Critical exponents in percolation via lattice animals

Alan Hammond*

September 7, 2007

1 Introduction

We examine the percolation model by an approach involving lattice animals, divided according to their surface-area-to-volume ratio. Throughout, we work with the bond percolation model in $\mathbb{Z}^d$. However, the results apply to the site or bond model on any infinite transitive amenable graph with inessential changes.

For any given $p \in (0,1)$, two lattice animals with given size are equally likely to arise as the cluster $C(0)$ containing the origin provided that they have the same surface-area-to-volume ratio. For given $\beta \in (0,\infty)$, there is an exponential growth rate in the number of edges for the number of lattice animals up to translation that have surface-area-to-volume ratio very close to β. This growth rate $f(\beta)$ may be studied as a function of β. To illustrate the connection between the percolation model and the combinatorial question of the behaviour of f, note that the probability that the cluster containing the origin contains a large number n of edges is given by

$$\mathbb{P}_p(|C(0)| = n) = \sum_m \sigma_{n,m} p^n (1-p)^m, \tag{1}$$

where $\sigma_{n,m}$ is the number of lattice animals that contain the origin, have n edges and m outlying edges. We rewrite the right-hand-side to highlight the role of the surface-area-to-volume ratio, m/n:

$$\mathbb{P}_p(|C(0)| = n) = \sum_m \left(f_n(m/n) p (1-p)^{m/n}\right)^n. \tag{2}$$

Here $f_n(\beta) = (\sigma_{n,\lfloor \beta n \rfloor})^{1/n}$ is a rescaling that anticipates the exponential growth that occurs. We examine thoroughly the link between percolation and combinatorics provided by (2): how do the quantities $f_n(\beta)$ scale for high n, and in which range of values of m is the expression in (2) carrying most of its weight? These questions are hardly new, and the techniques of proof we have employed

*This work was supported by Merton College, Oxford.

in seeking answers are not strikingly novel. However, in pursuing answers, we have found two critical exponents that arise naturally in this approach, and some inequalities that this pair must satisfy. Trying to determine the relationships satisfied by such exponents is a central task in understanding a phase transition, and we believe that what is of most interest in this re-examination of the lattice-animal based approach to percolation are these exponents and their relation to those more conventionally defined in percolation theory.

We outline the approach in more detail, describing the presentation of the results as we do so. In Section 2, we describe the model, and define notations, before stating the combinatorial results that we will use. Theorem 2.2 asserts the existence of the function f and describes aspects of its behaviour, and is a pretty standard result. The details of its proof are however cumbersome, and are not given here: they appear in [4]. Theorem 2.3 implies that

$$\log f(\beta) \leq (\beta+1)\log(\beta+1) - \beta\log\beta \text{ for } \beta \in (0, 2(d-1)). \tag{3}$$

F.Delyon [2] showed that equality holds for $\beta \in (0, 1/p_c - 1)$. Theorem 2.3 implies that the inequality is strict for higher values of β. The marked change, as β passes through $1/p_c - 1$, in the structure of large lattice animals of surface-area-to-volume ratio β is a combinatorial manifestation of the phase transition in percolation at criticality: a large lattice animal with surface-area-to-volume $\beta < 1/p_c - 1$ presumably has an internal structure resembling a typical portion of the infinite cluster at the supercritical value $p = 1/(1+\beta)$ that corresponds to β, whereas, if $\beta > 1/p_c - 1$, there are too few large animals with surface-area-to-volume ratio β to enable the formation of infinite structure in the percolation model at this corresponding value of p. We mention also that the notion of a collapse transition for animals has been explored in [3].

In Section 3, two scaling hypotheses are introduced, each postulating the existence of a critical exponent. One of the exponents, ς, describes how quickly $f(\beta)$ drops away from the explicit form given on the right-hand-side of (3) as β rises above $1/p_c - 1$. The other, λ, describes how rapidly decaying in n is the discrepancy between the critical value and that value on the subcritical interval at which the probability of observing an n-edged animal as the cluster to which the origin belongs is maximal. The first main result, Theorem 3.4, is then proved: the inequalities $\lambda < 1/2$ and $\varsigma\lambda < 1$ cannot both be satisfied. In outline, this is because $\lambda < 1/2$ implies that, for values of p just less than p_c, most of the weight in the sum in (2) is carried by terms indexed by $m >> n\alpha + n^{1/2}$, while $\varsigma < 1/\lambda$ implies that the limiting function $f(\beta)$ has dropped enough in this range of $\beta = m/n$ that the probability of such lattice animals is decaying quickly: we have reached beyond the low side of the critical scaling window. This decay rate is quick enough that it implies that the mean cluster size is uniformly bounded on the subcritical interval, but this contradicts known results. In fact, by a similar approach, it may be shown that $\varsigma < 2$ or $\lambda > 1/2$ provide sufficient conditions for the continuity of the percolation probability (c.f. [4]).

In Section 4, we relate the value of ς to an exponent of a more conventional nature in the scaling theory of percolation, that of correlation size (see Theorem

4.2). Suppose that we perform an experiment in which the surface-area-to-volume ratio of the cluster to which the origin belongs is observed, conditional on its having a very large number of edges, for a p-value slightly below p_c. How does the typical measurement, β_p, in this experiment behave as p tends to p_c? The value β_p tends to lie somewhere on the interval $(1/p_c - 1, 1/p - 1)$. In Theorem 4.3, we determine that there are two possible scaling behaviours. The inequality $\varsigma < 2$ again arises, distinguishing the two possibilities. If $\varsigma < 2$, then β_p scales much closer to $1/p_c - 1$ while if $\varsigma > 2$, it is found to be closer to $1/p - 1$. It would be of much interest further to understand the relation of λ and ς to other exponents.

2 Notations and combinatorial results

Throughout, we work with the bond percolation model on $\mathbb{Z}^d$, for any given $d \geq 2$. This model has a parameter p lying in the interval $[0,1]$. Nearest neighbour edges of $\mathbb{Z}^d$ are declared to be open with probability p, these choices being made independently between distinct edges. For any vertex $x \in \mathbb{Z}^d$, there is a cluster $C(x)$ of edges accessible from x, namely the collection of edges that lie in a nearest-neighbour path of open edges one of whose members contains x as an endpoint.

Definition 2.1 *A lattice animal is the collection of edges of a finite connected subgraph of $\mathbb{Z}^d$. An edge of $\mathbb{Z}^d$ is said to be outlying to a lattice animal if it is not a member of the animal, and if there is an edge in the animal sharing an endpoint with this edge. We adopt the notations:*

- *for $n, m \in \mathbb{N}$, set $\Gamma_{n,m}$ equal to the collection of lattice animals in $\mathbb{Z}^d$ one of whose edges contains the origin, having n edges, and m outlying edges. Define $\sigma_{n,m} = |\Gamma_{n,m}|$. The surface-area-to-volume ratio of any animal in $\Gamma_{n,m}$ is said to be m/n.*
- *for each $n \in \mathbb{N}$, define the function $f_n : [0,\infty) \to [0,\infty)$ by*

$$f_n(\beta) = \left(\sigma_{n,\lfloor \beta n \rfloor}\right)^{1/n}$$

On another point of notation, we will sometimes write the index set of a sum in the form nS, with $S \subseteq (0,\infty)$, by which is meant $\{m \in \mathbb{N} : m/n \in S\}$.
We require some results about the asymptotic exponential growth rate of the number of lattice animals as a function of their surface-area-to-volume ratio. We now state these results, noting that the following theorem appears as Theorem 2.1 in [4], which paper gives its proof.

Theorem 2.2

1. *For $\beta \in [0,\infty) \setminus \{2(d-1)\}$, $f(\beta)$ exists, being defined as the limit $\lim_{n\to\infty} f_n(\beta)$.*
2. *for $\beta > 2(d-1)$, $f(\beta) = 0$.*

3. *for* $\beta \in (0, 2(d-1)), n \in \mathbb{N}$, f_n *satisfies* $f_n(\beta) \leq L^{1/n} n^{1/n} f(\beta)$, *where the constant* L *may be chosen uniformly in* $\beta \in (0, 2(d-1))$.

4. f *is log-concave on the interval* $(0, 2(d-1))$.

Remark: The proof involves concatenating two large lattice animals of the same surface-area-to-volume ratio $\beta \in (0, 2(d-1))$ by translating one so that it just touches the other. This operation produces a new lattice animal with roughly this same surface-area-to-volume ratio. In this way, some part of the set $\Gamma_{2n,\lfloor 2\beta n \rfloor}$ is composed of concatenated pairs of animals each lying in $\Gamma_{n,\lfloor \beta n \rfloor}$, if we overlook the errors that the joining produces. It would follow that $\sigma_{2n,\lfloor 2\beta n \rfloor} \geq 2n^{-1} \sigma^2_{n,\lfloor \beta n \rfloor}$, the factor of $2n^{-1}$ occurring because, in performing the argument carefully, we work with the space of lattice animals up to translation rather than those containing the origin, this alteration producing some factors of n. The proof demonstrates that the errors involved in joining are indeed negligible. In this way, we obtain the existence of the limit $f(\beta)$, and the bound in the third part of the theorem. The proof of the fourth part is a reprisal of the same argument that instead involves concatenating two lattice animals of differing surface-area-to-volume ratio, with their relative sizes chosen to ensure a given surface-area-to-volume ratio for the resulting lattice animal. More specifically, for $\beta_1, \beta_2 \in \big(0, 2(d-1)\big)$ satisfying $\beta_1 < \beta_2$, and for $\lambda \in (0,1)$, we would show that

$$\log f\big(\lambda\beta_1 + (1-\lambda)\beta_2\big) \geq \lambda \log f\big(\beta_1\big) + (1-\lambda)\log f(\beta_2),$$

by concatenating a pair of animals drawn from the sets $\Gamma_{\lfloor \lambda n \rfloor, \lfloor \beta_1 \lfloor \lambda n \rfloor \rfloor}$ and $\Gamma_{\lfloor (1-\lambda) n \rfloor, \lfloor \beta_2 \lfloor (1-\lambda) n \rfloor \rfloor}$, the resulting animal lying in $\Gamma_{n, \lfloor \left(\lambda\beta_1 + (1-\lambda)\beta_2\right) n \rfloor}$, provided that we again tolerate the slight discrepancy caused by the joining mechanism, as well as some rounding errors.

The second part of the theorem follows from the fact, easily proved by an induction on n, that a lattice animal in $\mathbb{Z}^d$ of size n may have at most $2(d-1)n + 2d$ outlying edges.

Theorem 2.3 *Introducing* $g : (0, 2(d-1)) \to [0, \infty)$ *by means of the formula*

$$f(\beta) = g(\beta) \frac{(\beta+1)^{\beta+1}}{\beta^\beta},$$

we have that

$$g(\beta) \begin{cases} = 1 & \text{on } (0, \alpha], \\ < 1 & \text{on } (\alpha, 2(d-1)), \end{cases}$$

where throughout $\alpha = 1/p_c - 1$, p_c *being the critical value of the model.*

Remark: The assertion that $g = 1$ on $(0, \alpha]$ was originally proved by Delyon [2]. For the sake of completeness, however, we will give a proof of this result. We will make use of the following weaker result in the proof of Theorem 2.3.

Lemma 2.4 *The function* $f : (0, 2(d-1)) \to [0, \infty)$ *satisfies*

$$\log f(\beta) \leq (\beta + 1) \log(\beta + 1) - \beta \log \beta$$

Proof: We give a probabilistic proof, that uses the percolation model. The link between the random and combinatorial models is provided by (1), which we restate for convenience:

$$\mathbb{P}_p(|C(0)| = n) = \sum_m \sigma_{n,m} p^n (1-p)^m$$

Let $\beta \in (0, 2(d-1))$. Choosing $p = 1/(1+\beta)$, and noting that the right-hand-side of the above equation is bounded above by one, yields

$$\sigma_{n,\lfloor \beta n \rfloor} \leq \left(\frac{(\beta+1)^{\beta+1}}{\beta^\beta} \right)^n \max\big\{1, \beta\big\},$$

which may be rewritten

$$f_n(\beta) \leq \frac{(\beta+1)^{\beta+1}}{\beta^\beta} \max\big\{1, \beta^{\frac{1}{n}}\big\}.$$

Taking the limit as $n \to \infty$ gives that

$$f(\beta) \leq \frac{(\beta+1)^{\beta+1}}{\beta^\beta},$$

as required. □
We will also require the following lemma, to handle some expressions corresponding to a few miscellaneous lattice animals whose surface-area-to-volume ratio is almost maximal.

Lemma 2.5 *There exists* $r \in (0,1)$ *such that, for* n *sufficiently large and for* $m \in \{2(d-1)n, \ldots, 2(d-1)n + 2d\}$*, we have that*

$$\sigma_{n,m} \leq \frac{(1 + \frac{m}{n})^{n+m}}{(\frac{m}{n})^m} r^n.$$

Proof: The inequality $p_c > 1/(2d-1)$ is known, a proof is supplied on [4, page 13]. There is an exponential decay rate in n for the probability of observing an n-edged cluster containing the origin in the subcritical phase [1]. Hence, for some $r_j \in (0,1)$ and for n sufficiently large, we have that

$$\sigma_{n,2(d-1)n+j} \leq \Big(\frac{(2d-1)n + j}{n} \Big)^n \Big(\frac{(2d-1)n + j}{2(d-1)n + j} \Big)^{2(d-1)n+j} r_j^n.$$

Setting $r = \max_{j \in \{0,\ldots,2d\}} r_j$ gives the result. □
Proof of Theorem 2.3: We firstly prove Delyon's result, that $g(\beta) = 1$ for

$\beta \in [0,\alpha]$. By Theorem 2.2, we know that f is log-concave on $(0,\alpha)$ and, by Lemma 2.4, that on that interval, it satisfies

$$f(\beta) \leq \frac{(\beta+1)^{\beta+1}}{\beta^\beta}.$$

From these statements and the assumption that Delyon's result fails, it follows that there exist $\beta_0 \in (0,\alpha)$ and $\epsilon > 0$ such that

$$f(\beta) \leq \frac{(\beta+1)^{\beta+1}}{\beta^\beta} - \epsilon \quad \text{on } (\beta_0-\epsilon, \beta_0+\epsilon). \tag{4}$$

Set $p = 1/(1+\beta_0)$, and note that $p > p_c$. We rewrite the right-hand-side of (1) as follows:

$$\sum_m \sigma_{n,m} p^n (1-p)^m = \sum_m \left(f_n(m/n) \frac{\beta_0^{\frac{m}{n}}}{(1+\beta_0)^{1+\frac{m}{n}}} \right)^n.$$

It follows that

$$\begin{aligned}
\mathbb{P}_p(|C(0)| = n) \quad &= \quad \sum_{m \in nS_1} \left(f_n(m/n) \frac{\beta_0^{\frac{m}{n}}}{(1+\beta_0)^{1+\frac{m}{n}}} \right)^n \\
&\quad + \sum_{m \in nS_2} \left(f_n(m/n) \frac{\beta_0^{\frac{m}{n}}}{(1+\beta_0)^{1+\frac{m}{n}}} \right)^n \\
&\quad + \sum_{m=2(d-1)n}^{2(d-1)n+2d} \left(f_n(m/n) \frac{\beta_0^{\frac{m}{n}}}{(1+\beta_0)^{1+\frac{m}{n}}} \right)^n
\end{aligned}$$

where

$$S_1 = (\beta_0 - \epsilon, \beta_0 + \epsilon) \quad \text{and} \quad S_2 = \big(0, 2(d-1)\big) \setminus \big(\beta_0 - \epsilon, \beta_0 + \epsilon\big).$$

The behaviour of the three sums above will now be analysed, under the assumption that Delyon's result is false. Firstly, we need a definition.

Definition 2.6 *Let the function* $\phi : (0,\infty)^2 \to \mathbb{R}$ *be given by*

$$\phi(\alpha,\beta) = (\beta+1)\log(\beta+1) - \beta\log\beta + \beta\log\alpha - (\beta+1)\log(\alpha+1).$$

Remark. That $\phi \leq 0$ is straightforward.

- The sum indexed by nS_1 We estimate

$$\begin{aligned}
&\sum_{m \in nS_1} \left(f_n(m/n) \frac{\beta_0^{\frac{m}{n}}}{(1+\beta_0)^{1+\frac{m}{n}}} \right)^n \\
\leq \quad & Ln \sum_{m \in nS_1} \left(f(m/n) \frac{\beta_0^{\frac{m}{n}}}{(1+\beta_0)^{1+\frac{m}{n}}} \right)^n
\end{aligned}$$

$$
\begin{aligned}
&\le Ln \sum_{m\in nS_1} \left(\left(\frac{(1+\frac{m}{n})^{1+\frac{m}{n}}}{(\frac{m}{n})^{\frac{m}{n}}} - \epsilon\right)\left(\frac{\beta_0{}^{\frac{m}{n}}}{(1+\beta_0)^{1+\frac{m}{n}}}\right)\right)^n \\
&\le Ln \sum_{m\in nS_1} \exp\Big(n\big[\phi(\beta_0, m/n) + \log(1-c\epsilon)\big]\Big) \\
&\le Ln(2\epsilon n+1)(1-c\epsilon)^n,
\end{aligned}
$$

where the third part of Theorem 2.2 was applied in the first inequality, with (4) being used in the second. The constant $c > 0$ is chosen to satisfy $\log c < (\beta_0+\epsilon)\log(\beta_0+\epsilon) - (\beta_0+1+\epsilon)\log(\beta_0+1+\epsilon)$.

- The sum indexed by nS_2

  In this case, note that

$$
\begin{aligned}
&\sum_{m\in nS_2} \left(f_n(m/n)\frac{\beta_0{}^{\frac{m}{n}}}{(1+\beta_0)^{1+\frac{m}{n}}}\right)^n \\
&\le Ln \sum_{m\in nS_2} \left(f(m/n)\frac{\beta_0{}^{\frac{m}{n}}}{(1+\beta_0)^{1+\frac{m}{n}}}\right)^n \\
&\le Ln \sum_{m\in nS_2} \left(\left(\frac{(1+\frac{m}{n})^{1+\frac{m}{n}}}{(\frac{m}{n})^{\frac{m}{n}}}\right)\left(\frac{\beta_0{}^{\frac{m}{n}}}{(1+\beta_0)^{1+\frac{m}{n}}}\right)\right)^n \\
&= Ln \sum_{m\in nS_2} \exp\Big(n\phi(\beta_0, m/n)\Big),
\end{aligned}
$$

  where Lemma 2.4 was applied in the second inequality. The fact that

$$
\frac{d}{d\gamma}\phi(\beta_0,\gamma) = \log(1+1/\gamma) - \log(1+1/\beta_0)
$$

  implies that there exists $\delta > 0$ such that $\phi(\beta_0,\gamma) < -\delta$ for $\gamma \in S_2$. Hence

$$
\sum_{m\in nS_2} \left(f_n(m/n)\frac{\beta_0{}^{\frac{m}{n}}}{(1+\beta_0)^{1+\frac{m}{n}}}\right)^n \le 2(d-1)Ln^2 \exp\big(-n\delta\big).
$$

- The sum indexed by $\{2(d-1)n, \ldots, 2(d-1)n+2d\}$

  Note that Lemma 2.5 implies that, for n sufficiently large,

$$
\begin{aligned}
&\sum_{m=2(d-1)n}^{2(d-1)n+2d} \left(f_n(m/n)\frac{\beta_0{}^{\frac{m}{n}}}{(1+\beta_0)^{1+\frac{m}{n}}}\right)^n \\
&\le \sum_{m=2(d-1)n}^{2(d-1)n+2d} \left(\left(\frac{(1+\frac{m}{n})^{1+\frac{m}{n}}}{(\frac{m}{n})^{\frac{m}{n}}}\right)\left(\frac{\beta_0{}^{\frac{m}{n}}}{(1+\beta_0)^{1+\frac{m}{n}}}\right)\right)^n r^n \\
&= \sum_{m=2(d-1)n}^{2(d-1)n+2d} r^n \exp\Big(n\phi(\beta_0, m/n)\Big). \\
&\le (2d+1)r^n.
\end{aligned}
$$

We have demonstrated that if Delyon's result fails, then

$$\liminf_{n\to\infty} \frac{-\log \mathbb{P}_p(|C(0)| = n)}{n} > 0. \tag{5}$$

The sharp first-order asymptotics in large size for the probability of observing a finite cluster in the supercritical phase of the percolation model were derived in [5]. This probability certainly decays at a subexponential rate. Since $p > p_c$ is in the supercritical phase, we find that the left-hand-side of (5) is zero. This contradiction completes the proof of Delyon's result.

We must also show that, for $\beta \in (\alpha, 2(d-1))$, $g(\beta)$ is strictly less than one. For such β, let $p = 1/(1+\beta)$. Note that $p < p_c$, and that

$$\begin{aligned}
\mathbb{P}_p(|C(0)| = n) &\geq \mathbb{P}_p(C(0) \in \Gamma_{n,\lfloor \beta n \rfloor}) \\
&= |\Gamma_{n,\lfloor \beta n \rfloor}| \frac{\beta^{\lfloor \beta n \rfloor}}{(1+\beta)^{n+\lfloor \beta n \rfloor}} \\
&= \big(f_n(\beta)\big)^n \frac{\beta^{\lfloor \beta n \rfloor}}{(1+\beta)^{n+\lfloor \beta n \rfloor}}.
\end{aligned}$$

Taking logarithms yields

$$\frac{\log \mathbb{P}_p\big(|C(0)| = n\big)}{n} \geq \log f_n(\beta) + \frac{\lfloor \beta n \rfloor \log \beta}{n} - \Big(1 + \frac{\lfloor \beta n \rfloor}{n}\Big) \log(1+\beta),$$

from which it follows that

$$\liminf_{n\to\infty} \frac{\log \mathbb{P}_p\big(|C(0)| = n\big)}{n} \geq \log f(\beta) + \beta \log \beta - (1+\beta)\log(1+\beta). \tag{6}$$

The right-hand-side of (6) is equal to $\log g(\beta)$, by definition. The exponential decay rate for the probability of observing a large cluster in the subcritical phase was established in [1]. Since $p < p_c$, this means the left-hand-side of (6) is negative. This implies that $g(\beta) < 1$, as required. □

3 Critical exponents and inequalities

We introduce two scaling hypotheses, each of which postulates the existence of a critical exponent. We then state and prove the first main theorem, which demonstrates that a pair of inequalities involving the two exponents cannot both be satisfied.

3.1 Hypothesis (λ)

Definition 3.1 *For each $n \in \mathbb{N}$, let $t_n \in (0, p_c)$ denote the least value satisfying the condition*

$$\sum_m \sigma_{n,m} t_n^n (1-t_n)^m = \sup_{p \in (0,p_c]} \sum_m \sigma_{n,m} p^n (1-p)^m. \tag{7}$$

That is, t_n is some point at or below the critical value at which the probability of observing an n-edged animal as the cluster to which the origin belongs is maximal. It is reasonable to suppose that t_n is slightly less than p_c, and that the difference decays polynomially in n as n tends to infinity.

Definition 3.2 *Define* $\Omega_+^{(\lambda)} = \{\mu \geq 0 : \liminf_{n\to\infty} (p_c - t_n)/n^{-\mu} = \infty\}$*, and* $\Omega_-^{(\lambda)} = \{\mu \geq 0 : \limsup_{n\to\infty} (p_c - t_n)/n^{-\mu} = 0\}$*.*
If $\sup \Omega_-^{(\lambda)} = \inf \Omega_+^{(\lambda)}$*, then hypothesis* (λ) *is said to hold, and* λ *is defined to be equal to the common value.*

So, if hypothesis (λ) holds, then $p_c - t_n$ behaves like $n^{-\lambda}$, for large n. We remark that it would be consistent with the notion of a scaling window about criticality that the probability of observing the cluster $C(0)$ with n-edges achieves its maximum on the subcritical interval on a short plateau whose right-hand endpoint is the critical value. If this is the case, then t_n should lie at the left-hand endpoint of the plateau. To be confident that $p_c - t_n$ is of the same order as the length of this plateau, the definition of the quantities t_n could be changed, in such a way that a small and fixed constant multiplies the right-hand-side of (7). In this paper, any proof of a statement involving the exponent λ is valid if it is defined in terms of this altered version of the quantities t_n.

3.2 Hypothesis (ς)

This hypothesis is introduced to describe the behaviour of f for values of the argument just greater than α. Theorem 2.3 asserts that the value α is the greatest for which $\log f(\beta) = (\beta+1)\log(\beta+1) - \beta\log\beta$; the function g was introduced to describe how $\log f$ falls away from this function as β increases from α. Thus, we phrase hypothesis (ς) in terms of g.

Definition 3.3 *Set* $\Omega_-^{(\varsigma)} = \{\mu \geq 0 : \liminf_{\delta\to 0} (g(\alpha+\delta) - g(\alpha))/\delta^{\mu} = 0\}$*, and* $\Omega_+^{(\varsigma)} = \{\mu \geq 0 : \limsup_{\delta\to 0} (g(\alpha+\delta) - g(\alpha))/\delta^{\mu} = -\infty\}$*.*
If $\sup \Omega_-^{(\varsigma)} = \inf \Omega_+^{(\varsigma)}$*, then hypothesis* (ς) *is said to hold, and* ς *is defined to be equal to the common value.*

If hypothesis (ς) holds, then greater values of ς correspond to a smoother behaviour of f at α. For example, if ς exceeds N for $N \in \mathbb{N}$, then f is N-times differentiable at α.

3.3 Relation between λ and ς

Theorem 3.4 *Suppose that hypotheses* (ς) *and* (λ) *hold. If* $\lambda < 1/2$*, then* $\varsigma\lambda \geq 1$*.*

Proof: We prove the Theorem by contradiction, assuming that the two hypotheses hold, and that $\lambda < 1/2$, $\varsigma\lambda < 1$. We will arrive at the conclusion that the mean cluster size, given by $\sum_n n\mathbb{P}_p(|C(0)| = n)$, is bounded above, uniformly for $p \in (0, p_c)$. That this is not so is proved in [1]. Note that

$$\sup_{p\in(0,p_c)} \sum_n n\mathbb{P}_p(|C(0)| = n) \leq \sum_n n\mathbb{P}_{t_n}(|C(0)| = n).$$

We write

$$\mathbb{P}_{t_n}(|C(0)| = n) = \sum_m \sigma_{n,m} t_n^n (1 - t_n)^m, \tag{8}$$

and split the sum in (8). To do so, we use the following definition.

Definition 3.5 *For $n \in \mathbb{N}$, let α_n be given by $t_n = 1/(1+\alpha_n)$. For $G \in \mathbb{N}$, let $D_n (= D_n(G))$ denote the interval*

$$D_n = \Big(\alpha_n - G\big(\log(n)/n\big)^{1/2}, \alpha_n + G\big(\log(n)/n\big)^{1/2}\Big).$$

Now,

$$\sum_m \sigma_{n,m} t_n^n (1 - t_n)^m = C_1(n) + C_2(n) + C_3(n),$$

where the terms on the right-hand-side are given by

$$\begin{aligned} C_1(n) &= \sum_{m \in nD_n} \sigma_{n,m} t_n^n (1 - t_n)^m, \\ C_2(n) &= \sum_{m \in n\big((0,2(d-1)) - D_n\big)} \sigma_{n,m} t_n^n (1 - t_n)^m \end{aligned}$$

and

$$C_3(n) = \sum_{m \in \{2(d-1)n, \ldots, 2(d-1)n+2d\}} \sigma_{n,m} t_n^n (1 - t_n)^m.$$

Lemma 3.6 *The function ϕ specified in Definition 2.6 satisfies*

$$\phi\big(\alpha, \alpha + \gamma\big) = -\frac{\gamma^2}{2\alpha(\alpha+1)} + O(\gamma^3).$$

Proof: We compute

$$\begin{aligned} \phi(\alpha, \alpha+\gamma) &= -(\alpha+\gamma)\log(\alpha+\gamma) + (\alpha+1+\gamma)\log(\alpha+1+\gamma) \\ &\quad - (\alpha+1+\gamma)\log(\alpha+1) + (\alpha+\gamma)\log\alpha \\ &= -(\alpha+\gamma)\log(1+\gamma/\alpha) \\ &\quad + (\alpha+1+\gamma)\log(1+\gamma/(\alpha+1)) \\ &= -(\alpha+\gamma)[\gamma/\alpha - \gamma^2/2\alpha^2] \\ &\quad + (\alpha+1+\gamma)[\gamma/(\alpha+1) - \gamma^2/2(\alpha+1)^2] + O(\gamma^3) \\ &= -\gamma^2/[2\alpha(\alpha+1)] + O(\gamma^3), \end{aligned}$$

giving the result. □.
We have that

$$\sum_n C_2(n) = \sum_n \sum_{m \in n((\alpha,2(d-1))-D_n)} \Big(f_n(m/n) \frac{\alpha_n^{m/n}}{(1+\alpha_n)^{1+m/n}} \Big)^n$$
$$\leq L \sum_n n \sum_{m \in n((\alpha,2(d-1))-D_n)} \exp\big(n\phi_{\alpha_n,m/n}\big),$$

where the inequality is valid by virtue of Theorem 2.2 and the fact that $g \leq 1$. Lemma 3.6 implies that

$$\sum_{m \in n((\alpha,2(d-1))-D_n)} \exp\big(n\phi_{\alpha_n,m/n}\big) \leq (2(d-1)-\alpha)n^{-K},$$

where K may be chosen to be arbitrarily large by an appropriate choice of G. It is this consideration that determines the choice of G. The miscellaneous term C_3 is treated by Lemma 2.5. We find that the m-indexed summand in $C_3(n)$ is at most $r^n \exp\big(n\phi_{\alpha_n,m/n}\big)$: thus $C_3(n) \leq (2d+1)r^n$. Note that C_1 satisfies

$$C_1(n) = \sum_{m \in nD_n} \Big(f_n(m/n) \frac{\alpha_n^{m/n}}{(1+\alpha_n)^{1+m/n}} \Big)^n$$
$$\leq Ln \sum_{m \in nD_n} g(m/n)^n \exp(n\phi_{\alpha_n,m/n}),$$

where the inequality is a consequence of Theorems 2.2 and 2.3. The fact that the function ϕ is nowhere positive implies that

$$C_1(n) \leq Ln \sum_{m \in nD_n} g(m/n)^n.$$

Hence the desired contradiction will be reached if we can show that

$$\sum_n n \sum_{m \in nD_n} g(m/n)^n \tag{9}$$

is finite. As such, the proof is completed by the following lemma.

Lemma 3.7 *Assume hypotheses* (ς) *and* (λ). *Suppose that* $\lambda < 1/2$ *and that* $\varsigma\lambda < 1$. *Then, for* $\epsilon \in (0, 1-\varsigma\lambda)$ *and* $n \in \mathbb{N}$ *sufficiently large,*

$$\sum_{m \in nD_n} g(m/n)^n \leq \exp\Big(-n^{1-\varsigma\lambda-\epsilon}\Big). \tag{10}$$

Proof: Let $\varsigma^* > \varsigma$ and $\lambda^* > \lambda$ be such that $\lambda^* < 1/2$ and $\varsigma^*\lambda^* < \varsigma\lambda + \epsilon$. By hypothesis ($\varsigma$), there exists $\epsilon' > 0$ such that

$$\delta \in (0,\epsilon') \text{ implies } g(\alpha+\delta) - g(\alpha) < -\delta^{\varsigma^*}.$$

From Theorems 2.2 and 2.3, it follows that $\sup_{\beta \in [\alpha+\epsilon', 2(d-1)]} g(\beta) < 1$, which shows that the contribution to the sum in (10) from all those terms indexed by m for which $m/n > \alpha + \epsilon'$ is exponentially decaying in n. Thus, we may assume that there exists N_1 such that for $n \geq N_1$, if $m \in D_n^*$ then $m/n - \alpha < \epsilon'$. Note that, by hypothesis (λ), $\alpha_n - \alpha \geq n^{-\lambda^*}$ for sufficiently large. Hence, there exists N_2 such that, for $n \geq N_2$,

$$\alpha_n - G(\log(n)/n)^{1/2} \geq \alpha + n^{-\lambda^*} - G(\log(n)/n)^{1/2} \geq \alpha + (1/2)n^{-\lambda^*}.$$

For $n \geq \max\{N_1, N_2\}$ and $m \in nD_n^*$,

$$\begin{aligned} g(m/n) &\leq 1 - (m/n - \alpha)^{\varsigma^*} \\ &\leq 1 - (\alpha_n - G(\log(n)/n)^{1/2} - \alpha)^{\varsigma^*} \\ &\leq 1 - ((1/2)n^{-\lambda^*})^{\varsigma^*}. \end{aligned}$$

So, for $n \geq max(N_1, N_2)$,

$$\sum_{m \in nD_n^*} g(m/n)^n \leq (2G(n\log(n))^{1/2})[1 - C'n^{-\lambda^*\varsigma^*}]^n,$$

for some constant $C' > 0$. There exists $g \in (0,1)$, such that for large n,

$$[1 - C'n^{-\lambda^*\varsigma^*}]^n \leq g^{n^{1-\lambda^*\varsigma^*}}.$$

This implies that

$$\sum_{m \in nD_n^*} g(m/n)^n \leq h^{n^{1-\lambda^*\varsigma^*}} \text{ for large } n \text{ and } h \in (g,1).$$

From $\varsigma^*\lambda^* < \varsigma\lambda + \epsilon$, we find that

$$\sum_{m \in nD_n^*} g(m/n)^n \leq \exp -n^{1-\varsigma\lambda-\epsilon} \text{ for large } n,$$

as required. □

Remark: It may be similarly shown that $\varsigma < 2$ or $\lambda > 1/2$ are sufficient conditions for the absence of infinite open clusters at the critical value p_c. Except for some borderline cases, this leaves the region of the (λ, ς)-plane specified by $\varsigma > 2$ and $\varsigma\lambda > 1$. Here, such a sufficient condition may be phrased in terms of the extent to which $f_n(\beta)$ underestimates $f(\beta)$ for a restricted range of values of β, namely those that are roughly of the size $\alpha + n^{\lambda}$. In this regard, bounds on the entropic exponent are relevant (see [6]). These results are stated and proved as [4, Theorems 4.1 and 4.3].

4 Scaling law

In this section, we examine the exponential decay rate in n for the probability of the event $\{C(0) = n\}$ for p slightly less than p_c by our combinatorial approach.

In doing so, we relate the quantity ς to the exponent for correlation size, and see how the scaling behaviour for the typical surface-area-to-volume ratio of unusually large clusters in the marginally subcritical regime depends on the value of ς.

Definition 4.1 *Let $q : (0, p_c) \to [0, \infty)$ be given by*

$$q(p) = \lim_{n \to \infty} \frac{-\log \mathbb{P}_p(|C(0)| = n)}{n}.$$

Define $\Omega_+^{(\varrho)} = \{\gamma \geq 0 : \liminf_{p \uparrow p_c} \frac{q(p)}{(p_c - p)^\gamma} = \infty\}$ and
$\Omega_-^{(\varrho)} = \{\gamma \geq 0 : \limsup_{p \uparrow p_c} \frac{q(p)}{(p_c - p)^\gamma} = 0\}$. If $\sup \Omega_-^{(\varrho)} = \inf \Omega_+^{(\varrho)}$, then hypothesis (ϱ) is said to hold, and ϱ is defined to be equal to the common value.

Remark: The existence of q follows from a standard subadditivity argument. The quantity ϱ might reasonably be called the exponent for 'correlation size'.

Theorem 4.2 *There exists $\delta' > 0$ and $p_0 \in (0, p_c)$ such that $p \in (p_0, p_c)$ implies that $q(p)$ is given by*

$$q(p) = \inf_{\beta \in (\alpha, \alpha + \delta')} \Big(-\log g(\beta) - \phi\big(1/p - 1, \beta\big)\Big),$$

where the function $\phi : (0, \infty)^2 \to \mathbb{R}$ was specified in Definition 2.6. Recall also that $\alpha = 1/p_c - 1$.

Proof: We may write $\mathbb{P}_p(|C(0)| = n) = H_1 + H_2$, where

$$\begin{aligned} H_1 &= \sum_{m=0}^{2(d-1)n-1} \sigma_{n,m} p^n (1-p)^m \\ \text{and } H_2 &= \sum_{m=2(d-1)n}^{2(d-1)n+d} \sigma_{n,m} p^n (1-p)^m. \end{aligned} \tag{11}$$

Note that, by Lemma 2.5, there exists $r \in (0, 1)$ such that, for all $p \in (0, 1)$, $H_2 \leq (2d + 1) r^n$. To treat the quantity H_1, note that

$$H_1 = \sum_{m=0}^{2(d-1)n-1} a_n(m/n) \exp\Big(n\big(\log g(m/n) + \phi(1/p - 1, m/n)\big)\Big), \tag{12}$$

where the quantity $a_n(\beta)$ for $\beta \in \big(0, 2(d-1)\big)$ is given by the relation

$$\sigma_{n, \lfloor \beta n \rfloor} = a_n(\beta) f(\beta)^n.$$

Note that, by the third part of Theorem 2.2, $a_n(\beta) \leq Ln$ for such values of β. From this bound, and (12), it follows that

$$H_1 \leq 2(d-1) L n^2 \exp\big(-n\gamma_p\big),$$

where γ_p is given by

$$\gamma_p = \inf_{\beta \in [0,2(d-1))} \Big(-\log g(\beta) - \phi\big(1/p - 1, \beta\big)\Big). \tag{13}$$

From Theorem 2.3, we see that the function $\beta \to -\log g(\beta) - \phi(1/p-1,\beta)$ is continuous on $(0,2(d-1))$. Allied with the fact that for $\beta \in (0,2(d-1))$, $\limsup \frac{-\log a_n(\beta)}{n} \leq 0$, it follows that, for $\epsilon > 0$, and for n sufficiently large, $H_1 \geq \exp\big(-n(\gamma_p + \epsilon)\big)$. Hence,

$$\lim_{n\to\infty} \frac{-\log H_1}{n} = \gamma_p. \tag{14}$$

We now make the claim that there exists $p_0 \in (0,p_c)$ and $\delta' > 0$ such that, for $p \in (p_0, p_c)$, γ_p is given by

$$\gamma_p = \inf_{\beta \in [\alpha, \alpha+\delta']} \Big(-\log g(\beta) - \phi(1/p-1,\beta)\Big). \tag{15}$$

Note that from

$$-\frac{d}{d\beta}\phi(1/p-1,\beta) = \log\big(1 - 1/(1+\beta)\big) - \log(1-p),$$

and $p_c = 1/(1+\alpha)$, it follows that the expression $-\phi(1/p-1,\beta)$ is decreasing in β on $[0,\alpha]$, for $p \in (0,p_c)$. Recalling from Theorem 2.3 that $g(\beta) = 1$ for each $\beta \in [0,\alpha]$, it follows that, for $p \in (0,p_c)$,

$$-\log g(\alpha) - \phi(1/p-1,\alpha) \leq -\log g(\beta) - \phi(1/p-1,\beta). \tag{16}$$

The fourth part of Theorem 2.2 implies that $g : \big(0, 2(d-1)\big) \to [0,\infty)$ is continuous. Using this fact and Theorem 2.3, we may, for a given $\epsilon > 0$, find $\delta' > 0$ such that

$$\beta \in \big(\alpha + \delta', 2(d-1)\big) \text{ implies that } g(\beta) < 1 - \epsilon. \tag{17}$$

Using $g(\alpha) = 1$, we may similarly choose $\delta \in (0,\delta')$ such that

$$\beta \in (\alpha, \alpha+\delta) \text{ implies that } g(\beta) > 1 - \epsilon/2.$$

Let $p_0 = 1/(\alpha + 1 + \delta)$. For $p \in (p_0, p_c)$,

$$-\log g(1/p-1) - \phi(1/p-1, 1/p-1) < -\log(1-\epsilon/2), \tag{18}$$

the second term on the left-hand-side being zero. From (17), we have that, for $\beta \in \big(\alpha+\delta', 2(d-1)\big)$,

$$-\log g(\beta) - \phi(1/p-1,\beta) \geq -\log(1-\epsilon), \tag{19}$$

since ϕ is nowhere positive. We learn from (13), (16), (18) and (19) that, for $p \in (p_0,p_c)$, (15) holds, as claimed. Note that

$$\liminf_{n\to\infty} \frac{-\log H_2}{n} \geq -\log r,$$

whereas, for $p \in (p_0, p_c)$, it follows from (14), (15) and (18) that

$$\limsup_{n\to\infty} \frac{-\log H_1}{n} < -\log(1-\epsilon/2).$$

By choosing $\epsilon < 2(1-r)$, we obtain for such values of p,

$$\lim_{n\to\infty} \frac{-\log \mathbb{P}_p(|C(0)| = n)}{n} = \inf_{\beta\in[\alpha,\alpha+\delta']} \Big(-\log g(\beta) - \phi(1/p-1,\beta)\Big),$$

as required. $\square$
Theorem 4.2 allows us to deduce a scaling law that relates the combinatorially defined exponent ς to one which is defined directly from the percolation model.

Theorem 4.3 *Assume hypothesis* (ς).

- *Suppose that* $\varsigma \in (1,2)$. *Then hypothesis* (ϱ) *holds and* $\varrho = 2$.
- *Suppose that* $\varsigma \in (2,\infty)$. *Then hypothesis* (ϱ) *holds and* $\varrho = \varsigma$.

Proof: Suppose that $\varsigma \in (1,2)$. Choose $\epsilon > 0$ so that $1 < \varsigma - \epsilon < \varsigma + \epsilon < 2$. There exists constants $C_1, C_2 > 0$ such that, for $p \in (p_0, p_c)$ and $\beta \in (\alpha, \alpha+\delta')$,

$$\begin{aligned}(\beta-\alpha)^{\varsigma+\epsilon} + C_1\big(\beta-(1/p-1)\big)^2 &\leq -\log g(\beta) + -\phi\big(1/p-1,\beta\big) \\ &\leq (\beta-\alpha)^{\varsigma-\epsilon} + C_2\big(\beta-(1/p-1)\big)^2.\end{aligned} \tag{20}$$

Applying Theorem 4.2, we find that

$$(\beta_p-\alpha)^{\varsigma+\epsilon} + C_1\big(\beta_p-(1/p-1)\big)^2 \leq q(p), \tag{21}$$

where $\beta_p \in [\alpha, \alpha+\delta']$ denotes a value at which the infimum in the interval $[\alpha, \alpha+\delta']$ of the first term in (20) is attained. Let $y_p = 1/p - 1 - \alpha$, and let σ_p satisfy $\beta_p = \alpha + y_p^{\sigma_p}$. Then β_p and σ_p satisfy

$$\begin{aligned}(\varsigma+\epsilon)(\beta_p-\alpha)^{\varsigma+\epsilon-1} &= -2C_1\big(\beta_p-(1/p-1)\big) \\ (\varsigma+\epsilon)y_p^{\sigma_p(\varsigma+\epsilon-1)} &= 2C_1\big(y_p-y_p^{\sigma_p}\big)\end{aligned} \tag{22}$$

Since $\beta_p \leq 1/p-1$, $\sigma_p \geq 1$. From this and (22) follows $\liminf_{p\uparrow p_c}\sigma_p \geq 1/(\varsigma+\epsilon-1)$. Applying (22) again, we deduce that $\lim_{p\uparrow p_c}\sigma_p = 1/(\varsigma+\epsilon-1)$. Substituting σ_p in (21) yields

$$y_p^{\sigma_p(\varsigma+\epsilon)} + C_1\big(y_p - y_p^{\sigma_p}\big)^2 \leq q(p).$$

The facts that $\lim_{p\uparrow}\sigma_p > 1$ and $\lim_{p\uparrow}\sigma_p(\varsigma+\epsilon) = (\varsigma+\epsilon)/(\varsigma+\epsilon-1) > 2$ imply that, for a small constant c, $c(p_c-p)^2 \leq q(p)$ for values of p just less than p_c. A similar analysis in which $q(p)$ is bounded below by the infimum on the interval $[\alpha, \alpha+\delta']$ of the third expression in (20) implies that for large C, $q(p) \leq C(p_c-p)^2$, in a similar range of values of p. Thus hypothesis (ϱ) holds, and $\varrho = 2$.

In the case where $\varsigma > 2$, let $\epsilon > 0$ be such that $\varsigma > 2 + \epsilon$. Defining σ'_p by $\beta_p = 1/p - 1 - y_p^{\sigma'_p}$, we find that

$$(\varsigma + \epsilon)\big(y_p - y_p^{\sigma'_p}\big)^{\varsigma+\epsilon-1} = 2C_1 y_p^{\sigma'_p}. \tag{23}$$

Note that $\beta_p \geq \alpha$ implies that $\sigma'_p \geq 1$. From (23), it follows that $\liminf_{p\uparrow p_c} \sigma'_p \geq \varsigma + \epsilon - 1$. Since $\varsigma + \epsilon - 1 > 1$, applying (23) again shows that the limit $\lim_{p\uparrow p_c} \sigma'_p$ exists and infact equals $\varsigma + \epsilon - 1$. Substituting σ'_p in (20) yields

$$\big(y_p - y_p^{\sigma'_p}\big)^{\varsigma+\epsilon} + C_1 y_p^{2\sigma'_p} \leq q(p).$$

The fact that $\liminf_{p\uparrow p_c} \sigma'_p > 1$ implies that $c(p_c - p)^{\varsigma+\epsilon} \leq q(p)$ for values of p just less than p_c. Making use of the inequality $\varsigma > 2 + \epsilon$ in considering the infimum of the third term appearing in (20) yields in this case $q(p) \leq C(p_c - p)^{\varsigma-\epsilon}$ for similar values of p. Thus, since ϵ may be chosen to be arbitrarily small, we find that, if $\varsigma > 2$, then hypothesis (ϱ) holds, and that $\varrho = \varsigma$. $\square$

Acknowledgements Financial support was provided by a Domus Graduate Scholarship (Competition B) of Merton College, Oxford. I would like to thank Terry Lyons for stimulating and helpful discussions. I thank John Cardy, Amir Dembo and Mathew Penrose for their helpful comments.

References

[1] Michael Aizenman and Charles M. Newman. Tree graph inequalities and critical behavior in percolation models. *J. Statist. Phys* **36 (1-2):** 107 - 143, 1984.

[2] F. Delyon. Taille, forme et nombre des amas dans les problemes de percolation. These de 3eme cycle, Universite Pierre et Marie Curie, Paris, 1980.

[3] S. Flesia, D.S. Gaunt, C.E. Soteros and S.G.Whittington. Statistics of collapsing lattice animals. *J. Phys A* **27(17)**:5831-5846, 1994.

[4] Alan Hammond. A lattice animal approach to percolation. Available at www.arxiv.org/math.PR/0402026 .

[5] Harry Kesten and Yu Zhang. The probability of a large finite cluster in supercritical Bernoulli percolation. *Ann. Probab.*, **18 (2)**:537-555, 1990.

[6] Neal Madras. A rigorous bound on the critical exponent for the number of lattice tress, animals, and polygons. *J. Statist. Phys.*, **78(3-4)**:681-699, 1995.