f(x).$$

Let $\{n_k\}_{k \in \mathbb{N}}$ be a subsequence along which $x_y^{n_k}$ converges weakly to x_y^∞ . By Theorem 6.5 and the Skorokhod representation theorem, we have on a common probability space that $\text{Wgt}_n[(x, 0) \rightarrow (y, 1)] \rightarrow \text{Wgt}_\infty[(x, 0) \rightarrow$

$(y, 1]$ uniformly on compact sets and $x_y^{n_k} \rightarrow x_y^\infty$ almost surely, with Wgt_{n_k} and $x_y^{n_k}$ having the correct joint distribution. Then, by the uniform convergence on compact sets,

$$\lim_{k \rightarrow \infty} \text{Wgt}_{n_k}[(x_y^{n_k}, 0) \rightarrow (y, 1)] = \text{Wgt}_\infty[(x_y^\infty, 0) \rightarrow (y, 1)].$$

Now, since f is upper semi-continuous,

$$f(x_y^\infty) \geq \lim_{k \rightarrow \infty} f(x_y^{n_k}).$$

Taking $n \rightarrow \infty$ along the mentioned subsequence in (64), applying the last two displays, and recalling the definition of Wgt_∞^f completes the proof. \square

PROOF OF PROPOSITION 6.6. In the course of the proof we will also verify that $\text{Wgt}_\infty^f[(*, 0) \rightarrow (y, 1)]$ is finite for each y , so that the function is well-defined.

By Theorem 6.5 and the Skorokhod representation theorem, we have on a common probability space that $\text{Wgt}_n[(x, 0) \rightarrow (y, 1)] \rightarrow \text{Wgt}_\infty[(x, 0) \rightarrow (y, 1)]$ uniformly on compact sets almost surely. As in Lemma 6.7, let

$$\begin{aligned} x_1^n &= \operatorname{argmax}\left\{\text{Wgt}_n[(x, 0) \rightarrow (1, 1)] + f(x) : x \in \mathbb{R}\right\} \quad \text{and} \\ x_{-1}^n &= \operatorname{argmax}\left\{\text{Wgt}_n[(x, 0) \rightarrow (-1, 1)] + f(x) : x \in \mathbb{R}\right\}. \end{aligned}$$

Since $\{x_1^n\}_{n \in \mathbb{N}}$ and $\{x_{-1}^n\}_{n \in \mathbb{N}}$ are tight by Lemma 6.7, let K be a random compact interval such that, for all n , $x_1^n, x_{-1}^n \in K$. Note that Lemma 6.8 implies that $x_y^n \in K$ for all $y \in [-1, 1]$ and $n \in \mathbb{N}$ almost surely.

By uniform convergence on compact sets and the definitions of x_y^n , Wgt_n^f and Wgt_∞^f , we obtain

$$\begin{aligned} & \text{Wgt}_n^f[(*, 0) \rightarrow (y, 1)] - \text{Wgt}_\infty^f[(*, 0) \rightarrow (y, 1)] \\ & \leq \text{Wgt}_n[(x_y^n, 0) \rightarrow (y, 1)] + f(x_y^n) - \text{Wgt}_\infty[(x_y^n, 0) \rightarrow (y, 1)] - f(x_y^n) \\ & \leq \sup_{(x, y) \in K \times [-1, 1]} \left| \text{Wgt}_n[(x, 0) \rightarrow (y, 1)] - \text{Wgt}_\infty[(x, 0) \rightarrow (y, 1)] \right| \rightarrow 0, \end{aligned}$$

as $k \rightarrow \infty$. Now we bound $\text{Wgt}_\infty^f - \text{Wgt}_n^f$. Let x_y^∞ be any limit point of $\{x_y^n\}_{n \in \mathbb{N}}$, which implies that $x_y^\infty \in K$ and, by Lemma 6.9, that $\text{Wgt}_\infty^f[(*, 0) \rightarrow (y, 1)] = \text{Wgt}_\infty[(x_y^\infty, 0) \rightarrow (y, 1)] + f(x_y^\infty)$; note that this implies that $\text{Wgt}_\infty^f[(*, 0) \rightarrow (y, 1)]$ is finite as $x_y^\infty \in K$. Then, similar to above,

$$\begin{aligned} & \text{Wgt}_\infty^f[(*, 0) \rightarrow (y, 1)] - \text{Wgt}_n^f[(*, 0) \rightarrow (y, 1)] \\ & \leq \text{Wgt}_\infty[(x_y^\infty, 0) \rightarrow (y, 1)] + f(x_y^\infty) - \text{Wgt}_n[(x_y^\infty, 0) \rightarrow (y, 1)] - f(x_y^\infty) \\ & \leq \sup_{(x, y) \in K \times [-1, 1]} \left| \text{Wgt}_n[(x, 0) \rightarrow (y, 1)] - \text{Wgt}_\infty[(x, 0) \rightarrow (y, 1)] \right| \rightarrow 0, \end{aligned}$$

This completes the proof of Proposition 6.6. \square

Now we turn to the result that the patchwork quilt description is preserved in the limit. Note that, when considering the Brownian motion patchwork quiltability of a limiting family of processes $\{X_{\infty, \alpha}\}_{\alpha \in \mathcal{I}}$ for some index set \mathcal{I} , i.e., $n = \infty$, the constants β_3 and β_4 do not play any role; this is because β_3 controls the probability of the error event E_n , which is zero when $n = \infty$ for any $\beta_3 > 0$, while β_4 controls the scale above which the comparison to Brownian motion is made, which is also zero when $n = \infty$ for any $\beta_4 > 0$. So as a formal device, we may set both the parameters β_3 and β_4 to ∞ for the limiting family of processes, regardless of their values in the prelimiting family.

Proposition 6.10. — *Suppose the collection $\{X_{n, \alpha} : [a, b] \rightarrow \mathbb{R} : (n, \alpha) \in \mathbb{N} \times \mathcal{I}\}$ is uniformly Brownian motion patchwork $(\beta_1, \beta_2, \beta_3, \beta_4)$ -quiltability, and that $\{X_{n, \alpha}\}_{n \in \mathbb{N}}$ is a tight sequence of random functions for*

each $\alpha \in \mathcal{I}$. Let $\{X_{\infty,\alpha}\}_{\alpha \in \mathcal{I}}$ be a collection of weak limit points of this collection. Then $\{X_{\infty,\alpha}\}_{\alpha \in \mathcal{I}}$ is uniformly Brownian motion patchwork $(\beta_1, \beta_2, \infty, \infty)$ -quilttable.

More explicitly, $\{X_{\infty,\alpha}\}_{\alpha \in \mathcal{I}}$ being a uniformly Brownian motion patchwork $(\beta_1, \beta_2, \infty, \infty)$ -quilttable collection means that there exist, for each $\alpha \in \mathcal{I}$, a collection of random continuous functions $\bar{F}_\alpha = \{F_{\alpha,i} : i \in \mathbb{N}\}$ defined on $[a, b]$ and a stitch point set S_α such that

- the collection $\{F_{\alpha,i} : (\alpha, i) \in \mathcal{I} \times \mathbb{N}\}$ uniformly withstands $L^{\beta_2^-}$ -comparison to Brownian motion above scale zero;
- it holds that $\mathbb{P}(|S_\alpha| \geq \ell) \leq p_\ell$ for each $\ell \in \mathbb{N}$, with $p_j \leq j^{-\beta_1 + \varepsilon}$ for each $\varepsilon > 0$ and j sufficiently large; and
- in such a way that $X_{\infty,\alpha} = \text{Quilt}[\bar{F}_\alpha, S_\alpha]$ on $[a, b]$ almost surely.

As an immediate implication of Proposition 6.10 and Theorem 6.4, we have the following theorem.

Theorem 6.11. — *Let $\bar{\Psi} \in (0, \infty)^3$ satisfy $\Psi_2 \geq 1$. The collection of random continuous functions*

$$[-1, 1] \rightarrow \mathbb{R} : y \mapsto \text{Wgt}_\infty^f[(*, 0) \rightarrow (y, 1)]$$

indexed by $f \in \mathcal{I}_{\bar{\Psi}}$ (see Definition 3.8) is uniformly Brownian motion patchwork $(2, 3, \infty, \infty)$ -quilttable.

PROOF OF PROPOSITION 6.10. To prove the proposition and verify the three points listed after its statement, we take appropriate weak limits of the collections $\bar{F}_{n,\alpha}$ and sets $S_{n,\alpha}$ for every α , which we now fix for the remainder of the proof. We assume without loss of generality that $X_n \xrightarrow{d} X_\infty$, focusing our attention, if needed, on the subsequence along which this convergence occurs.

We start by fixing α and showing that there exists a single subsequence of n along which, for each $i \in \mathbb{N}$, $F_{n,\alpha,i} \xrightarrow{d} F_{\infty,\alpha,i}$, where $\{F_{\infty,\alpha,i} : i \in \mathbb{N}\}$ is some collection of random continuous functions. Since the construction of the patchwork quilt entails a vertical shift of all $F_{n,\alpha,i}$ for $i \geq 2$, we may assume that $F_{n,\alpha,i}(a) = 0$ for all $i \geq 2$ and all $n \in \mathbb{N}$. The tightness of this collection follows from a standard diagonalisation argument if we can show the subsequential convergence for each individual i , i.e., we must show tightness of $\{F_{n,\alpha,i} : n \in \mathbb{N}\}$ for each fixed $i \in \mathbb{N}$. This is a consequence of the uniform $L^{\beta_2^-}$ comparison to Brownian motion that $\bar{F}_{n,\alpha}$ enjoys. So let $K \subseteq \mathcal{C}_{0,*}([a, b], \mathbb{R})$ be a compact set such that $\mathcal{B}_{0,*}^{[a,b]}(K^c) \leq \varepsilon$. Then using the $L^{\beta_2^-}$ -comparison with $\eta = (1 - \beta_2^{-1})/2$, we get, for $i \geq 2$,

$$\mathbb{P}(F_{n,\alpha,i} \notin K) \leq C_0 \cdot \mathcal{B}_{0,*}^{[a,b]}(K^c)^{(1-\beta_2^{-1})/2} \leq C_0 \cdot \varepsilon^{(1-\beta_2^{-1})/2}.$$

Since $\varepsilon > 0$ is in our control, this establishes tightness of $\{F_{n,\alpha,i} : n \in \mathbb{N}\}$ for each $i \geq 2$.

For $i = 1$ we cannot assume $F_{n,\alpha,1}(a) = 0$, and so we must use the tightness of $\{X_{n,\alpha}\}_{n \in \mathbb{N}}$ as well. Recall that to establish tightness of $\{F_{n,\alpha,1} : n \in \mathbb{N}\}$, it is sufficient to show that $\{F_{n,\alpha,1}(a) : n \in \mathbb{N}\}$ is tight, and to show a uniform modulus of continuity; i.e., that, given $\rho > 0$ and $\varepsilon > 0$, we can find r and n_0 such that for $n > n_0$ we have

$$\mathbb{P}(\omega(F_{n,\alpha,1}, r) \leq \rho) \geq 1 - \varepsilon,$$

where for a continuous function $f : [a, b] \rightarrow \mathbb{R}$,

$$\omega(f, r) = \sup_{\substack{|x-y| < r \\ x, y \in [a, b]}} |f(x) - f(y)|.$$

That $\{F_{n,\alpha,1}(a) : n \in \mathbb{N}\}$ is tight follows from the fact that $F_{n,\alpha,1}(a) = X_n(a)$ on the event $E_{n,\alpha}^c$, which has probability less than q_n , and the assumed tightness of $\{X_n\}_{n \in \mathbb{N}}$. The uniform modulus of continuity follows

again from $L^{\beta_2^-}$ -comparison with Brownian motion, and the fact that the modulus of continuity event does not change by considering $F_{n,\alpha,1}(\cdot) - F_{n,\alpha,1}(a)$ instead of $F_{n,\alpha,1}(\cdot)$:

$$\mathbb{P}(\omega(F_{n,\alpha,1}, r) \geq \rho) \leq C_0 \cdot \mathbb{P}(\omega(B, r) \geq \rho)^{(1-\beta_2^-)/2},$$

where B is a Brownian motion on $[a, b]$ and C_0 is a constant independent of n and α . Since this right-hand side can be made less than ε by taking r sufficiently small, and since this inequality holds for all large enough n , the tightness of $\{F_{n,\alpha,1} : n \in \mathbb{N}\}$ follows.

Having established convergence of $F_{n,\alpha,i}$ along a subsequence simultaneously for all i , we may assume without loss of generality that this subsequence is the entire sequence. We now turn to the collection $\{S_{n,\alpha}\}$. This is a collection of random finite sets, and regarding them as $\{0, 1\}$ -valued random measures (i.e., as simple point processes), we see that $\{S_{n,\alpha}\}_{n \in \mathbb{N}}$ is a tight sequence of random measures, as these laws are all defined on the compact set $[a, b]$.

At this point we have assumed $X_{n,\alpha} \xrightarrow{d} X_{\infty,\alpha}$ as $n \rightarrow \infty$ and have established tightness of $\{S_{n,\alpha} : n \in \mathbb{N}\}$ as well as that $\bar{F}_{n,\alpha} \xrightarrow{d} \bar{F}_{\infty,\alpha}$ as $n \rightarrow \infty$. It is an immediate consequence by Prokhorov's theorem (using that \mathbb{R} , the space of sigma-finite Borel measures, and $\mathcal{C}_{*,*}([a, b], \mathbb{R})$ are Polish spaces) that there exists a subsequence n_k such that $(X_{n_k,\alpha}, \bar{F}_{n_k,\alpha}, S_{n_k,\alpha}) \xrightarrow{d} (X_{\infty,\alpha}, \bar{F}_{\infty,\alpha}, S_{\infty,\alpha})$, where $S_{\infty,\alpha}$ is some integer-valued random measure; we make no statement about the joint distribution of the limiting triple. We have to establish the properties of these objects contained in the three bullet points at the beginning of the proof, but to do so we will need to slightly modify $\bar{F}_{\infty,\alpha,i}$ and $S_{\infty,\alpha}$.

Note that it may not be the case that we can identify $S_{\infty,\alpha}$ with a finite set, as $S_{\infty,\alpha}$ may assign integer measure greater than 1 to singletons. So we define S_α to be the finite set support of $S_{\infty,\alpha}$. Now since $|S_\alpha|$ and $|S_{\infty,\alpha}|$ are integer-valued random variables (where the $|S_{\infty,\alpha}|$ is the measure assigned to $[a, b]$ by the associated integer-valued measure), we have, by the Portmanteau theorem,

$$(65) \quad \mathbb{P}(|S_\alpha| \geq \ell) \leq \mathbb{P}(|S_{\infty,\alpha}| \geq \ell) = \lim_{n \rightarrow \infty} \mathbb{P}(|S_{n,\alpha}| \geq \ell) \leq p\ell,$$

thus verifying the second bullet point from the beginning of the proof.

Because of the possibility that points in $S_{\infty,\alpha}$ have multiplicity greater than 1, we will later remove some entries of $\bar{F}_{\infty,\alpha}$ to obtain the final \bar{F}_α .

Now we verify the first bullet point, namely that $\bar{F}_{\infty,\alpha}$ is uniformly $L^{\beta_2^-}$ -comparable to Brownian motion. Fix $i \in \mathbb{N}$, $\eta \in (0, 1 - \beta_2^-)$, and let A be a given measurable subset of $\mathcal{C}_{0,*}([a, b], \mathbb{R})$ such that $\mathcal{B}_{0,*}^{[a,b]}(A) > 0$. Since all Borel probability measures on Polish spaces are regular, in particular outer regular, we have that

$$\mathbb{P}(F_{\infty,\alpha,i} \in A) = \inf \{ \mathbb{P}(F_{\infty,\alpha,i} \in O) : A \subseteq O \text{ open} \}.$$

Since the O are open, we have by the Portmanteau theorem

$$(66) \quad \begin{aligned} \inf \{ \mathbb{P}(F_{\infty,\alpha,i} \in O) : A \subseteq O \text{ open} \} &\leq \inf \left\{ \lim_{n \rightarrow \infty} \mathbb{P}(F_{n,\alpha,i} \in O) : A \subseteq O \text{ open} \right\} \\ &\leq \inf \left\{ C_0 \cdot \left(\mathcal{B}_{0,*}^{[a,b]}(O) \right)^{1-\beta_2^{-1}-\eta} : A \subseteq O \text{ open} \right\} \\ &= C_0 \cdot \left(\inf \left\{ \left(\mathcal{B}_{0,*}^{[a,b]}(O) \right) : A \subseteq O \text{ open} \right\} \right)^{1-\beta_2^{-1}-\eta} \\ &= C_0 \cdot \left(\mathcal{B}_{0,*}^{[a,b]}(A) \right)^{1-\beta_2^{-1}-\eta}. \end{aligned}$$

Here C_0 is an η -dependent constant independent of n, α, i which comes (for large enough n) from the uniform $L^{\beta_2^-}$ -comparison to Brownian motion enjoyed by the collection $\{F_{n,\alpha,i} : n \in \mathbb{N}, \alpha \in \mathcal{I}, i \in \mathbb{N}\}$. In the last

equality we have used that $\mathcal{B}_{0,*}^{[a,b]}$ is an outer regular measure. This establishes the uniform $L^{\beta_2^-}$ -comparison of $F_{n,\alpha,i}$ to Brownian motion above scale zero for every i .

Finally we define \bar{F}_α so that the third bullet point, i.e., $X_{\infty,\alpha} = \text{Quilt}[\bar{F}_\alpha, S_\alpha]$ on $[a, b]$, holds almost surely. Label the points in S_α as $\{s_1, \dots, s_N\}$; let $m(s_i)$ be the multiplicity of s_i in $S_{\infty,\alpha}$; and let $r(i) = \sum_{1 \leq j \leq i} m(s_j)$. Then we let

$$F_{\alpha,i} = F_{\infty,\alpha,r(i)}$$

for $i = 1, \dots, N$. For $i > N$, $F_{\alpha,i} = F_{\infty,\alpha,i}$. In words, if the multiplicity of s_i is greater than 1, this means that a corresponding number of entries of $F_{\infty,\alpha,i}$ were squished together into an interval of size zero in the stitching, and so must be removed from the fabric collection.

As we have $(X_{n_k,\alpha}, \bar{F}_{n_k,\alpha}, S_{n_k,\alpha}) \xrightarrow{d} (X_{\infty,\alpha}, \bar{F}_{\infty,\alpha}, S_{\infty,\alpha})$, by the Skorokhod representation theorem (recalling again that \mathbb{R} , the space of Borel measures on $[a, b]$ and $\mathcal{C}_{*,*}([a, b], \mathbb{R})$ are Polish spaces) we may assume we are working on a probability space where this convergence happens almost surely. Then it is immediate from the fact that $X_n = \text{Quilt}[\bar{F}_{n,\alpha}, S_{n,\alpha}]$ on E_n^c and from our definitions of \bar{F}_α and S_α that $X_\infty = \text{Quilt}[\bar{F}_\alpha, S_\alpha]$ almost surely. Since the right-hand sides of equations (65) and (66) are independent of $\alpha \in \mathcal{I}$, this establishes Proposition 6.10. \square

With the existence of the limiting weight profile and the knowledge that it enjoys a description as a Brownian motion patchwork quilt, we may prove the next corollary. It is a stronger version of Theorem 1.9, stated with the constants that are uniform once the space of admissible initial conditions has been fixed.

Corollary 6.12. — *Let $f \in \mathcal{I}_{\bar{\Psi}}$ and $0 < \eta < \frac{1}{2}$. Then there exist constants $G = G(\bar{\Psi}, \eta) < \infty$ and $y_0 > 0$ such that, for $|y| < y_0$,*

$$\mathbb{E} \left[\left| \text{Wgt}_\infty^f[(*, 0) \rightarrow (y, 1)] - \text{Wgt}_\infty^f[(*, 0) \rightarrow (y, 1)] \right|^{2-\eta} \right] \leq G|y|^{1-\eta/2}.$$

In the case where $f \equiv 0$, corresponding to flat initial conditions, $\text{Wgt}_\infty^f[(*, 0) \rightarrow (y, 1)]$ is strongly believed to be the Airy_1 process (in analogy with the exponential and Poissonian LPP models where such a result has been proven), though we have been unable to find a proof in the literature.

As was qualitatively described in Section 1.4, to upgrade this result from the $2 - \eta$ moment to the second moment would require the knowledge that the first parameter of 2 in the Brownian motion patchwork quilt parameters in Theorem 6.4 may be increased to $2 + \rho$ for some $\rho > 0$.

PROOF OF COROLLARY 6.12. We assume without loss of generality that $y > 0$, as the same argument works for the other case. The constant G may vary from line to line, and may depend only on η and $\bar{\Psi}$. For ease of notation, we will write $\text{Wgt}(y)$ for $\text{Wgt}_\infty^f[(*, 0) \rightarrow (y, 1)]$. Now, by Theorem 6.4 and Propositions 6.6 and 6.10, we may write

$$\text{Wgt}(y) = \text{Quilt}[\bar{F}, S](y),$$

and we label the points of the stitch point set S which are strictly smaller than y as $s_1 < s_2 < \dots < s_{N-1} < y$. We set $s_0 = 0$ and $s_N = y$ by convention, though they need not lie in S . Then, using that the convexity of the mapping $x \mapsto x^{2-\eta}$ implies $(\sum_{i=1}^n a_i)^{2-\eta} \leq n^{1-\eta} \sum_{i=1}^n a_i^{2-\eta}$, we get

$$\begin{aligned} \mathbb{E} \left[\left| \text{Wgt}(y) - \text{Wgt}(0) \right|^{2-\eta} \right] &= \mathbb{E} \left[\left| \sum_{i=1}^N (F_i(s_i) - F_i(s_{i-1})) \right|^{2-\eta} \right] \leq \mathbb{E} \left[\left(\sum_{i=1}^N |F_i(s_i) - F_i(s_{i-1})| \right)^{2-\eta} \right] \\ &\leq \mathbb{E} \left[N^{1-\eta} \sum_{i=1}^N |F_i(s_i) - F_i(s_{i-1})|^{2-\eta} \right]. \end{aligned}$$

We write the last sum as $\sum_{i=1}^{\infty} \mathbb{E} \left[N^{1-\eta} |F_i(s_i) - F_i(s_{i-1})|^{2-\eta} \mathbb{1}_{i \leq N} \right]$. We will bound each individual summand using the generalised Hölder's inequality, with $p = 2 + 2\eta$, $q = 2 - \eta$, and r defined by

$$\frac{1}{r} = 1 - \frac{1}{2 + 2\eta} - \frac{1}{2 - \eta},$$

which is positive for $0 < \eta < 1/2$. Doing so, we get that $\mathbb{E} \left[N^{1-\eta} |F_i(s_i) - F_i(s_{i-1})|^{2-\eta} \mathbb{1}_{i \leq N} \right]$ is bounded by

$$(67) \quad \mathbb{E} \left[N^{(1-\eta)p} \right]^{1/p} \cdot \mathbb{P}(N \geq i)^{1/q} \cdot \mathbb{E} \left[|F_i(s_i) - F_i(s_{i-1})|^{(2-\eta)r} \right]^{1/r}.$$

We require this to be summable and bounded by $Gy^{1-\eta/2}$. Note that $(1-\eta)p = 2(1-\eta^2) < 2$; since the tail bound on N ensures it has finite $2 - \varepsilon$ moment for any $\varepsilon > 0$, the first factor in (67) is finite and bounded by an η -dependent quantity.

Now we look at the third factor of (67). This is clearly bounded by

$$G \cdot \mathbb{E} \left[\sup_{s \in [0, y]} |F_i(s) - F_i(0)|^{(2-\eta)r} \right]^{1/r},$$

as $s_i \leq y$. Let Z be the Radon-Nikodym derivative of F_i with respect to standard Brownian motion on $[0, 1]$, denoted B , so that we know that Z has finite $3 - \varepsilon$ moment for any $\varepsilon > 0$. Thus we get

$$\mathbb{E} \left[\sup_{s \in [0, y]} |F_i(s) - F_i(0)|^{(2-\eta)r} \right] = \mathbb{E} \left[\sup_{s \in [0, t]} |B(s)|^{(2-\eta)r} Z \right] \leq \mathbb{E} \left[\sup_{s \in [0, y]} |B(s)|^{3(2-\eta)r} \right]^{1/3} \cdot \mathbb{E}[Z^{3/2}]^{2/3},$$

again using Hölder's inequality with Hölder conjugates 3 and $3/2$. Using the symmetry of Brownian motion and the reflection principle, we obtain

$$\mathbb{E} \left[\sup_{s \in [0, y]} |B(s)|^{3(2-\eta)r} \right] \leq G \cdot \mathbb{E} \left[|X|^{3(2-\eta)r} \right] \leq Gy^{3(1-\eta/2)r},$$

where X is a normal random variable with mean 0 and variance y , and the last inequality is by a standard expression for the moments of the normal distribution. Tracing the steps back, we see that the third factor of (67) is bounded by $Gy^{(1-\eta/2)}$, since $\mathbb{E}[Z^{3/2}]$ is finite and depends only on $\bar{\Psi}$.

Thus all that remains to be shown is that the second factor of (67) is summable in i . Taking the ε in Definition 6.3 to be $\eta/2$, we see that for large enough i , we have

$$\mathbb{P}(N \geq i)^{1/(2-\eta)} \leq i^{-(2-\eta/2)/(2-\eta)},$$

which is summable. This completes the proof of Corollary 6.12. \square

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APPENDIX A

BROWNIAN MEANDER CALCULATIONS

In this appendix we carry out the calculations concerning Brownian meander whose conclusions were used in Chapter 2; these calculations were recorded as Lemmas 2.4, 2.7, 2.8, and 2.9. We adopt the notation for the density and modified distribution function of mean zero normal random variables with variance σ^2 :

$$\varphi_{\sigma^2}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) \quad \text{and} \quad \tilde{\Phi}_{\sigma^2}(x) = \int_0^x \varphi_{\sigma^2}(w) \, dw.$$

We recall that the standard Brownian meander B_{me} is a non-homogeneous Markov process on $[0, 1]$, started at 0, which can be obtained as a weak limit, as $\varepsilon \searrow 0$, of standard Brownian motion on $[0, 1]$ conditioned to stay above $-\varepsilon$ [DIM77]. Its transition probabilities are given—see, for example, [DIM77]—for $x, y > 0$ and $0 < s < t \leq 1$ by

$$\begin{aligned} \mathbb{P}\left(B_{\text{me}}(t) \in dy \mid B_{\text{me}}(s) = x\right) &= \left(\varphi_{t-s}(y-x) - \varphi_{t-s}(y+x)\right) \frac{\tilde{\Phi}_{1-t}(y)}{\tilde{\Phi}_{1-s}(x)} \, dy, \\ \mathbb{P}\left(B_{\text{me}}(t) \in dy\right) &= 2\sqrt{2\pi} \cdot \frac{y}{t} \cdot \varphi_t(y) \tilde{\Phi}_{1-t}(y) \, dy. \end{aligned}$$

Before turning to the proof of Lemma 2.4, we state and prove the following simple estimate.

Lemma A.1. — *Let $X \geq 0$ be a random variable with density f , and suppose that f is non-increasing on $[0, \infty)$. Then we have, for all $x, y \in \mathbb{R}$ with $0 \leq x < y$,*

$$\mathbb{P}(X \geq x \mid X \leq y) \leq \frac{y-x}{y}.$$

PROOF. Let F be the distribution function of X . We have that

$$\mathbb{P}(X \geq x \mid X \leq y) = \frac{\mathbb{P}(x \leq X \leq y)}{\mathbb{P}(X \leq y)} = \frac{F(y) - F(x)}{F(y)} = 1 - \frac{F(x)}{F(y)}.$$

We wish to show that $F(x)/F(y) \geq x/y$ for all $x \leq y$, which, by rearranging, is equivalent to showing that the function $x \mapsto F(x)/x$ is non-increasing. By considering the derivative of this function, we see that the desired bound is equivalent to showing that for all $x \geq 0$, $xf(x) \leq F(x)$.

It is easy to see that our assumptions imply this inequality holds, for we have

$$F(x) \geq F(x) - F(0) = \int_0^x f(x) \, dx \geq xf(x),$$

as f is non-increasing on $[0, \infty)$. □

As in earlier parts of this paper, we will use the notation $N(m, \sigma^2)$ to denote a Gaussian random variable with mean m and variance σ^2 .

PROOF OF LEMMA 2.4. We may assume $\eta \leq 1$ as we are working on $[0, 1]$. We use the well-known fact that the distribution of B_{me} on an interval $[r, 1]$, conditionally on $B_{\text{me}}(r) = x$, is that of a Brownian motion on $[r, 1]$ started at x and conditioned to stay positive which is independent of B_{me} on $[0, r]$. Thus,

$$(68) \quad \mathbb{P}(\text{NZ}(B_{\text{me}}, \eta, a) \mid B_{\text{me}}(\eta)) = \mathbb{P}\left(\inf_{x \in [\eta, 1]} B(x) < a\eta^{1/2} - B_{\text{me}}(\eta) \mid B_{\text{me}}(\eta), \inf_{x \in [\eta, 1]} B(x) \geq -B_{\text{me}}(\eta)\right) \cdot \mathbb{1}_{B_{\text{me}}(\eta) \geq a\eta^{1/2}} + \mathbb{1}_{B_{\text{me}}(\eta) \leq a\eta^{1/2}},$$

where B is a standard Brownian motion started at 0. The reflection principle for Brownian motion asserts that $\inf_{x \in [\eta, 1]} B(x) \stackrel{d}{=} -|N(0, 1 - \eta)|$. Thus, letting Z be distributed as $|N(0, 1 - \eta)|$, we may write

$$\begin{aligned} & \mathbb{P}\left(\inf_{x \in [\eta, 1]} B(x) < a\eta^{1/2} - B_{\text{me}}(\eta) \mid B_{\text{me}}(\eta), \inf_{x \in [\eta, 1]} B(x) \geq -B_{\text{me}}(\eta)\right) \\ &= \mathbb{P}\left(Z > B_{\text{me}}(\eta) - a\eta^{1/2} \mid B_{\text{me}}(\eta), Z \leq B_{\text{me}}(\eta)\right) \leq \frac{a\eta^{1/2}}{B_{\text{me}}(\eta)}; \end{aligned}$$

here we were able to use Lemma A.1, as the law of Z clearly satisfies this lemma's hypotheses. We now take expectations in (68), using that $a\eta^{1/2}/B_{\text{me}}(\eta) \geq 1$ when $B_{\text{me}}(\eta) \leq a\eta^{1/2}$ and that the density of $B_{\text{me}}(\eta)$ at x is $2\sqrt{2\pi}\eta^{-1}x\varphi_\eta(x)\tilde{\Phi}_{1-\eta}(x)$, to obtain

$$(69) \quad \mathbb{P}(\text{NZ}(B_{\text{me}}, \eta, a)) \leq \int_0^\infty 2\sqrt{2\pi}\frac{x}{\eta}\varphi_\eta(x)\tilde{\Phi}_{1-\eta}(x) \cdot \frac{a\eta^{1/2}}{x} dx = 2\sqrt{2\pi}a\eta^{-1/2} \int_0^\infty \varphi_\eta(x)\tilde{\Phi}_{1-\eta}(x) dx.$$

There are two cases, depending on whether $\eta \in (0, \frac{1}{2})$ or $\eta \in [\frac{1}{2}, 1]$. In the second case we bound $\tilde{\Phi}_{1-\eta}(x)$ by $\frac{1}{2}$ for all x , and use that $\int_0^\infty \varphi_\eta(x) dx = 1/2$ and $\eta^{-1/2} \leq 2^{1/2}$ to obtain that

$$\mathbb{P}(\text{NZ}(B_{\text{me}}, \eta, a)) \leq 2\sqrt{2\pi}a2^{1/2} \times \frac{1}{4} = \sqrt{\pi}a \leq 2a.$$

In the first case, i.e., $\eta \in (0, \frac{1}{2})$, we observe that

$$\tilde{\Phi}_{1-\eta}(x) \leq x \cdot (2\pi(1-\eta))^{-1/2} \leq \pi^{-1/2}x.$$

Substituting this into the integral in (69) gives that

$$\mathbb{P}(\text{NZ}(B_{\text{me}}, \eta, a)) \leq \sqrt{2\pi}a\eta^{-1/2}\pi^{-1/2} \int_0^\infty 2x\varphi_\eta(x) dx = \sqrt{2}a\eta^{-1/2}\mathbb{E}[|N(0, \eta)|] = 2\pi^{-1/2}a \leq 2a,$$

where we used that $\eta^{-1/2}\mathbb{E}|N(0, \eta)| = \mathbb{E}|N(0, 1)| = \sqrt{2/\pi}$. \square

PROOF OF LEMMA 2.7.

$$\mathbb{P}(B_{\text{me}}(\eta) < 1.1\eta^{1/2}) = \frac{2\sqrt{2\pi}}{\eta} \int_0^{1.1\eta^{1/2}} y\varphi_\eta(y)\tilde{\Phi}_{1-\eta}(y) dy.$$

Note that

$$\tilde{\Phi}_{1-\eta}(y) \leq (2\pi(1-\eta))^{-1/2}y \quad \text{and} \quad \varphi_\eta(y) \leq (2\pi\eta)^{-1/2}.$$

Substituting these in, we get

$$\mathbb{P}(B_{\text{me}}(\eta) < 1.1\eta^{1/2}) \leq \frac{2\sqrt{2\pi}}{2\pi\eta^{3/2}(1-\eta)^{1/2}} \int_0^{1.1\eta^{1/2}} y^2 dy \leq \frac{(1.1)^3}{3\sqrt{\pi}} < \frac{1}{2},$$

the last inequality by numerical evaluation. We have used that $\eta \leq 1/2$. \square

PROOF OF LEMMA 2.8. We need to bound $\mathbb{P}(B_{\text{me}}(t) < 1.1\eta^{1/2} \mid B_{\text{me}}(t-\eta) = x)$ for all values of $0 < x \leq 1.1\eta^{1/2}$ and $t \in (\eta, 1-\eta)$. Let $s = t - \eta$ and note that $s \leq 1 - 2\eta$.

We have

$$\begin{aligned}
\mathbb{P}\left(B_{\text{me}}(t) < 1.1\eta^{1/2} \mid B_{\text{me}}(s) = x\right) &= \int_0^{1.1\eta^{1/2}} (\varphi_\eta(y-x) - \varphi_\eta(y+x)) \cdot \frac{\tilde{\Phi}_{1-t}(y)}{\tilde{\Phi}_{1-s}(x)} dy \\
&= \int_0^{1.1\eta^{1/2}} \varphi_\eta(y-x) (1 - \exp(-2xy/\eta)) \cdot \frac{\tilde{\Phi}_{1-t}(y)}{\tilde{\Phi}_{1-s}(x)} dy \\
(70) \qquad \qquad \qquad &\leq \frac{2}{\eta} \cdot \int_0^{1.1\eta^{1/2}} \varphi_\eta(y-x) \cdot xy \cdot \frac{\tilde{\Phi}_{1-t}(y)}{\tilde{\Phi}_{1-s}(x)} dy,
\end{aligned}$$

using in the last line that $1 - e^{-r} \leq r$ for any $r \geq 0$. Now we bound the fraction involving $\tilde{\Phi}$. Note that the numerator satisfies

$$\tilde{\Phi}_{1-t}(y) \leq \frac{y}{\sqrt{2\pi(1-t)}},$$

while for the denominator we have, using that the standard normal density is decreasing on the positive real line,

$$\tilde{\Phi}_{1-s}(x) = \mathbb{P}\left(N(0, 1) \in (1-s)^{-1/2}[0, x]\right) \geq \frac{x}{\sqrt{2\pi(1-s)}} \exp\left(-\frac{x^2}{2(1-s)}\right) \geq \frac{x}{\sqrt{2\pi(1-s)}} \cdot e^{-3/8},$$

as $x \leq 1.1\eta^{1/2} \leq (1.5)^{1/2}\eta^{1/2}$ and $1-s \geq 2\eta$. We also have the bound $\varphi_\eta(y-x) \leq (2\pi\eta)^{-1/2}$. Substituting these inequalities in (70), we get

$$\mathbb{P}\left(B_{\text{me}}(t) < 2\eta^{1/2} \mid B_{\text{me}}((k-1)\eta) = x\right) \leq \frac{2e^{3/8}}{\eta^{3/2}} \cdot \int_0^{1.1\eta^{1/2}} y^2(1-s)^{1/2}(2\pi(1-t))^{-1/2} dy.$$

Since $s = t - \eta$, we have $\sqrt{(1-s)/(1-t)} \leq \sqrt{2}$. Thus we may bound the last expression as

$$\frac{2e^{3/8}}{\eta^{3/2}\pi^{1/2}} \cdot \int_0^{1.1\eta^{1/2}} y^2 dy = \frac{2e^{3/8}(1.1)^3\eta^{3/2}}{3\eta^{3/2}\pi^{1/2}}.$$

Overall, this gives that

$$\mathbb{P}\left(B_{\text{me}}(t) < 1.1\eta^{1/2} \mid B_{\text{me}}(s) = x\right) \leq \frac{2 \times (1.1)^3 e^{3/8}}{3\sqrt{\pi}} \leq \frac{3}{4},$$

the last inequality by numerical evaluation. \square

PROOF OF LEMMA 2.9. We again use that, conditionally on $B_{\text{me}}(t-\eta) = x$, the distribution of B_{me} on $[t-\eta, 1]$ is that of a Brownian motion B started at x conditioned to stay positive. This implies

$$\mathbb{P}\left(\inf_{s \in [t-\eta, 1]} B_{\text{me}}(s) < \eta^{1/2} \mid B_{\text{me}}(t-\eta) = x\right) = \mathbb{P}\left(\inf_{s \in [t-\eta, 1]} B(s) < \eta^{1/2} - x \mid \inf_{s \in [t-\eta, 1]} B(s) > -x\right),$$

where B is a standard Brownian motion on $[t-\eta, 1]$ started at 0. Now by the reflection principle and the fact that the normal density decreases on the positive real line, we know that

$$\mathbb{P}\left(\inf_{s \in [t-\eta, 1]} B(s) > -x\right) = \mathbb{P}(|N(0, 1 - (t-\eta))| < x) \geq \frac{2 \times 1.1\eta^{1/2}}{\sqrt{2\pi(1 - (t-\eta))}} \exp\left(-\frac{(1.1)^2\eta}{2(1 - (t-\eta))}\right)$$

since $x \geq 1.1\eta^{1/2}$; and also

$$\begin{aligned}
\mathbb{P}\left(\inf_{s \in [t-\eta, 1]} B(s) \in [-x, -x + \eta^{1/2}]\right) &= \mathbb{P}\left(|N(0, 1 - (t-\eta))| \in [x - \eta^{1/2}, x]\right) \\
&\leq \mathbb{P}\left(|N(0, 1 - (t-\eta))| \in [0.1\eta^{1/2}, 1.1\eta^{1/2}]\right) \\
&\leq \frac{2\eta^{1/2}}{\sqrt{2\pi(1 - (t-\eta))}} \exp\left(-\frac{\frac{1}{4}[(0.1 + 1.1)^2]\eta}{2(1 - (t-\eta))}\right)
\end{aligned}$$

$$= \frac{2\eta^{1/2}}{\sqrt{2\pi(1-(t-\eta))}} \exp\left(-\frac{(0.6)^2\eta}{2(1-(t-\eta))}\right),$$

using Jensen's inequality and that the normal density $f_{\mu,\sigma^2}(x)$ is concave for $|x| < \sigma$ for the penultimate inequality (and that $1.1\eta^{1/2} < (1-(t-\eta))^{1/2}$ since $t < 1-4\eta$). So we obtain that the probability in the statement of Lemma 2.9 is bounded above by

$$\frac{1}{1.1} \exp\left(\frac{(1.1)^2\eta - (0.6)^2\eta}{2(1-(t-\eta))}\right) \leq \frac{1}{1.1} \exp\left(\frac{0.85}{10}\right) < 1 - \delta$$

for a $\delta > 0$ as can be checked by numeric evaluation; in fact, $\frac{1}{1.1} \exp(0.85/10) \approx 0.9897 < 1 - 0.01$, so $\delta = 0.01$ works. We have also used that $t < 1 - 4\eta$. \square

APPENDIX B

MEASURABILITY OF THE JUMP ENSEMBLE

In this appendix we collect the statements and proofs needed to verify measurability of the jump ensemble construction, several of which were referenced in Chapter 4. In particular, we need to show that the pole set P is a measurable function of $\mathcal{L}_n(k+1, \cdot)$, and so is a well-defined random object.

To set this up, we need the notion of a random closed set. We follow [Mol05] for the definition, and we will make use of results from that monograph.

Definition B.1 (Definition 1.1 of [Mol05]). — For an interval $[a, b] \subseteq \mathbb{R}$, let \mathcal{K} be the collection of closed (therefore compact) subsets of $[a, b]$. For a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, a mapping $A : \Omega \rightarrow \mathcal{K}$ is called a *random closed set* if, for every compact subset K of $[a, b]$,

$$\{\omega : A(\omega) \cap K \neq \emptyset\} \in \mathfrak{F}.$$

Equivalently, we endow \mathcal{K} with the σ -algebra $\mathcal{B}(\mathcal{K})$ generated by the collection of sets of the form $\{K \in \mathcal{K} : K \cap K' \neq \emptyset\}$ as K' varies over \mathcal{K} , and $A : \Omega \rightarrow \mathcal{K}$ is a random closed set if it is a measurable map.

The analogous definition holds for \mathbb{R} in place of $[a, b]$.

Recall that P is defined through $\text{xExt}(\mathbf{c}^+)$, the set of extreme points of the least concave majorant \mathbf{c}^+ of $\mathcal{L}_n(k+1, \cdot)$. We will first need to know that the mapping taking a function f to its least concave majorant $\text{LCM}(f)$ is measurable; in fact, it is (Lipschitz) continuous.

Lemma B.2. — *Let $[a, b] \subset \mathbb{R}$. The mapping which takes a continuous function f to its least concave majorant $\text{LCM}(f)$ is a continuous mapping (indeed, Lipschitz with parameter 1) on $\mathcal{C}_{*,*}([a, b], \mathbb{R})$ with the uniform topology.*

PROOF. It is easy to verify the following formula for $\text{LCM}(f)$ which is stated, for example, in [FKS17]:

$$(71) \quad (\text{LCM}(f))(x) = \sup \left\{ \frac{\beta - x}{\beta - \alpha} f(\alpha) + \frac{x - \alpha}{\beta - \alpha} f(\beta) : a \leq \alpha \leq x \leq \beta \leq b \right\}$$

for $x \in [a, b]$.

Let $f, g \in \mathcal{C}_{*,*}([a, b], \mathbb{R})$ and fix $x \in [a, b]$. Now, since the expression in the brackets in the previous display is a continuous function of (α, β) and $[a, x] \times [x, b]$ is compact, it follows that there exists (α_f, β_f) such that

$$(\text{LCM}(f))(x) = \frac{\beta_f - x}{\beta_f - \alpha_f} f(\alpha_f) + \frac{x - \alpha_f}{\beta_f - \alpha_f} f(\beta_f).$$

We also have that

$$(\text{LCM}(g))(x) \geq \frac{\beta_f - x}{\beta_f - \alpha_f} g(\alpha_f) + \frac{x - \alpha_f}{\beta_f - \alpha_f} g(\beta_f).$$

Thus we see that

$$\begin{aligned} (\text{LCM}(f))(x) - (\text{LCM}(g))(x) &\leq \frac{\beta_f - x}{\beta_f - \alpha_f} (f(\alpha_f) - g(\alpha_f)) + \frac{x - \alpha_f}{\beta_f - \alpha_f} (f(\beta_f) - g(\beta_f)) \\ &\leq \sup_{y \in [a, b]} |f(y) - g(y)|. \end{aligned}$$

The same argument applies with f and g switched, and, since x is arbitrary, we obtain that

$$\sup_{x \in [a, b]} |(\text{LCM}(f))(x) - (\text{LCM}(g))(x)| \leq \sup_{x \in [a, b]} |f(x) - g(x)|. \quad \square$$

The statements from [Mol05] regarding the measurability of several common operations on sets that we will need are collected in the next lemma.

Lemma B.3. — *Let A, B be random closed sets (and subsets of either $[a, b]$ or \mathbb{R}) and $X : [c, d] \rightarrow \mathbb{R}$ a stochastic process. Then*

1. $A \cup B$ and $A \cap B$ are random closed sets.
2. $X^{-1}(A)$ is a random closed set.
3. $\inf_{x \in A} X(x)$ is a random variable on the event that $A \neq \emptyset$.

PROOF. See [Mol05, Theorems 2.25, 2.26, and 2.27]. □

For a concave function f defined on an interval $[a, b]$ and an interval $[\ell, r] \subseteq [a, b]$, define $\text{xExt}(f) = \text{xExt}^{\ell, r}(f)$ to be the set of x -coordinates of the extreme points of the convex set $\{(x, y) : x \in [\ell, r], y \leq f(x)\}$. Note that $\ell, r \in \text{xExt}(f)$.

Lemma B.4. — *For a continuous function $f : [a, b] \rightarrow \mathbb{R}$ and an interval $[\ell, r] \subseteq [a, b]$,*

$$\text{xExt}(\text{LCM}(f)) = \left\{ x \in [\ell, r] : \text{LCM}(f) \text{ is not linear in any neighbourhood of } x \right\} \cup \{\ell, r\}.$$

PROOF. That $\text{LHS} \subseteq \text{RHS}$ is obvious from the definition of extreme points. So we need to prove $\text{RHS} \subseteq \text{LHS}$. Obviously $\ell, r \in \text{LHS}$. Now suppose $x \notin \text{LHS}$ (so $x \neq \ell, r$). Then there exist $\ell \leq y < x < z \leq r$ such that $(\text{LCM}(f))(x)$ is a nontrivial convex combination of $(\text{LCM}(f))(y)$ and $(\text{LCM}(f))(z)$ (where we make use of the standard fact that the extreme points of a convex set lie on its boundary). Since $\text{LCM}(f)$ is concave, $\text{LCM}(f)$ must be linear on $[y, z]$. This completes the proof. □

Proposition B.5. — *If $X : [a, b] \rightarrow \mathbb{R}$ is a continuous stochastic process and $[\ell, r]$ is a random closed set, then $\text{xExt}(\text{LCM}(X))$ is a random closed set.*

PROOF. For a continuous function $f : [a, b] \rightarrow \mathbb{R}$, consider the mapping

$$f \mapsto \text{Lin}(f), \quad (\text{Lin}(f))(x) = \limsup_{h \rightarrow 0} \mathbb{1}_{f(x+h)+f(x-h)=2f(x)}.$$

We claim that $f \mapsto \text{Lin}(f)$ is a measurable mapping. Indeed, this is immediate for the mapping $f \mapsto \text{Lin}_h(f)$ with $(\text{Lin}_h(f))(x) = \mathbb{1}_{f(x+h)+f(x-h)=2f(x)}$ for any fixed $h > 0$, and so, taking $h \rightarrow 0$, $f \mapsto \text{Lin}(f)$ is also measurable.

Now, it is easy to check using concavity that

$$(\text{Lin}(\text{LCM}(f)))^{-1}(0) = \{x \in [a, b] : \text{LCM}(f) \text{ is not linear in any neighbourhood of } x\}.$$

(In fact by concavity the lim sup in the definition of $\text{Lin}(\text{LCM}(f))$ is a limit.) So, by Lemma B.4,

$$\left[(\text{Lin}(\text{LCM}(X)))^{-1}(0) \cap [\ell, r] \right] \cup \{\ell, r\} = \text{xExt}(\text{LCM}(X)).$$

The measurability of Lin and LCM (from Lemma B.2) and Lemma B.3 implies that the LHS of the previous display is a random closed set, which completes the proof. \square

Recall the definition of \mathcal{F} from Section 4.1.1.5.

Corollary B.6. — $\text{xExt}(\mathcal{L}_n(k+1, \cdot))$, with $\ell = \mathfrak{l}$ and $r = \mathfrak{r}$, is a \mathcal{F} -measurable random variable.

PROOF. This follows by Proposition B.5 since $\mathcal{L}_n(k+1, \cdot)$, \mathfrak{l} , and \mathfrak{r} are \mathcal{F} -measurable. \square

Proposition B.7. — The pole set P is a \mathcal{F} -measurable random closed set.

PROOF. Recall that P is defined by iteratively adding elements of $\text{xExt}(\mathcal{L}_n(k+1, \cdot))$ with separation at least d_{ip} , except at the end, where we ensure \mathfrak{r} lies in the set (and is separated by at least d_{ip} from the previously added element). We can encode one round of adding an element as in this description by considering the mapping $U : \mathcal{K}([a, b]) \rightarrow \mathcal{K}([a, b])$ given by

$$A \mapsto A \cup \left\{ \inf \left(\text{xExt}(\mathcal{L}_n(k+1, \cdot)) \cap [\inf A + d_{\text{ip}}, \mathfrak{r} - d_{\text{ip}}] \right) \right\}.$$

This is measurable by Lemma B.3 (technically, the infimum operation is only measurable on the collection of non-empty sets, but this is not an issue: the event of non-emptiness is measurable, and, on the event that $\text{xExt}(\mathcal{L}_n(k+1, \cdot)) \cap [\inf A + d_{\text{ip}}, \mathfrak{r} - d_{\text{ip}}] = \emptyset$, the above mapping is the identity) and since xExt is a random closed set by Proposition B.5.

For $j \geq 1$, let $U^{(j)}$ be the composition of U with itself j times. Then, since $|P| \leq 2T/d_{\text{ip}}$ almost surely, we consider $U^{(\lceil 2T/d_{\text{ip}} \rceil)}(\{\mathfrak{l}, \mathfrak{r}\})$. From our definition of P it is easy to see that this set is exactly P . Thus, by the measurability of U , we conclude that P is a random closed set; further, it is \mathcal{F} -measurable since \mathfrak{l} , \mathfrak{r} , and $\mathcal{L}_n(k+1, \cdot)$ are \mathcal{F} -measurable. \square

Lemma B.8. — For A a random closed set, $\text{NonIntLow}_{\mathcal{L}_n(k+1, \cdot)}^A$ (as defined on page 43) and NonInt_A^f (as defined in (9)) are measurable.

PROOF. Writing out the definition of $\text{NonIntLow}_{\mathcal{L}_n(k+1, \cdot)}^A$, we see

$$\begin{aligned} \text{NonIntLow}_{\mathcal{L}_n(k+1, \cdot)}^A &= \left\{ B(x) > \mathcal{L}_n(k+1, x) \quad \forall x \in A \right\} \\ &= \left\{ \min_{x \in A} (B(x) - \mathcal{L}_n(k+1, x)) > 0 \right\}. \end{aligned}$$

Now, $B(x) - \mathcal{L}_n(k+1, x)$ is a continuous stochastic process, and Lemma B.3 asserts that the minimum of a continuous stochastic process over a random closed set is a random variable. Thus the last displayed set is measurable.

Similarly, we may write

$$\text{NonInt}_A^f = \left\{ \min_{1 \leq i \leq k} \min_{x \in A} (B(i, x) - B(i+1, x)) > 0 \right\},$$

letting $B(i+1, \cdot) = f(\cdot)$ by convention. Again by Lemma B.3 we conclude that NonInt_A^f is measurable (recall from (9) that f is assumed to be continuous). \square

Lemma B.9 (Theorem 8.1 of [Par05]). — Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space, where Ω is Polish and \mathfrak{F} is the associated Borel σ -algebra, and let (T, \mathcal{T}) be a measure space. Let $X : (\Omega, \mathfrak{F}) \rightarrow (T, \mathcal{T})$ be measurable. Then, there exists a probability kernel $\mu : T \times \mathfrak{F} \rightarrow [0, 1]$ such that, almost surely,

$$\mathbb{P}(\cdot \mid X) = \mu(X, \cdot).$$