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# Traces, Fixed Points and Quantization of Symmetric Spaces

Alban Jago

Supervisor:
Prof. Pierre Bieliavsky

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Examination commitee:

Prof. Victor Gayral (Université de Reims Champagne-Ardenne, France)
Prof. Simone Gutt (Université Libre de Bruxelles)
Prof. Florin Radulescu (Università degli studi di Roma “Tor Vergata”, Italy)
Prof. Jean Van Schaftingen (UCLouvain)
Prof. Pedro Vaz (UCLouvain)
Prof. Michel Willem (UCLouvain)

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# Remerciements

> *" L'univers (que d'autres appellent la Bibliothèque) se compose d'un nombre indéfini, et peut-être infini, de galeries hexagonales [...] Chacun des murs de chaque hexagone porte cinq étagères ; chaque étagère comprend trente-deux livres, tous de même format ; chaque livre a quatre cent dix pages ; chaque page, quarante lignes, et chaque ligne, environ quatre-vingts caractères noirs. [...] un bibliothécaire de génie [...] déduisit que la Bibliothèque est totale, et que ses étagères consignent toutes les combinaisons possibles des vingt et quelques symboles orthographiques (nombre, quoique très vaste, non infini), c'est-à-dire tout ce qu'il est possible d'exprimer, dans toutes les langues."*
>
> Jorge Luis Borges – *La Bibliothèque de Babel* [1]

En nous promenant au hasard des couloirs de la Bibliothèque, Borges nous rappelle qu'après tout, un texte n'est jamais formé que par un agencement de lettres placées les unes après les autres. Il en est de même pour cette thèse, dont un exemplaire se trouve d'ailleurs déjà dans une des étagères d'une des galeries hexagonales. Mais si on pousse la réflexion, ce qui distingue nos livres – ceux qui constituent notre littérature – des innombrables volumes de la Bibliothèque composés de permutations aléatoires de symboles, c'est le travail qui fut déployé pour choisir minutieusement l'ordre de chacun de leurs symboles, afin de leur donner un sens cohérent et intelligible. L'énergie qui a façonné ce texte, si elle est certes pour partie le fruit des tours et détours de la pensée de son auteur, ne serait rien sans toutes les relations que ce dernier a eues avec son entourage. Entre toutes ces lignes, transpire une véritable aventure humaine, dont l'aboutissement doit au moins tout autant à celui qui l'a vécue, qu'à celles et ceux qu'il a croisés, de près ou de loin, ici ou ailleurs, en chair et en os ou au travers de livres, ...

[1] Traduit de l'espagnol par Ibarra.

S'il faut un doctorant pour mener à bien une thèse de doctorat, il faut bien évidemment un promoteur. En sortant de mes études de physique, j'ai eu le plaisir de faire mon mémoire de mathématiques sous la direction de Pierre Bieliavsky, et de découvrir un domaine à la frontière entre les deux disciplines. Je le remercie chaleureusement d'avoir accepté de poursuivre l'aventure en devenant mon promoteur de thèse. Son enthousiasme débordant fut d'une aide très précieuse, tant lorsqu'il félicitait le moindre petit pas en avant, que lorsqu'il rallumait l'espoir quand une idée s'avérait ne pas fonctionner. J'ai également beaucoup appris de son souci de sans cesse tisser des liens entre des domaines de recherche très divers, en étant capable de traduire le propos des autres en son propre langage. Ce fut l'occasion d'explorer de nombreuses voies sans rester confiné à un cadre bien précis, et de goûter à l'un des grands plaisir des mathématiques, qui est de connecter des sujets à priori sans rapport apparent. Mais la personnalité de Pierre ne s'arrête pas aux mathématiques, et ce fut un plaisir de pouvoir échanger avec lui sur bien d'autres sujets entre deux équations, sur une table asiatique, dans un jardin au pied de la forêt, ou en l'accompagnant en conférence.

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par leurs cours passionnants et inspirants, qui resteront toujours gravés dans ma mémoire. Bien évidemment, je ne pourrais oublier l'équipe administrative. En particulier, merci à Carine Baras, Cathy Brichard et Martine Furnémont pour leur aide précieuse en d'innombrables circonstances. J'ai également beaucoup apprécié de pouvoir découvrir les rouages de l'université en m'impliquant au sein du Corps Scientifique. Ce fut l'occasion de rencontrer, à tous niveaux, des personnes ouvertes et dévouées, n'hésitant pas à investir de leur temps à faire fonctionner et à améliorer ces grosses machines que sont notre institution et la Science en général.

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# Conventions and Notations

## Conventions

In this text, a manifold is always taken to be real, Hausdorff, second-countable, smooth and without boundary. Vector bundles are also always considered to be smooth. Unless stated otherwise, a vector bundle means a complex vector bundle. Hilbert spaces are always considered to be separable, and their inner product are taken to be linear in the first argument.

## Notations

In the following list:

- $M$ and $N$ are two manifolds
- $E \to M$ is a (real or complex) vector bundle over $M$
- $V$ is a (real or complex) vector space
- $f : M \to N$ is a map
- $\mathcal{H}$ is a (separable) Hilbert space

Here is a list of notations that will be commonly used throughout the text:

- $\{\star\}$: the set containing one point
- $\mathcal{B}(V)$: set of bases of $V$
- $\rangle X_1, \ldots, X_n \langle$: vector subspace generated by $X_1, \ldots, X_n \in V$
- $GL(V)$: group of invertible linear transformations of $V$
- $V^*$: dual vector space of $V$
- $V'$: topological dual of a topological vector space $V$

- $|V|^\alpha$: vector space of $\alpha-$densities of a real vector space $V$
- $T(M)$: tangent bundle of $M$
- $N(Z)$: normal bundle of a submanifold $Z \subset M$
- $|TM|^\alpha$: vector bundle of $\alpha-$densities on $M$
- $|E|^\alpha$: vector bundle of $\alpha-$densities of a real vector bundle $E$ over $M$
- $\Lambda^k(M)$: vector bundle of differential $k-$forms on $M$
- $E^*$: dual vector bundle of $E$
- $E^\vee$: functional bundle of a complex vector bundle $E$
- $E_{|Z}$: restriction of $E$ to a submanifold $Z \subset M$ (i.e. the pullback of $E$ on $Z$)
- $\mathcal{C}^0(M)$: continuous functions on $M$
- $\mathcal{C}^\infty(M)$: smooth functions on $M$
- $\mathcal{C}^0_c(M)$: compactly supported continuous functions on $M$
- $\mathcal{C}^\infty_c(M)$: compactly supported smooth functions on $M$
- $\Gamma^0(M,E)$: continuous sections of $E$
- $\Gamma^\infty(M,E)$: smooth sections of $E$
- $\Gamma^0_c(M,E)$: compactly supported continuous sections of $E$
- $\Gamma^\infty_c(M,E)$: compactly supported smooth sections of $E$
- $\mathcal{D}(M)$: topological vector space of compactly supported smooth functions on $M$
- $\mathcal{E}(M)$: topological vector space of smooth functions on $M$
- $\mathcal{D}(M,E)$: topological vector space of compactly supported smooth sections of $E$
- $\mathcal{E}(M,E)$: topological vector space of smooth sections of $E$
- $\mathcal{D}'(M,E)$: generalized sections of $E$
- $\mathcal{E}'(M,E)$: compactly supported generalized sections of $E$
- $\mathrm{Diff}(M)$: group of diffeomorphisms of $M$
- $\partial^\alpha f$: partial derivative of $f$ corresponding to the multi-index $\alpha$

- $f_{*_x} : T_xM \to T_{f(x)}M$: differential at $x \in M$ of a smooth map $f : M \to N$
- $f_{|U}$: restriction of a map $f : M \to N$ to a subset $U \subset M$
- $\mathrm{supp}(u)$: support of a generalized section or a section $u$ of $E$
- $u_{|U}$: restriction of a generalized section $u$ to a open subset $U \subset M$
- $\langle u, \rho \rangle$: evaluation of a generalized section $u$ on a section $\rho$
- $\langle \xi, X \rangle$: evaluation of a linear form $\xi \in V^*$ on a vector $X \in V$
- $G \times_\chi V$: associated vector bundle over $M$ corresponding to a character $\chi$ of a Lie group $B$ on a vector space $V$ and a $B-$principal bundle $G \to M$
- $\mathcal{L}(\mathcal{H})$: the space of continuous linear operators on $\mathcal{H}$
- $\mathcal{U}(\mathcal{H})$: the space of unitary linear operators on $\mathcal{H}$
- $\mathcal{L}^2(\mathcal{H})$: the space of Hilbert-Schmidt operators on $\mathcal{H}$
- $\mathcal{L}^1(\mathcal{H})$: the space of trace-class operators on $\mathcal{H}$

# Introduction

From the very beginning, mathematics and physics have been deeply entangled, and many examples in the history of science show that both fields have benefited from the developments of each other. Without opening the question of “Who has influenced who?”, which is usually difficult to settle, let us point out, for instance, how physical problems have been a constant inspiration for the theory of differential equations. On the other hand, the mathematical development of non-Euclidean geometries has been crucial to provide a proper setting for Einstein’s theory of general relativity. Another example of the interplay between mathematics and physics is given by quantum mechanics and noncommutative geometry, which we suggest to explore a bit deeper.

## Quantization and noncommutativity

In classical physics, we consider that the observables of a physical system (that is, the physical quantities that can be measured, such as the energy or the position of a particle) correspond to the real valued functions on that system. The possible measurement outcomes of an experiment are then given by all the possible values of those functions. However it has been observed that this setting fails to describe the physics at microscopic scales. For instance, the energy spectrum of an atom might take only discrete values, a behaviour which cannot be reproduced by continuous functions on continuous spaces. Heisenberg has been one of the first to realize that the cure to this problem was to describe observables by noncommutative objects, such as matrices, instead of real valued functions. The possible measurement outcomes of an experiment are then described by the spectrum of those objects, which can now be discrete. More generally, the framework of quantum mechanics is the one of linear operators on Hilbert spaces. To a physical system is associated a Hilbert space, and to each observable on that system, a linear operator on that Hilbert space. Successive measurements then correspond to the composition of the corresponding operators. The observables are thus described by an algebra which is not commutative anymore (unlike the algebra of functions). As we will see later on, the idea of describing a system by a noncommutative algebra has lead to a change of paradigm in several fields, including geometry.

Naturally associated to this mathematical description of the microscopic world is the question of understanding how to pass from the quantum world to the classical one. Although this is a very subtle and still unsettled question, it has been suggested that, as some scale in a quantum system growths – like the number of particles, or the action of the system –, that system should tend to some classical one. However, since we obviously are more familiar with the classical world than the quantum one, a more practical approach is to work the other way around, and to build a quantum system starting from a classical one, in such a way that the latter is a limit of the former. This is known as quantization, and has lead to many different quantization programmes, which try to make the above idea precise.

For instance, let us consider the free particle on the real line. As a classical system, it is described by the phase space $\mathbb{R}^2 = \{(q,p)\}$ endowed with the symplectic form $\omega := dq \wedge dp$. The set of observables corresponds to the (smooth) functions on $\mathbb{R}^2$, which carries an additional structure given by the Poisson bracket $\{\cdot,\cdot\}$ corresponding to $\omega$. The celebrated Weyl quantization gives a way to associate a quantum system to that classical one. It was introduced by Weyl [Wey27] and has been extensively studied since then, to finally evolve into its modern formulation. Let us consider the Hilbert space $L^2(\mathbb{R})$ of square-integrable functions on $\mathbb{R}$, and fix some real number $\theta \neq 0$. To any observable $f \in \mathcal{S}(\mathbb{R}^2)$ ($\mathcal{S}(\mathbb{R}^2)$ denotes the Schwartz space, i.e. the space of rapidly decreasing smooth functions), we associate a bounded linear operator on $L^2(\mathbb{R})$ defined, for all $\varphi \in L^2(\mathbb{R})$, by

$$(\Omega_\theta(f)\varphi)(q_0) = \frac{1}{2\pi\theta} \int_{\mathbb{R}^2} e^{\frac{i}{\theta}(q_0-q)p} f\left(\frac{q_0+q}{2}, p\right) \varphi(q)\ dq\, dp. \tag{1}$$

That way, we get a so-called *quantization map*:

$$\Omega_\theta : \mathcal{S}(\mathbb{R}^2) \to \mathcal{L}\left(L^2(\mathbb{R})\right)$$

which associates to a classical observable (a function), a quantum observable, that is, a bounded operator on a Hilbert space. The function $f$ is called the *symbol* of the Weyl operator $\Omega_\theta(f)$. An important point is that the composition of $\Omega_\theta(f_1)$ and $\Omega_\theta(f_2)$ is again a Weyl operator, which means that the quantized observables form an algebra. Indeed, we have that

$$\Omega_\theta(f_1) \circ \Omega_\theta(f_2) =: \Omega_\theta(f_1 \star_\theta f_2), \tag{2}$$

where the function $f_1 \star_\theta f_2$ is given by an integral formula

$$(f_1 \star_\theta f_2)(x) = \frac{1}{2\pi\theta} \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{\frac{2i}{\theta}(\omega(x,y)+\omega(y,z)+\omega(z,x))} f_1(y)\, f_2(z)\ dy\, dz, \tag{3}$$

which is known as the *Weyl product*. Since the composition of operators is associative but not commutative, $\star_\theta$ gives a noncommutative associative product on $\mathcal{S}(\mathbb{R}^2)$. Furthermore, if we Taylor expand (3) with respect to $\theta$, we obtain

$$\begin{aligned} f_1 \star_\theta f_2 = f_1 f_2 &+ \frac{\theta}{2i}\{f_1, f_2\} \\ &+ \sum_{k=2}^{+\infty} \frac{1}{k!}\left(\frac{\theta}{2i}\right)^k \sum_{\substack{i_1\dots i_k=1\\ j_1\dots j_k=1}}^{2} \omega^{i_1 j_1}\dots\omega^{i_k j_k}\,\partial_{i_1\dots i_k} f_1\,\partial_{j_1\dots j_k} f_2, \end{aligned} \tag{4}$$

where $\omega^{ij}$ are the components of the inverse matrix of $\omega$. This suggests to see the product $\star_\theta$ as a deformation of the usual product of functions in the direction of the Poisson bracket, in the sense that

$$f_1 \star_\theta f_2 \xrightarrow{\theta\to 0} f_1 f_2 \qquad \text{and} \qquad \frac{1}{i\theta}(f_1 \star_\theta f_2 - f_2 \star_\theta f_1) \xrightarrow{\theta\to 0} \{f_1, f_2\}.$$

Together with the fact that it might seem unnatural that the objects used in classical physics – functions – and in quantum physics – operators – are so radically different in nature, this has been a motivation for the development of *deformation quantization* initiated by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [BFF$^+$78a, BFF$^+$78b]. Quoting them, they *"suggest that quantization be understood as a deformation of the structure of the algebra of classical observables, rather than as a radical change in the nature of the observables"*. In that spirit and in analogy with (4), a deformation quantization of a Poisson manifold $(M, \{\cdot,\cdot\})$ is (roughly) defined to be a map

$$\star : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)[\![\theta]\!],$$

where $\mathcal{C}^\infty(M)[\![\theta]\!]$ denotes the formal series in $\theta$ with coefficients in $\mathcal{C}^\infty(M)$, such that

$$f_1 \star f_2 = f_1 f_2 + \sum_{k=1}^{\infty} \theta^k\, c_k(f_1, f_2), \tag{5}$$

with the $c_k$ being bidifferential operators satisfying $c_1(f_1, f_2) - c_1(f_2, f_1) = \{f_1, f_2\}$ and an additional condition corresponding to formal associativity. Let us emphasize that (5) is only a formal expression: there is no requirement on the convergence of the series with respect to $\theta$. Such a $\star$ is called a *formal star-product* on $(M, \{\cdot,\cdot\})$.

The subject of deformation quantization has by far exceeded the realm of physics and quantum mechanics. From a mathematical point of view, the natural question to know whether there exists formal star-products on a given Poisson manifold has been gradually answered. The first existence theorems concerned symplectic manifolds and were given by De Wilde and Lecomte [DWL83], Gutt [Gut83], Omori, Maeda and Yoshioka [OMY91] and Fedosov [Fed94]. It has culminated with the work of Kontsevich [Kon03], from which the existence and a complete classification for arbitrary Poisson manifolds follows. It is interesting to mention that, after bringing many important contributions to mathematics, deformation quantization is now again increasingly used in contemporary physics, for instance in formulating quantum field theories. See [Wal16] for a recent review.

## Non-formal star-products

The idea of describing a system by a noncommutative algebra has also lead to the development of noncommutative geometry, whose origin lies in the correspondence between geometrical spaces and commutative algebras. More precisely, the commutative version of the theorem of Gelfand and Naimark [GN43] establishes an equivalence between the category of locally compact Hausdorff spaces and the category of commutative $C^*-$algebras. In analogy with the way quantum mechanics generalizes the notion of a physical system as being described by noncommutative operators, the latter correspondence suggests to interpret a noncommutative $C^*-$algebra as the data defining a noncommutative topological space. This kind of move can be done as soon as we have some duality between a category of geometrical spaces, and a category of commutative algebraic objects. For instance, Connes [Con95] has realized that many concepts of differential geometry – such as the notion of a Riemannian metric – can be expressed in an algebraic way, without referring to the underlying space. This allows to make sense of these notions also in the noncommutative setting, which leads to noncommutative differential geometry.

However, these constructions usually involve operator algebras, hence topological algebras. In the setting of deformation quantization, it implies that formal star-products are not the end of the story since, being only formal series in the parameter of deformation, they do not carry a satisfactory topological structure. This has lead to search for *non-formal* star-products, where $f_1 \star_\theta f_2$ is an actual function, at least for small real values of the parameter $\theta$. Let us notice that, contrary to the case of formal star-products, the non-formal setting is still under heavy development and far from being well understood. A systematic study was initiated by Rieffel [Rie89]. He has suggested a definition of what should be a continuous non-formal deformation of an algebra of functions, by defining the notion of *strict deformation quantization.* However, as Rieffel notices in [Rie90], the requirements of his definition are very tight, leading to very few known examples. As a consequence, several different – sometimes competing – definitions have arisen since then (see for instance [Lan93]), and there seems to be no consensus yet on which one should be taken as a general framework. Despite of this, applications of non-formal deformation quantization have already flourished in other fields. For instance, let us mention the work of Lechner and co-workers on constructing quantum field theories by using Rieffel's techniques [BLS11].

The present work takes part in this ongoing exploration into the world of non-formal star-products, and we will now introduce the specific questions that are addressed here. Let us recall that the Weyl product provides an example of a non-formal star-product since $f_1 \star_\theta f_2$, given by (3), is indeed a genuine function. Guided by this formula[2], if $M$ is a manifold, we might search for a star-product on $M$ of the form

$$(f_1 \star_\theta f_2)(x) = \int_{M\times M} K_\theta(x,y,z)\, f_1(y)\, f_2(z)\; d_M(y)\, d_M(z), \tag{6}$$

for some function $K_\theta(x,y,z)$ which is called the *three-point kernel* of the star product, and some measure $d_M$ on $M$.[3] In order for the star-product to be associative, the function $K_\theta(x,y,z)$ must satisfy some specific relations that make it difficult to be built out of the box. Also, if we consider some symmetries of $M$, we would like $\star_\theta$ to be compatible with those symmetries. More precisely, suppose that a Lie group $G$ acts on $M$. We require the star-product to be $G-$equivariant in the sense that, for all $g \in G$,

$$({}^g f_1) \star_\theta ({}^g f_2) = {}^g(f_1 \star_\theta f_2), \tag{7}$$

where $({}^g f)(x) := f\left(g^{-1}\cdot x\right)$. Notice that the Weyl product is indeed equivariant under the group of transformations of $\mathbb{R}^2$ that leave the symplectic form $\omega$ invariant. In [Wei94], Weinstein gives some heuristic arguments to suggest an interesting ansatz for the function $K_\theta(x,y,z)$, which takes the form of a fixed point formula. Let us first present it in the case of the Weyl product. To this aim, we need to exhibit an additional structure on $\mathbb{R}^2$, which turns out to be central in the construction. To each point $x \in \mathbb{R}^2$, we can associate a transformation of $\mathbb{R}^2$: the central symmetry around $x$, given by

$$s_x(y) = 2x - y.$$

Then, for each triple of points $(x,y,z) \in \mathbb{R}^{3\times 2}$, the transformation $s_z \circ s_y \circ s_x$ admits a unique fixed point $p$, which is given by $p = x-y+z$. Corresponding to that fixed point, there is a so-called *double triangle*, the triangle which admits $x$, $y$ and $z$ as the midpoints of its edges. The situation is pictured in Figure 1.

[2]and also by the fact that, by the Schwartz kernel theorem, any continuous bilinear functional on smooth compactly supported functions has a kernel.

[3]It is interesting to mention that, illustrating the various exploratory paths followed in non-formal deformation quantization, other approaches to non-formal star-products do not rely on an integral formula as in (6). For instance, motivated by the infinite dimensional case, Schötz and Waldmann [SW] rather use purely topological techniques to construct deformations of some locally convex vector spaces.

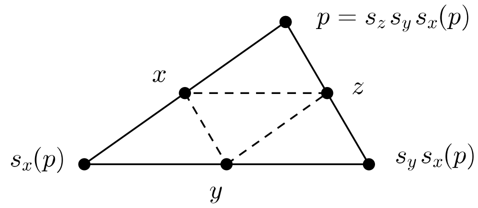



Figure 1: Double triangle defined by $x$, $y$ and $z$ in $\mathbb{R}^2$.

The key observation is now that the Weyl product can be written as

$$(f_1 \star_\theta f_2)(x) = \frac{1}{2\pi\theta} \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{\frac{i}{\theta} S(x,y,z)}\, f_1(y)\, f_2(z)\; dy\, dz,$$

where $S(x, y, z)$ is equal to the area of the double triangle defined by $x$, $y$ and $z$. This situation can be generalized in the following way. We define a *symmetric space* to be a connected manifold $M$ such that for each $x \in M$, there is an involutive diffeomorphism $s_x : M \to M$, called the symmetry at $x$, which admits $x$ as an isolated fixed point. We also require that $s_x$ depends smoothly on $x$, and that, for all $x, y \in M$, $s_x \circ s_y \circ s_x = s_{s_x(y)}$. This definition generalizes in some sense the notion of central symmetry in $\mathbb{R}^2$. A symplectic symmetric space is a symmetric space endowed with a symplectic form which is invariant under all symmetries. On a symmetric space, there is a natural connection invariant under all symmetries, so the notions of geodesic and double triangle make sense, as is represented in Figure 2.

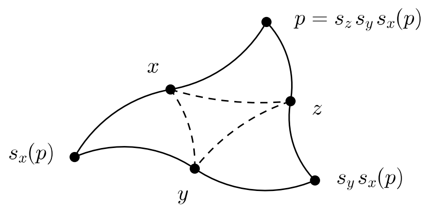



Figure 2: Double triangle defined by $x$, $y$ and $z$ in $M$.

Notice, however, that, contrary to the case of $\mathbb{R}^2$, the fixed points of $s_x$ and of $s_z \circ s_y \circ s_x$ need not be unique. Within this context, the conjecture of Weinstein is that the three-point kernel $K_\theta(x, y, z)$ should take the form

$$K_\theta(x, y, z) = \sum_{p(x,y,z) \in \mathrm{Fix}(s_z s_y s_x)} a_\theta(p, x, y, z)\, e^{\frac{i}{\theta} S_p(x,y,z)},$$

where the sum is taken over the fixed points of $s_z \circ s_y \circ s_x$, the "phase" $S_p(x, y, z)$ is equal to the symplectic area of any double triangle determined by the fixed point $p(x,y,z)$, and $a_\theta(p,x,y,z)$ is some "amplitude" function. Notice that it is indeed the case for the Weyl product, the fixed point of $s_z \circ s_y \circ s_x$ being unique in that case.

Regarding the explicit construction of equivariant star-products, one of the results of the work of Bieliavsky and Gayral in [BG15] is to provide a non-formal star-product $\star_\theta$ on the *elementary normal* $\mathbf{j}-$*groups* – which correspond to the Iwasawa factors $AN$ of the groups $SU(1,n)$. They are symplectic symmetric spaces, and the star-product is equivariant for the full group of automorphisms of $M$ – that is, the group of transformations that intertwine the symmetries and leave the symplectic form invariant. Also, the expression of $\star_\theta$ is of the form (6) and, being entirely explicit, it allows to see that Weinstein's conjecture about the fixed points – which in this case are unique – and the phase of the kernel is indeed verified. However, it does not make transparent *why* it holds.

One of the motivations behind this thesis is to get a better grasp on when and why Weinstein's conjecture holds. More specifically, although the exact form of the phase and the amplitude won't be investigated in general, we would like to understand the appearance of the fixed points in the expression of the kernel of the star-product. As a main tool towards that objective, we will need to prove a fixed point formula for the distributional trace of a family of operators. Let us therefore leave for a moment the world of quantization and star-products, in order to introduce this notion.

## The distributional trace

In group representation theory, the character of a finite-dimensional representation $\pi$ of a group $G$ is the function on $G$ given by the trace of the operators, that is, for $g \in G$, by $\chi_\pi(g) := \mathrm{Tr}\,(\pi(g))$. The study of characters of a group turns out to be a very powerful tool as they carry a lot of information about the structure of that group. For instance, character theory is essential in the classification of finite simple groups, as well as in the classification of representations of groups. We would naturally like to have such a tool for infinite-dimensional representations, but it is not readily available. Indeed, if $U$ is a unitary irreducible infinite-dimensional representation of a Lie group $G$ on some Hilbert space $\mathcal{H}$, for $g \in G$, the operator $U(g)$ is in general not trace-class[4]. However, if $\rho$ is a smooth compactly supported function on $G$, $dg$ the Haar measure on $G$ and $\varphi \in \mathcal{H}$, we can consider

$$U(\rho)(\varphi) := \int_G \rho(g)\, U(g)(\varphi)\, dg,$$

[4]For instance, the operator corresponding to the neutral element of the group is the identity operator.

which, in the good cases, gives a well-defined trace-class operator. This leads to the notion of a "distributional trace", defined as the linear mapping

$$\rho \mapsto \operatorname{Tr}\big(U(\rho)\big).$$

In the case of semisimple Lie groups for instance, this distributional trace has been extensively studied by Harish-Chandra (see, for instance, [HC54, HC55, HC66]), leading to some results that generalize theorems about finite-dimensional representations.

Although the previous construction shows the interest to consider a distributional trace and already has a lot of applications, the notion of distributional trace still makes sense far beyond the world of group theory and Hilbert spaces. More generally, it can be considered as soon as we have a (nice) family of operators – not necessarily on a Hilbert space – indexed by some manifold $M$ – which is not necessarily a group. Without caring too much about the details (see Chapter 1 for precise definitions and statements), here is how it goes. Let $M$ and $Q$ be two manifolds, let $dx$ and $dq$ be two measures on $M$ and $Q$ respectively, and let

$$\begin{aligned} \tau &: M \times Q \to Q \ ; \ (x,q) \mapsto \tau_x(q), \\ r &: M \times Q \to \mathbb{C} \ ; \ (x,q) \mapsto r_x(q) \end{aligned} \tag{8}$$

be smooth maps[5]. For each $x \in M$, we can consider the endomorphism $\tilde{\Omega}(x)$ of $\mathcal{C}^\infty(Q)$ defined, for all $\varphi \in \mathcal{C}^\infty(Q)$ and $q \in Q$, by

$$(\tilde{\Omega}(x)\varphi)(q) = r_x(q)\,\varphi\big(\tau_x(q)\big). \tag{9}$$

Then, for all $f \in \mathcal{C}_0^\infty(M)$, we define the endomorphism $\Omega(f)$ of $\mathcal{C}^\infty(Q)$ by

$$\Omega(f)\varphi = \int_M f(x)\,\tilde{\Omega}(x)\varphi \; dx. \tag{10}$$

If $\Omega(f)$ admits a smooth kernel, that is, a smooth function $k_f(q,q')$ such that

$$(\Omega(f)\varphi)(q) = \int_Q k_f(q,q')\,\varphi(q') \; dq', \tag{11}$$

we can consider its *smooth trace* $\operatorname{tr}\big(\Omega(f)\big) := \int_Q k_f(q,q) \; dq$. The distributional trace of the family $\tilde{\Omega}$ is then defined as

$$\operatorname{tr}_\Omega : \mathcal{C}_0^\infty(M) \to \mathbb{C} \ ; \ f \mapsto \operatorname{tr}\big(\Omega(f)\big). \tag{12}$$

A natural question is then to know whether $\operatorname{tr}_\Omega$ gives a genuine distribution – that is, whether it is continuous –, and, moreover, whether there exists a function $\operatorname{tr}_\Omega(x)$ such that $\operatorname{tr}_\Omega(f) = \int_M f(x) \operatorname{tr}_\Omega(x) \; dx$. We will see that this question is of particular relevance for the computation of the kernel of a star-product.

[5] $\tau$ might be, for instance, the action of a Lie group $M$ on a manifold $Q$.

## Goals of the thesis

- The goal of the first chapter is, given a family of operators as in (9), to show that the map (12) defines a distribution. Moreover, under some conditions on the fixed points of $\tau$, that distribution is smooth, and its kernel is given by a fixed point formula:

$$\operatorname{tr}_\Omega(f) = \int_M f(x) \left( \sum_{p=\tau_x(p)} \frac{r_x(p)}{|\det(\mathrm{id} - (\tau_x)_{*_p})|} \right) dx, \tag{13}$$

  where the sum is taken over the fixed points of $\tau_x$. We will actually prove such a result in the more general context where $\tilde{\Omega}(x)$ is acting on sections of a vector bundle over $Q$, and we won't need to fix a measure on $M$ and on $Q$.

  Fixed point formulas appear in many places in the mathematical literature. Among them, a very much celebrated result is the Atiyah-Bott-Lefschetz fixed point formula that Atiyah and Bott have proved in [AB67]. In [AB68, Section 5], they apply this formula to express the distributional trace of some group representations as a fixed point formula, leading to something similar to (13). We could be tempted to use the same approach to handle our situation, but the Atiyah-Bott-Lefschetz fixed point formula only covers the case of transformations of a *compact* manifold $Q$, and their arguments would be difficult to extend to the non-compact case. We will therefore follow a different approach, based on the work of Guillemin and Sternberg [GS90]. Besides its interest on its own, this result will also be a crucial ingredient in order to solve the next question.

- The aim of the second chapter is, in the spirit of Weinstein's conjecture, to understand when a fixed point formula for the kernel of a star-product on a symmetric space can actually hold, and to prove it, at least in a particular framework. To this end, we define a setting where an equivariant quantization map à la Weyl can be constructed (we will give more details on this below). We then identify some hypotheses under which we can show that this quantization map allows to define a non-formal, equivariant, associative star-product. Then, our main result is to prove, in this setting, an explicit expression for the kernel of the star product, as a fixed point formula. Finally, as an example, we show that for elementary normal $\mathsf{j}$-groups, all our hypotheses are satisfied, which sheds a new light on the appearance of the fixed points in the star-product of [BG15].

The relevance of the result about distributional traces to address our second question lies in the following observation. The computation of the kernel of the star-product boils down to the computation of the trace of some operator. We will show that this operator is of the form $\Omega(f)$ as in (10), and that its trace coincides with its smooth trace. Therefore, applying (13) will provide an expression of the kernel as a sum over the fixed points.

## Structure of the thesis

The thesis is divided into two chapters, each of them corresponding to one of the objectives previously stated. We give here a brief overview of their structure, and refer to each of them for precise definitions and statements.

### Chapter 1

This chapter is dedicated to the study of the distributional trace of a family of operators, in order to express it as a fixed point formula.

In Section 1.2, we first investigate the subject of integration on manifolds. We recall the notion of densities, that are objects that can be naturally integrated on any manifold, without any further choice (such as an orientation). Densities are here defined for any real vector bundle, not only the tangent bundle.

Section 1.3 introduces the notion of distributions on manifolds and, more generally, generalized sections of vector bundles. Standing as generalizations of functions, they are the objects we need to define a distributional trace in the same way as in equation (12). Generalized sections are also very useful to study linear operators on functional spaces.

This is what we explore in Section 1.4, where general operators are introduced. They provide a more general setting to handle linear operators between functional spaces than operators on Hilbert or Banach spaces, but they still admit those as particular cases. We discuss the Schwartz kernel theorem which (very roughly) asserts that, like in (11), any general operator admits a kernel, although it might be a generalized section instead of a function. The kernel of an operator will be a key tool for us since a critical step of our construction will be to express the distributional trace as a sequence of operations on the kernel of the operators. Also, the study of the regularity of the kernel of an operator reveals a lot of its properties. For instance, a particular class of operators is formed by those whose kernel is a smooth function. In this case, we define the smooth trace as the integral along the diagonal, and discuss the delicate question of its link with the usual trace of bounded linear operators on Hilbert spaces.

In Section 1.5, we introduce the operations we need to manipulate the kernel of operators. We recall how the usual notions of pullback and pushforward of a function by a smooth map $f$ can be extended to generalized sections. However, this extension is not completely general since we have to restrict the kind of map $f$ we consider if we want the definition to work for any generalized section[6]. For instance, the pullback is only defined if $f$ is a submersion.

In Section 1.6, we introduce a particularly important class of generalized sections – called $\delta-$sections. They correspond to the integration over a submanifold, and are described by their so-called symbol, a smooth section on the submanifold. We show that the pullback of a $\delta-$section can be defined for more general maps than submersions. We also explicitly describe, in terms of its symbol, the transformation of a $\delta-$section under the pullback and pushforward operations. This is a powerful feature of $\delta-$sections since their symbol, being an actual section, is by far easier to manipulate than the corresponding generalized section.

$\delta-$sections turn out to be crucial in our construction because the kernels of the general operators that we deal with – namely, pullback of sections of vector bundles, that have a form similar to (9) – are precisely given by $\delta-$sections, as we show in Section 1.7. We also define a notion of trace for those operators, and express it as a fixed point formula.

Finally, in Section 1.8, we introduce the notion of a family of geometric morphisms as a data similar to (8), to which we can associate a family of pullback operators as in (9). We then construct the corresponding distributional trace as in (12). We show that it is a distribution which, under some conditions, is smooth. Moreover, it is shown that its kernel is given by a fixed point formula similar to (13).

**Chapter 2**

The second chapter comes back to the world of quantization, and aims to understand the appearance of fixed points in the construction of non-formal star-products on symmetric spaces.

In Section 2.1, we recall some elementary facts about symmetric spaces, which are the kind of spaces we are interested in. In particular, we present three different points of view, each of them shedding another light on this notion.

Section 2.2 is dedicated to the construction of an equivariant quantization map. It is based on the work of [BG15], adapted to a more general context. We first specify the kind of spaces we are working with and the additional structure that we ask for. This leads to the definition of a nearly-quantum symmetric space, and its local version. In particular, it underlies a symmetric space $M$ and a group $G$ acting on $M$. Then, we identify a Hilbert space $\mathcal{H}_\chi$ naturally associated to a nearly-quantum symmetric space, and we give several equivalent

[6] We will see however that we can consider more general maps $f$ if some compatibility between $f$ and the generalized section is satisfied.

realizations of that Hilbert space. In the same spirit as (1) in the Weyl quantization, we construct a first quantization map $\Omega : L^1(M) \to \mathcal{L}(\mathcal{H}_\chi)$, which gives bounded operators on $\mathcal{H}_\chi$. It is $G$-equivariant in the sense that, for all $g \in G$, $\Omega({}^g f) = U(g)\Omega(f)U(g)^{-1}$ for some unitary representation $U$ of $G$.[7]
However, this quantization map, although very natural, turns out to be not very convenient. We thus turn to the construction of a slightly modified quantization map $\Omega_{\mathbf{m}}$, depending on a functional parameter $\mathbf{m}$. A major difference with $\Omega$ is that the associated operators do not give bounded operators on $\mathcal{H}_\chi$, but are rather defined as linear operators acting on smooth sections. We are then able to realize those operators as the pullback operators associated to a family of geometric morphism, as defined in the previous chapter.

In Section 2.3, we recall several notions about trace-class and Hilbert-Schmidt operators. We also briefly discuss when our quantization map gives genuine Hilbert-Schmidt operators.

Now we have built a quantization map $\Omega_{\mathbf{m}}(f)$, we would like to use it to define a star-product by the formula

$$\Omega_{\mathbf{m}}(f_1 \star_{\mathbf{m}} f_2) := \Omega_{\mathbf{m}}(f_1) \circ \Omega_{\mathbf{m}}(f_2).$$

This "dequantization procedure" is the subject of Section 2.4, where we look for an inverse of the quantization map. In order to do so, we require that the quantization map gives Hilbert-Schmidt operators. This allows the definition of a symbol map $\sigma_{\mathbf{m}}$ which is the inverse of the quantization map if the latter extends to a unitary operator between $L^2(M)$ and the Hilbert space $\mathcal{L}^2(\mathcal{H}_\chi)$ of Hilbert-Schmidt operators on $\mathcal{H}_\chi$. We are then able to define a deformed associative product $\star_{\mathbf{m}}$ on $L^2(M)$. The latter is $G$-equivariant because the quantization map is.
The second part of the section leads to the main result of the chapter, which is to give an explicit expression of the kernel of the previously constructed star-product. That kernel is given by computing the trace of an operator, which we compute using the results of the previous chapter. We rely on the fact that the operator is associated to a family of geometric morphisms $(\tau, r)$ like in (8), so we can compute its trace using the results proved in Chapter 1, which leads to a fixed-point formula of the kind

$$(f_1 \star_{\mathbf{m}} f_2)(x) = \int_{M^2} f_1(y) f_2(z) \left( \sum_{p=\tau_{(x,y,z)}(p)} \frac{r_{(x,y,z)}(p)}{|\det(\mathrm{id} - (\tau_{(x,y,z)})_{*_p})|} \right) dy\, dz,$$

where the sum is taken over the fixed points of $\tau_{(x,y,z)}$. Notice that we have an explicit expression of $\tau$ and $r$ in terms of the data of the nearly-quantum symmetric space.

[7] See (7) for the definition of ${}^g f$.

Finally, in Section 2.5, we apply the previous results to the particular case of elementary normal **j**-groups. After reviewing their definition and structure, we associate to each of them a nearly-quantum symmetric space. We then show that all the hypotheses needed in our previous construction are satisfied. This leads to an explicit expression of the star-product in terms of the fixed points, which coincides with the star-product constructed in [BG15].

# Chapter 1

# A fixed-point formula for the distributional trace

## 1.1 Introduction

In this chapter, we are going to study the distributional trace of a family of operators, in order to express it as a fixed point formula. Let us begin with an introductory example, which illustrates what is going on.

*Example* 1.1.1. Let $M := \mathbb{R}^2$ and $Q := \mathbb{R}$ and consider the smooth maps

$$\begin{aligned} \tau : M \times Q \to Q \;;\; \big((a,l),q\big) &\mapsto \tau_{(a,l)}(q) := 2a - q, \\ r : M \times Q \to \mathbb{C} \;;\; \big((a,l),q\big) &\mapsto r_{(a,l)}(q) := e^{2i(a+q)l}. \end{aligned} \tag{1.1}$$

This datum gives a family $\{\Omega(x)\}_{x\in M}$ of linear operators $\Omega(x) : \mathcal{C}^\infty(Q) \to \mathcal{C}^\infty(Q)$ given, for every $(a,l) \in M$, $\varphi \in \mathcal{C}^\infty(Q)$ and $q \in Q$, by

$$\begin{aligned} (\Omega(a,l)\varphi)(q) &= r_{(a,l)}(q) \cdot \varphi\big(\tau_{(a,l)}(q)\big) \\ &= e^{2i(a+q)l}\varphi(2a-q). \end{aligned} \tag{1.2}$$

Then, for every $\rho \in \mathcal{C}_c^\infty(M)$, we can form the linear operator $\Omega(\rho) : \mathcal{C}^\infty(Q) \to \mathcal{C}^\infty(Q)$ defined, for every $(a,l) \in M$, $\varphi \in \mathcal{C}^\infty(Q)$ and $q \in Q$, by

$$\begin{aligned} (\Omega(\rho)\varphi)(q) &= \int_{\mathbb{R}^2} \rho(a,l)\ (\Omega(a,l)\varphi)(q)\ da\,dl \\ &= \int_{\mathbb{R}^2} \rho(a,l)\ e^{2i(a+q)l}\varphi(2a-q)\ da\,dl \\ &= \int_{\mathbb{R}} \left(\frac{1}{2}\int_{\mathbb{R}} e^{i(q'+3q)l}\rho\left(\frac{q+q'}{2}, l\right) dl\right)\ \varphi(q')\,dq', \end{aligned} \tag{1.3}$$

where we have made the change of variable $q' = 2a - q$. If we define $k_\rho(q,q') := \frac{1}{2}\int_{\mathbb{R}} e^{i(q'+3q)l}\rho\left(\frac{q+q'}{2}, l\right) dl$, this operator can be written as

$$(\Omega(\rho)\varphi)(q) = \int_{\mathbb{R}} k_\rho(q,q')\, \varphi(q')\, dq', \tag{1.4}$$

The function $k_\rho(q,q')$ is called the *kernel* of the operator $\Omega(\rho)$. Inspired by the finite-dimensional situation, we can think of $k_\rho(q,q')$ as the matrix coefficients of the operator $\Omega(\rho)$. Following that analogy, its trace would be the sum of the diagonal elements, that is, the integral over the diagonal (we denote it by tr instead of Tr to emphasize that those two notions do not coincide in general, as we discuss in Subsection 1.4.4):

$$\mathrm{tr}\,(\Omega(\rho)) := \int_{\mathbb{R}} k_\rho(q,q)\, dq = \int_{\mathbb{R}^2} \frac{e^{4ial}}{2}\, \rho\,(a,l)\ da\, dl.$$

The linear map

$$\mathrm{tr}_{\,\Omega} : \mathcal{C}_c^\infty(M) \to \mathbb{C}\ ;\ \rho \mapsto \mathrm{tr}\,(\Omega(\rho)) \tag{1.5}$$

turns out to be continuous for some topology on $\mathcal{C}_c^\infty(M)$ and is therefore called a *distribution* on $M$. This is the definition of the distributional trace of the family of operators $\{\Omega_x\}_{x\in M}$. In this case, it has the functional form

$$\mathrm{tr}\,(\Omega(\rho)) = \int_{\mathbb{R}^2} \mathrm{tr}_{\,\Omega}(a,l)\,.\,\rho\,(a,l)\ da\, dl, \tag{1.6}$$

for the function, $\mathrm{tr}_{\,\Omega}(a,l) := e^{4ial}/2$. The striking point is that this function is smooth, and given by a fixed point formula:

$$\mathrm{tr}_{\,\Omega}(a,l) = \sum_{p\,=\,\tau_{(a,l)}(p)} \frac{r_{(a,l)}(p)}{\left|\det(\mathrm{id} - (\tau_{(a,l)})_{*_p})\right|}, \tag{1.7}$$

where the sum is taken over the fixed points of $\tau_{(a,l)} : Q \to Q$ (in this example, we only have one such fixed point). $\Diamond$

The goal of this chapter is to show that this situation is not restricted to this particular example, and is even valid in the more general context of operators between sections of vector bundles.

As we already mentioned in the introduction, formula (1.7) is very similar to the Atiyah-Bott-Lefschetz fixed point formula [AB67, AB68]. However, their result only concerns compact manifolds $Q$ and, in the next chapter, we will have to consider transformations of non-compact manifolds $Q$ – this was already the case in Example 1.1.1. The arguments of Atiyah and Bott being difficult to extend to the non-compact case, we will follow a different approach, based on the work of Guillemin and Sternberg [GS90, Chapter 6]. Although their fixed point formula is also restricted to transformations of a compact manifold, we identify some conditions that allow to extend it to the non-compact case. We give here a detailed exposition of the construction which, in our opinion, renders the appearance of the fixed points in the computation of traces very transparent.

Before getting to the heart of the matter, let us summarize how the construction of the introductory example will be generalized in this chapter, and how we will prove the fixed point formula. This is just a sketchy description, full details and precise definitions and hypotheses will be given later on. Given a manifold $M$ and a vector bundle $E \to Q$ over a manifold $Q$, suppose that we have a locally transitive[1] smooth map:

$$\tau : M \times Q \to Q \ ; \ \big(x, q\big) \mapsto \tau_x(q)$$

and, for each $x \in M$ and $q \in Q$, a linear map

$$r_x(q) : E_{\tau_x(q)} \to E_q$$

such that the dependence on $x$ and $q$ is smooth. We call this datum a family of geometric morphisms of $E$ indexed by $M$. Then, we can consider the family of operators $\{\Omega(x)\}_{x \in M}$ acting on smooth sections of $E$ by pullback, that is, for $\varphi \in \Gamma^\infty(Q, E)$ and $q \in Q$:

$$(\Omega(x)\varphi)(q) = r_x(q)\, \varphi\big(\tau_x(q)\big).$$

We will see that the kernel of those operators are given by $\delta-$sections. The latter are a special class of distributions – more generally, generalized sections –, which are described by their so-called symbol, which is a genuine section of a vector bundle. Their main advantage is that several operations on $\delta$-sections – such as the pullback and the pushforward – can be described in terms of their symbol, which is a lot easier to manipulate (this will be the subject of Section 1.6). We will be able to make sense of the "trace" of $\Omega(x)$ as a sequence of operations on the corresponding $\delta-$section. By tracking how its symbol changes under those operations, we will show that the trace, when well-defined, is given by a fixed point formula:

$$\text{“ tr}\,(\Omega(x))\text{ ”} = \sum_{p\,=\,\tau_x(p)} \frac{\operatorname{Tr}\,(r_x(p))}{\big|\det(\mathrm{id}_p - (\tau_x)_{*_p})\big|},$$

where the sum is taken over the fixed points of $\tau_x : Q \to Q$, $\mathrm{id}_p$ is the identity map on $T_p(Q)$ and $\operatorname{Tr}\,(r_x(p))$ is the (algebraic) trace of the homomorphism $r_x(p)$ of the finite-dimensional vector space $E_p$. Next, for every compactly supported density $\rho$ on $M$, we will form the operators $\Omega(\rho)$ defined, for $\varphi \in \Gamma^\infty(Q, E)$

[1] A smooth map $\tau : M \times Q \to Q$ is *locally transitive* if and only if, for every $(x, q) \in M \times Q$, the linear map $T_x(M) \to T_{\tau_x(q)}(Q)\ ;\ X \mapsto \tau_{*_{(x,q)}}(X, 0)$ is surjective.

and $q \in Q$, by:[2]

$$(\Omega(\rho)\varphi)(q) = \int_M \rho(x) \otimes (\Omega(x)\varphi)(q) = \int_M \rho(x) \otimes \big( r_x(q)\, \varphi\big(\tau_x(q)\big) \big).$$

We will show that, because of the local transitivity of $\tau$, these operators have a smooth kernel, i.e. there exists a smooth section $k_\rho$ of some vector bundle over $Q \times Q$ such that $(\Omega(\rho)\varphi)(q) = \int_Q k_\rho(q, q')\, \varphi(q')$.[3] If we suppose that, for each $\rho$, $\mathrm{Tr}\,(k_\rho)$ is integrable along the diagonal, $\Omega(\rho)$ has a well-defined smooth trace $\mathrm{tr}\,(\Omega(\rho)) := \int_Q \mathrm{Tr}\,(k_\rho(q,q))$. This smooth trace can be expressed as a sequence of pullback and pushforward operations on some $\delta-$section. Using the results of Section 1.6, this will allow us to show that the linear map $\rho \mapsto \mathrm{tr}\,(\Omega(\rho))$ is a distribution on $M$, which in addition is smooth. That is, there exists a smooth function $\mathrm{tr}_\Omega$ on $M$ such that

$$\mathrm{tr}\,(\Omega(\rho)) = \int_M \mathrm{tr}_\Omega(x)\, \rho(x).$$

Finally, we will identify that function with $\mathrm{tr}\,(\Omega(x))$, which will lead us to the fixed point formula

$$\mathrm{tr}\,(\Omega(\rho)) = \int_M \left( \sum_{p\,=\,\tau_x(p)} \frac{\mathrm{Tr}\,(r_x(p))}{\left|\det(\mathrm{id} - (\tau_x)_{*_p})\right|} \right) \rho(x).$$

## 1.2 Intrinsic integration on manifolds

On $\mathbb{R}^n$, the Lebesgue measure provides a canonical way to integrate functions. On a generic $n$ dimensional manifold, one can consider the measures such that, in each coordinate charts, their pushforward by the chart is equivalent to the Lebesgue measure by a smooth non-vanishing function. We call them Lebesguian measures on the manifold. However, there are many such Lebesguian measures, and no canonical one in general. This implies that there is no canonical way to integrate functions on a manifold. The workaround is usually to work with differential $n-$forms, that are objects that can be integrated in a natural way without the need to fix a measure. However, this requires the choice – and the existence – of an orientation on the manifold. This can be avoided by introducing $\alpha-$densities. Like $n-$forms, they are scalar functions on the space of bases of a vector space but which are transformed under a change of basis by the *absolute value* of the determinant taken to the power

[2]The tensor product inside the integral gives a density on $M$ valued in $E_q$, which is an object that can be naturally integrated to give an element of $E_q$.

[3]In order to explain what the integrand means, let us say, for the moment that $k_\rho(q, q')$ is a homomorphism from $E_{q'}$ to the densities on $Q$ valued in $E_q$, so the integral is an element of $E_q$. This will be made more precise later on.

$\alpha$. When that power is equal to one and the vector space is the tangent space at a point, we can make sense of the integral of a density in a way similar to the integration of differential forms. Furthermore, $1/2-$densities allow to define an intrinsic Hilbert space of square-integrable sections associated to the manifold. We will first define and study $\alpha-$densities on real vector spaces, and then extend the notion to manifolds and real vector bundles.

### 1.2.1 Densities on real vector spaces

Let $V$ be a real vector space of dimension $n$. We denote by $\mathcal{B}(V)$ the set of bases of $V$. $GL(n)$ acts on the right on $\mathcal{B}(V)$ by matrix multiplication. For $\mathbf{e} = (e_1, \ldots, e_n) \in \mathcal{B}(V)$ and $A \in GL(n)$, this action is defined by

$$\mathbf{e} \mapsto \mathbf{e} \cdot A := (e_1, \ldots, e_n) \cdot A. \tag{1.8}$$

**Definition 1.2.1.** *Let $V$ be a real vector space and $\alpha \in \mathbb{R}$. An $\alpha-$*density *on $V$ – or a* density of order $\alpha$ *on $V$ – is a map*

$$\lambda : \mathcal{B}(V) \to \mathbb{C}$$

*such that, for all $\mathbf{e} \in \mathcal{B}(V)$ and $A \in GL(n)$, we have:*

$$\lambda(\mathbf{e} \cdot A) = |\det A|^{\alpha} \lambda(\mathbf{e}). \tag{1.9}$$

*The set of all $\alpha-$densities on $V$ forms a complex vector space, which will be denoted by $|V|^{\alpha}$. An $\alpha-$density $\lambda$ on $V$ is said to be* positive *if, for every $\mathbf{e} \in \mathcal{B}(V)$, $\lambda(\mathbf{e}) \in \mathbb{R}$ and $\lambda(\mathbf{e}) > 0$. A $1-$density on $V$ is simply called a density on $V$ and the space of densities on $V$ is denoted by $|V|$.*

*Remark* 1.2.2. Since $GL(n)$ acts transitively on $\mathcal{B}(V)$, the transformation law (1.9) implies that an $\alpha-$density is completely determined by its value on one basis. Hence, $|V|^{\alpha}$ is a one-dimensional complex vector space. $\triangleleft$

*Remark* 1.2.3. For every $\omega \in \Lambda^n(V)$, we can define an $\alpha-$density $|\omega|^{\alpha}$ on $V$ by the formula $|\omega|^{\alpha}(\mathbf{e}) := |\omega(e_1, \ldots, e_n)|^{\alpha}$ for all $\mathbf{e} = (e_1, \ldots, e_n) \in \mathcal{B}(V)$. It is a positive density if $\omega$ is not zero. $\triangleleft$

**Lemma 1.2.4.** *Let $\alpha \in \mathbb{R}$ and $A$, $B$ and $C$ real vector spaces. Suppose we have a short exact sequence*

$$0 \to A \xrightarrow{\beta} B \xrightarrow{\gamma} C \to 0.$$

*Then, there is a canonical isomorphism*

$$|B|^{\alpha} \simeq |A|^{\alpha} \otimes |C|^{\alpha}.$$

*Remark* 1.2.5. Before going into the proof, let us make a comment on how this lemma should be understood. At first sight, it might look trivial because the space of densities on a vector space is $1-$dimensional, so the two sides are clearly isomorphic. However, it is not canonical without additional data. The statement is that, in this situation, there is a natural isomorphism associated to the maps $\beta$ and $\gamma$. The proof is a basic exercise in linear algebra. We detail it here in order to explicitly show the construction of the isomorphism, which we will need several times throughout this text. $\triangleleft$

*Proof.* Let $\alpha \in \mathbb{R}$, $\lambda_1 \in |A|^\alpha$ and $\lambda_2 \in |C|^\alpha$. To define an $\alpha-$density $\lambda$ on $B$, it is sufficient to define it on a basis of $B$. Let us choose a basis $(a_1, \ldots, a_m)$ of $A$ and denote $e_i = \beta(a_i)$. Then, because $\beta$ is injective, $(e_1, \ldots, e_m)$ is a tuple of linearly independent vectors, which can be extended to a basis of $B$

$$\mathbf{e} = (e_1, \ldots, e_m, e_{m+1}, \ldots, e_n).$$

Then, because the sequence is exact, $(\gamma(e_{m+1}), \ldots, \gamma(e_n))$ forms a basis of $C$. This allows to define

$$\lambda(\mathbf{e}) := \lambda_1(e_1, \ldots, e_m)\ \lambda_2(\gamma(e_{m+1}), \ldots, \gamma(e_n)).$$

Let us see that, as a density, $\lambda$ does not depend on the choice of $(a_1, \ldots, a_m)$ and $(e_{m+1}, \ldots, e_n)$. Another choice would lead to a basis $\mathbf{f} = (f_1, \ldots, f_n)$ of $B$ that would be related to $\mathbf{e}$ by a transformation $A \in GL(n)$ of the form

$$\mathbf{f} = \mathbf{e} \cdot \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix},$$

where $A_1 \in GL(m), A_2 \in GL(n-m)$ and $A_{12} \in \mathrm{Mat}(m, n-m)$. Notice that $\det A = \det A_1 \det A_2$. We would have

$$(\gamma(f_{m+1}), \ldots, \gamma(f_n)) = (\gamma(e_{m+1}), \ldots, \gamma(e_n)) \cdot A_2$$

and, therefore:

$$\begin{aligned}
&\lambda_1(f_1, \ldots, f_m)\ \lambda_2(\gamma(f_{m+1}), \ldots, \gamma(f_n)) \\
&= \lambda_1((e_1, \ldots, e_m) \cdot A_1)\ \lambda_2((\gamma(e_{m+1}), \ldots, \gamma(e_n)) \cdot A_2) \\
&= |\det A_1|^\alpha\ |\det A_2|^\alpha\ \lambda_1(e_1, \ldots, e_m)\ \lambda_2(\gamma(e_{m+1}), \ldots, \gamma(e_n)) \\
&= |\det A|^\alpha\ \lambda(\mathbf{e}) =: \lambda(\mathbf{f}),
\end{aligned}$$

which shows that the definition does not depend on the choice of basis. This construction gives a non-zero bilinear map $|A|^\alpha \times |C|^\alpha \to |B|^\alpha$ which induces an isomorphism $|A|^\alpha \otimes |C|^\alpha \to |B|^\alpha$. $\square$

In several occasions, we will need to decompose a density with respect to a vector subspace decomposition, which is possible as a consequence of the previous lemma.

**Corollary 1.2.6.** *Let $\alpha \in \mathbb{R}$, $W$ a real vector space and $U, V \subset W$ vector subspaces such that $W = U \oplus V$. Then there are canonical isomorphisms*

$$|W|^\alpha \simeq |U|^\alpha \otimes |W/U|^\alpha,$$
$$|W|^\alpha \simeq |U|^\alpha \otimes |V|^\alpha.$$

*Proof.* We apply the previous lemma to the short exact sequences

$$0 \to U \xrightarrow{\iota_U} W \xrightarrow{\pi_{W/U}} W/U \to 0,$$
$$0 \to U \xrightarrow{\iota_U} W \xrightarrow{\pi_V} V \to 0,$$

where $\iota_U$ denotes the inclusion of $U$ in $W$, $\pi_{W/U}$ is the natural projection and $\pi_V$ is the projection corresponding to the direct sum $W = U \oplus V$. □

An isomorphism between real vector spaces allows to select a particular isomorphism between their spaces of $\alpha-$densities. Again, the proof only deals with basic algebra, but since we will need the explicit form of the isomorphism several times, we detail it here.

**Lemma 1.2.7** (Pushforward of densities by isomorphisms)**.** *Let $V$ and $W$ be real vector spaces, $j : V \to W$ an isomorphism and $\alpha \in \mathbb{R}$. Then, $j$ induces a canonical isomorphism*

$$|j| : |V|^\alpha \to |W|^\alpha.$$

*Proof.* Let $\lambda \in |V|^\alpha$. For any basis $\mathbf{e} = (e_1, \dots, e_n)$ of $W$, we denote by $j^{-1}(\mathbf{e})$ the basis of $V$ given by $(j^{-1}(e_1), \dots, j^{-1}(e_n))$. We define $|j|(\lambda) : \mathcal{B}(W) \to \mathbb{C}$ by the formula

$$|j|(\lambda)(\mathbf{e}) := \lambda(j^{-1}(\mathbf{e})),$$

for any $\mathbf{e} \in \mathcal{B}(W)$. By linearity of $j$, for all $A \in GL(n)$ and $\mathbf{e} \in \mathcal{B}(V)$, we have $j^{-1}(\mathbf{e} \cdot A) = j^{-1}(\mathbf{e}) \cdot A$, which implies that $|j|(\lambda)$ is an $\alpha-$density on $W$. $|j|$ is an isomorphism since it is a non-zero linear map between one dimensional complex vector spaces. □

*Remark* 1.2.8 (Multiplication and conjugation of densities). Let $V$ be a real vector space and $\alpha, \beta \in \mathbb{R}$.
The product of $\lambda \in |V|^\alpha$ and $\mu \in |V|^\beta$ is defined by

$$\lambda.\mu : \mathcal{B}(V) \to \mathbb{C} \; ; \; \mathbf{e} \mapsto \lambda(\mathbf{e}).\mu(\mathbf{e}).$$

It is readily verified that it is a density of order $\alpha + \beta$ on $V$. This induces a linear map

$$|V|^\alpha \otimes |V|^\beta \xrightarrow{\sim} |V|^{\alpha+\beta}$$

which is an isomorphism since it is a non-zero linear map between one dimensional complex vector spaces.
The complex conjugation of $\lambda \in |V|^\alpha$ is defined by

$$\bar{\lambda} : \mathcal{B}(V) \to \mathbb{C} \; ; \; \mathbf{e} \mapsto \overline{\lambda(\mathbf{e})}.$$

It is also a density of order $\alpha$ on $V$. $\lhd$

### 1.2.2 Densities on manifolds

Let $A \to M$ be a real vector bundle of rank $n$ over a manifold $M$ and $\alpha \in \mathbb{R}$. We will define a complex line bundle over $M$ whose fiber at $x$ is $|A_x|^\alpha$. The construction is as follows. Let $\mathcal{B}(A) \to M$ be the frame bundle of $A$. It is a $GL(n)-$principal bundle for the action (1.8) whose fiber at $x$ is $\mathcal{B}(A_x)$. Consider the representation of $GL(n)$ on $\mathbb{C}$ given by the multiplication by the character

$$\delta^\alpha : GL(n) \to \mathbb{C} \ ; \ a \mapsto |\det a|^{-\alpha}.$$

**Definition 1.2.9.** *Let $A \to M$ be a real vector bundle of rank $n$ over a manifold $M$ and $\alpha \in \mathbb{R}$. The complex vector bundle $|A|^\alpha$ over $M$ is defined as the associated vector bundle*[4]

$$|A|^\alpha := \mathcal{B}(A) \times_{\delta^\alpha} \mathbb{C}.$$

**Proposition 1.2.10.** *Let $A \to M$ be a real vector bundle of rank $n$ over a manifold $M$ and $\alpha \in \mathbb{R}$. Then, $|A|^\alpha$ is a trivial line bundle over $M$, whose fiber at $x \in M$ is $|A_x|^\alpha$.*

*Proof.* By construction, $|A|$ is a complex line bundle. Let $x \in M$. To any $[(p, z)] \in (|A|^\alpha)_x$, we can associate the $\alpha-$density $\lambda \in |A_x|^\alpha$ on $A_x$ defined by $\lambda(p) := z$. It is well defined since any other representative of the equivalence class would be of the form $(p \cdot a, |\det a|^\alpha z)$ and would give the same $\alpha-$density. This way, we get a non-zero linear map $(|A|^\alpha)_x \to |A_x|^\alpha$ between one dimensional vector spaces, hence an isomorphism. To see that $|A|$ is a trivial line bundle, notice that its transition functions are all positive by definition of the character $\delta^\alpha$. Using a partition of unity associated to a trivialization of $|A|^\alpha$, we can thus construct a smooth positive section of $|A|^\alpha$. It is a nonvanishing smooth section, and $|A|^\alpha$ is therefore trivial. $\square$

**Definition 1.2.11.** *Let $A \to M$ be a real vector bundle of rank $n$ over a manifold $M$ and $\alpha \in \mathbb{R}$. A section $\rho$ of the vector bundle $|A|^\alpha$ is called positive if $\rho(x)$ is positive for all $x \in M$. From the previous Proposition, there exists a smooth positive section of $|A|^\alpha$.*

**Lemma 1.2.12.** *Let $M$ be a manifold, $\alpha \in \mathbb{R}$. If we have an exact sequence of real vector bundles over $M$*

$$0 \to A \to B \to C \to 0,$$

[4]Recall that $\mathcal{B}(A) \times_{\delta^\alpha} \mathbb{C} := \frac{\mathcal{B}(A)\times\mathbb{C}}{\sim}$, where $(p, z) \sim (p \cdot a, \delta^\alpha(a^{-1})z)$ for all $p \in \mathcal{B}(A)$, $z \in \mathbb{C}$ and $a \in GL(n)$.

*then, we have a canonical isomorphism of complex vector bundles over $M$*

$$|B|^{\alpha} \simeq |A|^{\alpha} \otimes |C|^{\alpha}.$$

*In particular, the result holds if $B = A \oplus C$.*

*Proof.* This is an immediate consequence of Lemma 1.2.4 and 1.2.6. □

A particularly important case of this construction is when it is applied to the tangent bundle $TM$.

**Definition 1.2.13.** *Let $M$ be a manifold and $\alpha \in \mathbb{R}$. The complex vector bundle $|TM|^{\alpha}$ is called the* $\alpha-$density bundle *of $M$. A section of $|TM|^{\alpha}$ is called an* $\alpha-$density *on $M$. A* positive $\alpha-$density *is a section $\rho$ of $|TM|^{\alpha}$ such that $\rho(x)$ is positive for all $x \in M$. An $\alpha-$density is smooth (resp. continuous) if the section is smooth (resp. continuous). In the case $\alpha = 1$, we drop the $\alpha$ from the terminology and simply talk about* densities.

*Remark* 1.2.14. By Proposition 1.2.10, $|TM|^{\alpha}$ is a trivial line bundle, i.e. there exists a non-vanishing smooth $\alpha-$density, but not a canonical one. However, some specific contexts allow to choose a preferred non-vanishing smooth density:

- If the manifold is orientable, a non-vanishing smooth volume form $\nu \in \Gamma^{\infty}(M, \Lambda^n(M))$ gives a non-vanishing smooth density $|\nu|^{\alpha}$ by the formula

  $$|\nu|^{\alpha}(x)((e_1, \dots, e_n)) := |\nu(x)(e_1, \dots, e_n)|^{\alpha}$$

  for all $x \in M$ and $(e_1, \dots, e_n) \in \mathcal{B}(T_xM)$.

- On a symplectic manifold $(M, \omega)$ of dimension $2n$, $\omega^{\wedge n}$ is a non-vanishing smooth volume form, so $|\omega^{\wedge n}|^{\alpha}$ is a non-vanishing smooth $\alpha-$density.

- Let $U \subset \mathbb{R}^n$ an open set with coordinates $(x_1, \dots, x_n)$. We denote by $|dx_1 \dots dx_n|^{\alpha}$ the smooth $\alpha-$density corresponding to the Lebesgue volume form $dx_1 \wedge \dots \wedge dx_n$. For any smooth $\alpha-$density $\rho$ on $U$, there exists a unique complex valued function $f_{\rho}$ on $U$ such that $\rho = f_{\rho}\, |dx_1 \dots dx_n|^{\alpha}$.

◁

**Definition 1.2.15.** *Let $\Phi : N \to M$ be a smooth map between two manifolds of dimension $n$ and $\alpha \in \mathbb{R}$. The pullback by $\Phi$ of an $\alpha-$density $\rho$ on $M$ is the $\alpha-$density $\Phi^*\rho$ on $N$ defined, for $y \in N$ and $\mathbf{e} \in \mathcal{B}(T_yN)$, by*

$$(\Phi^*\rho)(y)(\mathbf{e}) := \rho(\Phi(y))(\Phi_{*_y}(\mathbf{e})).$$

*Remark* 1.2.16. If $\Phi$ is a local diffeomorphism and $\rho$ is smooth, then $\Phi^*\rho$ is also smooth. When $N = M$, this gives a right action of the group $\mathrm{Diff}(M)$ on $\Gamma^{\infty}(M, |TM|^{\alpha})$. ◁

In the special case of open subsets of $\mathbb{R}^n$, the transformation can be computed more explicitly. In particular, this allows to describe how $\alpha-$densities transform under smooth maps in local coordinates.

**Proposition 1.2.17.** *Let $U$ and $V$ be two open subsets of $\mathbb{R}^n$, $\Phi : U \to V$ a smooth map, $\alpha \in \mathbb{R}$ and $f : V \to \mathbb{C}$ a function. Denote by $x_1, \dots, x_n$ (resp. $y_1, \dots, y_n$) the coordinates on $U$ (resp. on $V$). Then,*

$$\Phi^*\big(f\,.\,|dy_1 \dots dy_n|^\alpha\big) = (f \circ \Phi)\,|\mathrm{Jac}_\Phi|^\alpha\,|dx_1 \dots dx_n|^\alpha. \tag{1.10}$$

*Proof.* This readily follows from Definition 1.2.15 and from the expression of the pullback of the Lebesgue volume form $dy_1 \wedge \cdots \wedge dy_n$. □

### 1.2.3 Integration of densities

Because of equation (1.10), it is possible to define the integral of a $1-$density in a coordinate independent way, in very much the same way as for differential forms. We recall here how the construction works.

First, let $U \in \mathbb{R}^n$ be an open subset and $\rho$ a compactly supported continuous density on $U$. By Remark 1.2.14, there is a unique continuous function $f_\rho$ such that $\rho = f_\rho|dx_1 \dots dx_n|$ and we define

$$\int_U \rho := \int_U f_\rho(x_1, \dots, x_n)\, dx_1 \dots dx_n.$$

Next, we turn to the case of a manifold $M$. Let $(U, \phi)$ be a coordinate chart on $M$ and $\rho$ a compactly supported continuous density on $M$ with support in $U$. We define

$$\int_M \rho := \int_{\phi(U)} (\phi^{-1})^* \rho. \tag{1.11}$$

To verify that this expression does not depend on the chart, let $(V, \psi)$ be another coordinate chart such that $\rho$ is supported in $V$. Without loss of generality, we can suppose that $U = V$. Denote by $|dx_1 \dots dx_n|$ and $|dy_1 \dots dy_n|$ the Lebesgue measures on $\phi(U)$ and $\psi(U)$ respectively. We have $(\psi^{-1})^*\rho(y) = f(y)|dy_1 \dots dy_n|$ for some continuous function $f$ on $U$ and, by Proposition 1.2.17:

$$\begin{aligned}(\phi^{-1})^*\rho(x) &= ((\phi^{-1} \circ \psi)^*(\psi^{-1})^*\rho)(x)\\ &= f\big((\phi^{-1} \circ \psi)(x)\big)\,|\mathrm{Jac}_{\phi^{-1}\circ\psi}|\,|dx_1 \dots dx_n|.\end{aligned}$$

Therefore:

$$\begin{aligned}\int_{\phi(U)} (\phi^{-1})^*\rho &= \int_{\phi(U)} f\big((\phi^{-1} \circ \psi)(x)\big)\,|\mathrm{Jac}_{\phi^{-1}\circ\psi}|\,|dx_1 \dots dx_n|\\ &= \int_{\psi(U)} f(y)\,|dy_1 \dots dy_n| = \int_{\psi(U)} (\psi^{-1})^*\rho.\end{aligned}$$

To extend this to densities with arbitrary support, let us take $\{U_i\}$ a locally finite open cover of $M$ by relatively compact subsets and $\{\kappa_i\}$ a smooth partition of unity subordinate to $\{U_i\}$. Then, for each $i$, $\kappa_i\rho$ is a continuous density compactly supported in the domain of a single coordinate chart, whose integral is defined by (1.11).

**Definition 1.2.18.** *Let $M$ be a manifold and $\{U_i\}$ a locally finite open cover of $M$ by relatively compact subsets and $\{\kappa_i\}$ a smooth partition of unity subordinate to $\{U_i\}$. A continuous density $\rho$ on $M$ is* integrable *if and only if the following series converges*

$$\sum_{i=1}^{\infty} \int_M \kappa_i \, |\rho| \,. \tag{1.12}$$

*In this case, $\sum_{i=1}^{\infty} \int_M \kappa_i\rho$ converges and we define*

$$\int_M \rho := \sum_{i=1}^{\infty} \int_M \kappa_i\rho.$$

**Lemma 1.2.19.** *The previous definition does not depend on the choice of the open cover and the partition of unity.*

*Proof.* Let $\{V_i\}$ be another locally finite open cover of $M$ by relatively compact subsets and $\{\tilde{\kappa}_i\}$ a smooth partition of unity subordinate to $\{V_i\}$. Let $N \in \mathbb{N}$. For each $1 \leq j \leq N$, the support of $\tilde{\kappa}_j|\rho|$ is compact and meets only a finite number of $U_i$'s, so there is a $m_j$ such that $\tilde{\kappa}_j|\rho| = \sum_{i=1}^{m_j} \kappa_i\tilde{\kappa}_j \, |\rho|$. Let $m := \max_j(m_j)$, we have:

$$\begin{aligned}
\sum_{j=1}^{N} \int_M \tilde{\kappa}_j \, |\rho| &= \sum_{j=1}^{N} \int_M \sum_{i=1}^{m} \kappa_i\tilde{\kappa}_j \, |\rho| = \sum_{j=1}^{N} \sum_{i=1}^{m} \int_M \kappa_i\tilde{\kappa}_j \, |\rho| \\
&= \sum_{i=1}^{m} \int_M \sum_{j=1}^{N} \kappa_i\tilde{\kappa}_j \, |\rho| \leq \sum_{i=1}^{m} \int_M \kappa_i \, |\rho| \leq \sum_{i=1}^{\infty} \int_M \kappa_i \, |\rho| \,.
\end{aligned}$$

Therefore, the sum $\sum_{j=1}^{\infty} \int_M \tilde{\kappa}_j \, |\rho|$ is also convergent. The value of $\int_M \rho$ does not depend on the various choices neither since

$$\begin{aligned}
\sum_{i=1}^{\infty} \int_M \kappa_i \, |\rho| &= \sum_{i=1}^{\infty} \int_M \sum_{j=1}^{\infty} \tilde{\kappa}_j\kappa_i \, |\rho| = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_M \tilde{\kappa}_j\kappa_i \, |\rho| \\
&= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \int_M \tilde{\kappa}_j\kappa_i \, |\rho| = \sum_{j=1}^{\infty} \int_M \sum_{i=1}^{\infty} \tilde{\kappa}_j\kappa_i \, |\rho| = \sum_{j=1}^{\infty} \int_M \tilde{\kappa}_j \, |\rho| \,.
\end{aligned}$$

The commutation of the sum and the integral signs are justified because there are only a finite number of non-vanishing terms in the sum. The fact that the series is absolutely convergent allows to rearrange its terms. □

In particular, if $\rho$ is a compactly supported continuous density, then it is integrable. Indeed, its support, being compact, meets only a finite number of $U_i$'s, which implies that there are only a finite number of non-vanishing terms in (1.12). Therefore, we get a $\mathbb{C}-$linear functional on the space of compactly supported continuous densities on $M$

$$\int_M : \Gamma^0_c(M, |TM|) \to \mathbb{C} \; ; \; \rho \mapsto \int_M \rho \tag{1.13}$$

which has the following properties, which we borrow from [Lee13, Chapter 16].

**Proposition 1.2.20** (Properties of integration of densities)**.** *Let $M$ and $N$ be manifolds, and $\mu, \nu$ compactly supported continuous densities on $M$. Then*

*(a) For all $a, b \in \mathbb{C}$, $\int_M (a\mu + b\nu) = a \int_M \mu + b \int_M \nu$;*

*(b) If $\mu$ is positive, then $\int_M \mu > 0$;*

*(c) For all diffeomorphism $\Phi : N \to M$, $\int_M \mu = \int_N \Phi^*\mu$.*

*Remark* 1.2.21. Let $V$ be a complex vector space and denote by $E$ the trivial vector bundle $M \times V$ over the manifold $M$. The previous construction can be extended to $V-$valued densities, that is, sections of the vector bundle $E \otimes |TM|$. The integral is computed componentwise after a choice of basis of $V$ and this value does not depend on that choice because of the linearity of the integral. ◁

*Remark* 1.2.22. The construction we have presented here does not involve any choice from the start, making it clear that integration of densities is an intrinsic and canonical process. However, there is a more measure theoretical approach – which is the one followed by Dieudonné in [Die13] – that we now briefly describe. ◁

**Definition 1.2.23.** *Let $M$ be a manifold. A* measure *on $M$ is a linear functional on $\mathcal{C}^0_c(M)$ with the following property: for every compact subset $K \subset M$, there exists $a_K \geq 0$ such that, for all $f \in \mathcal{C}^0_c(M)$ supported in $K$,*

$$|u(f)| \leq a_K. \sup_{x \in K} |f(x)|.$$

*A measure $\mu$ on $M$ is a* Lebesguian measure *if, for every coordinate chart $(U, \phi)$ on $M$, the pushworward measure $\phi^*\mu$ is smoothly equivalent to the Lebesgue measure on $U$. That is, there exists a non-vanishing smooth function $f$ on $U$ such that $\phi^*\mu = f.dx_U$, where $dx_U$ is the Lebesgue measure on $U$.*

If we fix a smooth non-vanishing density $\rho$ on $M$, then the map (1.13) induces a Lebesguian measure $\mu_\rho$ on $M$ by the rule

$$f \in \mathcal{C}^0_c(M) \mapsto \int_M f\rho. \tag{1.14}$$

Since any continuous density $\lambda$ on $M$ is of the form $\lambda = g.\rho$ for some continuous function $g$, we can define that $\lambda$ is integrable if $g$ is, in which case we set $\int_M \lambda := \int_M g \, d\mu_\rho$. This definition turns out to be independent on the choice of $\rho$ and is equivalent to our construction.

*Remark* 1.2.24. Notice that in fact, every Lebesguian measure is of the form (1.14) for some non-vanishing smooth density $\rho$ (see [Die13, 23.4.2]). The density is positive if the measure is. $\lhd$

### 1.2.4 The intrinsic Hilbert space

Let us close this section by discussing half-densities and introducing a Hilbert space intrinsically associated to a manifold. Recall that, according to Remark 1.2.8, the complex conjugate of a half-density is still a half-density and that the product of two half-densities gives a one density, which can be integrated (at least if it has a compact support). This manipulation allows to define a Hermitian product on compactly supported smooth densities.

$$\Gamma_c^\infty(M, |TM|^{1/2}) \times \Gamma_c^\infty(M, |TM|^{1/2}) \to \mathbb{C} \; ; \; (\rho, \mu) \mapsto \int_M \rho.\bar{\mu} \; .$$

This leads to the following definition.

**Definition 1.2.25.** *Let $(E, \langle \cdot, \cdot \rangle_E)$ be a Hermitian vector bundle over a manifold $M$. On $\Gamma_c^\infty(M, E \otimes |TM|^{1/2})$, an inner product $\langle \cdot, \cdot \rangle$ is defined, for $r \otimes \rho, s \otimes \mu \in \Gamma_c^\infty(M, E \otimes |TM|^{1/2})$, by*

$$\langle r \otimes \rho, s \otimes \mu \rangle := \int_M \langle r, s \rangle_E \;\; \rho.\bar{\mu}. \tag{1.15}$$

*The* intrinsic Hilbert space *of square-integrable sections of $E$ is the completion of this pre-Hilbert space and it is denoted $L^2(M, E, \langle \cdot, \cdot \rangle_E)$, or $L^2(M, E)$ when there is no possible confusion about the Hermitian structure on $E$. When $E = M \times \mathbb{C}$, it is called the* intrinsic Hilbert space *of $M$ and it is denoted by $L^2(M)$. The norm on $L^2(M)$ is denoted by $|| \cdot ||_{L^2}$.*

*Remark* 1.2.26. Through the action given in Remark 1.2.16 and because of Proposition 1.2.20(c), the group $\mathrm{Diff}(M)$ acts on $L^2(M)$ by unitary transformations. $\lhd$

Although the previous construction is completely intrinsic, it is sometimes useful to work with a particular positive density – like in the Riemannian or symplectic framework –, for which we can also consider square-integrable functions. The following result shows that both constructions are naturally equivalent.

**Proposition 1.2.27.** *Let $M$ be a manifold, $\mu$ a positive Lebesguian measure on $M$, $\rho_\mu$ the corresponding positive density (see Remark 1.2.24) and $L^2(M, \mu)$*

*the Hilbert space of square-integrable functions on $M$ with respect to $\mu$. Then, the linear operator*

$$U : L^2(M,\mu) \to L^2(M) \; ; \; f \mapsto f.(\rho_\mu)^{1/2}$$

*is unitary.*

*Proof.* Since $(\rho_\mu)^{1/2}$ is a positive section of the bundle $|TM|^{1/2}$, any $\rho \in \Gamma_c^\infty(M, |TM|^{1/2}) \subset L^2(M)$ is of the form $f(\rho_\mu)^{1/2}$ for some $f \in \mathcal{C}_c^\infty(M)$. Since $||f(\rho_\mu)^{1/2}||_{L^2} = \int_M |f|^2 \rho_\mu < +\infty$, $f \in L^2(M,\mu)$. This shows that $U$ has a dense image. Next, for $f, g \in L^2(M,\mu)$, we have $\langle f, g\rangle_{L^2(M,\mu)} = \int_M f\bar{g}\,\rho_\mu = \int_M (f(\rho_\mu)^{1/2}).\overline{g(\rho_\mu)^{1/2}} = \left\langle f(\rho_\mu)^{1/2}, g(\rho_\mu)^{1/2}\right\rangle_{L^2(M)}$, which shows that $U$ is unitary. $\square$

## 1.3 Distributions on manifolds and generalized sections of vector bundles

Appearing in many areas of mathematics, physics and other fields of science, the $\delta-$function on $\mathbb{R}$ associates to a function $f$ on $\mathbb{R}$ the number $f(0)$. It is a basic example of what is called a distribution on $\mathbb{R}$, which is some kind of generalization of functions. This notion can be extended to vector bundles, in which case we call them generalized sections. For our purpose, generalized sections will turn out to be useful in mainly two ways. First, they allow to extend some linear operators on sections to a broader class of sections (or even to generalized sections). For instance, the Fourier transform of $e^{ix}$ is not a function, but it makes sense as a distribution. Second, generalized sections are a very powerful tool to describe those linear operators thanks to the Schwartz kernel theorem. It allows to study linear operators on functional spaces by looking at and manipulating their so-called kernel, which is a generalized section.

Generalized sections are defined as continuous linear functionals on some topological vector spaces of sections of a vector bundle. We will first consider the local theory of functions on open subsets of $\mathbb{R}^n$ in order to motivate the definitions. Then, we will extend the discussion to manifolds and vector bundles. The topological spaces we will define will be locally convex vector spaces. A short reminder on the related notions can be found in Appendix A and we refer to [Trè06] for the details. After these definitions, we will discuss the localization and support of generalized sections.

### 1.3.1 Local theory

In this subsection, let $U \subset \mathbb{R}^n$ be an open subset. The following family of seminorms will be central for all the subsequent definitions.

**Definition 1.3.1.** *Let $U \subset \mathbb{R}^n$ be an open subset. To each compact set $K \subset U$ and $r \in \mathbb{N}$, we associate a seminorm $||\cdot||_{K,r}$ on $\mathcal{C}^\infty(U)$ by*

$$||\cdot||_{K,r} : \mathcal{C}^\infty(U) \to \mathbb{R}^+ \; ; \; f \mapsto ||f||_{K,r} := \sup\left\{ |\partial^\alpha f(x)| \;\middle|\; x \in K, |\alpha| \leq r \right\}.$$

We now introduce several vector spaces of functions on $U$ and endow them with a specific topology to turn them into locally convex vector spaces. The proofs of the stated properties can be found in [Rud91].

### Smooth functions

The space of smooth functions on $U$

$$\mathcal{E}(U) := \mathcal{C}^\infty(U)$$

is endowed with the locally convex topology given by the family of seminorms

$$\left\{ ||\cdot||_{K,r} \;\middle|\; K \subset U \text{ compact}, r \in \mathbb{N} \right\}.$$

Using a countable exhaustion of $U$ by compact sets and the restriction property of families of seminorms, we can show that it is a Fréchet space.

### Smooth functions supported in a fixed compact set

Let $K \subset U$ be a compact set. The space of smooth functions on $U$ supported in $K$

$$\mathcal{E}_K(U) := \left\{ f \in \mathcal{C}^\infty(U) \;\middle|\; \mathrm{supp}(f) \subset K \right\}$$

is endowed with the locally convex topology given by the family of seminorms

$$\left\{ ||\cdot||_{K,r} \;\middle|\; r \in \mathbb{N} \right\}.$$

It is a Fréchet space. This topology is the same as the topology induced by the inclusion $\mathcal{E}_K(U) \hookrightarrow \mathcal{E}(U)$.

### Smooth functions of compact support

Let us denote the space of compactly supported smooth functions on $U$ by

$$\mathcal{D}(U) := \left\{ f \in \mathcal{C}^\infty(U) \;\middle|\; \mathrm{supp}(f) \text{ is compact} \right\}.$$

$\mathcal{D}(U)$ is not complete for the topology induced by $\mathcal{D}(U) \hookrightarrow \mathcal{E}(U)$ since a sequence of compactly supported functions may converge to a non compactly supported function. Let $K_1 \subset K_2 \subset \cdots \subset U$ be a countable exhaustion of $U$ by compact sets.[5] We then have

$$\mathcal{D}(U) = \bigcup_{k=1}^{\infty} \mathcal{E}_{K_k}(U)$$

and we endow $\mathcal{D}(U)$ with the inductive limit topology. Proposition A.17 shows that for a sequence to converge in that topology, there must exist a compact set such that every function of the sequence is supported in that compact set. $\mathcal{D}(U)$ is a complete space in the sense that any Cauchy sequence does converge in $\mathcal{D}(U)$. However, it is not a Fréchet space since it is not metrizable by Proposition A.18. It is worth noticing that the topology of $\mathcal{E}_K(U)$ coincides with the subspace topology corresponding to the inclusion $\mathcal{E}_K(U) \hookrightarrow \mathcal{D}(U)$.

#### Distributions

**Definition 1.3.2.** *Let $U \subset \mathbb{R}^n$ be an open subset. A* distribution *on $U$ is a continuous linear map*

$$u : \mathcal{D}(U) \to \mathbb{C}.$$

*The space of all distributions on $U$ is denoted by $\mathcal{D}'(U)$. For $u \in \mathcal{D}'(U)$ and $\varphi \in \mathcal{D}(U)$, we will use the following pairing notation*

$$\langle u, \varphi \rangle := u(\varphi).$$

*Example* 1.3.3. Any smooth function $f$ on $U$ induces a distribution on $U$ by the following definition:

$$u_f : \mathcal{D}(U) \to \mathbb{C} \; ; \; \varphi \mapsto \int_U f(x)\varphi(x) \, \mathrm{d}x, \tag{1.16}$$

where $\mathrm{d}x$ denotes the Lebesgue measure on $U$. This gives an inclusion $\mathcal{E}(U) \hookrightarrow \mathcal{D}'(U)$ which justifies the fact that distributions are considered as generalized functions. $\Diamond$

### 1.3.2 Global theory

We will now extend the previous discussion to the case of a vector bundle over a manifold. As a first step, we will consider various spaces of sections and use local charts to define topologies that turn them into locally convex vector spaces. Then, since we want to think about generalized sections precisely as a generalization of sections, we will have to make sense of formula (1.16) in the

[5] That is, for all $n = 1, 2, \ldots, +\infty$, $K_n$ is a compact subset of $M$ and $K_n$ is contained in the interior of $K_{n+1}$, and $M = \cup_{n=1}^{+\infty} K_n$.

context of sections of vector bundles. This will lead us to use the dual vector bundle to define the product of sections inside the integral, and to use densities over the manifold to be able to carry out the integration process.

Let $E \to M$ be a complex vector bundle of rank $p$ over a manifold $M$. As in the local case, we begin by introducing a family of seminorms on the space of smooth sections.

**Definition 1.3.4.** *Let $E \to M$ be a complex vector bundle of rank $p$ over a manifold $M$. Let $\mathcal{U} := \{U_i, \kappa_i, \tau_i\}_{i\in I}$ be a total trivialization*[6] *of $E$. Then, for each $i \in I$, we have an isomorphism of vector spaces*

$$\Phi_i : \Gamma^\infty\left(U_i, E_{|U_i}\right) \to \mathcal{C}^\infty\left(\kappa_i(U_i)\right)^p.$$

*To each $i \in I$, $1 \leq l \leq p$, $K \subset \kappa_i(U_i)$ compact and $r \in \mathbb{N}$, we associate a seminorm on $\Gamma^\infty(M, E)$ defined by:*

$$||\cdot||_{i,l,K,r} : \Gamma^\infty(M,E) \to \mathbb{R}^+ \ ; \ s \mapsto ||\Phi_i(s_{|U_i})^l||_{K,r} \ .$$

We can now consider several spaces of sections of $E$ and endow them with a structure of locally convex vector space.

### Smooth sections

The space of smooth sections of $E$ is denoted by

$$\mathcal{E}(M,E) := \Gamma^\infty(M,E)$$

when it is endowed with the topology induced by the family of seminorms

$$\left\{||\cdot||_{i,l,K,r} \ \middle| \ i \in I, 1 \leq l \leq p, K \subset \kappa_i(U_i) \text{ compact}, r \in \mathbb{N}\right\}.$$

It is a Fréchet space and the topology is independent on the choice of trivialization.

### Smooth sections supported in a fixed compact set

Let $K \subset M$ be a compact subset. We denote the space of smooth sections of $E$ supported on $K$ by

$$\mathcal{E}_K(M,E) := \left\{s \in \Gamma^\infty(M,E) \ \middle| \ \mathrm{supp}(s) \subset K\right\}$$

when it is endowed with the topology induced by the inclusion $\mathcal{E}_K(M,E) \hookrightarrow \mathcal{E}(M,E)$. It is a Fréchet space.

[6] Recall that it means that $\{U_i, \kappa_i\}_{i\in I}$ is an atlas of $M$, and that for each $i \in I$, $\tau_i : E_{|U_i} \to U_i \times \mathbb{C}^p$ is a local trivialization of $E \to M$.

### Smooth sections of compact support

The space of compactly supported sections of $E$ is denoted by

$$\mathcal{D}(M,E) := \left\{ s \in \Gamma^\infty(M,E) \,\middle|\, \operatorname{supp}(s) \text{ is compact in } M \right\}$$

when it is endowed with the following topology, whose definition depends on whether $M$ is compact or not.

- If $M$ is compact, then $\mathcal{D}(M,E) = \mathcal{E}(M,E)$ and we use the previously defined topology, which turns $\mathcal{D}(M,E)$ into a Fréchet space.
- In the noncompact case, let $K_1 \subset K_2 \subset \cdots \subset M$ be a countable exhaustion of $M$ by compact sets.[7] Then:

$$\mathcal{D}(M,E) = \bigcup_{k=1}^{\infty} \mathcal{E}_{K_k}(M,E).$$

  We endow $\mathcal{D}(M,E)$ with the inductive limit topology as in the local case. $\mathcal{D}(M,E)$ is complete but not Fréchet.

The inclusion $\mathcal{D}(M,E) \hookrightarrow \mathcal{E}(M,E)$ is continuous and dense.

*Remark* 1.3.5. Although $\mathcal{D}(M,E)$ and $\mathcal{E}(M,E)$ coincides respectively with $\Gamma_c^\infty(M,E)$ and $\Gamma^\infty(M,E)$ as vector spaces, we will usually use the notations $\mathcal{D}$ and $\mathcal{E}$ only when their topology is involved. $\triangleleft$

### Generalized sections

We are now ready to define the space $\mathcal{D}'(M,E)$ of "generalized sections" of $E$. As in the local case, we would like to have a natural inclusion $\mathcal{E}(M,E) \hookrightarrow \mathcal{D}'(M,E)$, by generalizing formula (1.16). Making sense of the product inside the integral as well as of the integration itself requires to define generalized sections as linear functionals on sections not of $E$ but of the so-called functional bundle.

**Definition 1.3.6.** *Let $E \to M$ be a vector bundle over a manifold $M$. The* functional bundle *of a vector bundle $E \to M$ over a manifold $M$ is the vector bundle over $M$*

$$E^\vee := E^* \otimes |TM|.$$

**Definition 1.3.7.** *Let $E \to M$ be a vector bundle over a manifold $M$. A* generalized section *of $E$ – or a* distribution *on $E$ – is a continuous linear functional on $\mathcal{D}(M,E^\vee)$. We denote by $\mathcal{D}'(M,E)$ the space of generalized section of $E$ – which is the continuous dual of $\mathcal{D}(M,E^\vee)$ –, endowed with the strong topology. The evaluation of $u \in \mathcal{D}'(M,E)$ on any $\rho \in \mathcal{D}(M,E^\vee)$ is denoted by $\langle u, \rho \rangle$.*

[7] That is, for all $n = 1, 2, \ldots, +\infty$, $K_n$ is a compact subset of $M$ and $K_n$ is contained in the interior of $K_{n+1}$, and $M = \cup_{n=1}^{+\infty} K_n$.

*Example* 1.3.8. As was previously advertised, any smooth section of the vector bundle $E \to M$ naturally gives a generalized section of $E$. Indeed, at each $x \in M$, we have $\left(E^* \otimes |TM|\right)_x \simeq \mathrm{Hom}\left(E_x, |T_x M|\right)$, so there is a pairing

$$(\cdot,\cdot) \ : \ \mathcal{E}(M,E) \times \mathcal{D}(M,E^\vee) \to \mathcal{D}(M,|TM|). \tag{1.17}$$

This gives a continuous inclusion

$$\mathcal{E}(M,E) \hookrightarrow \mathcal{D}'(M,E) \ ; \ s \mapsto \left[ s_0 \in \mathcal{D}(M,E^\vee) \mapsto \langle s, s_0 \rangle := \int_M (s,s_0) \right]. \tag{1.18}$$

$\diamondsuit$

There is an important subset of $\mathcal{D}'(M,E)$ which is formed by the generalized sections that can be defined not only on compactly supported smooth sections but on all smooth sections.

**Definition 1.3.9.** *Let $E \to M$ be a vector bundle over a manifold $M$. A* generalized section of $E$ of compact support *is a continuous linear functional on $\mathcal{E}(M,E^\vee)$. We denote by $\mathcal{E}'(M,E)$ the space of generalized section of $E$ of compact support – which is the continuous dual of $\mathcal{E}(M,E^\vee)$ –, endowed with the strong topology. The evaluation of $u \in \mathcal{E}'(M,E)$ on any $\rho \in \mathcal{E}(M,E^\vee)$ is denoted by $\langle u,\rho\rangle$.*

*Remark* 1.3.10. To verify that a linear functional on $\mathcal{D}(M,E^\vee)$ or $\mathcal{E}(M,E^\vee)$ is continuous, it is sufficient to check whether it is sequentially continuous. Indeed, $\mathcal{E}$ is a Fréchet space and $\mathcal{D}$ is also a Fréchet space if $M$ is compact, and an inductive limit of Fréchet spaces if $M$ is not compact. The claim then follows from Proposition A.19. $\lhd$

*Remark* 1.3.11. We should emphasize that we choose here to work with the strong topology on the continuous duals $\mathcal{D}'(M,E)$ and $\mathcal{E}'(M,E)$, which is different from the weak* topology chosen by Hörmander in [Hör03]. This choice will be of importance when we will state the Schwartz kernel theorem in Section 1.4.2. However, for some applications, it does not matter which topology we consider, as is shown in the next two lemmas. $\lhd$

**Lemma 1.3.12.** *Let $E \to M$ be a vector bundle over a manifold $M$. Then, in $\mathcal{D}'(M,E)$ and in $\mathcal{E}'(M,E)$, every sequence that converges in the weak* topology*[8] *also converges in the strong topology.*

*Proof.* From [Trè06, pp. 357-358], we know that $\mathcal{D}(M,E)$ and $\mathcal{E}(M,E)$ are Montel spaces and that, in the dual of a Montel space, every weakly convergent sequence is strongly convergent. $\square$

**Lemma 1.3.13.** *Let $E \to M$ be a vector bundle over a manifold $M$, and $V$ a locally convex vector space. Suppose that $V$ is a Fréchet space, or an inductive*

[8] Recall that the weak* convergence of linear functionals is the pointwise convergence.

*limit of Fréchet spaces. Then a linear map $P : V \to \mathcal{D}'(M, E)$ is continuous if it is sequentially continuous for the weak* topology on $\mathcal{D}'(M, E)$. Also, a linear map $Q : V \to \mathcal{E}'(M, E)$ is continuous if it is sequentially continuous for the weak* topology on $\mathcal{E}'(M, E)$.*

*Proof.* Let $P : V \to \mathcal{D}'(M, E)$ and $Q : V \to \mathcal{E}'(M, E)$. Proposition A.19 implies that $P$ and $Q$ are continuous if they are sequentially continuous, and Lemma 1.3.12 allows to conclude. □

The following result implies that every Cauchy sequence in the space of generalized sections converges to a generalized section.

**Theorem 1.3.14.** *Let $E \to M$ be a vector bundle over a manifold and $\{u_i\}_{i\in\mathbb{N}}$ a sequence in $\mathcal{D}'(M, E)$ such that*

$$u(\varphi) := \lim_{i\to\infty} u_i(\varphi) \tag{1.19}$$

*exists for every $\varphi \in \mathcal{D}(M, E)$. Then $u \in \mathcal{D}'(M, E)$.*

*Proof.* Let $\varphi_j \to 0$ in $\mathcal{D}(M, E)$. Then, there exists a compact $K \subset M$ such that $\mathrm{supp}(\varphi_j) \subset K$ for every $j$. Since $u_i$ is continuous on $\mathcal{D}(M, E)$, it is continuous on $\mathcal{E}_K(M, E)$. The latter being a Fréchet space, let us denote by $\{p_k\}_{k\in\mathbb{N}}$ a countable family of seminorms on $\mathcal{E}_K(M, E)$. Because of (1.19), the principle of uniform boundedness implies that $\{u_i\}$ is equicontinuous, that is, there exists $C > 0$ and $k \in \mathbb{N}$ such that $|u_i(\varphi_j)| < C \,.\, p_k(\varphi_j)$ for all $j$. Passing to the limit in $i$, we get that this inequality holds for $u$, so $u(\varphi_j) \to 0$, which shows that $u$ is continuous on $\mathcal{D}(M, E)$ by Remark 1.3.10. □

**Definition 1.3.15.** *Let $M$ be a manifold. A generalized section of the trivial bundle $M \times \mathbb{C}$ is called a* generalized function *or a* distribution *on $M$. It is given by a continuous linear functional on the space of compactly supported densities.*

*Example* 1.3.16. A well-known example is given by the $\delta-$function on $\mathbb{R}^n$. Any compactly supported density on $\mathbb{R}^n$ is given by $\rho(x) = f(x)\,|dx_1 \dots dx_n|$ where $f \in \mathcal{D}(\mathbb{R}^n)$ and $|dx_1 \dots dx_n|$ is the standard density corresponding to the standard coordinates $x = (x_1, \dots, x_n)$. Then, the $\delta-$function is defined by $\langle \delta, \rho\rangle := f(0)$. For a generic point $a \in \mathbb{R}^n$, we similarly define $\langle \delta_a, \rho\rangle := f(a)$. Notice however that on a generic manifold, there is no canonical way of defining a $\delta-$function at a point since there is no canonical positive density on that manifold. We will come back to that point later. ♢

**Definition 1.3.17.** *Let $M$ be a manifold. A generalized section of the density bundle is called a* generalized density*. If $E \to M$ is a vector bundle over $M$, a* generalized density of $E$ *is a generalized section of $E^* \otimes |TM|$.*

*Remark* 1.3.18. Let us give a more precise description of a generalized density. Since the density bundle of a manifold $M$ is a complex line bundle, $|TM|^* \otimes |TM|$ is canonically isomorphic to the trivial line bundle. Therefore, the space of generalized densities is canonically isomorphic to the space of continuous linear functionals on $\mathcal{D}(M, \mathbb{C})$. $\triangleleft$

*Remark* 1.3.19. We should warn about a possible confusion between generalized functions and generalized densities, since some authors define distributions as linear functionals on compactly supported functions. Recall that since the density bundle is trivial, both notions are completely equivalent as soon as we fix a non-vanishing density on the manifold. On $\mathbb{R}^n$, this is usually done using the Lebesgue density. This is why the $\delta-$function is usually defined on functions rather than on densities. $\triangleleft$

### 1.3.3 Localization and support

In this subsection, let $M$ be a manifold, and $E \to M$ be a vector bundle over $M$. We will see that generalized sections can be restricted to arbitrary open subsets of $M$. This allows to define two notions of support for a generalized section, one that describes the points where it is not vanishing (similarly to the support of a section), and another that describes its singularities.

Let $U \subset M$ be an open subset. Then, $U$ is itself a manifold and we can consider generalized sections of $E_{|U}$. Since we have the natural identification ${E^\vee}_{|U} \simeq (E_{|U})^\vee$, there is a natural inclusion

$$\mathcal{D}(U, (E_{|U})^\vee) \hookrightarrow \mathcal{D}(M, E^\vee) \; ; \; \rho \mapsto \hat{\rho} \tag{1.20}$$

given by extending a compactly supported section on $U$ by zero outside of $U$. This allows to restrict to $U$ a generalized section on $M$ in the following manner.

**Definition 1.3.20.** *Let $E \to M$ be a vector bundle over a manifold $M$ and $U \subset M$ an open subset. The* restriction to $U$ *of generalized sections of $E$ is defined by*

$$\mathcal{D}'(M, E) \to \mathcal{D}'(U, E_{|U}) \; ; \; u \mapsto u_{|U},$$

*where $\langle u_{|U}, \rho \rangle := \langle u, \hat{\rho} \rangle$ for all $\rho \in \mathcal{D}(U, (E_{|U})^\vee)$.*

The following theorem shows that a generalized section is completely determined by its local restrictions.

**Theorem 1.3.21.** *Let $E \to M$ be a vector bundle over a manifold $M$ and $u \in \mathcal{D}'(M, E)$. If for every $x \in M$, there exists an open neighbourhood $U$ of $x$ such that $u_{|U} = 0$, then $u = 0$.*

*Proof.* Let $\rho \in \mathcal{D}(M, E^\vee)$. For every $x \in \mathrm{supp}(\rho)$, let $U_x$ be an open neighbourhood of $x$ such that $u_{|U_x} = 0$. Since $\mathrm{supp}(\rho)$ is compact, it can be covered by a finite number $U_1, \ldots U_k$ of such $U_x$'s. Let $U_0 := M \backslash \mathrm{supp}(\rho)$ and let $\psi_0, \ldots, \psi_k$ be a partition of unity subordinate to $U_0, \ldots, U_k$. Then, for all $i = 1, \ldots, k$, $\mathrm{supp}(\psi_i \rho) \subset U_i$ and $u(\psi_i \rho) = u_{|U_i}(\psi_i \rho) = 0$. Finally, since $u$ is linear and $\rho = \sum_{i=1}^k \psi_i \rho$, it implies that $u(\rho) = 0$. $\square$

**Corollary 1.3.22.** *Let $E \to M$ be a vector bundle over a manifold $M$ and $\{U_i\}_{i \in I}$ an open cover of $M$. If $u, v \in \mathcal{D}'(M, E)$ are such that $u_{|U_i} = v_{|U_i}$ for every $i \in I$, then $u = v$.*

*Remark* 1.3.23. The previous results will be very useful to us in the sequel since they allow to study generalized sections in local coordinates. Also, they imply that to show that two generalized sections are equal, it is sufficient to show that they agree on sections of arbitrary small supports. $\triangleleft$

**Definition 1.3.24.** *Let $E \to M$ be a vector bundle over a manifold $M$ and $u$ a generalized section of $E$. We define*

$$M_u := \left\{ x \in M \quad | \, \exists \, U \, \textit{open neighbourhood of } x \textit{ such that } u_{|U} = 0 \right\}.$$

*It is the largest open subset of $M$ on which the restriction of $u$ is zero. The* support *of $u$ is defined as*

$$\mathrm{supp}(u) := M \backslash M_u.$$

*Remark* 1.3.25. The generalized sections of compact support as defined in 1.3.9 are exactly those such that $\mathrm{supp}(u)$ is compact in $M$. $\triangleleft$

*Remark* 1.3.26. More generally, the domain of definition of a generalized section $u$ can be extended to any smooth section $s$ such that $\mathrm{supp}(u) \cap \mathrm{supp}(s)$ is compact. Indeed, choose $\phi$ a compactly supported smooth function that equals 1 on a neighbourhood of $\mathrm{supp}(u) \cap \mathrm{supp}(s)$. Then, we can define $\langle u, s \rangle$ by $\langle u, \phi.s \rangle$, which does not depend on the choice of $\phi$ since $\langle u, \phi.s \rangle$ only depends on $(\phi.s)_{|\mathrm{supp}(u) \cap \mathrm{supp}(s)} = s_{|\mathrm{supp}(u) \cap \mathrm{supp}(s)}$. $\triangleleft$

When a generalized function is represented by a smooth section under the inclusion (1.18), it is called regular, and singular otherwise. Thanks to the localization property of generalized sections, we can be more precise in the description of the singularities of a generalized section by describing the points around which it cannot be represented by a smooth section.

**Definition 1.3.27.** *Let $E \to M$ be a vector bundle over a manifold $M$ and $u$ a generalized section of $E$. The* singular support *of $u$ is denoted* $\mathrm{sing}(u)$ *and is defined as follows. A point $x \in M$ does not lie in* $\mathrm{sing}(u)$ *if there exists an open neighbourhood $U$ of $x$ and a smooth section $s$ of $E$ on $U$ such that $u_{|U} = s$ under the inclusion (1.18).*

## 1.4 General operators and kernels

When studying linear operators between functions on manifolds – or more generally, between sections of vector bundles –, one often works with a specific functional space, that is, a specific subspace of sections and a corresponding topology. It might be for example $L^2(M)$ or some Sobolev space on $M$. In many cases, the functional space contains the space of compactly supported smooth sections and the inclusion is continuous and dense. On the other hand, in very much the same way as smooth sections are included in the space of generalized sections, those functional spaces are often continuously included in the space of generalized sections. Because the inclusions are continuous, any continuous linear operator between such functional spaces therefore gives a continuous linear operator from compactly supported sections to generalized sections. This is the motivation to introduce general operators as we will do now.

### 1.4.1 Definitions

**Definition 1.4.1.** *Let $M$ and $N$ be two manifolds and $E \to M$ and $F \to N$ two vector bundles. A* general operator *$P$ from $F$ to $E$ is a continuous linear map*

$$P : \mathcal{D}(N, F) \to \mathcal{D}'(M, E).$$

*We denote by $\mathcal{L}_b\big(\mathcal{D}(N, F) \to \mathcal{D}'(M, E)\big)$ the vector space of the general operators, endowed with the strong topology.*

*Remark* 1.4.2. To check whether a linear operator $P : \mathcal{D}(N, F) \to \mathcal{D}'(M, E)$ is continuous might seem to be complicated for two reasons. First, $\mathcal{D}'(M, E)$ is endowed with its strong topology, which is more complicated to deal with than the weak* one. Second, if $N$ is not compact, $\mathcal{D}(N, F)$ is not metrizable, so sequential continuity of $P$ does not imply that it is continuous. However, we can forget about these difficulties since Lemma 1.3.13 implies that we only need to check whether $P$ maps convergent sequences to weakly convergent sequences. $\triangleleft$

The following example illustrates the discussion at the beginning of this section, which led us to the definition of a general operator.

*Example* 1.4.3. Let $M$ be a manifold. Any continuous linear operator

$$L : L^2(M) \to L^2(M)$$

gives rise to a general operator

$$P_L : \mathcal{D}(M, |TM|^{1/2}) \to \mathcal{D}'(M, |TM|^{1/2}).$$

Indeed, we have the continuous inclusion $\mathcal{D}(M, |TM|^{1/2}) \hookrightarrow L^2(M)$. On the other hand, since $|TM| \simeq |TM|^{1/2} \otimes |TM|^{1/2}$, $(|TM|^{1/2})^* \otimes |TM| \simeq |TM|^{1/2}$, so $\mathcal{D}'(M, |TM|^{1/2})$ is the continuous dual of $\mathcal{D}(M, |TM|^{1/2})$. To any $\rho \in L^2(M)$, we can therefore associate the generalized section given by

$$\mathcal{D}(M, |TM|^{1/2}) \to \mathbb{C} \; ; \; \mu \mapsto \int_M \rho\mu.$$

This gives an inclusion $L^2(M) \hookrightarrow \mathcal{D}'(M, |TM|^{1/2})$ which is continuous. Indeed, let $\rho_n \to 0$ in $L^2(M)$. Then, for each $\mu \in \mathcal{D}(M, |TM|^{1/2})$, using the Cauchy-Schwartz inequality, we get that $|\langle \rho_n, \mu \rangle| = |\int_M \rho_n \mu| \leq ||\rho_n||_{L^2}.||\mu||_{L^2} \to 0$. By Lemma 1.3.13, the inclusion is continuous. Putting everything together, we can define the linear operator

$$P_L : \mathcal{D}(M, |TM|^{1/2}) \hookrightarrow L^2(M) \xrightarrow{L} L^2(M) \hookrightarrow \mathcal{D}'(M, |TM|^{1/2}),$$

which is continuous because all the inclusions and $L$ are continuous. $\lozenge$

Other examples of general operators, that will be central in our discussion, are given by the pullback of functions on a manifold and sections of a vector bundle.

*Example* 1.4.4. Let $h : M \to N$ be a smooth map between manifolds and consider the pullback

$$h^* : \mathcal{E}(N) \to \mathcal{E}(M) \; ; \; f \mapsto f \circ h,$$

which is a continuous linear map. Because of the continuous inclusions $\mathcal{D}(N) \hookrightarrow \mathcal{E}(N)$ and $\mathcal{E}(M) \hookrightarrow \mathcal{D}'(M)$, it gives a general operator by the rule

$$P^h : \mathcal{D}(N) \to \mathcal{D}'(M) \; ; \; f \mapsto \left[\rho \mapsto \int_M (h^* f)\, \rho = \int_M (f \circ h)\, \rho\right]. \tag{1.21}$$

Notice that if $h$ is proper, then $h^*$ maps compactly supported functions to compactly supported functions. $\lozenge$

The following notion of a geometric morphism allows to naturally generalize the pullback operation to sections of vector bundles.

**Definition 1.4.5.** *Let $M$ and $N$ be two manifolds, and $E \to M$ and $F \to N$ two vector bundles over those manifolds. A* geometric morphism *from $E$ to $F$ is a pair $\underline{h} = (h, r)$ where $h : M \to N$ is a smooth map and $r$ is a smooth section of* $\mathrm{Hom}(h^*F, E)$.[9] *In particular, for every $x \in M$, it gives a linear operator*

$$r(x) : F_{h(x)} \to E_x.$$

[9] Recall that the pullback bundle $h^*F$ is a vector bundle over $M$ whose fiber at a point $x \in M$ is $F_{h(x)}$, and that $\mathrm{Hom}(h^*F, E)$ is a vector bundle over $M$ whose fiber at a point $x \in M$ is $\mathrm{Hom}(F_{h(x)}, E_x)$.

*Remark* 1.4.6. Notice that this is not the same notion as a morphism of vector bundles since here, the linear maps $r(x)$ are going the other way around, to get back from the fibers of $F$ to those of $E$. This is because we want to define the pullback of a section. ◁

*Remark* 1.4.7. Let $F \to N$ be a vector bundle and $h : M \to N$ a smooth map. Then, there is an obvious natural geometric morphism from $h^*F$ to $F$, whose corresponding section of $\mathrm{Hom}(h^*F, h^*F)$ is the identity section. We will usually still denote this geometric morphism by $h$. ◁

*Example* 1.4.8. Let $M$ and $N$ be two manifolds, $E \to M$ and $F \to N$ two vector bundles over those manifolds, and $\underline{h} = (h, r)$ a geometric morphism from $E$ to $F$. We define the *pullback by* $\underline{h}$ as the operator

$$\underline{h}^* : \Gamma^\infty(N, F) \to \Gamma^\infty(M, E) \; ; \; s \mapsto \underline{h}^* s,$$

where, for every $x \in M$,

$$(\underline{h}^* s)(x) := r(x)\big(\, s(h(x))\,\big) \quad \in E_x.$$

As in the previous example, this defines a general operator

$$P^{\underline{h}} : \mathcal{D}(N, F) \to \mathcal{D}'(M, E) \; ; \; s \mapsto \left[\rho \in \mathcal{D}(M, E^\vee) \mapsto \int_M (\underline{h}^* s, \rho)\right], \tag{1.22}$$

where $(\cdot,\cdot)$ denotes the pairing (1.17). Again, if $h$ is proper, then $\underline{h}^*$ maps compactly supported sections to compactly supported sections. ◊

### 1.4.2 The Schwartz kernel theorem

There is a deep link between general operators and generalized sections, which is suggested by the following example.

*Example* 1.4.9. Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be two open subsets, and $k \in \mathcal{C}^\infty(U \times V)$. Then, we can associate to $k$ a continuous linear operator $P_k : \mathcal{D}(V) \to \mathcal{E}(U)$ defined, for every $\varphi \in \mathcal{D}(V)$ and $x \in U$, by the formula

$$P_k(\varphi)(x) := \int_V k(x, y)\, \varphi(y)\, dy.$$

Since $\mathcal{E}(U) \hookrightarrow \mathcal{D}'(U)$, $P_k$ induces a general operator $P_k : \mathcal{D}(V) \to \mathcal{D}'(U)$ given, for every $\varphi \in \mathcal{D}(V)$ and $\psi \in \mathcal{D}(U)$, by

$$\begin{aligned} \langle P_k(\varphi), \psi\rangle &= \int_{U\times V} k(x, y)\, \psi(x)\, \varphi(y)\, dx\, dy \\ &= \langle k, \mathrm{pr}_U^*(\psi) \otimes \mathrm{pr}_V^*(\varphi)\rangle \,, \end{aligned}$$ [10]

where in the last line, $k$ is seen as an element of $\mathcal{D}'(U \times V)$ and $\mathrm{pr}_U^*(\psi) \otimes \mathrm{pr}_V^*(\varphi) \in \mathcal{D}(U \times V)$. Since $\langle k, \mathrm{pr}_U^*(\psi) \otimes \mathrm{pr}_V^*(\varphi)\rangle$ only depends on $k$ as a generalized function on $U \times V$ and not as a smooth function, this suggests a way to associate a general operator to any generalized function on $U \times V$ by the same formula. This construction can be extended to a general manifold, although care must be taken to introduce densities at the right places. $\Diamond$

The so-called Schwartz kernel theorem asserts that the previous construction is in fact completely general, in the sense that any general operator can be represented in a unique way by a generalized section – the *kernel* of the operator. It also states that this association is a topological isomorphism[11]. As we shall see, there are several variants of the kernel theorem, the first versions of which are due to Schwartz [Sch57]. A proof in the setting of general operators between vector bundles can be found in [Tar12, Section 1.5].

**Definition 1.4.10.** *Let $E \to M$ and $F \to N$ be two complex vector bundles over the manifolds $M$ and $N$. The* external tensor product *is the vector bundle over $M \times N$ defined by*

$$E \boxtimes F := \mathrm{pr}_M^*(E) \otimes \mathrm{pr}_N^*(F).$$

*Notice that its fiber over a point $(x, y) \in M \times N$ is given by*

$$(E \boxtimes F)_{(x,y)} = E_x \otimes F_y.$$

**Theorem 1.4.11** (Schwartz kernel theorem)**.** *Let $M$ and $N$ be two manifolds and $E \to M$ and $F \to N$ two complex vector bundles. There is a topological isomorphism*

$$\mathcal{D}'(M \times N, E \boxtimes F^\vee) \xrightarrow{\sim} \mathcal{L}_b\big(\mathcal{D}(N, F) \to \mathcal{D}'(M, E)\big)$$

*that associates to a generalized section $k \in \mathcal{D}'(M \times N, E \boxtimes F^\vee)$ the general operator $P_k$ given by*

$$P_k : \mathcal{D}(N, F) \to \mathcal{D}'(M, E)\ ;\ \varphi \mapsto [\psi \mapsto \langle k, \mathrm{pr}_M^*\psi \otimes \mathrm{pr}_N^*\varphi\rangle]\,. \tag{1.23}$$

*The generalized section $k$ is called the* kernel *of the operator $P_K$.*

To better see how this isomorphism works, let us explain what is meant by formula (1.23). There is an isomorphism

$$(E \boxtimes F^\vee)^\vee \simeq E^\vee \boxtimes F. \tag{1.24}$$

[10] $\mathrm{pr}_U$ and $\mathrm{pr}_V$ denote the projection of $U \times V$ onto $U$ and $V$ respectively.

[11] The fact that the isomorphism is topological is true only if we consider the strong topology on both spaces.

Indeed, we have:

$$\begin{aligned}(E \boxtimes F^\vee)^\vee &\simeq (\mathrm{pr}_M^*(E) \otimes \mathrm{pr}_N^*(F^*) \otimes \mathrm{pr}_N^*(|TN|))^* \otimes |T(M \times N)| \\ &\simeq \mathrm{pr}_M^*(E^*) \otimes \mathrm{pr}_N^*(F) \otimes \mathrm{pr}_N^*(|TN|^*) \\ &\quad \otimes \mathrm{pr}_M^*(|TM|) \otimes \mathrm{pr}_N^*(|TN|) \\ &\simeq \mathrm{pr}_M^*(E^*) \otimes \mathrm{pr}_M^*(|TM|) \otimes \mathrm{pr}_N^*(F) \simeq E^\vee \boxtimes F.\end{aligned}$$

Since for $\varphi \in \mathcal{D}(N, F)$ and $\psi \in \mathcal{D}(M, E^\vee)$, $\mathrm{pr}_M^*\psi \otimes \mathrm{pr}_N^*\varphi$ is a section of $E^\vee \boxtimes F$, this isomorphism allows to evaluate $k \in \mathcal{D}'\,(M \times N, E \boxtimes F^\vee)$ on this section.

*Example* 1.4.12. In the case of operators between functions on manifolds, a kernel is a generalized section $k \in \mathcal{D}'(M \times N, \mathrm{pr}_N^*(|TN|))$. The isomorphism (1.24) corresponds to

$$\begin{aligned}(\mathrm{pr}_N^*(|TN|))^\vee &= (\mathrm{pr}_N^*(|TN|))^* \otimes |T(M \times N)| \\ &\simeq (\mathrm{pr}_N^*(|TN|))^* \otimes \mathrm{pr}_N^*(|TN|) \otimes \mathrm{pr}_M^*(|TM|) \\ &\simeq \mathrm{pr}_M^*(|TM|).\end{aligned}$$

The operator $P_k : \mathcal{D}(N) \to \mathcal{D}'(M)$ associated to $k$ is therefore defined, for $\varphi \in \mathcal{D}(N)$ and $\psi \in \mathcal{D}(M, |TM|)$, by

$$P_k(\varphi)(\psi) := \langle k, \mathrm{pr}_N^*\varphi \,.\, \mathrm{pr}_N^*\bar{\mu}_0 \otimes (\mathrm{pr}_N^*\mu_0 \otimes \mathrm{pr}_M^*\psi)\rangle\,, \tag{1.25}$$

where $\mu_0$ is any non-vanishing density on $N$, $\bar{\mu}_0$ the corresponding dual density, and $(\mathrm{pr}_N^*\mu_0 \otimes \mathrm{pr}_M^*\psi)$ is seen as a density on $M \times N$. ◊

*Example* 1.4.13. In the case of general operators between vector bundles, let us give the expression of the operator $P_k : \mathcal{D}(N) \to \mathcal{D}'(M)$ associated to a kernel $k \in \mathcal{D}'(M \times N, E \boxtimes F^\vee)$. Let $\varphi \in \mathcal{D}(N, F)$ and $\psi \in \mathcal{D}(M, E^* \otimes |TM|)$. Since $|TM|$ is a trivial complex line bundle, $\psi$ can be written as $\psi = \psi_1 \otimes \psi_2$ with $\psi_1 \in \mathcal{D}(M, E^*)$ and $\psi_2 \in \mathcal{E}(M, |TM|)$. Then, we have

$$P_k(\varphi)(\psi) := \langle k, \mathrm{pr}_N^*\varphi \otimes \mathrm{pr}_M^*\psi_1 \otimes \mathrm{pr}_N^*\bar{\mu}_0 \otimes (\mathrm{pr}_N^*\mu_0 \otimes \mathrm{pr}_M^*\psi_2)\rangle\,, \tag{1.26}$$

where $\mu_0$ is any non-vanishing density on $N$, $\bar{\mu}_0$ the corresponding dual density, and $(\mathrm{pr}_N^*\mu_0 \otimes \mathrm{pr}_M^*\psi_2)$ is seen as a density on $M \times N$. ◊

*Remark* 1.4.14. We have seen in Example 1.4.3 that a continuous linear operator on $L^2(M)$ naturally gives a general operator. A related but in some sense opposite question is to know whether a general operator can be continuously extended to a functional space larger than $\mathcal{D}(M, E)$. Similarly, we can ask whether its range consists in a functional space smaller than the whole $\mathcal{D}'(N, F)$, while still being continuous for the topology on that functional space. We will see that these questions can sometimes be answered solely from properties of the kernel of the operator – such as regularity or integrability – showing the power of the Schwartz kernel theorem. An extreme situation corresponds to smoothing operators, that we now introduce. ◁

### 1.4.3 Smoothing operators

**Definition 1.4.15.** *Let $M$ and $N$ be two manifolds and $E \to M$ and $F \to N$ two vector bundles. A general operator $P$ from $F$ to $E$ is* smooth *if its kernel is an element of $\mathcal{E}(M \times N, E \boxtimes F^\vee)$.*

**Definition 1.4.16.** *Let $M$ and $N$ be two manifolds and $E \to M$ and $F \to N$ two vector bundles. A* smoothing operator *$P$ from $F$ to $E$ is a general operator $P : \mathcal{D}(N, F) \to \mathcal{D}'(M, E)$ such that*

- *the range of $P$ is contained in $\mathcal{E}(M, E)$;*
- *$P$ extends to a continuous linear map $\mathcal{E}'(N, F) \to \mathcal{E}(M, E)$.*

*We denote by $\mathcal{L}_b\big(\mathcal{E}'(N, F) \to \mathcal{E}(M, E)\big)$ the vector space of the smoothing operators, endowed with the strong topology.*

This variant of the Schwartz kernel theorem – they follow from the same theorem proved in [Tar12, Section 1.5] – characterizes those smoothing operators.

**Theorem 1.4.17.** *Let $M$ and $N$ be two manifolds and $E \to M$ and $F \to N$ two vector bundles. There is a topological isomorphism*

$$\mathcal{L}_b\big(\mathcal{E}'(N, F) \to \mathcal{E}(M, E)\big) \xrightarrow{\sim} \mathcal{E}(M \times N, E \boxtimes F^\vee)$$

*given by sending a smoothing operator to its kernel. In particular, a general operator $P : \mathcal{D}(N, F) \to \mathcal{D}'(M, E)$ is smooth if and only if it is a smoothing operator.*

*Remark* 1.4.18. On a compact manifold $M$, $\mathcal{E} = \mathcal{D}$, which implies that smoothing operators can be composed together. The kernel of the composition is given by the convolution of the kernels. On a non-compact manifold, this is not true anymore. $\triangleleft$

Let $E \to M$ be a vector bundle over a manifold $M$. We will now define a notion of trace for smooth operators from $E$ to $E$. Let $k \in \mathcal{E}(M \times M, E \boxtimes E^\vee)$ be the kernel of a smooth operator from $E$ to $E$. Then, for all $x \in M$, $k(x, x) \in E_x \otimes E_x^* \otimes |T_xM| = \mathrm{Hom}(E_x, E_x) \otimes |T_xM|$. Taking the trace of the homomorphism thus gives a density at $x$. Therefore, $[x \mapsto \mathrm{Tr}\,\big(k(x, x)\big)]$ is a smooth density on $M$, that can be integrated if it is integrable.

**Definition 1.4.19.** *Let $E \to M$ be a vector bundle over a manifold $M$ and $P_k$ a smooth operator from $E$ to $E$ with kernel $k \in \mathcal{E}(M \times M, E \boxtimes E^\vee)$. We say that $P_k$ is* smooth-traceable *if $[x \mapsto \mathrm{Tr}\,\big(k(x, x)\big)]$ is integrable. Then, the* smooth trace *of $P_k$ is defined as*

$$\mathrm{tr}\,(P_k) := \int_M \mathrm{Tr}\,\big(k(x, x)\big).$$

### 1.4.4 The smooth trace of trace-class operators

We will now see how the smooth trace is related to the usual trace of operators on Hilbert spaces of square-integrable functions. Notice that this section aims only at suggesting some motivation to study the smooth trace. It will not be needed in the following part of this chapter. The reader who would appreciate a quick recap on Hilbert-Schmidt and trace-class operators might find useful to first refer to Section 2.3 before going on.

Let $M$ be a manifold, $\mu$ a measure on $M$, and $A : L^2(M, \mu) \to L^2(M, \mu)$ a continuous linear operator. Suppose that $A$ is integral, that is, there exists some measurable function $K$ on $M \times M$ such that for every $\phi \in L^2(M, \mu)$ and almost every $x \in M$:

$$(A\phi)(x) = \int_M K(x, y)\phi(y)\, d\mu(y).$$

$K$ is called the kernel of $A$. It is well-known that $A$ is Hilbert-Schmidt if and only if its kernel belongs to $L^2(M \times M, \mu \times \mu)$. However, the question of determining whether $A$ is trace-class and computing its trace, solely from its kernel, turns out to be much more subtle.

A useful result in that direction, first due to Duflo [Duf72] and then generalized by Brislawn [Bri91], gives some conditions to express the trace as the integral of the kernel over the diagonal. Specializing Brislawn's result to measures on manifolds, we have the following theorem.

**Theorem 1.4.20.** *Let $\mu$ be a measure on a manifold $M$, and let $K$ be a trace-class operator on $L^2(M, \mu)$. If the kernel $K(x, y)$ is continuous at $(x, y)$ for almost every $x$, then*

$$\mathrm{Tr}\,(K) = \int_M K(x, x) d\mu(x).$$

It should be emphasized that in this theorem, the operator has to be known to be trace-class. The integrability of the kernel along the diagonal is not sufficient to ensure that the operator is trace-class, even when the kernel is continuous and integrable. Carleman [Car16] has given an example of an operator on $L^2(S^1)$ with a continuous kernel (hence integrable since the manifold is compact) which is not trace-class. However, more can be said if we impose more regularity on the kernel. Delgado and Ruzhansky [DR14] give a simple regularity condition on the kernel of an integral operator on the square-integrable functions on a compact manifold, that ensures that it is trace-class. As a particular case, we have:

**Theorem 1.4.21.** *Let $M$ be a compact manifold endowed with a positive measure $\mu$. Let $k \in \mathcal{C}^\infty(M \times M)$. Then, the integral operator $P$ on $L^2(M, \mu)$*

*defined, for* $\varphi \in L^2(M,\mu)$, *by*

$$(P\varphi)(x) := \int_M k(x,y)\varphi(y)d\mu(y)$$

*is trace-class and its trace is given by*

$$\mathrm{Tr}\,(P) = \int_M k(x,x)d\mu(x).$$

Theorems 1.4.20 and 1.4.21 give the relation between our smooth trace and the usual trace of linear operators on $L^2(M,\mu)$.

## 1.5 Operations on generalized sections

Functions on a manifold and sections of a vector bundle can be manipulated in a variety of ways: multiplication by a function, pullback of a function, pushforward of a vector field by a diffeomorphism, etc. In this section, we would like to extend such kind of operations to generalized sections. For instance, since functions are particular generalized functions, we can ask whether the pullback can be defined for every generalized function. This turns out to be possible only with restrictions because of the singularities exhibited by generalized sections. We thus have to limit either the set of generalized functions we consider, or the set of maps by which we want to pullback. In this section, the question of extending those operations will be addressed using duality, that is, using the fact that generalized sections are linear functionals on sections. However, we should mention that there are other ways to carry on the extension of the pullback and pushforward from sections to generalized sections, such as extension by continuity – which is the approach of Hörmander [Hör03] – or more specific definitions – like we will do in Section 1.6.

In the following discussion, $M$ and $N$ will denote two manifolds, and $E \to M$ and $F \to N$ vector bundles over those manifolds.

### 1.5.1 Multiplication by a function

As a warm-up, let $f : M \to \mathbb{C}$ be a smooth function. For any $\rho \in \mathcal{D}(M, E^\vee)$, we can consider the section $f\rho$ given by the pointwise multiplication. It is still a compactly supported section of $E^\vee$, which allows the following definition.

**Definition 1.5.1.** *Let* $f : M \to \mathbb{C}$ *be a smooth function over a manifold* $M$ *and* $u$ *a generalized section of a vector bundle* $E \to M$*. The* multiplication *of* $u$ *by* $f$ *is the generalized section* $fu \in \mathcal{D}'(M,E)$ *defined, for* $\rho \in \mathcal{D}(M,E^\vee)$, *by*

$$\langle fu, \rho\rangle := \langle u, f\rho\rangle\,.$$

### 1.5.2 Pushforward of a density by a submersion

We will now define the pushforward of a density by a submersion through the process of "integration along the fibers". For the integrals to be finite, we will require a properness condition relative to the support of the density.

**Definition 1.5.2.** *Let $f : X \to Y$ be a continuous map between two topological spaces, and $E \subset X$. We say that $f_{|E}$ is* proper *if and only if, for all $K \subset Y$ compact, $f^{-1}(K) \cap E$ is compact.*

Let $\rho \in \Gamma^\infty(N, |TN|)$ and $h : N \to M$ a submersion such that $h_{|\mathrm{supp}(\rho)}$ is proper. To explain how the integration along the fibers works, let $x \in h(N) \subset M$. Since $h$ is a submersion, $Z_x := h^{-1}(\{x\})$ is an embedded submanifold of $N$. For any $z \in Z_x$, we have the short exact sequence

$$0 \to T_z Z_x \xrightarrow{(\iota_{Z_x})_{*z}} T_z N \xrightarrow{h_{*z}} T_x M \to 0,$$

where $\iota_{Z_x}$ denotes the inclusion of $Z_x$ in $N$. Lemma 1.2.4 gives an isomorphism

$$|T_z N| \simeq |T_z Z_x| \otimes |T_x M|. \tag{1.27}$$

Therefore, when restricted to $Z_x$, the density $\rho$ can be seen as a density $\rho_{Z_x}$ over $Z_x$ valued in $|T_x M|$. $h_{|\mathrm{supp}(\rho)}$ being proper, $Z_x \cap \mathrm{supp}(\rho)$ is compact, so $\rho_{Z_x}$ is of compact support. It can thus be integrated to give an element of $|T_x M|$, to which we set $(h_*\rho)(x)$. For any $x \in M$ outside of the range of $h$, we set $(h_*\rho)(x) = 0$.

**Proposition 1.5.3.** *Let $h : N \to M$ be a submersion between two manifolds. Then, for any $\rho \in \Gamma^\infty(N, |TN|)$ such that $h_{|\mathrm{supp}(\rho)}$ is proper, the pushforward $h_*\rho$ defined as above is a smooth density on $M$. Furthermore,* $\mathrm{supp}(h_*\rho) \subset h(\mathrm{supp}(\rho))$. *In particular, if $\rho$ is compactly supported, so is $h_*\rho$.*

*Proof.* Let $\rho \in \Gamma^\infty(N, |TN|)$. First, the claim regarding the support of $h_*\rho$ follows from the facts that, from the definition, $(h_*\rho)(x) = 0$ if $x \notin h(\mathrm{supp}(\rho))$ and that $h(\mathrm{supp}(\rho))$ is closed since $h_{|\mathrm{supp}(\rho)}$ is proper[12].

Next, suppose that $\rho$ is supported in a coordinate patch

$$(V, \varphi = (y_1, \dots, y_m, z_1, \dots, z_k))$$

such that $h$ has the local expression

$$h(y_1, \dots, y_m, z_1, \dots, z_k) = (y_1, \dots, y_m)$$

[12] To show this last claim, let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence in $h(\mathrm{supp}(\rho))$ converging to $x$. We can choose a sequence $\{y_k\}_{k\in\mathbb{N}}$ in $\mathrm{supp}(\rho)$ such that $h(y_k) = x_k$. Since $\{x\} \cup \{x_k\}_{k\in\mathbb{N}}$ is compact, there exists a convergent subsequence $\{y_{k_i}\}$. Let $y$ be its limit, which must belong to $\mathrm{supp}(\rho)$ since the latter is closed. By continuity of $h$, $h(y) = x$, which shows that $x \in h(\mathrm{supp}(\rho))$, hence $h(\mathrm{supp}(\rho))$ is closed.

for some local coordinates $(U, \psi = (x_1, \dots, x_m))$ on $M$ such that $h(V) \subset U$. Then, $\rho$ has the local form

$$\rho_V(y_1, \dots, y_m, z_1, \dots, z_k)\, |dz_1 \dots dz_k\, dy_1 \dots dy_m|$$

for some smooth function $\rho_V$ on $V$. Let $x \in h(V)$ with coordinates $(x_1, \dots, x_m)$ and $Z_x := h^{-1}(\{x\})$. Since $(z_1, \dots, z_k)$ are local coordinates on $Z_x$ the density $\rho_{Z_x}$ corresponding to the splitting (1.27) has the form

$$\rho_U(x_1, \dots, x_m, z_1, \dots, z_k)\, |dz_1 \dots dz_k| \otimes |dx_1 \dots dx_m|.^{13}$$

Integration over $Z_x$ finally leads to

$$\begin{aligned} &(h_*\rho)(x_1, \dots, x_m) \\ &= \left( \int_{\varphi(Z_x \cap V)} \rho_V(x_1, \dots, x_m, z_1, \dots, z_k)\, dz_1 \dots dz_k \right) |dx_1 \dots dx_m|. \end{aligned} \tag{1.28}$$

As mentioned before, the integral is well-defined because $h_{|\mathrm{supp}(\rho)}$ is proper. The properness also ensures that $h_*\rho$ is smooth with respect to $x$. Indeed, if we fix an open subset $W$ in $h(V)$ with compact closure, the properness allows to choose a common compact integration domain to replace $\varphi(Z_x \cap V)$ in (1.28) for every $x$ in $W$. This implies that $h_*\rho$ is smooth on $W$, hence on $h(V)$. $h_*\rho$ is also smooth outside of $h(V)$ since we have seen that its support is contained in $h(V)$.

Finally, if $\rho$ is not supported in such a coordinate patch, we can use a partition of unity and the properness assumption to express locally $h_*\rho$ as a finite sum of smooth densities. □

*Remark* 1.5.4. It is clear from formula (1.28) that $\int_N \rho = \int_M h_*\rho$. ◁

This construction can be readily extended to the case of vector bundles. Let $\underline{h} = (h, r)$ be a geometric morphism from $F$ to $E$, such that $h : N \to M$ is a submersion and $\rho \in \Gamma^\infty(N, F^* \otimes |TN|)$ such that $h_{|\mathrm{supp}(\rho)}$ is proper. For each $x \in M$ and $y \in h^{-1}(\{x\}) \subset N$, composition with the map $r(y)^* : F_y^* \to E_{h(y)}^*$ allows to see $\rho$ as a section of $h^*(E^*) \otimes |TN|$. As before, this gives a density along the fiber $h^{-1}(\{x\})$ valued in $E_x^* \otimes |T_xM|$, which is compactly supported because of the properness condition. Then, integration along the fiber can be performed, in order to get a section $\underline{h}_*\rho$ of $E^* \otimes |TM|$. Using local charts as in the proof of Proposition 1.5.3 and trivializations of the vector bundles, we get the following result.

[13] To be completely explicit, let $\partial y_i$, $\partial z_j$ and $\partial x_k$ be the vectors tangent to coordinates $y_i$, $z_j$ and $x_k$ respectively. Then, $h_{*(x,z)}(\partial y_i) = \partial x_i$ and, by the explicit construction of the isomorphism (1.27) in the proof of Lemma 1.2.4, the density corresponding to $|dz_1 \dots dz_k| \otimes |dx_1 \dots dx_m|$ evaluated on the basis $(\partial z_1, \dots, \partial z_k, \partial y_1, \dots, \partial y_k)$ must be equal to

$$|dz_1 \dots dz_k|(\partial z_1, \dots, \partial z_k).|dx_1 \dots dx_m|(h_{*(x,z)}(\partial y_1), \dots, h_{*(x,z)}(\partial y_m)) = 1.$$

This is precisely the density $|dz_1 \dots dz_k\, dy_1 \dots dy_m|$.

**Proposition 1.5.5.** *Let $E \to M$ and $F \to N$ be vector bundles over manifolds $M$ and $N$, and $\underline{h} = (h, r)$ a geometric morphism from $F$ to $E$ such that $h : N \to M$ is a submersion. Then, for any $\rho \in \Gamma^\infty(N, F^* \otimes |TN|)$ such that $h_{|\mathrm{supp}(\rho)}$ is proper, the pushforward $\underline{h}_*\rho$ defined as above is a smooth section of $E^* \otimes |TM|$. Furthermore, $\mathrm{supp}(\underline{h}_*\rho) \subset h(\mathrm{supp}(\rho))$. In particular, if $\rho$ is compactly supported, so is $\underline{h}_*\rho$.*

*Remark* 1.5.6. In the case of a compactly supported section, the properness condition is always fulfilled, so the pushforward of a compactly supported section of $F^* \otimes |TN|$ is defined for any submersion. On the other hand, the pushforward by a proper submersion is defined for any section of $F^* \otimes |TN|$, without any restriction on its support. ◁

### 1.5.3 Pullback of a generalized section by a submersion

As advertised in the beginning of this section, we can now use the pushforward of a density to define the pullback of a generalized section by duality.

**Definition 1.5.7.** *Let $E \to M$ and $F \to N$ be vector bundles over the manifolds $M$ and $N$, and $\underline{h} = (h, r)$ a geometric morphism from $F$ to $E$ such that $h : N \to M$ is a submersion. The* pullback *by $\underline{h}$ of a generalized section $u \in \mathcal{D}'(M, E)$ is the generalized section $\underline{h}^*u \in \mathcal{D}'(N, F)$ defined, for $\rho \in \mathcal{D}(N, F^* \otimes |TN|)$, by:*

$$\langle \underline{h}^* u, \rho \rangle := \langle u, \underline{h}_* \rho \rangle .$$

*Remark* 1.5.8. This definition of the pullback extends the pullback of functions as considered in Example 1.4.4. Indeed, let $f : M \to \mathbb{C}$ be a smooth function on a manifold $M$, $h : N \to M$ a smooth map between manifolds and $\rho \in \mathcal{D}(N, |TN|)$. Because $f \circ h$ is constant along the fibers of $h$, we have $f.(h_*\rho) = h_*((f \circ h)\rho)$. Therefore:

$$\begin{aligned} \langle h^* f, \rho \rangle := \langle f, h_* \rho \rangle &= \int_M f.h_*\rho = \int_M h_*((f \circ h)\rho) \\ &= \int_N (f \circ h).\rho = \langle f \circ h, \rho \rangle , \end{aligned}$$

which is indeed the usual pullback of functions. The same argument shows that the definition also extends the pullback of sections of a vector bundle. ◁

*Remark* 1.5.9. Let us emphasize once more that there is no universal notion of pullback for generalized sections in the sense that it depends on the kind of generalized sections we consider, as well as on the kind of transformation by which we pullback. For instance, the pullback in Definition 1.5.7 is valid for any generalized section, but only for transformations that are submersions. On the other extreme, the pullback of sections of Example 1.4.8 can be seen as a pullback operation that is valid for any transformation, but only for generalized sections that are actual sections. As an intermediate case, we will be able to define the pullback for a subset of generalized sections, but for more general (although not all) transformations than submersions. We should also stress that although this seems to give many different notions of pullback, they all agree on their common cases, as is shown for instance by the previous Remark 1.5.8. ◁

### 1.5.4 Pushforward of generalized densities

We have seen how to pushforward a density – or more generally a section of $F^* \otimes |TN|$ – by a submersion. It is in fact a special case of the pushforward of a generalized section of $F^* \otimes |TN|$, which can be defined by duality. Indeed, such a generalized section is a linear functional on sections of $(F^* \otimes |TN|)^* \otimes |TN| \simeq F$, which can be pullbacked by a geometric morphism.

**Definition 1.5.10.** *Let $E \to M$ and $F \to N$ be vector bundles over the manifolds $M$ and $N$, $u \in \mathcal{D}'(N, F^* \otimes |TN|)$ and $\underline{h} = (h, r)$ a geometric morphism from $F$ to $E$ such that $h_{|\mathrm{supp}(u)}$ is proper. The* pushforward *of $u$ by $\underline{h}$ is the generalized section $\underline{h}_* u \in \mathcal{D}'(M, E^* \otimes |TM|)$ defined, for $\rho \in \mathcal{D}(M, E^\vee \otimes |TM|) \simeq \mathcal{D}(M, E)$, by:*

$$\langle \underline{h}_* u, \rho \rangle := \langle u, \underline{h}^* \rho \rangle \,. \tag{1.29}$$

*Remark* 1.5.11. Notice that $\underline{h}^* \rho$ might not be compactly supported. However, the condition that $h_{|\mathrm{supp}(u)}$ is proper implies that $\mathrm{supp}(u) \cap \mathrm{supp}(\underline{h}^* \rho)$ is compact, so $\langle u, \underline{h}^* \rho \rangle$ is well-defined by Remark 1.3.26. If $u$ is compactly supported, then this condition is always verified, and $\underline{h}_* u$ is also compactly supported. ◁

*Remark* 1.5.12. In the special case where $h$ is a proper submersion, and $\mu$ is a smooth density on $N$, this definition coincides with the previous one. Indeed, for any $f \in \mathcal{D}(M)$, we have $\langle h_* \mu, f \rangle = \int_M f.(h_* \mu)$ (where $h_* \mu$ is defined as the pushforward of a density as in Proposition 1.5.3). Since $h^* f$ is constant along the fibers of $h$, it can be entered into the integral of (1.28), so $\int_M f.(h_* \mu) = \int_N h^* f.\mu$, which is the definition 1.5.10 of the pushforward of $\mu$ as a generalized section. The same argument is still valid for a smooth section of $F^* \otimes |TN|$. The important point to note is therefore that under submersions, smooth densities – seen as generalized sections – pushforward to smooth densities. ◁

*Remark* 1.5.13. Since $h_{|\mathrm{supp}(u)}$ is proper, $h(\mathrm{supp}(u))$ is closed, and we get from the definition of $\underline{h}_* u$ by duality that $\mathrm{supp}(\underline{h}_* u) \subset h(\mathrm{supp}(u))$. ◁

*Remark* 1.5.14. As would be expected, the pushforward by a composition of geometric morphisms is the composition of the pushforwards. Indeed, let $E \to M$, $F \to N$, $G \to L$ be vector bundles over manifolds $M$, $N$ and $L$ and

$u \in \mathcal{D}'(L, G^* \otimes |TL|)$. Let $\underline{h}_1 = (h_1, r_1)$ be a geometric morphism from $G$ to $F$ such that $h_{1|\mathrm{supp}(u)}$ is proper, and $\underline{h}_2 = (h_2, r_2)$ a geometric morphism from $F$ to $E$ such that $h_{1|h_2(\mathrm{supp}(u))}$ is proper. Then, $(h_2 \circ h_1)_{|\mathrm{supp}(u)}$ is proper and the definition readily gives

$$(\underline{h}_2 \circ \underline{h}_1)_*(u) = \underline{h}_{2*}\, \underline{h}_{1*}(u). \tag{1.30}$$

$\lhd$

The following technical lemma will be used later on. As a particular case, it asserts that the pushforward of a generalized section by a geometric morphism coincides with the pushforward by the restriction of that morphism to any open subset that contains the support of the generalized section.

**Lemma 1.5.15.** *Let $E \to M$ and $F \to N$ be vector bundles over the manifolds $M$ and $N$, $u \in \mathcal{D}'(N, F^* \otimes |TN|)$ and $\underline{h} = (h, r)$ a geometric morphism from $F$ to $E$ such that $h_{|\mathrm{supp}(u)}$ is proper. Let $U \subset M$ be an open subset and $V \subset h^{-1}(U)$ open such that* $\mathrm{supp}(u) \cap h^{-1}(U) \subset V$. *Then,* $(\underline{h}_* u)_{|U} = (\underline{h}_{|V})_*(u_{|V})$.

*Proof.* First, notice that $h_{|V} : V \to U$ is proper on $\mathrm{supp}(u_{|V}) = \mathrm{supp}(u) \cap V$, so $(\underline{h}_{|V})_*(u_{|V})$ is well-defined.[14] From the definitions by duality of the restriction and the pushforward of generalized densities, we have for any $\rho \in \mathcal{D}(U, (E^* \otimes |TM|)_{|U})$:

$$\begin{aligned}
\left\langle (\underline{h}_* u)_{|U}, \rho \right\rangle &= \langle \underline{h}_* u, \hat{\rho} \rangle = \langle u, \underline{h}^* \hat{\rho} \rangle \\
&= \left\langle u, \widehat{(\underline{h}_{|h^{-1}(U)})^* \rho} \right\rangle = \left\langle u, \widehat{(\underline{h}_{|V})^* \rho} \right\rangle \\
&= \left\langle u_{|V}, (\underline{h}_{|V})^* \rho \right\rangle = \left\langle (\underline{h}_{|V})_*(u_{|V}), \rho \right\rangle,
\end{aligned} \tag{1.31}$$

where the "$\widehat{\ }$" sign denotes the extension by zero of compactly supported sections (see (1.20)), and (1.31) follows from the fact that $\widehat{(\underline{h}_{|h^{-1}(U)})^* \rho}$ and $\widehat{(\underline{h}_{|V})^* \rho}$ coincides on $\mathrm{supp}(u)$ since $\mathrm{supp}(u) \cap h^{-1}(U) = \mathrm{supp}(u) \cap V$. $\square$

*Remark* 1.5.16. As a particular case of the pushforward, let us consider the projection $\bar{\pi} : M \to \{\star\}$ of a manifold $M$ onto a point. Then, the pushforward by $\bar{\pi}$ of a compactly supported generalized density $u$ is a functional on $\mathbb{C}$, which we shall identify with the number $\langle \bar{\pi}_* u, 1 \rangle$. In the particular case of a compactly supported continuous density $u \in \Gamma^0_c(M, |TM|)$, it is given by:

$$\bar{\pi}_* u = \langle \bar{\pi}_* u, 1 \rangle = \langle u, 1 \rangle = \int_M u.$$

This suggests to think about the pushforward by $\bar{\pi}$ as the integration over $M$ of the generalized density. $\lhd$

[14] Indeed, for any $K \subset U$ compact, since $h^{-1}(U) \cap \mathrm{supp}(u) \subset V$, we have $h^{-1}(K) \cap \mathrm{supp}(u_{|V}) = h^{-1}(K) \cap \mathrm{supp}(u) \cap V = h^{-1}(K) \cap \mathrm{supp}(u) \cap h^{-1}(U) = h^{-1}(K) \cap \mathrm{supp}(u)$, which is compact since $h_{|\mathrm{supp}(u)}$ is proper.

We end this section by a handy application of the previous observation, which allows to express the smooth trace of a smooth operator in terms of the pullback and push forward operations.

**Lemma 1.5.17.** *Let $E \to M$ be a vector bundle over a manifold $M$ and $P_k$ a smooth operator from $E$ to $E$ with kernel $k \in \mathcal{E}(M \times M, E \boxtimes E^\vee)$. Denote by $\bar{\pi} : M \to \{\star\}$ the projection onto a point, and by $\Delta : M \to M \times M$ the diagonal map. If $\Delta^*(k)$ is of compact support, then $P_k$ is smooth-traceable and*

$$\operatorname{tr}(P_k) = \bar{\pi}_* \operatorname{Tr} \Delta^*(k). \tag{1.32}$$

*Proof.* Since $k$ is a smooth function, the pullback is just the composition. If $\Delta^*(k)$ is compactly supported, so is $x \mapsto (\operatorname{Tr} \Delta^* k)(x) = \operatorname{Tr}(k(x,x))$, which is therefore an integrable density on $M$. $P_k$ is thus smooth-traceable and the identity (1.32) follows from Remark 1.5.16. □

## 1.6 $\delta-$sections

Introduced in Example 1.3.16, the $\delta-$function is a distribution that associates to a function – more precisely, to a density – its value at a point. In this section, we are going to generalize this concept by associating to a submanifold generalized sections – called $\delta-$sections – given by integration over that submanifold. However, since there is no standard way of integrating over a submanifold – unlike on $\mathbb{R}^n$, where we have the standard Lebesgue density –, those generalized sections will carry an additional datum related to the direction transverse to the submanifold.

### 1.6.1 Definitions

The following elementary lemma is a key ingredient in the definition of a $\delta-$section. It allows to decompose the restriction of a density to a submanifold into densities on the submanifold and on the normal bundle.

**Lemma 1.6.1.** *Let $M$ be a manifold, $Z \subset M$ an embedded or immersed submanifold and $\iota : Z \hookrightarrow M$ the inclusion map. Then, canonically,*

$$\iota^*|TM| \simeq |NZ| \otimes |TZ|. \tag{1.33}$$

*Proof.* Since we have the exact sequence of vector bundles over $Z$

$$0 \to |TZ| \to \iota^*|TM| \to |NZ| \to 0,$$

this is an immediate consequence of Lemma 1.2.12. □

Now, let $E \to M$ be a vector bundle over a manifold $M$, and $Z \subset M$ a properly embedded submanifold.[15] Denote by $\iota : Z \hookrightarrow M$ the inclusion map. For all $\rho \in \mathcal{D}(M, E^\vee)$, $\rho_{|Z} := \iota^* \rho$ is of compact support. By Lemma 1.6.1, it can be written as $\rho_E \otimes \rho_N \otimes \rho_T$ for some $\rho_E \in \Gamma^\infty(Z, E^*_{|Z})$, $\rho_N \in \Gamma^\infty(Z, |NZ|)$ and $\rho_T \in \Gamma^\infty(Z, |TZ|)$.[16] Given a section $\sigma \in \Gamma^\infty\left(Z, E_{|Z} \otimes |NZ|^*\right)$, we can form

$$\langle \sigma, \rho_E \otimes \rho_N \rangle \, \rho_T,$$

which is a compactly supported density over $Z$. It is clear that this density does not depend on the particular choice of $\rho_E$, $\rho_N$ and $\rho_T$ but only on their tensor product $\rho_E \otimes \rho_N \otimes \rho_T$. We can finally integrate this density over $Z$ to get a number. This justifies the following definition.

**Definition 1.6.2.** *Let $E \to M$ be a vector bundle over a manifold $M$, $Z \subset M$ a properly embedded submanifold and $\sigma \in \Gamma^\infty\left(Z, E_{|Z} \otimes |NZ|^*\right)$. The $\delta-$*section *associated to the submanifold $Z$ and the symbol $\sigma$ is the generalized section of $E$ denoted by $\delta_{Z,\sigma}$ and defined on $\rho \in \mathcal{D}(M, E^\vee)$ by*

$$\langle \delta_{Z,\sigma}, \rho \rangle := \int_Z \langle \sigma, \rho_E \otimes \rho_N \rangle \, \rho_T, \tag{1.34}$$

*where $\rho_E \in \Gamma^\infty(Z, E^*_{|Z})$, $\rho_N \in \Gamma^\infty(Z, |NZ|)$ and $\rho_T \in \Gamma^\infty(Z, |TZ|)$ are such that $\rho_E \otimes \rho_N \otimes \rho_T$ is identified to $\rho_{|Z}$ through (1.33).*

*Remark* 1.6.3. Let us give another description of the symbol of a $\delta-$section that will be very useful later on. For every vector bundles $E$ and $F$, we have $E \otimes F^* \simeq \mathrm{Hom}(F, E)$. Therefore, the symbol of a $\delta-$section of the bundle $E$ along the submanifold $Z$ is equivalently given by a section $\sigma \in \Gamma^\infty(Z, \mathrm{Hom}(|NZ|, E_{|Z}))$. In this case, if $\rho_{|Z}$ is identified with $\rho_E \otimes \rho_N \otimes \rho_T$ where $\rho_E \in \Gamma^\infty(Z, E^*)$, $\rho_Z \in \Gamma^\infty(Z, |NZ|)$ and $\rho_N \in \Gamma^\infty(Z, |TZ|)$, then we can form a smooth section $\sigma(\rho_N)$ of $E_{|Z}$ given at $z \in Z$ by $\sigma(z)(\rho_N(z))$, and pair it with $\rho_E$. The value of $\delta_{Z,\sigma}$ on $\rho$ is then given by

$$\langle \delta_{Z,\sigma}, \rho \rangle = \int_Z \langle \rho_E, \sigma(\rho_N) \rangle \, \rho_Z. \tag{1.35}$$

$\triangleleft$

*Remark* 1.6.4. As a convention, we set the $\delta-$section associated to the empty submanifold to be the trivial generalized section assigning 0 to every section. $\triangleleft$

*Remark* 1.6.5. The support of a $\delta-$section $\delta_{Z,\sigma}$ coincides with $\mathrm{supp}(\sigma) \subset Z$. $\triangleleft$

*Example* 1.6.6. The $\delta-$function $\delta_a$ on $\mathbb{R}^n$ (Example 1.3.16) can be seen as a $\delta-$section of the trivial bundle. The corresponding submanifold is the point

[15] Notice that a properly immersed manifold is automatically properly embedded.

[16] We should stress that although $\rho_E$, $\rho_N$ and $\rho_T$ are not uniquely determined, the value of $\rho_E \otimes \rho_N \otimes \rho_T$ is canonical since the isomorphism (1.33) is.

$\{a\}$. The normal bundle is the whole tangent bundle, whose densities are generated by the Lebesgue density. Hence, the symbol of $\delta_a$ is $|dx_1 \dots dx_n|^*$, the dual element of the Lebesgue density. $\diamondsuit$

*Remark* 1.6.7. On a generic manifold $M$, there is no canonical density and therefore there is no canonical $\delta-$function at a point $x \in M$. We need to specify its symbol, which is given by a dual density at the point – which can be specified by a choice of a basis of the tangent space at that point. $\triangleleft$

**Definition 1.6.8.** *Let $E \to M$ be a vector bundle over a manifold $M$. A $\delta-$density of $E$ is a $\delta-$section of the vector bundle $E^* \otimes |TM|$.*

### 1.6.2 Pullback of a $\delta-$section

The following computation of the pullback of the $\delta-$function is a useful exercise to get more familiar with the calculus of $\delta-$sections and the splitting of densities.

*Example* 1.6.9. Let $M = \mathbb{R}^m$, $N = \mathbb{R}^n$ and $h : M \to N$ a smooth map. Consider the smooth map

$$H : M \times N \to \mathbb{R}^n \ ; \ (x, y) \mapsto y - h(x)$$

We denote by $x$, $y$ and $w$ the coordinates on $M$, $N$ and $\mathbb{R}^n$ respectively, and by $|dx|$, $|dy|$ and $|dw|$ the corresponding Lebesgue densities. Since $H$ is a submersion, we can form $H^*\delta$, where $\delta$ is the $\delta-$function on $\mathbb{R}^n$. Let us compute the value of $H^*\delta$ on the density $u(x)v(y)|dxdy|$ of $M \times N$ for some $u \in \mathcal{D}(M)$ and $v \in \mathcal{D}(N)$. By definition of the pullback,

$$\langle H^*\delta, u(x)v(y)|dxdy|\rangle = \langle \delta, H_*(u(x)v(y)|dxdy|)\rangle =: c,$$

where $c \in \mathbb{C}$ is such that $H_*\big(u(x)v(y)|dxdy|\big)(0) = c.|dw|(0)$. Let

$$Z := H^{-1}(\{0\}) = \{(x, y) \in M \times N \mid y = h(x)\} = \mathrm{graph}(h).$$

We have to identify the density $|dz|$ on $Z$ such that, for each $z \in Z$, $|dz|(z) \otimes |dw|(0)$ corresponds to $|dxdy|(z)$ through the isomorphism (1.27). Since

$$\begin{aligned} |dxdy|(z)\big((1,0),(0,1)\big) &= 1 \\ &= \big((\mathrm{gr}^{-1})^*(|dx|)\big)(z)\big(1,0\big) \,.\, |dw|(0)\left(H_{*(x,h(x))}(0,1)\right), \end{aligned}$$

where

$$\mathrm{gr} : M \xrightarrow{\sim} Z \subset M \times N \ ; \ x \mapsto (x, h(x)),$$

we have that $|dz| = (\mathrm{gr}^{-1})^*(|dx|)$. By definition of the pushforward of a smooth density, $H_*\big(u(x)v(y)|dxdy|\big)(0)$ is the integral over $Z$ of $(\mathrm{gr}^{-1})^*(|dx|) \otimes |dw|(0)$, so

$$c = \int_Z \mathrm{pr}_M^*(u) \,.\, \mathrm{pr}_N^*(v) \,.\, (\mathrm{gr}^{-1})^*(|dx|) \tag{1.36}$$
$$= \int_M \mathrm{gr}^*\mathrm{pr}_M^*(u) \,.\, \mathrm{gr}^*\mathrm{pr}_N^*(v) \,.\, |dx| = \int_M (v \circ h) \,.\, u|dx|.$$

This is usually written more suggestively as $(H^*\delta)(x, y) = \delta(x - h(y))$, i.e.

$$\int_N (H^*\delta)(x, y)\, v(y)\, dy = \int_N \delta(x - h(y))\, v(y)\, dy = v(h(x)).$$

Finally, equation (1.36) suggests that $H^*\delta$ is itself a $\delta-$section along the submanifold $Z$. This is indeed the case, as we shall see in this section. $\Diamond$

We have seen that for a generalized section, its pullback by a submersion can always be defined. For $\delta-$sections, this definition can be extended to more general maps, which is what we will now carry on. Let us first recall the notion of transversality which in some sense generalizes the notion of regular values of a smooth map.

**Definition 1.6.10.** *Let $h : N \to M$ be a smooth map between two manifolds, and $Z \subset M$ an embedded submanifold of $M$. We say that $h$ is* transverse *to $Z$ if for every $y \in h^{-1}(Z)$:*

$$T_{h(y)}M \simeq T_{h(y)}Z + h_{*_y}(T_yN).$$

*Remark* 1.6.11. We follow here the general convention of saying that if $h^{-1}(Z)$ is empty, then $h$ is trivially transverse to $Z$. $\lhd$

*Remark* 1.6.12. A submersion is transverse to any embedded submanifold. $\lhd$

The following theorem follows from the Preimage Theorem for submersions. We refer to [Lee13] for a proof.

**Theorem 1.6.13.** *Let $h : N \to M$ be a smooth map between two manifolds that is transverse to an embedded submanifold $Z \subset M$. Then, $h^{-1}(Z)$ is an embedded submanifold of $N$. Furthermore, if $h^{-1}(Z)$ is not empty, the codimension of $h^{-1}(Z)$ in $N$ is the same as the codimension of $Z$ in $M$.*

Now let us turn to the pullback of a $\delta-$section. Let $E \to M$ and $F \to N$ be vector bundles over manifolds $M$ and $N$, $Z \subset M$ be a properly embedded submanifold and $\underline{h} = (h, r)$ be a geometric morphism from $F$ to $E$ such that $h : N \to M$ is transverse to $Z$.

Let $W := h^{-1}(Z)$. It is an embedded submanifold of $N$ by Theorem 1.6.13 and it is also properly embedded since $Z$ is. The key point is that, since $h$ is transverse to $Z$, it induces an isomorphism between the normal bundle of

$Z$ and the normal bundle of $W$. This naturally gives a geometric morphism from $F_{|W} \otimes |NW|^*$ to $E_{|Z} \otimes |NZ|^*$, which allows to pullback the symbol of any $\delta-$section of $E$ along $Z$ to a symbol of a $\delta-$section of $F$ along $W$. Let us see how this works. For any $y \in W$, the map $h_{*_y} : T_yN \to T_{h(y)}(M)$ induces a map

$$T_yN \to T_{h(y)}(M)/T_{h(y)}(Z)$$

which is surjective since $T_{h(y)}(M) = (h_{*_y})(T_yN) + T_{h(y)}(Z)$ by transversality of $h$. Its kernel contains $T_yW$ because $(h_{*_y})(T_yW) \subset T_{h(y)}(Z)$. Therefore, we get a surjective map:

$$\bar{h}_{*_y} : N_yW \simeq T_yN/T_yW \to N_{h(y)}(Z) \simeq T_{h(y)}(M)/T_{h(y)}(Z).$$

It is in fact an isomorphism since $N_yW$ and $N_{h(y)}(Z)$ have the same dimension because the codimensions of $Z$ and $W$ are equal by Theorem 1.6.13. By Lemma 1.2.7, this induces an isomorphism $|\bar{h}_{*_y}|$ between densities, and we denote its dual by

$$|\bar{h}_{*_y}|^* : |N_{h(y)}(Z)|^* \xrightarrow{\sim} |N_yW|^*. \tag{1.37}$$

We can now define a section $\tilde{r}$ of $\mathrm{Hom}(h^*(E_{|Z} \otimes |NZ|^*), F_{|W} \otimes |NW|^*)$ given at $y \in W$ by

$$\tilde{r}(y) := r_{|Z}(y) \otimes |\bar{h}_{*_y}|^*. \tag{1.38}$$

The following Lemma shows that this section is smooth, so it defines a geometric morphism $\underline{\tilde{h}} = (h, \tilde{r})$ from $F_{|W} \otimes |NW|^*$ to $E_{|Z} \otimes |NZ|^*$.

**Lemma 1.6.14.** *Within the previous setting, $\underline{\tilde{h}} = (h, \tilde{r})$ defined by the formula (1.38) is a geometric morphism from $F_{|W} \otimes |NW|^*$ to $E_{|Z} \otimes |NZ|^*$. Therefore, for any $\delta-$section $\delta_{Z,\sigma}$ of $E$ along $Z$ with symbol $\sigma \in \Gamma^\infty\left(Z, E_{|Z} \otimes |NZ|^*\right)$, $\delta_{h^{-1}(Z),\underline{\tilde{h}}^*(\sigma)}$ is a well-defined $\delta-$section of $F$ along the submanifold $h^{-1}(Z)$.*

*Proof.* Let $(y_1, \dots, y_n)$ be local coordinates on an open set $V \subset N$ such that $h^{-1}(Z) \cap V$ is described by $y_1 = \dots = y_k = 0$, and local coordinates $(x_1, \dots, x_m)$ on an open set $U \subset M$ such that $h(V) \subset U$ and $Z \cap U$ is given by $x_1 = \dots = x_k = 0$. Then, $|NZ|^*$ is spanned by $|dx_1 \dots dx_k|^*$, $|NW|^*$ by $|dy_1 \dots dy_k|^*$ and the isomorphism (1.37) maps $|dx_1 \dots dx_k|^*$ to

$$\frac{1}{|\mathrm{Jac}_h^k|} |dy_1 \dots dy_k|^*,$$

where $\mathrm{Jac}_h^k$ denotes the partial Jacobian of $h$:

$$\mathrm{Jac}_h^k = \det \begin{pmatrix} \partial h_1/\partial y_1 & \dots & \partial h_k/\partial y_1 \\ \vdots & \ddots & \vdots \\ \partial h_1/\partial y_k & \dots & \partial h_k/\partial y_k \end{pmatrix}.$$

This shows that $\tilde{r}$ is indeed a smooth section. Let $\delta_{Z,\sigma}$ be a $\delta-$section of $E$ along $Z$ with symbol $\sigma \in \Gamma^\infty\left(Z, E_{|Z} \otimes |NZ|^*\right)$. Then, $\underline{\tilde{h}}^*(\sigma)$ is a well-defined symbol on $h^{-1}(Z)$. For a later use, let us give an explicit expression of it. The symbol $\sigma$ has the form $s \otimes |dx_1 \dots dx_k|^*$ for some smooth section $s$ of $E_{|Z}$, so, for $y \in W \cap V$, we have

$$\begin{aligned}\underline{\tilde{h}}^*(\sigma)(y) &= (r_{|Z}(y) \otimes |\bar{h}_{*_y}|^*)(\sigma(h(y)) \qquad \in F_y \otimes |N_y W|^* \\ &= \frac{r(y)(\, s(h(y))\,)}{|\mathrm{Jac}^k_h(y)|} \otimes |dy_1 \dots dy_k|^*.\end{aligned} \tag{1.39}$$

That is the local expression of the $\delta-$section $\delta_{h^{-1}(Z),\underline{\tilde{h}}^*(\sigma)}$ which we shorten by writing

$$\underline{\tilde{h}}^*(\sigma) = \frac{r(s \circ h)}{|\mathrm{Jac}^k_h|} \otimes |dy_1 \dots dy_k|^*. \tag{1.40}$$

□

**Lemma 1.6.15.** *Within the previous setting, if we suppose in addition that $h$ is a submersion, then the pullback $\underline{h}^*(\delta_{Z,\sigma})$ coincides with the $\delta-$section $\delta_{h^{-1}(Z),\underline{\tilde{h}}^*(\sigma)}$.*

*Proof.* Let us take some local coordinates as in the proof of Lemma 1.6.14. Because $h$ is a submersion, we can take them to be such that the local expression of $h$ is

$$h(y_1, \dots, y_n) = (y_1, \dots, y_m).$$

Let $\rho \in \mathcal{D}(N, F^* \otimes |TN|)$ supported in $V$. It has the form

$$\rho = l \otimes |dy_1 \dots dy_k| \otimes |dy_{k+1} \dots dy_n|$$

for some smooth section of compact support $l$ of $F^*$. The symbol $\sigma$ has the form $s \otimes |dx_1 \dots dx_k|^*$ for some smooth section $s$ of $E_{|Z}$. By definition of a $\delta-$section and the formula (1.40) for the symbol of $\delta_{h^{-1}(Z),\underline{\tilde{h}}^*(\sigma)}$, we have (notice that $|\mathrm{Jac}^k_h| = 1$):

$$\begin{aligned}&\left\langle \delta_{h^{-1}(Z),\underline{h}^*(\sigma)}, \rho \right\rangle \\ &:= \int \langle l, r(s \circ h) \rangle \,.\, \langle |dy_1 \dots dy_k|^*, |dy_1 \dots dy_k| \rangle \,.|dy_{k+1} \dots dy_n| \\ &= \int \langle l, r(s \circ h) \rangle \,.|dy_{k+1} \dots dy_n|.\end{aligned}$$

On the other hand, the definition of the pushforward of $\rho$ by $\underline{h}$ leads to

$$\underline{h}_* \rho := \left( \int r^* l \,.\, |dy_{m+1} \dots dy_n| \right) \otimes |dx_1 \dots dx_m| \in \mathcal{D}(M, E^* \otimes |TM|).$$

By definition of the pullback by a submersion, we thus have

$$
\begin{aligned}
&\langle \underline{h}^*(\delta_{Z,\sigma}), \rho\rangle := \langle \delta_{Z,\sigma}, \underline{h}_*\rho\rangle \\
&= \int \left\langle \int r^* l \,.\, |dy_{m+1} \dots dy_n|, s \right\rangle . \langle |dx_1 \dots dx_k|^*, |dx_1 \dots dx_k|\rangle \,.|dx_{k+1} \dots dx_m| \\
&= \int\int \langle l, r(s\circ h)\rangle \,.|dy_{m+1} \dots dy_n||dy_{k+1} \dots dy_m| \\
&= \int\int \langle l, r(s\circ h)\rangle \,.|dy_{k+1} \dots dy_n|.
\end{aligned}
$$

This indeed coincides with the value of $\left\langle \delta_{h^{-1}(Z),\underline{\tilde{h}}^*(\sigma)}, \rho\right\rangle$. □

The two previous results suggest a way to extend the definition of the pullback of $\delta-$sections to more general maps than submersions. Notice that in the following, we will write $\underline{h}$ instead of $\underline{\tilde{h}}$ since it should not introduce any confusion.

**Definition 1.6.16.** *Let $E \to M$ and $F \to N$ be vector bundles over manifolds $M$ and $N$. Let $\delta_{Z,\sigma}$ be a $\delta-$section of $E$ along a properly embedded submanifold $Z \subset M$ with symbol $\sigma \in \Gamma^\infty\left(Z, E_{|Z} \otimes |NZ|^*\right)$. Let $\underline{h} = (h, r)$ be a geometric morphism from $F$ to $E$ such that $h : N \to M$ is transverse to $Z$. The* pullback *$\underline{h}^*(\delta_{Z,\sigma})$ of $\delta_{Z,\sigma}$ by $\underline{h}$ is the $\delta-$section $\delta_{h^{-1}(Z),\underline{h}^*(\sigma)}$.*

*Remark* 1.6.17. By Remark 1.6.5 and (1.40), the support of the pullback is

$$\operatorname{supp}\left(\underline{h}^*(\delta_{Z,\sigma})\right) = h^{-1}(\operatorname{supp}(\sigma)) = h^{-1}(\operatorname{supp}(\delta_{Z,\sigma})). \qquad \lhd$$

The following lemma shows that the pullback of $\delta-$sections behaves as expected with respect to the composition of geometric morphisms.

**Lemma 1.6.18.** *Let $E \to M$, $F \to N$, $G \to L$ be vector bundles over manifolds $M$, $N$ and $L$. Let $\delta_{Z,\sigma}$ be a $\delta-$section of $E$ along a properly embedded submanifold $Z \subset M$ with symbol $\sigma \in \Gamma^\infty\left(Z, E_{|Z} \otimes |NZ|^*\right)$. Let $\underline{h}_1 = (h_1, r_1)$ be a geometric morphism from $F$ to $E$ such that $h_1 : N \to M$ is transverse to $Z$, and $\underline{h}_2 = (h_2, r_2)$ a geometric morphism from $G$ to $F$ such that $h_2 : L \to N$ is transverse to $h_1^{-1}(Z)$. Then, $(h_1 \circ h_2)$ is transverse to $Z$, and*

$$(\underline{h}_1 \circ \underline{h}_2)^*(\delta_{Z,\sigma}) = \underline{h}_2{}^*\, \underline{h}_1{}^*(\delta_{Z,\sigma}). \tag{1.41}$$

*Proof.* Let $z \in Z$, $y \in L$ such that $(h_1(h_2(y)) = z$ and $X \in T_z(M)$. Then, since $h_1$ is transverse to $Z$, there exists $X_Z \in T_z(Z)$ and $Y \in T_{h_2(y)}(N)$ such that $X = X_Z + (h_1)_{*_{h_2(y)}}(Y)$. Since $h_2$ is transverse to $h_1^{-1}(Z)$, there exists $Y_1 \in T_{h_2(y)}(h_1^{-1}(Z))$ and $Y_0 \in T_y(L)$ such that $Y = Y_1 + (h_2)_{*_y}(Y_0)$. Therefore,

$$
\begin{aligned}
X &= X_Z + (h_1)_{*_{h_2(y)}}\left(Y_1 + (h_2)_{*_y}(Y_0)\right) \\
&= X_Z + (h_1)_{*_{h_2(y)}}(Y_1) + (h_1 \circ h_2)_{*_y}(Y_0),
\end{aligned}
$$

which shows that $(h_1 \circ h_2)$ is transversal to $Z$ because $(h_1)_{*_{h_2(y)}}(Y_1) \in T_z(Z)$. $(h_1 \circ h_2)^*(\delta_{Z,\sigma})$ is thus a well-defined $\delta-$section, whose symbol is $(\underline{h}_1 \circ \underline{h}_2)^*(\sigma) = \underline{h}_2{}^* \underline{h}_1{}^*(\sigma)$ (since these are pullbacks of a smooth section), which shows (1.41). $\square$

### 1.6.3 Pushforward of a $\delta-$density

We have seen in Section 1.5.4 how to pushforward generalized densities, that is, generalized sections of $F^* \otimes |TN|$ for $F \to N$ a vector bundle over a manifold $N$. In the case of a $\delta-$density, we can describe its pushforward in terms of its symbol. Also, we will show that the pushforward of a $\delta-$density by a submersion gives a smooth section.

**Proposition 1.6.19.** *Let $E \to M$ and $F \to N$ be vector bundles over manifolds $M$ and $N$ and $\underline{h} = (h, r)$ a geometric morphism from $F$ to $E$. Let $\delta_{Z,\sigma}$ be a $\delta-$density of $F$ (that is, a $\delta-$section of $F^* \otimes |TN|$) along a properly embedded submanifold $Z \subset N$ with symbol $\sigma$. Suppose that $h_{|\mathrm{supp}(\sigma)}$ is proper. Then,*

$$\underline{h}_*(\delta_{Z,\sigma}) = (\underline{h}_{|Z})_*(\sigma), \tag{1.42}$$

*where, on the right hand side, $\sigma$ is considered as a (smooth) generalized section of $(F_{|Z})^* \otimes |TZ|$.*

*Proof.* First, let us make sense of formula (1.42). The symbol $\sigma$ is a smooth section of the vector bundle

$$\begin{aligned}(F^* \otimes |TN|)_{|Z} \otimes |NZ|^* &\simeq (F_{|Z})^* \otimes |TZ| \otimes |NZ| \otimes |NZ|^* \\ &\simeq (F_{|Z})^* \otimes |TZ|,\end{aligned} \tag{1.43}$$

so it can be seen as a smooth generalized section of $(F_{|Z})^* \otimes |TZ|$, which can be pushforwarded by $\underline{h}_{|Z}$ using formula (1.29). Because the support of $\delta_{Z,\sigma}$ is $\mathrm{supp}(\sigma)$, the condition that $h_{|\mathrm{supp}(\sigma)}$ is proper ensures that the pushforward is well-defined.

Now let us evaluate $\underline{h}_*(\delta_{Z,\sigma})$ on a compactly supported section $\rho$ of $(E^* \otimes |TM|)^\vee \simeq E$. By definition of the pullback, we have $\langle \underline{h}_*(\delta_{Z,\sigma}), \rho\rangle = \langle \delta_{Z,\sigma}, \underline{h}^*\rho\rangle$. Under the identification (1.43), we have:

$$\langle \delta_{Z,\sigma}, \underline{h}^*\rho\rangle = \int_Z (\sigma, (\underline{h}^*\rho)_{|Z})$$

where $(\sigma, (\underline{h}^*\rho)_{|Z})$ denotes the pairing (1.17) of the first component of $\sigma$ with $(\underline{h}^*\rho)_{|Z}$, which gives a density on $Z$. This last integral is $\left\langle \sigma, (\underline{h}^*\rho)_{|Z}\right\rangle$, the evaluation on $(\underline{h}^*\rho)_{|Z}$ of $\sigma$ seen as a generalized section. Since $(\underline{h}^*\rho)_{|Z} = (\underline{h}_{|Z})^*\rho$, it coincides with the definition of the pullback of $\sigma$. $\square$

The following result gives conditions for the pushforward of a $\delta-$density to be smooth.

**Proposition 1.6.20.** *Let $E \to M$ and $F \to N$ be vector bundles over manifolds $M$ and $N$. Let $\delta_{Z,\sigma}$ be a $\delta-$density of $F$. Let $\underline{h} = (h, r)$ be a geometric morphism from $F$ to $E$ such that $h_{|\text{supp}(\sigma)}$ is proper. Then $\underline{h}_*(\delta_{Z,\sigma})$ is smooth at every regular value of $h_{|Z}$. More precisely, for every regular value $x \in M$ of $h_{|Z}$, there exists an open neighbourhood $U \subset M$ of $x$ such that $\underline{h}_*(\delta_{Z,\sigma})_{|U}$ is smooth. In particular, if $h_{|Z}$ is a submersion, then $\underline{h}_*(\delta_{Z,\sigma})$ is a smooth section of $E^* \otimes |TM|$.*

*Proof.* Let $x \in M$ be a regular value of $h_{|Z}$ and denote $g := h_{|Z}$ and $\underline{g} := \underline{h}_{|Z}$. Notice that $g_{|\text{supp}(\sigma)}$ is proper since $Z$, being properly embedded, is closed in $N$.
First, let us show that, because $g_{|\text{supp}(\sigma)}$ is proper, there exists an open neighbourhood $U$ of $x$ such that every point of $\text{supp}(\sigma) \cap g^{-1}(U)$ is a regular point of $g$. Indeed, if it is not the case, one can find a sequence $x_n \to x$ in $U$ and a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \text{supp}(\sigma)$ such that $g(y_n) = x_n$ and $y_n$ is a singular point of $g$. Since $K := \{x_n\}_{n\in\mathbb{N}} \cup \{x\}$ is compact and $g_{|\text{supp}(\sigma)}$ is proper, $g^{-1}(K) \cap \text{supp}(\sigma)$ is compact and there is a convergent subsequence $y_{n_k} \to y$. By continuity, $g(y) = x$, so $y$ must be a regular point. We therefore have a sequence of singular points converging to a regular point, which contradicts the fact that being a regular point is an open condition.
Next, since being a regular point is an open condition, there exists an open subset $\tilde{V}$ containing $\text{supp}(\sigma) \cap g^{-1}(U)$ such that $g_{|\tilde{V}}$ is a submersion. Then, if we define $V := \tilde{V} \cap g^{-1}(U)$, Lemma 1.5.15 imply that

$$(\underline{g}_*\sigma)_{|U} = (\underline{g}_{|V})_*(\sigma_{|V}).$$

Therefore, since $g_{|V}$ is a submersion, it follows from Remark 1.5.12 and Proposition 1.5.5 that $(\underline{g}_*\sigma)_{|U}$ is smooth. By Proposition 1.6.19, this finally implies that $(\underline{h}_*(\delta_{Z,\sigma}))_{|U}$ is smooth. $\square$

We end up our considerations about the calculus of $\delta-$sections with a result about the commutation of pullback and pushforward. It should be interpreted as an instance of "integration commutes with restriction". The key point of the proof is to work with the symbol of the $\delta-$section, which is easier to manipulate thanks to the previous results.

**Proposition 1.6.21.** *Let $M$, $N$ and $Q$ be manifolds, $f : M \to N$ a smooth map and $Z \subset Q \times N$ a submanifold. Consider the following commutative diagram:*

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \pi \uparrow & & \uparrow p \\ W \subset Q \times M & \xrightarrow{g} & Q \times N \supset Z, \end{array}$$

*where:*

- $\pi : Q \times M \to M$ *and* $p : Q \times N \to N$ *are the projections onto the second components;*
- $g := \mathrm{id} \times f$ *and* $W := g^{-1}(Z)$.

*Let* $E \to M$ *and* $F \to N$ *be vector bundles. Suppose that:*

*i.* $g$ *is transverse to* $Z$*;*
*ii.* $p_{|Z}$ *is a submersion;*
*iii.* $f$ *is proper;*
*iv.* $f^*(F^* \otimes |TN|) \simeq E^* \otimes |TM|$.

*Then, at the level of the vector bundles, we have the following commutative diagram*[17]

$$\begin{array}{ccc} E^* \otimes |TM| & \xrightarrow{f} & F^* \otimes |TN| \\ \pi \uparrow & & \uparrow p \\ \pi^*(E)^* \otimes |T(Q \times M)| & \xrightarrow{g} & p^*(F)^* \otimes |T(Q \times N)|, \end{array} \tag{1.44}$$

*and for every* $\delta-$*section* $\delta_{Z,\sigma}$ *of the vector bundle* $p^*(F)^* \otimes |T(Q \times N)|$ *along* $Z$ *and of symbol* $\sigma$*, such that* $p_{|\mathrm{supp}(\sigma)}$ *is proper,* $f^*\, p_*(\delta_{Z,\sigma})$ *and* $\pi_*\, g^*(\delta_{Z,\sigma})$ *are well-defined smooth sections of* $E^* \otimes |TM|$ *and*

$$f^*\, p_*(\delta_{Z,\sigma}) = \pi_*\, g^*(\delta_{Z,\sigma}). \tag{1.45}$$

*Proof.* Let us first notice that, because of the definition of $\pi$, $p$ and $g$, for every subset $A \subset M$, we have that

$$g(\pi^{-1}(A)) = p^{-1}(f(A)), \tag{1.46}$$

which we will use later on. Also, for every $(q, x) \in Q \times M$, we have:

$$\begin{aligned} g^*\big(p^*(F^*) \otimes |T(Q \times N)|\big)_{(q,x)} &\simeq (p^*(F^*) \otimes |T(Q \times N)|)_{(q,f(x))} \\ &\simeq {F^*}_{f(x)} \otimes |TQ|_q \otimes |TN|_{f(x)} \\ &\simeq f^*\,(F^* \otimes |TN|)_x \otimes |TQ|_q \\ &\simeq (E^* \otimes |TM|)_x \otimes |TQ|_q \\ &\simeq \big(\pi^*(E^*) \otimes |T(Q \times M)|\big)_{(q,x)}, \end{aligned} \tag{1.47}$$

which justifies the diagram (1.44). Let us now verify that both generalized sections in (1.45) are well-defined. Since $p_{|\mathrm{supp}(\sigma)}$ is proper and $p_{|Z}$ is a submersion by assumption, Proposition 1.6.20 implies that $p_*(\delta_{Z,\sigma})$ is a smooth section of $F^* \otimes |TN|$ and the pullback by $f$ gives a smooth section $f^*\, p_*(\delta_{Z,\sigma})$ of $f^*(F^* \otimes |TN|) \simeq E^* \otimes |TM|$.

[17] The geometric morphisms associated to each arrows are the natural ones between a vector bundle and its pullback bundle as mentioned in Remark 1.4.7.

On the other hand, since $g$ is transverse to the properly embedded submanifold $Z$, $W$ is a properly embedded submanifold by Theorem 1.6.13, and $g^*(\delta_{Z,\sigma})$ is a $\delta-$section along $W$ of the vector bundle $g^*\big(p^*(F^*)\otimes|T(Q\times N)|\big)\simeq\pi^*(E^*)\otimes|T(Q\times M)|$. To show that $\pi_{|g^{-1}(\mathrm{supp}(\sigma))}$ is proper, let $K\subset M$ be a compact subset. Then, $f(K)$ is compact by continuity of $f$ and $p^{-1}(f(K))\cap\mathrm{supp}(\sigma)$ also by the properness of $p_{|\mathrm{supp}(\sigma)}$, and it is equal to $g(\pi^{-1}(K))\cap\mathrm{supp}(\sigma)$ by (1.46). Since $f$ is proper, $g$ is also proper, and $\pi^{-1}(K)\cap g^{-1}(\mathrm{supp}(\sigma))$ is compact, which proves the claim. We have in addition that $\pi_{|W}$ is a submersion. Indeed, let $(q,x)\in W\subset Q\times M$ and $X\in T_xM$. By definition of $W$, $(q,f(x))\in Z$ and, since $p_{|Z}$ is a submersion, there exists $X_Z\in T_{(q,f(x))}(Z)$ such that $p_{*_{(q,f(x))}}(X_Z)=f_{*_x}(X)$. By writing $T_{(q,f(x))}(Z)\subset T_q(Q)\times T_{f(x)}(N)$, $X_Z$ must be of the form $(Y,f_{*_x}(X))$ for some $Y\in T_qQ$. Then, $(Y,X)\in T_{(q,x)}(W)$ and it is such that $\pi_{*_{(q,x)}}(Y,X)=X$, which shows that $\pi_{|W}$ is a submersion. Putting everything together, Proposition 1.6.20 implies that $\pi_*\,g^*(\delta_{Z,\sigma})$ is a smooth section of $E^*\otimes|TM|$.

In order to finally show that the two sections coincides, we will track how the symbol of $\delta_{Z,\sigma}$ is transformed under the various operations. Let $x\in M$. Since $p_*(\delta_{Z,\sigma})$ is a smooth section, $f^*\,p_*(\delta_{Z,\sigma})(x)=p_*(\delta_{Z,\sigma})(f(x))$, which, by (1.42), is given by integration of $\sigma$ along the fibers of $p_{|Z}$, that is:

$$f^*\,p_*(\delta_{Z,\sigma})(x)=\int_{p^{-1}(\{f(x)\})\cap Z}\sigma. \tag{1.48}$$

On the other hand, by (1.42) and the definition of the pullback of a $\delta-$section,

$$\begin{aligned}\pi_*\,g^*(\delta_{Z,\sigma})(x)&=\int_{\pi^{-1}(\{x\})\cap W}g^*\sigma\\&=\int_{g(\pi^{-1}(\{x\})\cap W)}\sigma,\end{aligned}$$

where we have used (1.47) and the change of variable formula. (1.46) finally gives the equality with (1.48). □

## 1.7 Pullback by a geometric morphism of vector bundles

We will now compute the kernel of the operator corresponding to the pullback by a geometric morphism of vector bundles. We will see that it is given by a $\delta-$section, and identify its symbol. After introducing a transversality condition, we will then be able to define some "trace" for those operators.

Recall that given a smooth map $h:M\to N$ between manifolds, its graph is defined by

$$\mathrm{graph(h)}:=\{(x,h(x))\mid x\in M\}\subset M\times N.$$

Also, we will denote by $\mathrm{pr}_M : M \times N \to M$ and $\mathrm{pr}_N : M \times N \to N$ the two projections onto each component.

In order to keep notations simple, we first focus on the case of the pullback by a smooth map.

**Proposition 1.7.1.** *Let $h : M \to N$ be a smooth map between manifolds and denote by $P : \mathcal{D}(N) \to \mathcal{D}'(M)$ the general operator induced by the pullback (see Example 1.4.4)*

$$h^* : \mathcal{E}(N) \to \mathcal{E}(M).$$

*Then, the kernel of $P$ is a $\delta-$section of the vector bundle $\mathrm{pr}_N^*(|TN|) \to M \times N$ along the submanifold $Z := \mathrm{graph(h)}$ with a symbol $\sigma \in \Gamma^\infty(Z, \mathrm{pr}_N^*(|TN|)_{|Z} \otimes |NZ|^*)$.*

*Proof.* According to the Schwartz kernel theorem[18], the kernel of $P$ is an element of $\mathcal{D}'(M \times N, \mathrm{pr}_N^*(|TN|))$. We will first identify a $\delta-$section $k$ of $\mathrm{pr}_N^*(|TN|)$ along $Z$, and then verify explicitly that it is the kernel of $P$. Notice that it makes sense to define a $\delta-$section along $Z$ since, as the graph of a smooth map, it is a properly embedded submanifold of $M \times N$.

The proof is mainly a matter of correctly identifying the splitting of a density on $M \times N$ as a tensor product of densities on $Z$ and $NZ$, as well as on $M$ and $N$. Let $z = (x, h(x)) \in Z$. We have the two decompositions

$$\begin{aligned} T_z(M \times N) &= T_{h(x)}N \oplus T_xM, \\ T_z(M \times N) &= N_zZ \oplus T_zZ, \end{aligned}$$

to each of which corresponds, by Lemma 1.2.6, a canonical isomorphism:

$$|T_z(M \times N)| \simeq |T_{h(x)}N| \otimes |T_xM|, \tag{1.49}$$

$$|T_z(M \times N)| \simeq |N_zZ| \otimes |T_zZ|. \tag{1.50}$$

The diffeomorphism

$$\mathrm{gr} : M \xrightarrow{\sim} Z \subset M \times N \; ; \; x \mapsto (x, h(x)).$$

induces an isomorphism of vector spaces

$$\mathrm{gr}_{*_x} : T_xM \xrightarrow{\sim} T_zZ \; ; \; (X, h_{*_x}(X)),$$

hence, by Lemma 1.2.7, an isomorphism $|\mathrm{gr}_{*_x}| : |T_xM| \xrightarrow{\sim} |T_zZ|$.

Then, for every $\lambda_1 \in |T_{h(x)}N|$ and $\lambda_2 \in |T_xM|$, there is a unique density in $|N_zZ|$, that we denote by $|j_z|(\lambda_1)$, such that the elements $\lambda_1 \otimes \lambda_2$ and

[18] We should note that the kernel theorem is in fact not really needed here because we will explicitly compute the kernel of $P$. However, we still refer to it since we think it gives some more insight into the proof.

$|j_z|(\lambda_1) \otimes |\mathrm{gr}_{*_x}|(\lambda_2)$ have the same image under the isomorphisms (1.49) and (1.50). This defines an isomorphism

$$|j_z| : |T_{h(x)}N| \xrightarrow{\sim} |N_z Z|.$$

We can explicitly compute that it is induced from the isomorphism

$$j_z : T_{h(x)}N \xrightarrow{\sim} N_z Z \; ; \; Y \mapsto [(0,Y)] \,.^{19} \tag{1.51}$$

We define the symbol of our $\delta-$section $k$ as the section $\sigma$ of the vector bundle $\mathrm{Hom}(|NZ|, \mathrm{pr}_N^*(|TN|)_{|Z})$ over $Z$ given by $\sigma(z) := |j_z|^{-1}$. Before showing that this section is smooth, let us see that $k$ indeed gives the kernel of $P$.

Let $\varphi \in \mathcal{D}(N)$ and $\psi \in \mathcal{D}(M, |TM|)$. By definition of the operator $P_k$ associated to the kernel $k$ (see Example 1.4.12), we have:

$$\langle P_k \varphi, \psi \rangle := \langle k, \mathrm{pr}_N^* \varphi \,.\, \mathrm{pr}_N^* \bar{\mu}_0 \otimes (\mathrm{pr}_N^* \mu_0 \otimes \mathrm{pr}_M^* \psi) \rangle \,,$$

where $\mu_0$ is any non-vanishing density on $N$, $\bar{\mu}_0$ is the corresponding dual density and $(\mathrm{pr}_N^*\mu_0 \otimes \mathrm{pr}_M^*\psi)$ is considered as a section of $|T(M \times N)|$ through the isomorphism (1.49). When restricted to $Z$, this section canonically gives a section of the bundle $|NZ| \otimes |TZ|$ over $Z$ through the isomorphism (1.50). By the compatibility between $|j_z|$ and $|\mathrm{gr}_{*_x}|$, this section is given at a point $z \in Z$ by

$$|j_z|(\mathrm{pr}_N^*\mu_0(z)) \otimes |\mathrm{gr}_{*_x}|(\mathrm{pr}_M^*\psi(z)) = \sigma(z)^{-1}(\mathrm{pr}_N^*\mu_0(z)) \otimes ((\mathrm{gr}^{-1})^*\psi)(z). \tag{1.52}$$

By definition of a $\delta-$section (equation (1.35)), we therefore have

$$\begin{aligned}
&\langle k, \mathrm{pr}_N^* \varphi \,.\, \mathrm{pr}_N^* \bar{\mu}_0 \otimes (\mathrm{pr}_N^* \mu_0 \otimes \mathrm{pr}_M^* \psi) \rangle \\
&:= \int_Z (\mathrm{pr}_N^*\varphi)(z) \,.\, \left\langle (\mathrm{pr}_N^*\bar{\mu}_0)(z), \sigma(z)(\sigma(z)^{-1}(\mathrm{pr}_N^*\mu_0(z))) \right\rangle \,.\, ((\mathrm{gr}^{-1})^*\psi)(z) \\
&= \int_Z (\mathrm{pr}_N^*\varphi)(z) \,.\, ((\mathrm{gr}^{-1})^*\psi)(z) \\
&= \int_M (\mathrm{gr}^*\mathrm{pr}_N^*\varphi)(x) \,.\, \psi(x) = \int_M (\varphi \circ h) \,.\, \psi,
\end{aligned}$$

[19] Indeed, let $(e_1, \ldots, e_m)$ be a basis of $T_xM$ and $(f_1, \ldots, f_n)$ be a basis of $T_{h(x)}N$ and denote $\mu := |de_1 \ldots de_n| \in |T_xM|$ and $\lambda := |df_1, \ldots, df_n| \in |T_{h(x)}N|$ the densities that are equal to 1 one those bases. Then $\mathbf{b} := ((e_1, h_{*x}(e_1)), \ldots, (e_m, h_{*x}(e_m)), (0, f_1), \ldots, (0, f_n))$ is a basis of $T_z(M \times N)$ and, by the proof of Lemma 1.2.4, the evaluation on $\mathbf{b}$ of the image of $|j_z|(\lambda) \otimes |\mathrm{gr}_{*_x}|(\mu)$ under the isomorphism (1.50) is

$$\begin{aligned}
(|j_z|(\lambda) \otimes |\mathrm{gr}_{*_x}|(\mu))(\mathbf{b}) &:= |j_z|(\lambda)([(0,f_1)], \ldots, [(0,f_n)]) \\
&\quad .|\mathrm{gr}_{*_x}|(\mu)((e_1, h_{*x}(e_1)), \ldots, (e_m, h_{*x}(e_m)) \\
&= \lambda((f_1, \ldots, f_n)).\mu((e_1, \ldots, e_m)) = 1.
\end{aligned}$$

On the other hand, since $\mathbf{b}$ is related to the basis $\mathbf{c} := ((e_1, 0), \ldots, (e_m, 0), (0, f_1), \ldots, (0, f_n))$ by a matrix of determinant 1, the image of $\lambda \otimes \mu$ under the isomorphism (1.49) evaluates as:

$$(\lambda \otimes \mu)(\mathbf{b}) = (\lambda \otimes \mu)(\mathbf{c}) = \lambda((f_1, \ldots, f_n)).\mu((e_1, \ldots, e_m)) = 1,$$

which proofs the claim.

where, to get the last line from the previous one, we have performed a change of variables using the diffeomorphism $\mathrm{gr} : M \xrightarrow{\sim} Z$. This last expression coincides with the value (1.21) of $\langle P\varphi, \psi\rangle$, so $k$ is indeed the kernel of $P$.

Finally, let us get back to the symbol $\sigma$ and show that it is smooth by giving a local expression that will be useful later. Let $(x_1, \ldots, x_m)$ be local coordinates on an open neighbourhood $U \subset M$ of $x$ and $(y_1, \ldots, y_n)$ local coordinates on an open neighbourhood $V \subset N$ of $h(x)$. Denote by

$$(h_1(x_1, \ldots, x_m), \ldots, h_n(x_1, \ldots, x_m))$$

the expression of $h$ in those coordinates. Then,

$$(u_1 := y_1 - h_1(x_1, \ldots, x_m), \ldots, u_m := y_m - h_m(x_1, \ldots, x_m), x_1, \ldots, x_m)$$

gives local coordinates on $U \times V \subset M \times N$ such that $Z$ is described by the vanishing of the first $n$ coordinates. $|NZ|$ is generated by $|du_1 \ldots du_n|$, and, by equation 1.51, $|j_z|$ maps $|dy_1 \ldots dy_n|$ to $|du_1 \ldots du_n|$. Therefore, as an element of $|TN| \otimes |NZ|^*$, the symbol is

$$\sigma(z) := |j_z|^{-1} = |dy_1 \ldots dy_n| \otimes |du_1 \ldots du_n|^*, \tag{1.53}$$

which is clearly smooth. □

We can now generalize this result to vector bundles. Since the delicate point of identifying densities has already been carried on in the previous proof, there is nothing really new. It is mainly a matter of carefully introduce tensor products with sections and homomorphisms of $E$ and $F$ where they should be.

**Proposition 1.7.2.** *Let $E \to M$ and $F \to N$ be vector bundles over two manifolds $M$ and $N$. Let $\underline{h} = (h, r)$ be a geometric morphism from $E$ to $F$ and denote by $P : \mathcal{D}(N, F) \to \mathcal{D}'(M, E)$ the general operator induced by the pullback (see Example 1.4.8)*

$$\underline{h}^* : \mathcal{E}(N, F) \to \mathcal{E}(M, E).$$

*Then, the kernel of $P$ is a $\delta-$section of the vector bundle*

$$\mathrm{Hom}(\mathrm{pr}_N^* F, \mathrm{pr}_M^* E) \otimes \mathrm{pr}_N^*(|TN|) \to M \times N$$

*along the submanifold $Z := \mathrm{graph(h)}$ with a symbol*

$$\sigma \in \Gamma^\infty \left(Z, \mathrm{Hom}(\mathrm{pr}_N^* F, \mathrm{pr}_M^* E)_{|Z} \otimes \mathrm{Hom}\big(|NZ|, \mathrm{pr}_N^*(|TN|)_{|Z}\big)\right).$$

*The support of this $\delta-$section $k$ is given by*

$$\mathrm{supp}(k) = \{(x, h(x)) \mid x \in \mathrm{supp}(r)\} \subset \mathrm{graph}(h). \tag{1.54}$$

*Proof.* The proof follows pretty much the same line as the previous one. According to the Schwartz kernel theorem, the kernel of $P$ is a generalized section of the vector bundle over $M \times N$

$$E \boxtimes F^\vee \simeq \mathrm{Hom}(\mathrm{pr}_N^* F, \mathrm{pr}_M^* E) \otimes \mathrm{pr}_N^*(|TN|),$$

which we will identify as a $\delta-$section $k$ along the properly embedded submanifold $Z$ of $M \times N$.

Its symbol must be a section of the vector bundle over $Z$

$$\begin{aligned}&\big(\mathrm{Hom}(\mathrm{pr}_N^* F, \mathrm{pr}_M^* E) \otimes \mathrm{pr}_N^*(|TN|)\big)_{|Z} \otimes |NZ|^* \\ &\simeq \mathrm{Hom}(\mathrm{pr}_N^* F, \mathrm{pr}_M^* E)_{|Z} \otimes \mathrm{Hom}(|NZ|, \mathrm{pr}_N^*(|TN|)_{|Z}).\end{aligned}$$

Let $z = (x, h(x)) \in Z$. As in the proof of Proposition 1.7.1, we get an isomorphism

$$|j_z|^{-1} : |N_z Z| \xrightarrow{\sim} |T_{h(x)} N|.$$

The fiber at $z$ of $\mathrm{Hom}(\mathrm{pr}_N^* F, \mathrm{pr}_M^* E)_{|Z}$ is

$$\begin{aligned}\big(\mathrm{Hom}(\mathrm{pr}_N^* F, \mathrm{pr}_M^* E)_{|Z}\big)_z &\simeq \mathrm{Hom}\big((\mathrm{pr}_N^* F)_{(x,h(x))}, (\mathrm{pr}_M^* E)_{(x,h(x))}\big) \\ &\simeq \mathrm{Hom}(F_{h(x)}, E_x).\end{aligned}$$

From the geometric morphism $\underline{h}$, we get such a homomorphism as the element

$$r(x) : F_{h(x)} \to E_x.$$

All this allows us to define the symbol of our $\delta-$section $k$ as the section $\sigma$ of the vector bundle

$$\mathrm{Hom}(\mathrm{pr}_N^* F, \mathrm{pr}_M^* E)_{|Z} \otimes \mathrm{Hom}\big(|NZ|, \mathrm{pr}_N^*(|TN|)_{|Z}\big) \to Z$$

defined at $z$ by

$$\sigma(z) := r(\mathrm{pr}_M(z)) \otimes |j_z|^{-1}. \tag{1.55}$$

We have shown in the previous proposition that $|j_z|^{-1}$ is smooth in $z$ and since $r$ is a smooth section, $\sigma$ is also smooth. From (1.55), we also get that the support of $k$ is given by $(\mathrm{pr}_M)^{-1}(\mathrm{supp}(r))$, which coincides with (1.54).

Finally, to see that $k$ indeed coincides with the kernel of $P$, let us follow Example 1.4.13 and take $\varphi \in \mathcal{D}(N, F)$ and $\psi \in \mathcal{D}(M, E^* \otimes |TM|)$. $\psi$ can be written as $\psi = \psi_1 \otimes \psi_2$ with $\psi_1 \in \mathcal{D}(M, E^*)$ and $\psi_2 \in \mathcal{E}(M, |TM|)$. The value of the operator $P_k$ associated to the kernel $k$ is then given by equation (1.26):

$$P_k(\varphi)(\psi) := \langle k, \mathrm{pr}_N^* \varphi \otimes \mathrm{pr}_M^* \psi_1 \otimes \mathrm{pr}_N^* \bar{\mu}_0 \otimes (\mathrm{pr}_N^* \mu_0 \otimes \mathrm{pr}_M^* \psi_2) \rangle,$$

where $\mu_0$ is any non-vanishing density on $N$, $\bar{\mu}_0$ is the corresponding dual density and $(\mathrm{pr}_N^* \mu_0 \otimes \mathrm{pr}_M^* \psi_2)$ is considered as a section of $|T(M \times N)|$ through the isomorphism (1.49). To evaluate $k$ from its definition as a $\delta-$section, we proceed as in the previous proof, relying on equation (1.52):

$$\begin{aligned}
&\langle k, \mathrm{pr}_N^*\varphi \otimes \mathrm{pr}_M^*\psi_1 \otimes \mathrm{pr}_N^*\bar{\mu}_0 \otimes (\mathrm{pr}_N^*\mu_0 \otimes \mathrm{pr}_M^*\psi_2)\rangle \\
&= \int_Z \langle (\mathrm{pr}_M^*\psi_1)(z), r(\mathrm{pr}_M(z))(\mathrm{pr}_N^*\varphi)(z)\rangle \\
&\qquad . \left\langle (\mathrm{pr}_N^*\bar{\mu}_0)(z), |j_z|^{-1}(|j_z|(\mathrm{pr}_N^*\mu_0(z)))\right\rangle \ . ((\mathrm{gr}^{-1})^*\psi_2)(z) \\
&= \int_Z \langle (\mathrm{pr}_M^*\psi_1)(z), r(\mathrm{pr}_M(z))(\mathrm{pr}_N^*\varphi)(z)\rangle \ . ((\mathrm{gr}^{-1})^*\psi_2)(z).
\end{aligned}$$

Changing variables using the diffeomorphism $\mathrm{gr} : M \xrightarrow{\sim} Z$ leads to

$$\begin{aligned}
&\int_M \langle (\mathrm{pr}_M^*\psi_1)(\mathrm{gr}(x)), r(\mathrm{pr}_M(\mathrm{gr}(x)))(\mathrm{pr}_N^*\varphi)(\mathrm{gr}(x))\rangle \ . (\mathrm{gr}^*(\mathrm{gr}^{-1})^*\psi_2)(x) \\
&= \int_M \langle \psi_1(x), r(x)(\varphi(h(x)))\rangle \ . \psi_2(x) \\
&= \int_M \langle \psi_1(x), (\underline{h}^*\varphi)(x)\rangle \ . \psi_2(x) = \int_M (\underline{h}^*\varphi, \psi)\,,
\end{aligned}$$

where $(\cdot,\cdot)$ denotes the pairing (1.17). This is indeed the expression (1.22) for $P$. $\square$

**Definition 1.7.3.** *Let $h : M \to M$ be a smooth map on a manifold $M$. A point $p \in M$ is said to be a* simple fixed point *if it is a fixed point of $h$ (i.e. $h(p) = p$) such that* $\det(\mathrm{id} - h_{*_p}) \neq 0$.[20]

**Lemma 1.7.4.** *Let $h : M \to M$ be a smooth map on a manifold $M$. Then, the diagonal map $\Delta : M \to M \times M$ ; $x \mapsto (x, x)$ is transverse to* $\mathrm{graph}(h)$ *if and only if all the fixed points of $h$ are simple. Furthermore, in this case, all the fixed points of $h$ are isolated.*

*Proof.* First notice that a point $p \in M$ is in $\Delta^{-1}(\mathrm{graph}(h))$ if and only if $(p, p) = (p, h(p))$, that is if and only if it is a fixed point of $h$. Let $p \in M$ be a fixed point of $h$. We have

$$\begin{aligned}
T_{(p,p)}(\mathrm{graph}(h)) &\simeq \left\{(X, h_{*_p}(X)) \mid X \in T_pM\right\} \simeq T_pM, \\
\Delta_{*_p}(T_pM) &\simeq \{(Y, Y) \mid Y \in T_pM\} \simeq T_pM.
\end{aligned}$$

To say that $\Delta$ is transverse to $\mathrm{graph}(h)$ is to say that these vector spaces must span $T_{(p,p)}(M \times M)$. Because they both are of dimension $\dim(M)$, this is equivalent to their intersection being $\{0\}$. This is in turn equivalent to the linear map $(\mathrm{id} - h_{*_p}) : T_pM \to T_pM$ being injective, hence invertible, which is the condition for $p$ to be simple.
Regarding the second part, if $\Delta$ is transverse to $\mathrm{graph}(h)$, Theorem 1.6.13

[20] Here, $\mathrm{id} : T_pM \to T_pM$ denotes the identity map. Since $p$ is a fixed point, we have $h_{*_p} : T_pM \to T_pM$, so the condition is well-defined.

implies that $\Delta^{-1}(\mathrm{graph}(h))$, the set of fixed points of $h$, is an embedded submanifold of $M$ of codimension $\dim(M)$, that is, a union of isolated points. □

**Proposition 1.7.5.** *Let $E \to M$ be a vector bundle over a manifold $M$. Let $\underline{h} = (h, r)$ be a geometric morphism from $E$ to itself. Let $k_h$ be the kernel of the general operator induced by the pullback by $\underline{h}$ (see Example 1.4.8). Let us denote by $\Delta : M \to M \times M; x \mapsto (x, x)$ the diagonal map and by $\bar{\pi} : M \to \{\star\}$ the projection onto a point. Suppose that all the fixed points $p \in M$ of $h$ are simple and that $\mathrm{Tr}\,(r(p)) \neq 0$ for only a finite number of them.*
*Then, the quantity $\bar{\pi}_* \,\mathrm{Tr}\,\Delta^*\, k_h$ is well-defined and*

$$\bar{\pi}_* \,\mathrm{Tr}\,\Delta^*\, k_h = \sum_{p=h(p)} \frac{\mathrm{Tr}(r(p))}{\left|\det\left(\mathrm{id} - h_{*_p}\right)\right|}, \tag{1.56}$$

*where the sum is over the fixed points of $h$.*

*Remark* 1.7.6. The sum in (1.56) always has only a finite number of nonvanishing terms, even if $h$ has an infinite number of fixed points, because of the hypothesis that $\mathrm{Tr}(r(p)) \neq 0$ for finitely many of them. Notice that this condition is trivially always verified if $h$ has a finite number of fixed points. It is also the case if $M$ is compact since, all the fixed points being simple, they must be isolated by Lemma 1.7.4, hence in finite number. ◁

*Proof of Proposition 1.7.5.* Let $\mathrm{pr}_i : M \times M \to M$ be the projection onto the $i$th component. By Proposition 1.7.2, $k_h$ is a $\delta-$section of the vector bundle[21]

$$F := \mathrm{Hom}(\mathrm{pr}_2^* E, \mathrm{pr}_1^* E) \otimes \mathrm{pr}_2^*(|TM|) \to M \times M$$

along the submanifold graph(h). To pullback by $\Delta$, we need to define a geometric morphism that "enhances" $\Delta$ with a vector bundle morphism from $\Delta^* F$ to some vector bundle over $M$. Since $\mathrm{pr}_1 \circ \Delta = \mathrm{pr}_2 \circ \Delta = \mathrm{id}_M$, we have in fact that

$$\Delta^* F \simeq \mathrm{Hom}(E, E) \otimes |TM|.$$

We still denote by $\Delta$ the associated geometric morphism. Because all the fixed points of $h$ are simple, Lemma 1.7.4 implies that $\Delta$ is transverse to graph(h), so we can pullback $k_p$ by $\Delta$ using Definition 1.6.16. This gives a $\delta-$section of the bundle $\mathrm{Hom}(E, E) \otimes |TM|$ along the submanifold $W := \Delta^{-1}(\mathrm{graph(h)}) \subset M$, which is the set of fixed points of $h$. At each $p \in W$, the symbol of $\Delta^*\, k_h$ is an element of $\mathrm{Hom}(E_p, E_p) \otimes |T_p M| \otimes |N_p W|^*$. Taking the trace of the homomorphism gives a symbol of a $\delta-$section of $|TM|$ along $W$ that we denote $\mathrm{Tr}\,\Delta^*\, k_h$. Its support is exactly the set of fixed points $p$ such that $\mathrm{Tr}\,(r(p)) \neq 0$, which is finite by hypothesis. Therefore, $\mathrm{Tr}\,\Delta^*\, k_h$ is compactly supported and we can pushforward it by $\bar{\pi}$, to get the well defined number $\bar{\pi}_* \,\mathrm{Tr}\,\Delta^*\, k_h$.

[21] We denote by $\mathrm{pr}_i : M \times M \to M$ the projection onto the $i$th component.

To compute this number, let $p \in M$ be a fixed point of $h$ in $\operatorname{supp}(\operatorname{Tr} \Delta^* k_h)$, that is, such that $\operatorname{Tr}(r(p)) \neq 0$. Let $(x_1, \ldots, x_m)$ be local coordinates on an open neighbourhood $U \subset M$ of $p$. Denote by $(h_1(x_1, \ldots, x_m), \ldots, h_m(x_1, \ldots, x_m))$ the expression of $h$ in those coordinates. Then,

$$(u_1 := y_1 - h_1(x_1, \ldots, x_m), \ldots, u_m := y_m - h_m(x_1, \ldots, x_m), x_1, \ldots, x_m)$$

gives local coordinates on $U \times U \subset M \times N$ such that $Z$ is described by the vanishing of the first $n$ coordinates.[22] We know from equations (1.53) and (1.55) that the symbol of $k_h$ is $r(x) \otimes |dy_1 \ldots dy_n| \otimes |du_1 \ldots du_n|^*$. To pullback by $\Delta$, we notice that

$$\begin{aligned}&\Delta(x_1, \ldots, x_m)\\ &= (x_1 - h_1(x_1, \ldots, x_m), \ldots, x_m - h_m(x_1, \ldots, x_m), x_1, \ldots, x_m),\end{aligned}$$

and that $N_p W = T_p M$. From the proof of Lemma 1.6.14 and equation (1.40), we compute that the symbol of $\Delta^* k_h$ at $p$ is

$$\frac{r(p) \otimes |dx_1 \ldots dx_n|}{\left|\det\left(\mathrm{id} - h_{*_p}\right)\right|} \otimes |dx_1 \ldots dx_n|^* \in \operatorname{Hom}(E_p, E_p) \otimes |T_p M| \otimes |T_p M|^*.$$

Taking the trace gives $\operatorname{Tr}(r(p))\, |dx_1 \ldots dx_n| \otimes |dx_1 \ldots dx_n|^*$, a $\delta-$section of $|TM|$ over the fixed points of $h$. $\bar{\pi}$ being a submersion, Proposition 1.6.20 implies that the pushforward $\bar{\pi}_* \operatorname{Tr} \Delta^* k_h$ is a smooth density given by integration over the fiber. In this case, the fiber is the submanifold associated to the $\delta-$section. Integration is thus given by the sum over the fixed points $p$ of the pairing of $|dx_1 \ldots dx_n|^*$ with $\operatorname{Tr}(r(p))\, |dx_1 \ldots dx_n|$, which leads to the expected identity (1.56). As already noticed in Remark 1.7.6, we can extend the sum to all fixed points of $h$ since $\operatorname{Tr}(r(p)) = 0$ for the additional ones. $\square$

*Remark* 1.7.7. We have seen in Lemma 1.5.17 that if $k$ is the kernel of a smooth operator $P$ such that $\Delta^* k$ is compactly supported, then $P$ is smooth-traceable and $\operatorname{tr}(P) = \bar{\pi}_* \operatorname{Tr} \Delta^*(k)$. The previous result suggests thus to interpret the operation $\bar{\pi}_* \operatorname{Tr} \Delta^*$ as a generalized trace for pullback operators. However, we should mention that this analogy should be taken with care. Indeed, even when the pullback by $\underline{h}$ extends to a bounded operator on the intrinsic Hilbert space of square-integrable sections on $M$, it is usually not trace-class, although $\bar{\pi}_* \operatorname{Tr} \Delta^*(k_h)$ is well defined. $\triangleleft$

[22] Here, $(x_1, \ldots, x_m)$ are understood to be defined on the first component of $U \times U$, while $(y_1, \ldots, y_m)$ denotes the same coordinates as $(x_1, \ldots, x_m)$, but defined on the second component of $U \times U$.

# 1.8 Distributional trace of a family of geometric morphisms

## 1.8.1 Pullback by a family of geometric morphisms

Let $M$ and $Q$ be two manifolds, and suppose that we have a smooth map [23]

$$\tau : M \times Q \to Q \;;\; (x, q) \mapsto \tau_x(q).$$

This gives a family $\{\Omega(x)\}_{x \in M}$ of continuous linear operators on $\mathcal{E}(Q)$ given, for every $x \in M$, by

$$\Omega(x) : \mathcal{E}(Q) \to \mathcal{E}(Q) \;;\; \varphi \mapsto \varphi \circ \tau_x.$$

Then, for every compactly supported smooth density $\rho \in \mathcal{D}(M, |TM|)$, we can form the continuous linear operator

$$\Omega(\rho) : \mathcal{E}(Q) \to \mathcal{E}(Q) \;;\; \varphi \mapsto \int_M (\Omega(x)\varphi)\; \rho(x),$$

which is explicitly defined as $(\Omega(\rho)\varphi)(q) = \int_M \varphi(\tau_x(q))\; \rho(x)$. According to the discussion in section 1.4, $\Omega(\rho)$ induces a general operator $\mathcal{D}(Q) \to \mathcal{D}'(Q)$. The goal of this section is to show that, under some conditions, it is a smooth operator (that is, its kernel is a smooth density $k_\rho$ on $Q \times Q$). If in addition, it is smooth-traceable, we can define the distributional trace as the linear map

$$\mathrm{tr}_\Omega : \mathcal{D}(M) \to \mathbb{C} \;;\; \rho \mapsto \mathrm{tr}\,(\Omega(\rho)).$$

We will show that it is a generalized function on $M$ and express it by giving an explicit formula for the smooth trace $\int_Q k_\rho(q, q)$ in terms of the fixed points of $\tau_x$.

We will in fact work in the more general setting of a family of geometric morphisms of a vector bundle $E \to Q$ over $Q$. To motivate the next definition, we would like to think of a family of geometry morphisms of $E$ parametrized by $M$ as the datum, for each $x \in M$, of $\underline{\tau}_x = (\tau_x, r_x)$, where $\tau_x : Q \to Q$ is a smooth map and $r_x(q) : E_{\tau_x(q)} \to E_q$ is a linear map for each $q \in Q$. It would be a smooth family if everything depends smoothly on $x$. This can be encoded by the following definition.

**Definition 1.8.1.** *Let $M$ and $Q$ be two manifolds and $E \to Q$ a vector bundle over $Q$. Denote by $\mathrm{pr}_2 : M \times Q \to Q$ the projection onto the second component. A* smooth family of geometric morphisms of $E$ parametrized by $M$ *is a geometric morphism from $\mathrm{pr}_2^*(E)$ to $E$. In other words, it is a pair $\underline{\tau} = (\tau, r)$, where $\tau : M \times Q \to Q$ is a smooth map and $r$ is a smooth section of the vector bundle $\mathrm{Hom}(\tau^*E, \mathrm{pr}_2^*E)$ over $M \times Q$.*

[23] For instance, we can think about the case where $M$ is a Lie group, and $\tau$ is a smooth group action of $M$ on $Q$.

*Remark* 1.8.2. Let us see that the definition gives what we would like. For each $x \in M$ and $q \in Q$, we have

$$\begin{aligned}\mathrm{Hom}(\tau^* E, \mathrm{pr}_2^* E)_{(x,q)} &= \mathrm{Hom}\left((\tau^* E)_{(x,q)}, (\mathrm{pr}_2^* E)_{(x,q)}\right)\\ &= \mathrm{Hom}\left(E_{\tau_x(q)}, E_q\right),\end{aligned}$$

so $r_x(q)$ is a linear map from $E_{\tau_x(q)}$ to $E_q$ as expected. ◁

Given a smooth family of geometric morphisms $\underline{\tau} = (\tau, r)$, we can build a family of operators in the same spirit as before, and use it to associate an operator on $\mathcal{E}(Q,E)$ to every compactly supported density on $M$. For each $x \in M$, we define a continuous linear operator $\Omega(x) : \mathcal{E}(Q,E) \to \mathcal{E}(Q,E)$ by the rule, for $\varphi \in \mathcal{E}(Q,E)$ and $q \in Q$:

$$(\Omega(x)\varphi)(q) := r_x(q)\big(\varphi(\tau_x(q))\big) \;\in\; E_q. \tag{1.57}$$

**Definition 1.8.3.** *Let $M$ and $Q$ be two manifolds, $E \to Q$ a vector bundle over $Q$. Let $\underline{\tau} = (\tau, r)$ be a smooth family of geometric morphisms of $E$ parametrized by $M$. We call the operators $\{\Omega(x)\}_{x \in M}$ defined by (1.57) the* family of pullback operators *associated to the geometric morphism $\underline{\tau}$.*

Then, to each compactly supported smooth density $\rho \in \mathcal{D}(M, |TM|)$, we associate the continuous linear operator

$$\Omega(\rho) : \mathcal{E}(Q,E) \to \mathcal{E}(Q,E)\;;\; \varphi \mapsto \int_M (\Omega(x)\varphi) \otimes \rho(x), \tag{1.58}$$

which is more explicitly defined, for $\varphi \in \mathcal{E}(Q,E)$ and $q \in Q$, by

$$(\Omega(\rho)\varphi)(q) := \int_M r_x(q)\big(\varphi(\tau_x(q))\big) \otimes \rho(x). \tag{1.59}$$

The following hypothesis will turn out to be crucial for $\Omega(\rho)$ to be a smooth operator.

**Definition 1.8.4.** *Let $M$ and $Q$ be two manifolds. A smooth map $\tau : M \times Q \to Q$ is* locally transitive *if and only if, for every $(x,q) \in M \times Q$, the linear map*

$$T_x(M) \to T_{\tau_x(q)}(Q)\;;\; X \mapsto \tau_{*(x,q)}(X, 0)$$

*is surjective.*

**Proposition 1.8.5.** *Let $M$ and $Q$ be two manifolds, $E \to Q$ a vector bundle over $Q$. Let $\underline{\tau} = (\tau, r)$ be a smooth family of geometric morphisms of $E$ parametrized by $M$ such that $\tau$ is locally transitive. Then, for every $\rho \in \mathcal{D}(M, |TM|)$, the operator $\Omega(\rho) : \mathcal{E}(Q,E) \to \mathcal{E}(Q,E)$ defined by (1.59) is smooth.*

*Proof.* Here is the outline of the proof. We first show that the integrand of (1.59) corresponds to the pullback by a geometric morphism, whose kernel $K_\rho$ is thus a $\delta-$section. Then, we express the integration over $M$ as the pushforward by a projection $\pi_{23}$, and we compute that the kernel of $\Omega(\rho)$ is $(\pi_{23})_*(K_\rho)$. Finally, we show that $\pi_{23}$ is a submersion on the submanifold associated to the $\delta-$section $K_\rho$, which implies that $(\pi_{23})_*(K_\rho)$, hence $\Omega(\rho)$, is smooth.

Let us denote by $\mathrm{pr}_1 : M \times Q \to M$ and $\mathrm{pr}_2 : M \times Q \to Q$ the projections and consider the geometric morphism

$$\underline{\tau}_\rho = (\tau, r_\rho) \tag{1.60}$$

from the vector bundle $\mathrm{pr}_2^*(E) \otimes \mathrm{pr}_1^*(|TM|)$ over $M \times Q$ to the vector bundle $E$, where $r_\rho \in \mathrm{Hom}\,(\tau^*(E), \mathrm{pr}_2^*(E) \otimes \mathrm{pr}_1^*(|TM|))$ is defined, for $(x,q) \in M \times Q$ and $v \in E_{\tau_x(q)}$, by

$$r_{\rho,x}(q)(v) := r_x(q)(v) \otimes \rho(x).$$

Then, for every $\varphi \in \mathcal{E}(Q,E)$, the pullback by $\underline{\tau}_{\rho,x}$ is given at $(x,q) \in M \times Q$ by:

$$\begin{aligned}(\underline{\tau}_\rho{}^*\varphi)(x,q) &= r_{\rho,x}(q)\big(\,\varphi(\tau_x(q))\,\big)\\ &= r_x(q)\big(\,\varphi(\tau_x(q))\,\big) \otimes \rho(x),\end{aligned} \tag{1.61}$$

which coincides with the integrand of (1.59). By Proposition 1.7.2, the kernel of the corresponding general operator is given by a $\delta-$section, which we denote by $K_\rho$, of the following vector bundle over $M \times Q \times Q$. Let $\pi_i$ denote the projection of $M \times Q \times Q$ onto the $i$th component, $\pi_{12} : M \times Q \times Q \to M \times Q$ the projection onto the first two components and $\pi_{23} : M \times Q \times Q \to Q \times Q$ the projection onto the last two components. The vector bundle is

$$\begin{aligned}&\pi_{12}^*\big(\,\mathrm{pr}_2^*(E) \otimes \mathrm{pr}_1^*(|TM|)\,\big) \otimes (\pi_3^*(E))^* \otimes \pi_3^*(|TQ|)\\ &\simeq \mathrm{Hom}\big(\pi_3^*(E), \pi_2^*(E)\big) \otimes \pi_1^*(|TM|) \otimes \pi_3^*(|TQ|).\end{aligned}$$

We would like to pushforward $K_\rho$ by $\pi_{23}$, which would correspond to integration over $M$ in (1.59). First, notice that the submanifold associated to the $\delta-$section $K_\rho$ is $Z := \mathrm{graph}(\tau) \subset M \times Q \times Q$ and that the corresponding symbol has the form

$$\sigma_\rho(x,q,\tau_x(q)) = r_x(q) \otimes \rho(x) \otimes \sigma(x,q), \tag{1.62}$$

for some $\sigma \in \Gamma^\infty(Z, \mathrm{Hom}(|NZ|, \pi_3^*(|TQ|)_{|Z})$.[24] Therefore, since $\rho$ has compact support, $\pi_{23}$ is proper on $\mathrm{supp}(\sigma_\rho)$, hence on $\mathrm{supp}(K_\rho)$. Then, let us write

$$\begin{aligned}&\mathrm{Hom}\big(\pi_3^*(E), \pi_2^*(E)\big) \otimes \pi_1^*(|TM|) \otimes \pi_3^*(|TQ|)\\ &\simeq \big(\,\mathrm{Hom}\big(\pi_2^*(E), \pi_3^*(E)\big) \otimes \pi_2^*(|TQ|)\,\big)^* \otimes |T(M \times Q \times Q)|.\end{aligned}$$

[24]For a later purpose, notice that $r_x(q) \otimes \sigma(x,q)$ is the symbol of the $\delta-$section corresponding to the kernel of the pullback by $\tau$.

Since the vector bundle

$$\mathrm{Hom}\big(\pi_2^*(E), \pi_3^*(E)\big) \otimes \pi_2^*(|TQ|) \to M \times Q \times Q$$

is the pullback by $\pi_{23}$ of the vector bundle ($p_i : Q \times Q \to Q$ denotes the projection onto the $i$th component)

$$\mathrm{Hom}\big(p_1^*(E), p_2^*(E)\big) \otimes p_1^*(|TQ|) \to Q \times Q,$$

we have a natural geometric morphism $\underline{\pi}_{23} = (\pi_{23}, r_{23})$

$$\mathrm{Hom}\big(\pi_2^*(E), \pi_3^*(E)\big) \otimes \pi_2^*(|TQ|) \to \mathrm{Hom}\big(p_1^*(E), p_2^*(E)\big) \otimes p_1^*(|TQ|)$$

given by naturally identifying the fibers. We can thus pushforward $K_\rho$ by $\underline{\pi}_{23}$ to get a generalized section of the vector bundle over $Q \times Q$

$$\begin{aligned}&\big(\,\mathrm{Hom}\big(p_1^*(E), p_2^*(E)\big) \otimes p_1^*(|TQ|)\,\big)^* \otimes |T(Q \times Q)|\\ &\simeq \mathrm{Hom}\big(p_2^*(E), p_1^*(E)\big) \otimes p_2^*(|TQ|).\end{aligned}$$

Let us see that $(\underline{\pi}_{23})_*(K_\rho)$ is the kernel of $\Omega(\rho) : \mathcal{D}(Q, E) \to \mathcal{D}'(Q, E)$ by evaluating it on some $\varphi \in \mathcal{D}(Q, E)$ and $\bar{s} \otimes \mu_Q \in \mathcal{D}(Q, E^* \otimes |TQ|)$. We have

$$\begin{aligned}&\langle (\underline{\pi}_{23})_*(K_\rho), p_2^*(\varphi) \otimes (p_1^*(\bar{s}) \otimes p_1^*(\mu_Q))\rangle\\ &= \big\langle K_\rho, (\underline{\pi}_{23})^*\big(\,p_2^*(\varphi) \otimes p_1^*(\bar{s}) \otimes p_1^*(\mu_Q)\,\big)\big\rangle\\ &= \big\langle K_\rho, \pi_3^*(\varphi) \otimes \pi_2^*(\bar{s}) \otimes \pi_1^*(\bar{\mu}_M) \otimes \big(\pi_1^*(\mu_M) \otimes \pi_2^*(\mu_Q)\big)\big\rangle\,,\end{aligned}$$

where $\mu_M$ is some non-vanishing density on $M$ and $\bar{\mu}_M$ the corresponding dual one. Then, by definition of $K_\rho$ as the kernel of the pullback (1.61):[25]

$$\begin{aligned}&\langle K_\rho, \varphi \otimes (\bar{s} \otimes \bar{\mu}_M) \otimes (\mu_M \otimes \mu_Q)\rangle\\ &= \int_{M\times Q} \big\langle \bar{s}(q) \otimes \bar{\mu}_M(x), r_x(q)\big(\,\varphi(\tau_x(q))\,\big) \otimes \rho(x)\big\rangle\ (\mu_M(x) \otimes \mu_Q(q))\\ &= \int_{M\times Q} \big\langle \bar{s}(q), r_x(q)\big(\,\varphi(\tau_x(q))\,\big)\big\rangle\ (\langle \bar{\mu}_M(x), \rho(x)\rangle\, \mu_M(x) \otimes \mu_Q(q))\\ &= \int_{M\times Q} \big\langle \bar{s}(q), r_x(q)\big(\,\varphi(\tau_x(q))\,\big)\big\rangle\ (\rho(x) \otimes \mu_Q(q))\\ &= \int_Q \left\langle \bar{s}(q), \int_M r_x(q)\big(\,\varphi(\tau_x(q))\,\big)\,\rho(x)\right\rangle\ \mu_Q(q)\\ &= \int_Q\ \langle \bar{s}, \Omega(\rho)(\varphi)\rangle\ \mu_Q,\end{aligned}$$

which is indeed the operator $\Omega(\rho)$.

To see that the kernel $(\underline{\pi}_{23})_*(K_\rho)$ of $\Omega(\rho)$ is smooth, let $(q, q') \in Q \times Q$ and $(x, q, q') \in (\pi_{23})^{-1}(\{(q, q')\})$. Then,

$$T_{(x,q,q')}(Z) = \big\{(X, Y, \tau_{*(x,q)}(X, Y)) \mid X \in T_xM, Y \in T_qQ\big\}\,.$$

[25] We have dropped the pullbacks $\pi_i^*$ from the equations in order to simplify the notations.

For $X \in T_xM$ and $Y \in T_qQ$, we have

$$\begin{aligned}((\pi_{23})_{|Z})_{*_{(x,q,q')}}(X, Y, \tau_{*_{(x,q)}}(X, Y)) &= (Y, \tau_{*_{(x,q)}}(X, Y)) \\ &= (Y, \tau_{*_{(x,q)}}(X, 0) + \tau_{*_{(x,q)}}(0, Y))).\end{aligned}$$

Since the map $\tau_{*_{(x,q)}}(\cdot, 0)$ is surjective because $\tau$ is locally transitive, it implies that $(\pi_{23})_{|Z}$ is a submersion. Therefore, by Proposition 1.6.20, $(\underline{\pi}_{23})_*(K_\rho)$ is smooth. $\square$

*Example* 1.8.6. To make things a bit more concrete, let $M := \mathbb{R}^2$, $Q := \mathbb{R}$ and $E$ the trivial line bundle over $Q$. Consider the smooth family of geometric morphisms given by

$$\begin{aligned}&\tau : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R} \; ; \; \big((a, l), q\big) \mapsto 2a - q, \\ &r_{(a,l)}(q) : \mathbb{C} \to \mathbb{C} \; ; \; z \mapsto e^{2i(a-q)l} z.\end{aligned}$$

$\tau$ is locally transitive, and we have:

$$(\Omega(a, l)\varphi)(q) = e^{2i(a-q)l}\varphi(2a - q).$$

Any compactly supported smooth density on $\mathbb{R}^2$ is of the form $\rho(a, l)|da\, dl|$ for some $\rho \in \mathcal{D}(\mathbb{R}^2)$, so

$$\begin{aligned}(\Omega(\rho)\varphi)(q) &= \int_{\mathbb{R}^2} e^{2i(a-q)l}\varphi(2a - q)\; \rho(a, l)\; da\, dl \\ &= \int_{\mathbb{R}} \left( \frac{1}{2} \int_{\mathbb{R}} e^{i(q'-q)l} \rho\left(\frac{q + q'}{2}, l\right) dl \right)\; \varphi(q')\, dq'.\end{aligned}$$

The kernel of $\Omega(\rho)$ is therefore $k_\rho(q, q') = \frac{1}{2} \int_{\mathbb{R}} e^{i(q'-q)l} \rho\left(\frac{q+q'}{2}, l\right) dl$ which is indeed smooth. $\Diamond$

Intuitively, we should think that the integration against $\rho$ allows to "smooth out" the singularities of the pullback by $\underline{\tau}$. The local transitivity of $\tau$ ensures that the smoothing occurs in all directions. The next example gives some more insight into how $\Omega(\rho)$ fails to be smooth when $\tau$ is not locally transitive.

*Example* 1.8.7. Let $\tau : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ ; $(x, q) \mapsto q$, $\varphi \in \mathcal{E}(\mathbb{R})$, $x, q \in \mathbb{R}$, and $\rho|dx| \in \mathcal{D}(\mathbb{R})$. Then, $(\Omega(x)\varphi)(q) = \varphi(q)$ and

$$(\Omega(\rho)\varphi)(q) = \int_{\mathbb{R}} (\Omega(x)\varphi)(q)\rho(x)\, dx = \left( \int_{\mathbb{R}} \rho(x)\, dx \right)\; \varphi(q),$$

which is not smooth (the kernel of the identity operator is a $\delta-$section along the diagonal). $\Diamond$

*Example* 1.8.8. Our last example emphasizes that $\Omega(\rho)$ is smooth even when $\tau$ is locally transitive but not globally. Let $\tau : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ ; $(x, q) \mapsto \arctan(x)-q$,

$x, q \in \mathbb{R}$, $\varphi \in \mathcal{E}(\mathbb{R})$ and $\rho|dx| \in \mathcal{D}(\mathbb{R})$. Then, $(\Omega(x)\varphi)(q) = \varphi(\arctan(x) - q)$ and

$$(\Omega(\rho)\varphi)(q) = \int_{\mathbb{R}} \frac{\rho(\tan(q+q'))}{\cos^2(q+q')} \, \mathbf{1}_{[-\frac{\pi}{2}-q, \frac{\pi}{2}-q]}(q') \, \varphi(q')dq',$$

where $\mathbf{1}_A$ denotes the characteristic function of a subset $A$. This is indeed a smooth kernel since $\rho$ has compact support. $\diamondsuit$

### 1.8.2 The distributional trace

Let us now turn to the distributional trace and begin with a sketchy discussion which, although very formal and not rigorous, should help to understand what happens. If the kernel $k_x$ of $\Omega(x)$ were a smooth function[26], we would have:

$$(\Omega(x)\varphi)(q) = \int_Q k_x(q, q')\varphi(q').$$

Then, we would write[27]

$$\begin{aligned}(\Omega(\rho)\varphi)(q) &= \int_M \Omega(x)\varphi(q)\,\rho(x) = \int_M \int_Q k_x(q,q')\varphi(q')\,\rho(x) \\ &= \int_Q \left( \int_M k_x(q,q')\,\rho(x) \right) \varphi(q').\end{aligned}$$

The smooth trace of $\Omega(\rho)$ would therefore be given by

$$\mathrm{tr}\,(\Omega(\rho)) = \int_Q \int_M k_x(q,q)\,\rho(x) = \int_M \int_Q k_x(q,q)\,\rho(x). \tag{1.63}$$

Recall that we have seen in Section 1.7 that, if $\tau_x$ only has a finite number of fixed points and if they are all simple, the generalized trace of the pullback $\Omega(x)$ by $\underline{\tau}_x$ is a well-defined number $\mathrm{tr}\,(\Omega(x)) := \bar{\pi}_* \mathrm{Tr}\, \Delta^*(k_x)$ (see Proposition 1.7.5 and Remark 1.7.7). Furthermore, it is given by a fixed point formula (1.56). Since we would like to think about the integral along the diagonal as the trace, equation (1.63) suggests to write

$$\begin{aligned}\mathrm{tr}\,(\Omega(\rho)) &= \int_M \mathrm{tr}\,(\Omega(x))\,\rho(x) \\ &= \int_M \left( \sum_{p=\tau_x(p)} \frac{\mathrm{Tr}(r_x(p))}{\left|\det\left(\mathrm{id} - (\tau_x)_{*p}\right)\right|} \right) \rho(x).\end{aligned}$$

This would give an explicit fixed point formula for the distributional trace

$$\mathrm{tr}_{\,\Omega} : \mathcal{D}(M, |TM|) \to \mathbb{C}\,;\; \rho \mapsto \mathrm{tr}\,(\Omega(\rho)).$$

[26] It is obviously not the case, but it gives an insightful analogy.

[27] Again, this is only formal since it is not clear at all that the permutation of the integrals is justified.

In the case where the fixed points are not simple, we will see that $\operatorname{tr}(\Omega(x))$ still makes sense, but as a generalized function on $M$.

Before going on, let us see that the set of all fixed points of $\underline{\tau}$ is a submanifold. It will turn to be central in the study of the smoothness of the distributional trace.

**Lemma 1.8.9.** *Let $M$ and $Q$ be two manifolds, $E \to Q$ a vector bundle over $Q$. Let $\underline{\tau} = (\tau, r)$ be a smooth family of geometric morphisms of $E$ parametrized by $M$ such that $\tau$ is locally transitive. Then, the diagonal map*

$$\tilde{\Delta} : M \times Q \to M \times Q \times Q \ ; \ (x, q) \mapsto (x, q, q)$$

*is transverse to* $\operatorname{graph}(\tau)$ *and*

$$\tilde{\Delta}^{-1}\left(\operatorname{graph}(\tau)\right) = \{(x, q) \in M \times Q \quad | \quad \tau_x(q) = q\}$$

*is a properly embedded submanifold of $M \times Q$. If it is not empty, it is of dimension $M$.*

*Proof.* For $\tilde{\Delta}$ to be transverse to $\operatorname{graph}(\tau)$, we must have that, for all $(x, q) \in M \times Q$ such that $\tau_x(q) = q$,

$$T_{(x,q,q)}(M \times Q \times Q) = T_{(x,q,q)}(W) + \tilde{\Delta}_{*_{(x,q)}}(T_{(x,q)}(M \times Q)). \tag{1.64}$$

Let $(x, q) \in M \times Q$ such that $\tau_x(q) = q$, $X \in T_x(M)$ and $Y_1, Y_2 \in T_q(Q)$. Since $\tau$ is locally transitive, there is $X_0 \in T_x(M)$ such that $\tau_{*_{(x,q)}}(X_0, 0) = Y_2 - Y_1$, i.e. $(X_0, 0, Y_2 - Y_1) \in T_{(x,q,q)}(W)$. We compute:

$$\begin{aligned}
&(X_0, 0, Y_2 - Y_1) + \tilde{\Delta}_{*_{(x,q)}}\big((X - X_0, Y_1)\big) \\
&= (X_0, 0, Y_2 - Y_1) + (X - X_0, Y_1, Y_1) = (X, Y_1, Y_2),
\end{aligned}$$

which shows that (1.64) is verified. By Theorem 1.6.13, $\tilde{\Delta}^{-1}(\operatorname{graph}(\tau))$ is an embedded submanifold of $M \times Q$, which, if not empty, has the same codimension as $\operatorname{graph}(\tau)$. The latter being $\dim(Q)$, $\tilde{\Delta}^{-1}(\operatorname{graph}(\tau))$ is of dimension $\dim(M)$ if it is not empty. It is properly embedded because $\operatorname{graph}(\tau)$ is and $\tilde{\Delta}$ is continuous. $\square$

**Definition 1.8.10.** *Let $M$ and $Q$ be two manifolds, $E \to Q$ a vector bundle over $Q$, and $\underline{\tau} = (\tau, r)$ a smooth family of geometric morphisms of $E$ parametrized by $M$ such that $\tau$ is locally transitive. The* fixed point bundle *of $\tau$ is the properly embedded submanifold of $M \times Q$ given by*

$$\tilde{\Delta}^{-1}\left(\operatorname{graph}(\tau)\right) = \{(x, q) \in M \times Q \quad | \quad \tau_x(q) = q\}, \tag{1.65}$$

*where $\tilde{\Delta} : M \times Q \to M \times Q \times Q \ ; \ (x, q) \mapsto (x, q, q)$ is the diagonal map. It is either empty or of dimension* $\dim(M)$.

**Lemma 1.8.11.** *Let $M$ and $Q$ be two manifolds, $E \to Q$ a vector bundle over $Q$. Let $\underline{\tau} = (\tau, r)$ be a smooth family of geometric morphisms of $E$ parametrized by $M$ such that $\tau$ is locally transitive. Denote by $\mathrm{pr}_M : M \times Q \to M$ the projection onto $M$ and by $Z$ the fixed point bundle. Then, for all $x \in M$, $x$ is a regular value of $(\mathrm{pr}_M)_{|Z}$ if and only if all the fixed points of $\tau_x$ are simple.*

*Proof.* Let $x \in M$ and $q \in Q$ a fixed point of $\tau_x$. We get from (1.65) that for all $Y \in T_q(Q)$, $(0, Y) \in T_{(x,q)}(M \times Q)$ is tangent to $Z$ if and only if $Y = (\tau_x)_{*_q}(Y)$, that is, if and only if $(\mathrm{id} - (\tau_x)_{*_q})(Y) = 0$. Since $\dim(Z) = \dim(M)$ ($Z$ is not empty in this case) and $((\mathrm{pr}_M)_{|Z})_{*_{(x,q)}}(X, Y) = X$ for all $(X, Y) \in T_{(x,q)}(Z) \subset T_x(M) \times T_q(Q)$, $((\mathrm{pr}_M)_{|Z})_{*_{(x,q)}}$ is surjective if and only if $(0, Y) \notin T_{(x,q)}(Z)$ for all $Y \neq 0 \in T_q(Q)$, that is, if and only if $(\mathrm{id} - (\tau_x)_{*_q})$ is injective – hence invertible. We have thus shown that $(x, q)$ is a regular point of $(\mathrm{pr}_M)_{|Z}$ if and only if $q$ is a simple fixed point of $\tau_x$. Since ${(\mathrm{pr}_M)_{|Z}}^{-1}(\{x\})$ is the set of fixed points of $\tau_x$, the claim is proved. □

We are now able to state the main results of this chapter. Their proofs will be given later on.

**Theorem 1.8.12.** *Let $M$ and $Q$ be two manifolds, $E \to Q$ a vector bundle over $Q$. Let $\underline{\tau} = (\tau, r)$ be a smooth family of geometric morphisms of $E$ parametrized by $M$ such that $\tau$ is locally transitive. Let us denote by*

- $\tilde{\Delta} : M \times Q \to M \times Q \times Q$ ; $(x, q) \mapsto (x, q, q)$ *the diagonal map;*
- $\mathrm{pr}_M : M \times Q \to M$ *the projection onto $M$;*
- $Z := \tilde{\Delta}^{-1}(\mathrm{graph}(\tau)) = \{(x, q) \in M \times Q \quad | \quad \tau_x(q) = q\}$.

*Suppose that one of the following conditions is true:*

- $(\mathrm{pr}_M)_{|Z}$ *is proper;*
- $Q$ *is compact.*

*Then, for every $\rho \in \mathcal{D}(M, |TM|)$, the operator $\Omega(\rho) : \mathcal{E}(Q, E) \to \mathcal{E}(Q, E)$ defined by (1.59) is smooth-traceable and the linear map*

$$\mathrm{tr}_\Omega : \mathcal{D}(M, |TM|) \to \mathbb{C} \; ; \; \rho \mapsto \mathrm{tr}\,(\Omega(\rho))$$

*is a generalized function on $M$ which coincides with $(\mathrm{pr}_M)_* \mathrm{Tr}\, \tilde{\Delta}^* k$, where $k$ is the kernel of $\underline{\tau}^*$ (see Proposition 1.7.2). Furthermore, the set of all $x \in M$ such that all the fixed points of $\tau_x$ are simple is an open subset $U \subset M$, and the restriction $(\mathrm{tr}_\Omega)_{|U}$ is smooth and, for all $x \in U$:*

$$(\mathrm{tr}_\Omega)_{|U}(x) = \sum_{p = \tau_x(p)} \frac{\mathrm{Tr}(r_x(p))}{\left|\det\left(\mathrm{id} - (\tau_x)_{*_p}\right)\right|}, \tag{1.66}$$

*where the sum is over the fixed points of $\tau_x$, which are necessarily in finite number, and is equal to 0 if $\tau_x$ has no fixed point.*

**Corollary 1.8.13.** *In the setting of Theorem 1.8.12, suppose in addition that for every $x \in M$, the fixed points of $\tau_x$ are all simple. Then,* $\mathrm{tr}_\Omega$ *is smooth, given by (1.66) for $U = M$ and, for all $\rho \in \mathcal{D}(M, |TM|)$, we have:*

$$\mathrm{tr}\,(\Omega(\rho)) = \int_M \left( \sum_{p=\tau_x(p)} \frac{\mathrm{Tr}(r_x(p))}{\left|\det\left(\mathrm{id} - (\tau_x)_{*_p}\right)\right|} \right) \rho(x), \tag{1.67}$$

*where the sum is over the fixed points of $\tau_x$, and is equal to* 0 *if $\tau_x$ has no fixed point.*

If we know in advance that $\Omega(\rho)$ is smooth-traceable for all $\rho$, then we can drop the condition that the restriction of the projection to the fixed point bundle is proper, although the result is weaker.

**Theorem 1.8.14.** *Let $M$ and $Q$ be two manifolds, $E \to Q$ a vector bundle over $Q$. Let $\underline{\tau} = (\tau, r)$ be a smooth family of geometric morphisms of $E$ parametrized by $M$. Suppose that*

- *$\tau$ is locally transitive;*
- *for every $\rho \in \mathcal{D}(M, |TM|)$, the operator $\Omega(\rho) : \mathcal{E}(Q, E) \to \mathcal{E}(Q, E)$ defined by (1.59) is smooth-traceable.*

*Then, the linear map*

$$\mathrm{tr}_\Omega : \mathcal{D}(M, |TM|) \to \mathbb{C}\ ;\ \rho \mapsto \mathrm{tr}\,(\Omega(\rho))$$

*is a generalized function on $M$.*

*Furthermore, suppose that $U \subset M$ is an open subset such that for all $x \in U$, all the fixed points of $\tau_x$ are simple. Let $C_0 \subset C_1 \subset \cdots \subset Q$ be a countable exhaustion of $Q$ by compact sets, and for each $n \in \mathbb{N}$, let $\phi_n \in \mathcal{D}(Q)$ such that* $0 \leq \phi_n \leq 1$, $\mathrm{supp}(\phi_n) \subset C_{n+1}$ *and* $(\phi_n)_{|C_n} = 1$. *Then, for each $\rho \in \mathcal{D}(U, |TM|)$, we have:*

$$(\mathrm{tr}_\Omega)_{|U}(\rho) = \lim_{n\to\infty} \int_U \left( \sum_{p=\tau_x(p)} \frac{\phi_n(p)\,\mathrm{Tr}(r_x(p))}{\left|\det\left(\mathrm{id} - (\tau_x)_{*_p}\right)\right|} \right) \rho(x), \tag{1.68}$$

*where the sum is over the fixed points of $\tau_x$, and is equal to* 0 *if $\tau_x$ has no fixed point.*

### 1.8.3 Examples

We now give various examples of smooth families of geometric morphisms $\underline{\tau} = (\tau, r)$ of a vector bundle $E \to Q$ parametrized by $M$. The goal is to illustrate the previous results and to highlight why their hypotheses are important. If the reader prefers to directly dive into the proofs of the previous theorems, he should feel free to skip this part. In this subsection, we denote by $Z$ the fixed point bundle of $\tau$ and we always identify smooth densities on $\mathbb{R}^n$ with the smooth functions through the Lebesgue density.

Let us begin with two examples where the projection $(\mathrm{pr}_M)_{|Z}$ is proper and all the fixed points are simple, so the distributional trace is smooth by Corollary 1.8.13. The first example illustrates the noncompact case, and the second one the compact case, with multiple fixed points.

*Example* 1.8.15. Let $m, n \in \mathbb{N}$, $M := \mathbb{R}^n \times \mathbb{R}^m$, $Q := \mathbb{R}^n$, $r : \mathbb{R}^{n+m} \times \mathbb{R}^n \to \mathbb{C}$ a smooth map and $f : \mathbb{R}^n \to \mathbb{R}^n$ a diffeomorphism. We consider the family of geometric morphisms of the trivial bundle over $\mathbb{R}^n$ parametrized by $\mathbb{R}^{n+m}$ corresponding to $r$ and

$$\tau : \mathbb{R}^{n+m} \times \mathbb{R}^n \to \mathbb{R}^n \ ; \ ((a,l),q) \mapsto 2f(a) - q.$$

Notice that both Examples 1.1.1 and 1.8.6 fit into this context. For all $(a,l) \in \mathbb{R}^{n+m}$, the corresponding pullback operator $\Omega(a,l)$ is given, for $\varphi \in \mathcal{E}(\mathbb{R}^n)$ and $q \in \mathbb{R}^n$, by:

$$(\Omega(x)\varphi)(q) = r_{(a,l)}(q)\, \varphi\left(2f(a) - q\right).$$

Then, $\tau$ is locally transitive since $f$ is a diffeomorphism, and for every $(a,l) \in \mathbb{R}^{n+m}$, $\tau_{(a,l)}$ has a unique fixed point $q = f(a)$. The fixed point bundle is therefore given by

$$Z := \left\{((a,l), f(a)) \mid (a,l) \in \mathbb{R}^{n+m}\right\},$$

so $(\mathrm{pr}_M)_{|Z}$ is a diffeomorphism and is thus proper. By Lemma 1.8.11, it also implies that all fixed points are simple, as can also be seen from the fact that, for all $(a,l) \in \mathbb{R}^{n+m}$ and $q \in \mathbb{R}^n$:

$$\det\left(\mathrm{id} - (\tau_{(a,l)})_{*f(a)}\right) = \det\left(\mathrm{id} - (-\mathrm{id})\right) = 2 \neq 0.$$

By Corollary 1.8.13, the distributional trace is smooth and is given, for every $\rho \in \mathcal{D}(\mathbb{R}^n)$, by (1.67), that is:

$$\mathrm{tr}_{\,\Omega}(\rho) = \frac{1}{2} \int_{\mathbb{R}^{n+m}} r_{(a,l)}(f(a))\, \rho(a,l)\, da dl.$$

It is also an enlightening computation to get $\mathrm{tr}_{\,\Omega}(\rho)$ directly from the kernel of $\Omega(\rho)$ which is given, for $q, q' \in \mathbb{R}^n$, by

$$k_\rho(q,q') = \int_{\mathbb{R}^m} \frac{1}{2} |\mathrm{Jac}_q(f^{-1})|\, \rho\left(f^{-1}\left(\frac{q'+q}{2}\right), l\right)\, r_{\left(f^{-1}\left(\frac{q'+q}{2}\right), l\right)}(q)\, dl. \quad \diamond$$

*Example* 1.8.16. Let $M := \mathbb{R}$ and $Q := S^1$ and let us consider the family of geometric morphisms of the trivial bundle over $S^1$ given by any smooth map $r : \mathbb{R} \times S^1 \to \mathbb{C}$ and

$$\tau : \mathbb{R} \times S^1 \to S^1 \; ; \; (x, z) \mapsto e^{2\pi i x} z^{-1}.$$

For every $x \in \mathbb{R}$, the corresponding pullback operator $\Omega(x)$ reads, for $\varphi \in \mathcal{E}(S^1)$ and $z \in S^1$:

$$(\Omega(x)\varphi)(z) = r_x(z)\, \varphi\left(e^{2\pi i x} z^{-1}\right).$$

$\tau$ is locally transitive, and for every $x \in \mathbb{R}$, $\tau_x$ admits two fixed points $z_\pm = \pm e^{i\pi x}$, which are simple. We are thus in the setting of Corollary 1.8.13 which asserts that the distributional trace $\operatorname{tr}_\Omega$ is smooth and given, for $\rho \in \mathcal{D}(\mathbb{R})$, by:

$$\operatorname{tr}_\Omega(\rho) = \int_{\mathbb{R}} \frac{1}{2} \left( r_x\left(e^{i\pi x}\right) + r_x\left(-e^{i\pi x}\right)\right) \rho(x)\, dx. \qquad \Diamond$$

We now turn to some more pathological behaviours that show that the distributional trace may fail to be smooth, or locally-integrable, or might even not be a generalized function. Unless otherwise stated, we will take $M = Q = \mathbb{R}$ and consider the trivial bundle over $Q$. We denote by $\tilde{\Delta} : M \times Q \to M \times Q \times Q \; ; \; (x, q) \mapsto (x, q, q)$ the diagonal map. Let us first give a general result that helps building specific examples by defining a family of geometric morphisms whose fixed point bundle corresponds to the graph of a given function $f$.

*Remark* 1.8.17. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}_0$ be smooth maps. Then

$$\tau : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \; ; \; (x, q) \mapsto \tau_x(q) := g(q)^{-1}(x - f(q)) + q \tag{1.69}$$

is locally transitive and for every smooth map $r : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, $\underline{\tau} := (\tau, r)$ defines a smooth family of geometric morphisms of the trivial bundle over $\mathbb{R}$ whose fixed point bundle corresponds to the graph of $f$:

$$Z = \{(f(q), q) \mid q \in \mathbb{R}\}.$$

Furthermore, for every $x \in \mathbb{R}$, the fixed points of $\tau_x$ – which are therefore all $q \in \mathbb{R}$ such that $x = f(q)$ – are simple if and only if $f'(q) \neq 0$. Finally, for every $\rho \in \mathcal{D}(\mathbb{R})$, the kernel of the operator $\Omega(\rho)$ (see (1.59)) is given by

$$k_\rho(q, q') = |g(q)|\, r\big(g(q)(q' - q) + f(q), q\big)\, \rho\big(g(q)(q' - q) + f(q)\big). \tag{1.70}$$

*Proof.* The local transitivity is ensured by the equation $\partial_x \tau_x(q) = g(q) \neq 0$ and the claim about the fixed point bundle follows from the fact that $q \in \mathbb{R}$ is a fixed point of $\tau_x$ if and only if $g(q)^{-1}(x - f(q)) + q = q \Leftrightarrow x = f(q)$. Such a fixed point is simple if and only

$$0 \neq (1 - \partial_q \tau_x(q))_{|x=f(q)} = \left(1 - 1 - \frac{g'(q)}{g(q)^2}(x - f(q)) + \frac{f'(q)}{g(q)}\right)_{|x=f(q)},$$

that is, if and only if $f'(q) \neq 0$. Finally, for every $\rho \in \mathcal{D}(\mathbb{R})$, $\varphi \in \mathcal{E}(\mathbb{R})$ and $q \in \mathbb{R}$, we have:

$$\begin{aligned}(\Omega(\rho)\varphi)(q) &= \int_{\mathbb{R}} r(x,q)\, \varphi\big(\, g(q)(x - f(q)) + q\,\big)\, \rho(x)\, dx \\ &= \int_{\mathbb{R}} |g(q)|\, r\left(g(q)(q'-q) + f(q), q\right)\ \rho\left(g(q)(q'-q) + f(q)\right) \\ &\qquad \varphi(q')\, dq',\end{aligned}$$

by making the change of variable $q' = g(q)^{-1}(x - f(q)) + q$. □

The following example illustrates that the distributional trace might fail to be smooth if there exists some non simple fixed points.

*Example* 1.8.18. Let $r : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be any smooth map and

$$\tau : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\ ;\ (x,q) \mapsto (x - q^2) + q,$$

which has the form of (1.69). Its fixed point bundle is therefore

$$Z := \{(q^2, q) \mid q \in \mathbb{R}\}$$

and, for every $x > 0$, $\tau_x$ has two simple fixed points, $\tau_0$ has one fixed point, which is not simple, and for every $x < 0$, $\tau_x$ has no fixed point, as is illustrated on Figure 1.1.

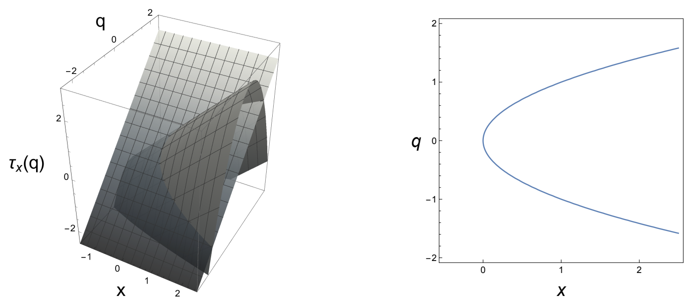



Figure 1.1: Graph of $\tilde{\Delta}$ and $\tau$ (left) and fixed point bundle of $\tau$ (right).

Since $(\mathrm{pr}_M)_{|Z}$, the restriction to $Z$ of the projection of $\mathbb{R} \times \mathbb{R}$ on the first component, is proper, we are in the setting of Theorem 1.8.12 and $\mathrm{tr}_{\,\Omega}$ is thus a generalized function, which is smooth on $\mathbb{R}_0$. Since for all $x > 0$, the fixed points of $\tau_x$ are $q = \pm\sqrt{x}$, (1.66) leads to

$$\mathrm{tr}_{\,\Omega}(x) = \begin{cases} \frac{r(x,\sqrt{x}) + r(x,-\sqrt{x})}{2\sqrt{x}} & x > 0 \\ 0 & x < 0. \end{cases}$$

This shows that for a generic $r$, $\mathrm{tr}_{\,\Omega}$ cannot be smooth at $x = 0$. However, it is a locally integrable function, as can be shown by explicitly computing

$\int_{\mathbb{R}} k_\rho(q,q)dq$ from (1.70):

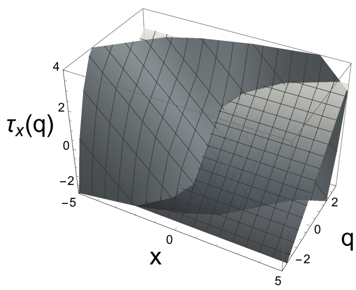
$$\operatorname{tr}_\Omega(\rho) = \int_{\mathbb{R}_0^+} \frac{r(x,\sqrt{x}) + r(x,-\sqrt{x})}{2\sqrt{x}}\, \rho(x)\, dx. \qquad \lozenge$$

The next example shows that the distributional trace might not be given by a locally integrable function. Here, this is due to the fact that the set of non simple fixed points is not negligible, so their contribution to the distributional trace leads to a $\delta-$function.

*Example* 1.8.19. In the setting of Remark 1.8.17, let us consider the family of geometric morphisms associated to the constant maps $g \equiv -1$ and $r \equiv 1$, and to

$$f : \mathbb{R} \to \mathbb{R}\ ;\ q \mapsto \begin{cases} \exp\left(\frac{1}{q+1} - (q+1)\right) & q < -1 \\ 0 & -1 \le q \le 1 \\ \exp\left((q-1) - \frac{1}{q-1}\right) & q > 1. \end{cases}$$

For every $x \neq 0$, $\tau_x$ admits a unique fixed point, which is simple. However, the set of fixed points of $\tau_0$ is $[-1,1]$ and they are all non simple. This is represented on Figure 1.2.



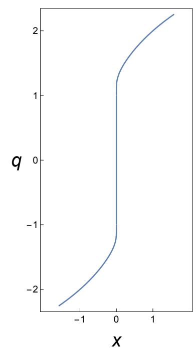




Figure 1.2: Graph of $\tilde{\Delta}$ and $\tau$ (left) and fixed point bundle of $\tau$ (right).

From (1.70), we compute that, for $\rho \in \mathcal{D}(\mathbb{R})$, we have:

$$\begin{aligned} \operatorname{tr}_\Omega(\rho) &= \int_{\mathbb{R}} k_\rho(q,q)\, dq = \int_{\mathbb{R}\setminus[-1,1]} \rho(f(q))\, dq + \int_{-1}^{1} \rho(0)\, dq \\ &= 2\,\rho(0) + \int_{\mathbb{R}_0} \frac{1}{2|x|}\left(1 + \frac{\log(|x|)}{\sqrt{4+\log(|x|)^2}}\right) \rho(x)\, dx. \end{aligned}$$

On $\mathbb{R}_0$, the distributional trace is therefore smooth as predicted by Theorem 1.8.12, but at $x = 0$, the non simple fixed points give rise to a multiple of the $\delta-$function. $\lozenge$

We now focus on the situation where $(\mathrm{pr}_M)_{|Z}$ is not proper. In that case, the distributional trace might fail to be a well-defined generalized function, as is illustrated by the following example.

*Example* 1.8.20. Still in the setting of Remark 1.8.17, we consider the family of geometric morphisms associated to a smooth map $g : \mathbb{R} \to \mathbb{R}_0^+$ and, the constant map $r \equiv 1$, and to

$$f : \mathbb{R} \to \mathbb{R} \; ; \; q \mapsto e^q.$$

Then, $\tau_x$ has a unique fixed point if $x > 0$, none otherwise, and all the fixed points are simple. For any $\rho \in \mathcal{D}(\mathbb{R})$, the kernel $k_\rho$ along the diagonal has the

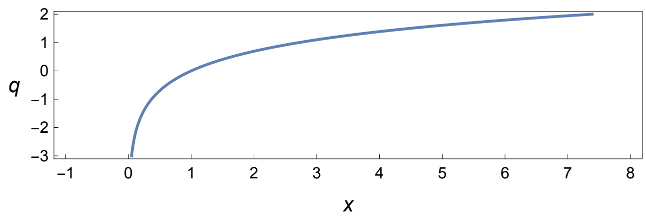



Figure 1.3: Fixed points bundle of $\tau$.

form $k_\rho(q, q) = g(q)\rho(e^q)$ by (1.70). For $g \equiv 1$ for instance, the distributional trace is thus not well-defined since $\Omega(\rho)$ is not smooth-traceable for every $\rho$. It is, however, as soon as we choose $g$ integrable since $\rho$, being compactly supported, is bounded. In that case, the distributional trace is smooth and given by

$$\mathrm{tr}\,_\Omega(\rho) = \int_{\mathbb{R}_0^+} \frac{g(\log(x))}{x} \rho(x)\, dx,$$

which coincides with the result (1.68) of Theorem 1.8.14. $\lozenge$

Our last example, for which we will just describe the fixed point bundle, is cooked up to show that, in the non proper case, the condition to have simple fixed points is not an open condition.

*Example* 1.8.21. In the setting of Remark 1.8.17, let us consider the family of geometric morphisms associated to the constant maps $g \equiv 1$ and $r \equiv 1$, and to

$$f : \mathbb{R} \to \mathbb{R} \; ; \; q \mapsto \frac{3}{4}\left(1 + \cos\left(\pi q - \arctan(\sqrt{2})\right)^2\right) e^{-\frac{\pi}{\sqrt{2}} q}.$$

Since $f$ is injective, surjective on $\mathbb{R}_0^+$ and $f'(q) = 0 \Leftrightarrow q \in \mathbb{N}$, for each $x > 0$, $\tau_x$ has a unique fixed point, which is simple if and only if $x \neq e^{-\frac{\pi n}{\sqrt{2}}}$ for some $n \in \mathbb{Z}$, and $\tau_x$ has no fixed point if $x \leq 0$. The situation is pictured on Figure 1.4. Therefore, the set of $x \in \mathbb{R}$ such that $\tau_x$ only has simple fixed points – that is, the set of regular values of $(\mathrm{pr}_M)_{|Z}$ (see Lemma 1.8.11) – is not an open set since $\left(e^{-\frac{\pi n}{\sqrt{2}}}\right)_{n \in \mathbb{N}}$ is a sequence of singular values of $(\mathrm{pr}_M)_{|Z}$ converging to the regular value $x = 0$. $\lozenge$

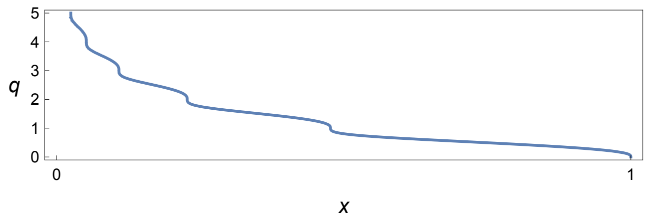



Figure 1.4: Fixed points bundle of $\tau$. Notice that the horizontal axis has been arbitrarily scaled in order to better show the stairlike behaviour.

### 1.8.4 The proofs

In order to prove Theorems 1.8.12 and 1.8.14, our strategy will consist of two main steps. First Lemma 1.8.22 will show that, even when the fixed points of $\tau_x$ are not all simple, $\mathrm{tr}\,(\Omega(x))$ still makes sense, but as a generalized function on $M$, that we will denote $\mathrm{tr}_{\underline{\tau}}$. Then, in Proposition 1.8.23, we will show that $\mathrm{tr}_{\underline{\tau}}$ coincides with $\mathrm{tr}_{\Omega}$. The proofs will mainly consist in playing with commutations of pullbacks and pushforwards of $\delta-$sections, and these manipulations will often rely on a technical properness condition. In Theorems 1.8.12 and 1.8.14 we will finally give settings where this properness condition can be simplified or relaxed. If at some point the reader feels a bit lost in the middle of all those commutations of pullbacks and pushforwards, he is invited to come back to the sketchy discussion of subsection 1.8.2 to interpret them as restrictions to diagonals and integrals respectively, by playing with the kernels as if they were smooth.

**Lemma 1.8.22.** *Let $M$ and $Q$ be two manifolds, $E \to Q$ a vector bundle over $Q$. Let $\underline{\tau} = (\tau, r)$ be a smooth family of geometric morphisms of $E$ parametrized by $M$. We denote by*

- *$k$ the kernel of $\underline{\tau}^*$ (see Proposition 1.7.2);*
- *$\tilde{\Delta} : M \times Q \to M \times Q \times Q$ ; $(x, q) \mapsto (x, q, q)$ the diagonal map;*
- *$\mathrm{pr}_M : M \times Q \to M$ the projection onto $M$.*

*Suppose that*

1. *$\tau$ is locally transitive;*
2. *$(\mathrm{pr}_M)_{|\tilde{\Delta}^{-1}(\mathrm{supp}(k))}$ is proper.*

*Then,*

$$\mathrm{tr}_{\underline{\tau}} := (\mathrm{pr}_M)_* \mathrm{Tr}\, \tilde{\Delta}^* k$$

*is a well-defined generalized function on $M$. Furthermore, we have the following result regarding the smoothness of $\mathrm{tr}_{\underline{\tau}}$:*

- $\operatorname{supp}(\operatorname{tr}_{\underline{\tau}}) \subset \operatorname{pr}_M(\tilde{\Delta}^{-1}(\operatorname{supp}(k)))$; [28]
- $\operatorname{tr}_{\underline{\tau}}$ *is smooth at each* $x \in M$ *such that all the fixed points of* $\tau_x$ *are simple;*
- *for all open subset* $U \subset M$ *such that for all* $x \in U$*, all the fixed points of* $\tau_x$ *are simple, the restriction* $(\operatorname{tr}_{\underline{\tau}})_{|U}$ *is smooth and, for all* $x \in U$*:*

$$(\operatorname{tr}_{\underline{\tau}})_{|U}(x) = \sum_{p=\tau_x(p)} \frac{\operatorname{Tr}(r_x(p))}{\left|\det\left(\operatorname{id} - (\tau_x)_{*_p}\right)\right|}, \tag{1.71}$$

*where the sum, which always has only a finite number of nonvanishing terms, is over the fixed points of* $\tau_x$*, and is equal to* $0$ *if* $\tau_x$ *has no fixed point.*[29]

*Proof.* Let us begin with some notations. Let $\pi_i$ denote the projection of $M \times Q \times Q$ onto the $i$th component, $\pi_{12} : M \times Q \times Q \to M \times Q$ the projection onto the first two components and $\pi_{23} : M \times Q \times Q \to Q \times Q$ the projection onto the last two components. Let also $\operatorname{pr}_M : M \times Q \to M$ and $\operatorname{pr}_Q : M \times Q \to Q$ be the projections.

Let us first verify that $(\operatorname{pr}_M)_* \operatorname{Tr} \tilde{\Delta}^* k$ is a well-defined generalized function on $M$. Recall that, by Proposition 1.7.2, the kernel $k$ of the pullback by $\underline{\tau}$ is a $\delta-$section of the vector bundle

$$F := \operatorname{Hom}\big(\pi_3^*(E), \pi_2^*(E)\big) \otimes \pi_3^*(|TQ|) \to M \times Q \times Q.$$

along the submanifold $\operatorname{graph}(\tau) \subset M \times Q \times Q$. By Lemma 1.8.9, $\tilde{\Delta}$ is transverse to $\operatorname{graph}(\tau)$ and $\tilde{\Delta}^* k$ is therefore a well-defined $\delta-$section of the vector bundle $\tilde{\Delta}^*(F)$, that is

$${\operatorname{pr}_Q}^*\big(\operatorname{Hom}(E,E) \otimes |TQ|\big) \to M \times Q,$$

Taking the trace of the homomorphism gives a $\delta-$section $\operatorname{Tr} \tilde{\Delta}^* k$ of the vector bundle ${\operatorname{pr}_Q}^*(|TQ|)$. The corresponding submanifold is the fixed point bundle $Z := \tilde{\Delta}^{-1}(\operatorname{graph}(\tau))$, that is:

$$Z = \{(x,q) \in M \times Q \mid \tau_x(q) = q\}, \tag{1.72}$$

which, by Lemma 1.8.9, is a properly embedded submanifold, either empty or of dimension $\dim(M)$. $(\operatorname{pr}_M)_{|\tilde{\Delta}^{-1}(\operatorname{supp}(k))}$ being proper by assumption and given that (see Remark 1.6.17)

$$\operatorname{supp}(\operatorname{Tr} \tilde{\Delta}^* k) \subset \operatorname{supp}(\tilde{\Delta}^* k) = \tilde{\Delta}^{-1}(\operatorname{supp}(k)), \tag{1.73}$$

$(\operatorname{pr}_M)_{|\operatorname{supp}(\operatorname{Tr} \tilde{\Delta}^* k)}$ is also proper. Since ${\operatorname{pr}_Q}^*(|TQ|) \simeq {\operatorname{pr}_M}^*(|TM|)^* \otimes |T(M \times Q)|$, $\operatorname{Tr} \tilde{\Delta}^* k$ can be seen as a $\delta-$density which we can thus pushforward by $\operatorname{pr}_M$

[28] In particular, any point $x \in M$ such that $\tau_x$ has no fixed point is outside of $\operatorname{supp}(\operatorname{tr}_{\underline{\tau}})$.

[29] Notice that, as we have seen in Example 1.8.21, the condition that the fixed points of $\tau_x$ are all simple is not an open condition in general.

to get $(\mathrm{pr}_M)_* \operatorname{Tr} \tilde{\Delta}^* k$, a well-defined generalized section of $|TM|^* \otimes |TM| \simeq \mathbb{C}$, that is, a generalized function on $M$. Let us denote it by $\operatorname{tr}_{\underline{\tau}}$.

Let us turn to the claims about the smoothness of $\operatorname{tr}_{\underline{\tau}}$. Regarding the support of this generalized function, by (1.73) and Remark 1.5.13, we have indeed that

$$\operatorname{supp}(\operatorname{tr}_{\underline{\tau}}) \subset \mathrm{pr}_M(\, \tilde{\Delta}^{-1}(\operatorname{supp}(k))\,).$$

Next, any $x \in M$ such that all the fixed points of $\tau_x$ are simple is a regular value of $(\mathrm{pr}_M)_{|Z}$ by Lemma 1.8.11. It follows from Proposition 1.6.20 that $(\mathrm{pr}_M)_* \operatorname{Tr} \tilde{\Delta}^* k$ is smooth at $x$. Finally, let $U \subset M$ be an open subset such that for all $x \in U$, all the fixed points of $\tau_x$ are simple. We have just seen that $\Big( (\mathrm{pr}_M)_* \operatorname{Tr} \tilde{\Delta}^* k \Big)_{|U}$ is smooth. To explicitly evaluate this function, let $x \in U$ and let us consider the pullback by $\underline{\tau}_{|U} = (\tau_{|U}, r_{|U})$, where $\tau_{|U}$ and $r_{|U}$ are respectively the restrictions of $\tau$ and $r$ to $U$ instead of $M$. Then, the whole previous discussion regarding $(\mathrm{pr}_M)_* \operatorname{Tr} \tilde{\Delta}^* k$ still holds if we replace $M$ by $U$ and we have $(\operatorname{Tr} \tilde{\Delta}^* k)_{|(\mathrm{pr}_M)^{-1}(U)} = \operatorname{Tr} \tilde{\Delta}^* k_U$, where on the right hand side, $\tilde{\Delta}$ is restricted to $U \times Q$ and $k_U$ denotes the kernel of the pullback by $\underline{\tau}_{|U}$, which is a $\delta-$section along the submanifold $Z_U := \{(x,q) \in U \times Q | \tau_x(q) = q\}$ . By Lemma 1.5.15, we have

$$\Big( (\mathrm{pr}_M)_* \operatorname{Tr} \tilde{\Delta}^* k \Big)_{|U} = (\mathrm{pr}_U)_* \operatorname{Tr} \tilde{\Delta}^* k_U, \tag{1.74}$$

where $\mathrm{pr}_U : U \times Q \to U$ is the projection. Let us introduce the following smooth maps[30]:

$$\begin{aligned}
&\bar{\pi} : Q \to \{\star\} \; ; \; q \mapsto \star \\
&\iota_x : \{\star\} \to U \; ; \; \star \mapsto x \\
&\tilde{\iota}_x : Q \to U \times Q \; ; \; q \mapsto (x,q) \\
&\hat{\iota}_x : Q \times Q \to U \times Q \times Q \; ; \; (q,q') \mapsto (x,q,q') \\
&\Delta : Q \to Q \times Q \; ; \; q \mapsto (q,q).
\end{aligned}$$

We also make the elementary observation that, for any function $f : U \to \mathbb{C}$, we can write the evaluation at $x$ as the pullback by $\iota_x$. Indeed, ${\iota_x}^* f$ is a function over a point, that is, a number, which is precisely $f(x)$. Therefore, we have:

$$\Big( (\mathrm{pr}_U)_* \operatorname{Tr} \tilde{\Delta}^* k_U \Big)(x) = {\iota_x}^* (\mathrm{pr}_U)_* \operatorname{Tr} \tilde{\Delta}^* k_U. \tag{1.75}$$

Now, notice that we have the following commutative diagram:

$$\begin{array}{ccc}
\{\star\} & \xrightarrow{\iota_x} & U \\
\bar{\pi} \uparrow & & \uparrow \mathrm{pr}_U \\
Q & \xrightarrow{\tilde{\iota}_x} & Z_U \subset U \times Q.
\end{array}$$

[30] $\{\star\}$ denotes the set containing one point.

Considering the trivial vector bundle over $\{\star\}$, the density bundle $|TU|$ over $U$ and the $\delta-$section $\mathrm{Tr}\ \tilde{\Delta}^* k_U$ of the vector bundle $\mathrm{pr}_Q{}^*(|TQ|) \simeq \mathrm{pr}_U{}^*(|TU|)^* \otimes |T(Q\times U)|$, we are in the setting of Proposition 1.6.21 (we identify $Q\times\{\star\}$ with $Q$). Indeed, we have that $\iota_x{}^*(|TU|^* \otimes |TU|) \simeq \mathbb{C}$ and that $\tilde{\iota}_x$ is transverse to $Z_U$ because we have seen in the proof of Lemma 1.8.11 that for all $X \in T_q(Q)$, $(0, X)$ is tangent to $Z_U$ if and only if $X = 0$ since the fixed points of $\tau_x$ are simple. Therefore,

$$\begin{aligned}
\iota_x{}^*(\mathrm{pr}_U)_*\ \mathrm{Tr}\ \tilde{\Delta}^* k_U &= \bar{\pi}_*\ \tilde{\iota}_x{}^*\ \mathrm{Tr}\ \tilde{\Delta}^* k_U && (1.76)\\
&= \bar{\pi}_*\ \mathrm{Tr}\ \tilde{\iota}_x{}^*\ \tilde{\Delta}^* k_U = \bar{\pi}_*\ \mathrm{Tr}\ (\tilde{\Delta}\circ\tilde{\iota}_x)^* k_U && (1.77)\\
&= \bar{\pi}_*\ \mathrm{Tr}\ (\hat{\iota}_x \circ \Delta)^* k_U = \bar{\pi}_*\ \mathrm{Tr}\ \Delta^*\ \hat{\iota}_x{}^* k_U && (1.78)\\
&= \bar{\pi}_*\ \mathrm{Tr}\ \Delta^*\ k_x, && (1.79)
\end{aligned}$$

where $k_x$ is the kernel of the pullback by $\underline{\tau}_x$. Line (1.77) is justified by Lemma 1.6.18. Line (1.78) as well because $\hat{\iota}_x$ is transverse to $\mathrm{graph}(\tau_{|U})$, and $\Delta$ is transverse to $\hat{\iota}_x{}^{-1}(\mathrm{graph}(\tau_{|U})) = \mathrm{graph}(\tau_x)$ by Lemma 1.7.4 since all the fixed points of $\tau_x$ are simple (because $x$ is a regular value of $(\mathrm{pr}_M)_{|Z}$). Line (1.79) follows from the observation that the symbol of $k_x$ coincides with the pullback by $\hat{\iota}_x$ of the symbol of $k_U$. Now, notice that, since all the fixed points of $\tau_x$ are simple, they are isolated by Lemma 1.7.4. The fixed points $p$ of $\tau_x$ such that $\mathrm{Tr}\,(r_x(p)) \neq 0$ are contained in $(\mathrm{pr}_M)^{-1}(\{x\})\cap\tilde{\Delta}^{-1}(\mathrm{supp}(k))$. The latter being compact since $(\mathrm{pr}_M)_{|\tilde{\Delta}^{-1}(\mathrm{supp}(k))}$ is proper, they must be in finite number. We can thus apply Proposition 1.7.5 to $\underline{\tau}_x$ to get that [31]

$$\bar{\pi}_*\ \mathrm{Tr}\ \Delta^*\ k_x = \sum_{p=\tau_x(p)} \frac{\mathrm{Tr}(r_x(p))}{\left|\det\left(\mathrm{id} - (\tau_x)_{*_p}\right)\right|}. \tag{1.80}$$

Putting equations (1.74), (1.75), (1.76) and (1.80) together finally gives

$$(\mathrm{tr}\,\underline{\tau})_{|U}(x) = \sum_{p=\tau_x(p)} \frac{\mathrm{Tr}(r_x(p))}{\left|\det\left(\mathrm{id} - (\tau_x)_{*_p}\right)\right|}.$$

□

**Proposition 1.8.23.** *Let $M$ and $Q$ be two manifolds, $E \to Q$ a vector bundle over $Q$. Let $\underline{\tau} = (\tau, r)$ be a smooth family of geometric morphisms of $E$ parametrized by $M$. Let us denote by*

- *$k$ the kernel of $\underline{\tau}^*$ (see Proposition 1.7.2);*
- *$\tilde{\Delta} : M \times Q \to M \times Q \times Q$ ; $(x, q) \mapsto (x, q, q)$ the diagonal map;*
- *$\mathrm{pr}_M : M \times Q \to M$ the projection onto $M$.*

*Suppose that*

[31] As before, we sum over all fixed points because it only adds vanishing terms to the sum.

*(H1)* $\tau$ *is locally transitive;*

*(H2)* $(\mathrm{pr}_M)_{|\tilde{\Delta}^{-1}(\mathrm{supp}(k))}$ *is proper.*

*Then, for every* $\rho \in \mathcal{D}(M, |TM|)$, *the operator* $\Omega(\rho) : \mathcal{E}(Q, E) \to \mathcal{E}(Q, E)$ *defined by (1.59) is smooth-traceable and the linear map*

$$\mathrm{tr}_{\,\Omega} : \mathcal{D}(M, |TM|) \to \mathbb{C} \; ; \; \rho \mapsto \mathrm{tr}\,(\Omega(\rho))$$

*is a generalized function on* $M$ *which coincides with* $\mathrm{tr}_{\,\underline{\tau}}$ *(see Lemma 1.8.22). Therefore, for all open subset* $U \subset M$ *such that for all* $x \in U$, *all the fixed points of* $\tau_x$ *are simple, the restriction* $(\mathrm{tr}_{\,\Omega})_{|U}$ *is smooth and, for all* $x \in U$*:*

$$(\mathrm{tr}_{\,\Omega})_{|U}(x) = \sum_{p=\tau_x(p)} \frac{\mathrm{Tr}(r_x(p))}{\left|\det\left(\mathrm{id} - (\tau_x)_{*_p}\right)\right|}, \tag{1.81}$$

*where the sum, which always has a finite number of nonvanishing terms, is over the fixed points of* $\tau_x$, *and is equal to* $0$ *if* $\tau_x$ *has no fixed point.*

*Proof.* In this proof, we will use the same notations for the projection maps as in the proof of Lemma 1.8.22.

Denote by $k$ the kernel of the pullback by $\underline{\tau}$. We know by Proposition 1.7.2 that it is a $\delta-$section along the submanifold graph$(\tau)$. Because of assumptions (H1) and (H2), we can apply Lemma 1.8.22 to get that $\mathrm{tr}_{\,\underline{\tau}} := (\mathrm{pr}_M)_* \mathrm{Tr}\,\tilde{\Delta}^* k$ is a well-defined generalized function on $M$.

Before going on, let us make the following observation. For all $u \in \mathcal{D}'(M)$ and $\rho \in \mathcal{D}(M, |TM|)$, we can define the compactly supported generalized density $u\rho$ by $\langle u\rho, f\rangle := \langle u, f\rho\rangle$ for all $f \in \mathcal{D}(M)$. Then, we have that (see Remark 1.5.16):

$$\langle u, \rho\rangle = \langle u\rho, 1\rangle = \overline{\Pi}_*(u\rho),$$

where we denote by $\overline{\Pi} : M \to \{\star\}$ the projection onto a point.

If we take $\rho \in \mathcal{D}(M, |TM|)$ and apply this to $(\mathrm{pr}_M)_* \mathrm{Tr}\,\tilde{\Delta}^* k$, we see that

$$\Big((\mathrm{pr}_M)_* \mathrm{Tr}\,\tilde{\Delta}^* k\Big)\rho = (\mathrm{pr}_M)_* \mathrm{Tr}\,\tilde{\Delta}^* K_\rho,$$

where $K_\rho$ is the kernel of the pullback by $\underline{\tau}_\rho$ as defined in the proof of Proposition 1.8.5 (see equations (1.60) and (1.61)).[32] Notice also that $\tilde{\Delta}^* K_\rho$ is

[32] Let us explain why. By (1.62) and footnote 24, the symbol of $k$ reads $r_x(q) \otimes \sigma(x, q)$ and the symbol of $K_\rho$ is $r_x(q) \otimes \rho(x) \otimes \sigma(x, q)$. Therefore, the symbols of $\mathrm{Tr}\,\tilde{\Delta}^* k$ and $\mathrm{Tr}\,\tilde{\Delta}^* K_\rho$ are respectively $\mathrm{Tr}\,(r_x(q)) \otimes \tilde{\Delta}^* \sigma(x, q)$ and $\mathrm{Tr}\,(r_x(q)) \otimes \rho(x) \otimes \tilde{\Delta}^* \sigma(x, q)$. From Proposition 1.6.19, we get that

$$\begin{aligned}\Big\langle (\mathrm{pr}_M)_* \mathrm{Tr}\,\tilde{\Delta}^* k, \rho \Big\rangle &= \Big\langle \mathrm{Tr}\,(r_x(q)) \otimes \tilde{\Delta}^* \sigma(x, q), (\mathrm{pr}_M)^* \rho \Big\rangle \\ &= \Big\langle \mathrm{Tr}\,(r_x(q)) \otimes \rho(x) \otimes \tilde{\Delta}^* \sigma(x, q), 1 \Big\rangle = \Big\langle (\mathrm{pr}_M)_* \mathrm{Tr}\,\tilde{\Delta}^* K_\rho, 1 \Big\rangle.\end{aligned}$$

compactly supported. Indeed, $\mathrm{supp}(\tilde{\Delta}^* K_\rho)$ is a closed subset contained in $(\mathrm{pr}_M)^{-1}(\mathrm{supp}(\rho)) \cap \tilde{\Delta}^{-1}(\mathrm{supp}(k))$, which is compact by assumption (H2). This implies that the following pushforwards and their permutations are justified since all the maps are proper on the supports of the generalized sections:

$$\begin{aligned}\Big\langle (\mathrm{pr}_M)_* \mathrm{Tr}\, \tilde{\Delta}^* k, \rho \Big\rangle &= \overline{\Pi}_* (\mathrm{pr}_M)_* \mathrm{Tr}\, \tilde{\Delta}^* K_\rho \\ &= (\overline{\Pi} \circ \mathrm{pr}_M)_* \mathrm{Tr}\, \tilde{\Delta}^* K_\rho \\ &= (\overline{\pi} \circ \mathrm{pr}_Q)_* \mathrm{Tr}\, \tilde{\Delta}^* K_\rho \\ &= \overline{\pi}_* (\mathrm{pr}_Q)_* \mathrm{Tr}\, \tilde{\Delta}^* K_\rho \\ &= \overline{\pi}_* \mathrm{Tr}\, (\mathrm{pr}_Q)_* \tilde{\Delta}^* K_\rho .\end{aligned} \tag{1.82}$$

Now, notice that we have the following commutative diagram:

$$\begin{array}{ccc} Q & \xrightarrow{\Delta} & Q \times Q \\ \mathrm{pr}_Q \uparrow & & \uparrow \pi_{23} \\ M \times Q & \xrightarrow{\tilde{\Delta}} & \mathrm{graph}(\tau) \subset M \times Q \times Q. \end{array}$$

Considering the vector bundles $\mathrm{Hom}\,(E,E)^* \otimes |T(Q)|$ over $Q$ and

$$\mathrm{Hom}\big(p_1^*(E), p_2^*(E)\big) \otimes p_1^*(|TQ|) \to Q \times Q$$

as well as the $\delta-$section $K_\rho$, the fact that

$$\begin{aligned}&\mathrm{Hom}\,(E,E)^* \otimes |T(Q)| \\ &\simeq \Delta^* \Big( \big( \mathrm{Hom}\big(p_1^*(E), p_2^*(E)\big) \otimes p_1^*(|TQ|) \big)^* \otimes |T(Q \times Q)| \Big)\end{aligned}$$

together with our previous discussions ensure that the assumptions of Proposition 1.6.21 are satisfied, from which it follows that $(\mathrm{pr}_Q)_* \tilde{\Delta}^* K_\rho = \Delta^* (\pi_{23})_* K_\rho$. Recall from the proof of Proposition 1.8.5 that $(\pi_{23})_* K_\rho$ is the kernel of $\Omega(\rho)$ and that it is smooth. Since $\Delta^*(\pi_{23})_* K_\rho$ has compact support, we have by Lemma 1.5.17 and equation (1.82) that

$$\begin{aligned}\mathrm{tr}\,(\Omega(\rho)) &= \overline{\pi}_* \mathrm{Tr}\, \Delta^* (\pi_{23})_* K_\rho \\ &= \overline{\pi}_* \mathrm{Tr}\, (\mathrm{pr}_Q)_* \tilde{\Delta}^* K_\rho \\ &= \Big\langle (\mathrm{pr}_M)_* \mathrm{Tr}\, \tilde{\Delta}^* k, \rho \Big\rangle .\end{aligned}$$

This means that

$$\mathrm{tr}_{\,\Omega} : \mathcal{D}(M, |TM|) \mapsto \mathbb{C}\ ;\ \rho \mapsto \mathrm{tr}\,(\Omega(\rho))$$

is a generalized function on $M$ which coincides with $\mathrm{tr}_{\,\underline{\tau}}$ and the proof is complete by Lemma 1.8.22. $\square$

*Proof of Theorem 1.8.12.* Notice first that if $Q$ is compact, then $\mathrm{pr}_M$ is proper, which implies that $(\mathrm{pr}_M)_{|Z}$ is also proper since $Z$ is a properly embedded submanifold of $M \times Q$ and is therefore a closed subset of $M \times Q$. Next, suppose that $(\mathrm{pr}_M)_{|Z}$ is proper. Since $\tilde{\Delta}^{-1}(\mathrm{supp}(k))$ is closed and contained in $Z$, $(\mathrm{pr}_M)_{|\tilde{\Delta}^{-1}(\mathrm{supp}(k))}$ is proper so we can apply Proposition 1.8.23 to get that, for every $\rho \in \mathcal{D}(M, |TM|)$, the operator $\Omega(\rho)$ is smooth-traceable and that $\mathrm{tr}_{\,\Omega}$ is a generalized function on $M$ which coincides with $(\mathrm{pr}_M)_* \mathrm{Tr}\, \tilde{\Delta}^* k$. By Lemma 1.8.11, the set $U$ of all $x \in M$ such that all the fixed points of $\tau_x$ are simple coincides with the set of regular values of $(\mathrm{pr}_M)_{|Z}$. Since $(\mathrm{pr}_M)_{|Z}$ is proper, it is an open subset. Therefore, by Proposition 1.8.23, $(\mathrm{tr}_{\,\Omega})_{|U}$ is smooth and given by (1.66). For all $x \in U$, the fixed points of $\tau_x$ are isolated by Lemma 1.7.4 and contained in ${(\mathrm{pr}_M)_{|Z}}^{-1}(\{x\})$, which is compact, so they are in finite number. □

We finally turn to the proof of Theorem 1.8.14. Let us notice that if $Q$ is compact, everything boils down to Theorem 1.8.12. In the case where $Q$ is not compact, the difficulty is that we can not apply straightforwardly Proposition 1.8.23 because of the properness condition (H2). We will thus have to first localize our operators to compact sets, and then globalize the result by using an exhaustion of the manifold by compact sets and passing to the limit.

*Proof of Theorem 1.8.14.* In this proof, we will use the same notations for the projection maps as in the proof of Lemma 1.8.22.

Let $C_0 \subset C_1 \subset \cdots \subset Q$ be a countable exhaustion of $Q$ by compact sets.[33] For each $n \in \mathbb{N}$, let $\phi_n \in \mathcal{D}(Q)$ such that $0 \le \phi_n \le 1$, $\mathrm{supp}(\phi_n) \subset C_{n+1}$ and $(\phi_n)_{|C_n} = 1$.

Let us consider the geometric morphism $\underline{\tau}_n = (\tau, r_n)$, with $r_{n,x}(q) := \phi_n(q) r(q)$. For each $x \in M$, the corresponding pullback operator reads, for $\varphi \in \mathcal{E}(M, E)$ and $q \in Q$:

$$\begin{aligned} (\Omega_n(x)\varphi)(q) &:= r_{n,x}(q)\varphi(\tau_x(q)) \\ &= \phi_n(q) r_x(q) \varphi(\tau_x(q)) = \phi_n(q)(\Omega(x)\varphi)(q). \end{aligned} \tag{1.83}$$

Denote by $k_n$ the kernel of the pullback by $\underline{\tau}_n$. We know by Proposition 1.7.2 that it is a $\delta-$section along the submanifold graph$(\tau)$. The expression (1.55) of its symbol and the definition of $r_n$ imply that $\mathrm{supp}(k_n) \subset M \times C_{n+1} \times Q$, so $\tilde{\Delta}^{-1}(\mathrm{supp}(k_n)) \subset M \times C_{n+1}$. Therefore, $(\mathrm{pr}_M)_{|\tilde{\Delta}^{-1}(\mathrm{supp}(k_n))}$ is proper and we can apply Proposition 1.8.23 to get that

$$\mathrm{tr}_{\,\Omega_n} : \mathcal{D}(M, |TM|) \mapsto \mathbb{C}\ ;\ \rho \mapsto \mathrm{tr}\,(\Omega_n(\rho))$$

[33]That is, for all $n = 0, 1, \ldots, +\infty$, $C_n$ is a compact subset of $Q$ and $C_n$ is contained in the interior of $C_{n+1}$, and $Q = \cup_{n=0}^{+\infty} C_n$.

is a generalized function on $M$ which coincides with $\mathrm{tr}_{\mathcal{T}_n}$, that is, for every $\rho \in \mathcal{D}(M, |TM|)$:

$$\mathrm{tr}\,(\Omega_n(\rho)) = \left\langle (\mathrm{pr}_M)_* \mathrm{Tr}\,\tilde{\Delta}^* k_n, \rho \right\rangle.$$

On the other hand, from (1.83), we get that, for all $q \in Q$:

$$(\Omega_n(\rho)\varphi)(q) = \phi_n(q)(\Omega(\rho)\varphi)(q).$$

If we denote by $k_\rho$ the (smooth) kernel of $\Omega(\rho)$, this shows that $k_{n,\rho}(q, q') = \phi_n(q)k_\rho(q, q')$ for all $q, q' \in Q$ and thus that $|\mathrm{Tr}\,(k_n(q, q))| \leq |\mathrm{Tr}\,(k(q, q))|$. Since $\Omega(\rho)$ is smooth-traceable by assumption – which means that $q \mapsto |\mathrm{Tr}\,(k(q, q))|$ is integrable –, Lebesgue's dominated convergence theorem implies that:

$$\mathrm{tr}_{\Omega}(\rho) = \int_Q \lim_{n\to\infty} \mathrm{Tr}\,(k_{n,\rho}(q, q)) = \lim_{n\to\infty} \mathrm{tr}_{\Omega_n}(\rho). \tag{1.84}$$

By Theorem 1.3.14, $\mathrm{tr}_{\Omega}(\rho)$ is thus a generalized function on $M$.

Now, suppose that $U \subset M$ is an open subset such that for all $x \in U$, all the fixed points of $\tau_x$ are simple. Then, formula (1.68) follows from (1.84) and the expression (1.81) of $(\mathrm{tr}_{\Omega_n})_{|U}$ given by Proposition 1.8.23. $\square$

# Chapter 2

# Quantization of symmetric spaces

In this chapter, we investigate the question of constructing non-formal star-products on a symmetric space. More specifically, we would like to understand the appearance of the fixed points in the kernel of such a star-product, motivated by the formula of Weinstein's conjecture. The first section recalls some facts about symmetric spaces. The next section is dedicated to the construction of a quantization map, and to the study of its properties, such as equivariance. In Section 2.3, some notions related to Hilbert-Schmidt and trace-class operators are recalled, and we briefly discuss when our quantization map fits into that setting. Then, in Section 2.4, we tackle the problems of "dequantization" and of defining a star-product. We also compute the kernel of the latter as a fixed point formula, which is the main result of this chapter. Finally, in Section 2.5, we apply the previous results to elementary normal **j**-groups.

## 2.1 Symmetric spaces

There are several ways of approaching symmetric spaces, each of them providing a new insight into that notion. A first one is to consider a symmetric space as a manifold for which it is possible to define some kind of *central symmetry* around each of its points. These central symmetries acquire a very geometric meaning as soon as we highlight the natural affine structure of a symmetric space, and its associated geodesics. Finally, the whole machinery of Lie theory can be used to study symmetric spaces since they can be realized as homogeneous spaces $G/K$ for some Lie group $G$ acting transitively on the symmetric space. In this section, we are going to briefly describe those different aspects. For a complete treatment, we refer to the classical references of Loos [Loo69], Kobayashi and Nomizu [KN09] and Helgason [Hel78]. A nice summary of the equivalence between the different definitions can also be found in Voglaire [Vog11, Section 1.4].

### A first definition

**Definition 2.1.1.** *A* symmetric space *is a pair $(M, s)$, where $M$ is a connected manifold* [1] *and $s : M \times M \to M$ is a smooth map such that:*

1. *For every $x \in M$, the map $s_x : M \to M$ ; $y \mapsto s(x, y)$ is an involutive (i.e. $s_x^2 = \mathrm{Id}_M$) diffeomorphism admitting $x$ as an isolated fixed point;*
2. *For every $x, y \in M$, $s_x \circ s_y \circ s_y = s_{s_x(y)}$.*

*The map $s_x$ is called the* symmetry at the point $x$.

**Definition 2.1.2.** *Let $(M, s)$ and $(M', s')$ be two symmetric spaces. A* morphism *between $(M, s)$ and $(M', s')$ is a smooth map $\phi : M \to M'$ such that, for all $x, y \in M$,*

$$s'_{\phi(x)}(\phi(y)) = \phi(s_x(y)).$$

*It is an* isomorphism *if $\phi$ is also a diffeomorphism. An* automorphism *of $(M, s)$ is an isomorphism of $(M, s)$ to itself.*

*Example* 2.1.3. The simplest example of a symmetric space is given by $\mathbb{R}^n$ endowed with the following symmetry:

$$s : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \; ; \; (x, y) \mapsto 2x - y.$$

The symmetry $s_x$ corresponds to the central symmetry around $x$. ◊

*Example* 2.1.4. Generalizing the previous example, any Lie group $G$ can be endowed with a symmetric space structure through the following map:

$$s : G \times G \to G \; ; \; (g, g') \mapsto g(g')^{-1}g. \qquad \Diamond$$

Let us briefly comment on the two conditions in the definition. As for the first one, $s_x$ being an involution implies that $(s_x)_{*_x}$ is an involutive automorphism of $T_xM$, hence is diagonalizable and admits only $+1$ and $-1$ as eigenvalues. The fact that $x$ is an isolated fixed point implies that $+1$ eigenvalues can not occur. Therefore, $(s_x)_{*_x} = -\mathrm{Id}_{T_xM}$, suggesting that $s_x$ is something like a central symmetry around $x$. The second condition is represented in Figure 2.1 and, again, is a generalization of a property of the central symmetries of $\mathbb{R}^n$.

[1] Notice that we restrict to connected symmetric space, which is not the case everywhere in the literature.

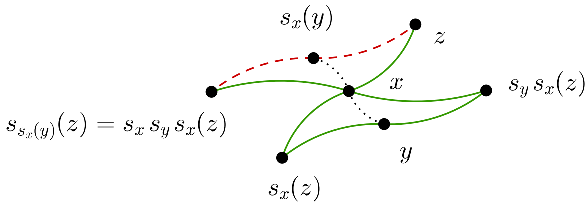



Figure 2.1: The point $z$ must be mapped on the same point by the sequence of transformations corresponding to the plain line, and by the one corresponding to the dashed line.

### Symmetric spaces and geodesics

The following theorem allows to highlight a geometric structure underlying a symmetric space.

**Theorem 2.1.5.** *Let $(M, s)$ be a symmetric space. There exists a unique affine connection (i.e. a covariant derivative)*

$$\nabla : \Gamma^\infty(M, TM) \times \Gamma^\infty(M, TM) \to \Gamma^\infty(M, TM)\ ;\ (X, Y) \mapsto \nabla_X Y$$

*that is invariant under all symmetries. It is called the* Loos connection. *Moreover, it is complete, torsion-free, its curvature tensor is parallel and it is explicitly given by the formula, for all $X, Y \in \Gamma^\infty(M, TM)$ and $x \in M$,*

$$(\nabla_X Y)_x = \frac{1}{2}\,[X, Y + s_{x_*} Y]_x\,.$$

The next result asserts that the symmetry at a point $x$ is in fact the geodesic symmetry around $x$ – at least for points that are connected to $x$ by a geodesic. This is again pictured in Figure 2.1.

**Proposition 2.1.6.** *Let $(M, s)$ be a symmetric space, and $\gamma : \mathbb{R} \to M\ ;\ t \mapsto \gamma(t)$ a maximal geodesic for the Loos connection. Then, for all $t, s \in \mathbb{R}$,*

$$s_{\gamma(t)}(\gamma(t + s)) = \gamma(t - s).$$

### Symmetric spaces as homogeneous spaces

**Proposition 2.1.7.** *Let $(M, s)$ be a symmetric space. The group of automorphisms of $(M, s)$, denoted by* $\mathrm{Aut}(M, s)$, *is a finite dimensional Lie group acting transitively on $M$.*

Let $(M, s)$ be a symmetric space, and fix $o \in M$. Let us denote by $G$ the identity component of $\mathrm{Aut}(M, s)$, and consider $K$ the stabilizer in $G$ of $o$, that is, $K := \{g \in G \mid g(o) = o\}$. It is a closed subgroup of $G$ and $M$ is diffeomorphic to the homogeneous space $G/K$ – recall that $M$ is connected by definition, so $G$ must act transitively on $M$. The second property in the definition of a symmetric space ensures that every symmetry is an automorphism of $M$. We can thus consider the following involutive homomorphism of $\mathrm{Aut}(M, s)$:

$$\sigma : \mathrm{Aut}(M, s) \to \mathrm{Aut}(M, s) \; ; \; g \mapsto s_o g s_o.$$

When restricted to $G$, $\sigma$ gives an involutive automorphism of $G$. Furthermore, if we denote $G^\sigma := \{g \in G \mid \sigma(g) = g\}$ and by $(G^\sigma)_0$ the identity component of $G^\sigma$, we have the inclusions

$$(G^\sigma)_0 \subset K \subset G^\sigma.$$

This motivates the following definition.

**Definition 2.1.8.** *A* symmetric triple *is a triple* $(G, K, \sigma)$*, where*

1. *$G$ is a connected Lie group;*
2. *$\sigma : G \to G$ is an involutive automorphism of $G$;*
3. *$K$ is a closed subgroup of $G$ such that*

$$(G^\sigma)_0 \subset K \subset G^\sigma := \{g \in G \mid \sigma(g) = g\},$$

   *where $(G^\sigma)_0$ denotes the identity component of $G^\sigma$.*

Given a symmetric triple $(G, K, \sigma)$, the homogeneous space $G/K$ can be endowed with the symmetric structure $\tilde{s} : G/K \times G/K \to G/K$ defined, for $g_xK, g_yK \in G/K$, by the formula

$$\tilde{s}_{g_xK}(g_yK) := g_x\sigma(g_x^{-1}g_y)K.$$

If we start with a symmetric space $(M, s)$ and consider the associated symmetric triple $(G, K, \sigma)$ as above, it turns out that $(G/K, \tilde{s})$ and $(M, s)$ are isomorphic as symmetric spaces. However, let us notice that there is generally not a unique symmetric triple associated to a symmetric space. There is for instance some freedom in the choice of $K$. But we could also have started the above construction from any connected subgroup of $\mathrm{Aut}(M, s)$ acting transitively on $M$ and stabilized by $\sigma$.

At the infinitesimal level, given a symmetric triple $(G, K, \sigma)$, the differential $\sigma_{*_e}$ of $\sigma$ at the neutral element is an involutive automorphism of the Lie algebra $\mathfrak{g}$ of $G$. We therefore have a decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ corresponding to the $(\pm 1)-$eigenspace decomposition (that is, $\sigma_{*_e} = \mathrm{Id}_{\mathfrak{g}_+} \oplus (-\mathrm{Id}_{\mathfrak{g}_-})$). Moreover, if $\mathfrak{k}$ denotes the Lie algebra of $K$, we have that $\mathfrak{g}_+ = \mathfrak{k}$, and

$$[\mathfrak{g}_+, \mathfrak{g}_+] \subset \mathfrak{g}_+ \; , \; [\mathfrak{g}_-, \mathfrak{g}_-] \subset \mathfrak{g}_+ \; , \; [\mathfrak{g}_+, \mathfrak{g}_-] \subset \mathfrak{g}_-.$$

Finally, notice that we have $T_{eK}(G/K) = \mathfrak{g}_-$.

## 2.2 Equivariant quantization map

### 2.2.1 The setting

In this subsection, we describe the setting we will use for the construction of the quantization map.

**Definition 2.2.1.** *A* nearly-quantum symmetric space *is a tuple* $(G, K, \sigma, B, \chi)$ *such that:*

1. $(G, K, \sigma)$ *is symmetric triple such that there exists a* $G-$*invariant smooth measure* $d_{G/K}$ *on* $G/K$*;*
2. $B$ *is a closed subgroup of* $G$ *such that* $K \subset B$*;*
3. $\chi : B \to U(1)$ *is a unitary character of* $B$ *which is* $\sigma-$*invariant, that is, for all* $b \in B$, $\chi(\sigma(b)) = \chi(b)$.

*A nearly-quantum symmetric space is said to be* local *if there exists a subgroup* $Q$ *of* $G$ *such that the map*

$$Q \times B \to G \; ; \; (q, b) \mapsto qb$$

*is a global diffeomorphism.*

*Remark* 2.2.2. Recall that, as we have seen in the previous Section, the symmetric triple $(G, K, \sigma)$ gives rise to a symmetric space $(M \simeq G/K, s)$, with $s$ given, for all $g_x K, g_y K \in G/K$, by

$$s_{g_x K}(g_y K) = g_x \sigma\left(g_x^{-1} g_y\right) K. \qquad \lhd$$

*Remark* 2.2.3. For a local quantum symmetric space $(G, K, \sigma, B, \chi)$, notice that $Q$ is a closed subgroup of $G$. For any element $g \in G$, there is a unique decomposition $g = qb$, with $q \in Q$ and $b \in B$ and we will use the following superscript notation to denote that decomposition:

$$g = g^Q g^B. \qquad \lhd$$

*Example* 2.2.4. This example endows the cylinder with a symmetric structure. Let us consider the group $G$ given by the semi-direct product

$$G := S^1 \ltimes_\rho \mathbb{C}$$

for the action $\rho$ defined, for $e^{ia} \in S^1$ and $z \in \mathbb{C}$, by $\rho(a)z := e^{ia}z$. The corresponding group law and inverse are respectively given, for $(e^{ia}, z), (e^{ia'}, z') \in G$, by:

$$(e^{ia}, z) \cdot (e^{ia'}, z') = (e^{i(a+a')}, z' + e^{ia'} z),$$
$$(e^{ia}, z)^{-1} = (e^{-ia}, -e^{-ia} z).$$

Let us consider the following involution of $G$:

$$\sigma : G \to G \; ; \; (e^{ia}, z) \mapsto (e^{-ia}, -\bar{z}),$$

and the subgroup $K := G^\sigma = \{g \in G \mid \sigma(g) = g\} = \{(0, iy) \mid y \in \mathbb{R}\} \subset G$. Then, $(G, K, \sigma)$ is a symmetric triple and we may consider the corresponding symmetric space $M \simeq G/K$. If we define the two subgroups of $G$

$$A := \left\{(e^{ia}, 0) \mid e^{ia} \in S^1\right\} \simeq S^1 \quad \text{and} \quad N := \{(0, x) \mid x \in \mathbb{R}\},$$

we have the decomposition $G = ANK$. This gives the following coordinates:

$$AN \xrightarrow{\sim} M \simeq G/K \; ; \; (e^{ia}, n) \mapsto (e^{ia}, n)K,$$

which we use to identify $M$ with $AN$. Under this identification, the action of $G$ on $M$ is given, for $(e^{ia}, x + iy) \in G$ and $(e^{ia_0}, n_0) \in M$, by

$$(e^{ia}, x + iy) \cdot (e^{ia_0}, n_0) = \left(e^{i(a_0+a)}, n_0 + x\cos(a_0) - y\sin(a_0)\right).$$

The symmetric structure $s : M \times M \to M$ on $M$ reads, for $(e^{ia}, n), (e^{ia'}, n') \in M$:

$$s_{(e^{ia},n)}(e^{ia'}, n') = (e^{i(2a-a')}, 2x\cos(a - a') - x').$$

The $G-$invariant measure on $M$ is $d_M = dadn$. Finally, we have some choice left for the subgroup $B$ and its character $\chi$. As a first choice, we could take $B := K$, and any unitary character of $K$ would do (since $\sigma$ is the identity on $K$). As another example, let us consider $B := NK$. Then, since $\sigma_{|N}$ is the inverse map, the $\sigma-$invariance of the character implies that $\chi_{|N} = 1$. It must thus be of the form $\chi_m(x + iy) = e^{imy}$ for some $m \in \mathbb{R}$ and all $x + iy \in NK$. Notice that in this case, the nearly-quantum symmetric space is local. $\Diamond$

For the following discussion, let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space. It will be specified when we take it to be local. The remaining of this section aims to construct a quantization map as in the Weyl quantization, which is $G$-equivariant. The construction is based on the one of [BG15, Chapter 7], adapted to the more general setting of nearly-quantum symmetric spaces.[2]

### 2.2.2 The Hilbert space and a first quantization map

As a first ingredient of the construction, we consider the unitary representation of $G$ induced by the character $\chi$ of $B$. It associates to the data of $G$, $B$ and $\chi$, a vector bundle that carries a natural left action of $G$, and whose sections allow to define a Hilbert space on which $G$ acts by pullback. Let us recall how

[2] We need, for instance, to use half-densities on $G/B$, since we do not require the existence of a $G$-invariant measure on $G/B$.

it works. Given the $B-$principal bundle $G \to G/B$, we consider the associated vector bundle over $G/B$

$$\pi : \left( E_\chi := G \times_\chi \mathbb{C} := \frac{G \times \mathbb{C}}{\sim} \right) \to G/B,$$

where, for all $g \in G$, $b \in B$ and $z \in \mathbb{C}$, the equivalence relation is defined by $(g, z) \sim \big(gb, \chi(b)^{-1}z\big)$ and the projection by $\pi([g, z]) := gB$. It is a complex line bundle and there is a natural left action of $G$ on $E_\chi$ defined, for all $g_0, g \in G$ and $z \in \mathbb{C}$, by:

$$g_0 \cdot [g, z] := [g_0 g, z]. \tag{2.1}$$

We can endow $E_\chi$ with a $G-$invariant Hermitian structure $h$ given, for all $g \in G$ and $z, z' \in \mathbb{C}$, by:

$$h_{gB}\left([g, z'], [g, z]\right) := z'\bar{z}. \tag{2.2}$$

We can now get a Hilbert space from that bundle by making use of its intrinsic Hilbert space, that we denote by $\mathcal{H}_\chi$. Recall from Definition 1.2.25 that it is built from sections of the tensor product of $E_\chi$ with the bundle of half-densities on $G/B$. More precisely, it is the completion of $\Gamma_c^\infty(G/B, E_\chi \otimes |T(G/B)|^{1/2})$ for the inner product

$$\langle \varphi \otimes \rho, \varphi' \otimes \mu \rangle := \int_{G/B} h(\varphi, \varphi')\ \rho.\bar{\mu}. \tag{2.3}$$

In order to define a representation of $G$ on $\mathcal{H}_\chi$, we consider the representation of $G$ on $\Gamma_c^\infty(G/B, E_\chi)$ given by the pullback, that is, for every $g_0, g \in G$ and $\varphi \in \Gamma_c^\infty(G/B, E_\chi)$:

$$(g \cdot \varphi)(g_0 B) := g \cdot \varphi(g^{-1} g_0 B). \tag{2.4}$$

Also, recall that, as mentioned in Remark 1.2.26, Diff$(G/B)$ acts by pullback unitarily on the Hilbert space of half-densities. Since $G$ is a subgroup of Diff$(G/B)$ via the left action of $G$ on $G/B$ given by left translations

$$\alpha : G \times G/B \to G/B\ ;\ (g, g_0 B) \mapsto \alpha_g(g_0) := g g_0 B, \tag{2.5}$$

$G$ acts also on the half-densities. These two considerations define a representation of $G$ on $\mathcal{H}_\chi$, which is unitary by $G-$invariance of the Hermitian structure on $E_\chi$.

**Definition 2.2.5.** *Let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space, the unitary representation of $G$*

$$U_\chi : G \to \mathcal{U}(\mathcal{H}_\chi)$$

*defined, for $g \in G$, $\varphi \otimes \rho \in \Gamma_c^\infty(G/B, E_\chi \otimes |T(G/B)|^{1/2})$, by*

$$U_\chi(g)(\varphi \otimes \rho) = (g \cdot \varphi) \otimes (\alpha^*_{g^{-1}} \rho)$$

*is called the* unitary representation of $G$ induced from $(B, \chi)$.

Before going on, let us take some time to give different descriptions of the Hilbert space $\mathcal{H}_\chi$. In each of the next remarks, we build an isomorphism between $\Gamma_c^\infty\left(G/B, E_\chi \otimes |T(G/B)|^{1/2}\right)$ and another vector space of sections. By transporting the pre-Hilbert structure (2.3) to that new vector space and taking the completion, we get a concrete realization of a Hilbert space isomorphic to $\mathcal{H}_\chi$. These different realizations of $\mathcal{H}_\chi$ will allow us to express our operators on one or another vector space, depending on which one is the most convenient.

For the first one, we will need the following lemma which expresses the density bundle as an associated bundle. Since there are multiple conventions for defining the modular function on a group, let us recall its definition.[3]

**Definition 2.2.6.** *Let $G$ be a Lie group. The* modular function *on $G$ is the smooth homomorphism $\Delta_G : G \to \mathbb{R}_0^+$ such that, for all left Haar measure $\lambda$ on $G$, $d\lambda(xy) = \Delta_G(y) d\lambda(x)$ for all $x, y \in G$. It is given explicitly by the formula $\Delta_G(x) = |\det \mathrm{Ad}_{y^{-1}}|$ for all $y \in G$.*

Let us also recall the following notion, coming from the theory of quasi-invariant integration on homogeneous spaces, for which we refer to Folland [Fol94].

**Definition 2.2.7.** *Let $G$ be a Lie group, $B$ a closed subgroup of $G$. A $\rho$-function associated to the pair $(G, B)$ is a positive continuous function $\rho : G \to \mathbb{R}_0^+$ such that, for all $g \in G$ and $b \in B$,*

$$\rho(gb) = \frac{\Delta_B(b)}{\Delta_G(b)} \rho(g).$$

**Lemma 2.2.8.** *Let $G$ be a Lie group, $B$ a closed subgroup of $G$ and $\delta^{1/2}$ the character of $B$ defined by*

$$\delta^{1/2} : B \to \mathbb{C} \ ; \ b \mapsto \delta^{1/2}(b) := \left(\frac{\Delta_G(b)}{\Delta_B(b)}\right)^{1/2}.$$

*Then, we have an isomorphism of vector bundles* [4]

$$|T(G/B)|^{1/2} \simeq G \times_{\delta^{1/2}} \mathbb{C},$$

*which induces a $G-$equivariant* [5] *isomorphism at the level of sections*

$$\Gamma^\infty(G/B, |T(G/B)|^{1/2}) \xrightarrow{\sim} \Gamma^\infty(G/B, G \times_{\delta^{1/2}} \mathbb{C}).$$

*Proof.* Let us fix a positive half-density $\mu$ on $G/B$. By the theory of quasi-invariant integration on homogeneous spaces (see for instance Folland [Fol94,

[3] We follow the convention of Folland [Fol94].

[4] $G \times_{\delta^{1/2}} \mathbb{C}$ is the associated vector bundle corresponding to the $B-$principal bundle $G \to G/B$ and the character $\delta^{1/2}$

[5] The action of $G$ on $G \times_{\delta^{1/2}} \mathbb{C}$ is the natural one, given by the same formula (2.1), and the induced action on the sections is given by (2.4).

Section 2.6]), we know that there exists an associated so called $\rho-$function, that is, some smooth (because $\mu$ is) positive function $\rho : G \to \mathbb{R}_0^+$ such that, for all $g \in G$, $g_0B \in G/B$ and $b \in B$:

$$(\alpha_{g^{-1}}^*\mu)(g_0B) = \left(\frac{\rho(g^{-1}g_0)}{\rho(g_0)}\right)^{1/2} \mu(g_0B), \tag{2.6}$$

$$\rho(gb) = \frac{\Delta_B(b)}{\Delta_G(b)}\,\rho(g). \tag{2.7}$$

Let $g_0B \in G/B$. Recall that the half-density bundle is a complex line bundle, so any element of $(|T(G/B)|^{1/2})_{g_0B}$ is of the form $z.\mu(g_0B)$ for some $z \in \mathbb{C}$. We can thus consider the linear map between the fibers

$$(|T(G/B)|^{1/2})_{g_0B} \xrightarrow{\sim} (G \times_{\delta^{1/2}} \mathbb{C})_{g_0B} \; ; \; z.\mu(g_0B) \mapsto [g_0, \rho(g_0)^{1/2}z].$$

It is well-defined because of (2.7), and it is an isomorphism, being a surjection between one-dimensional spaces. Also, it depends smoothly on $g_0B$ since $\mu$ and $\rho$ are smooth. This gives the isomorphism of vector bundles

$$A : |T(G/B)|^{1/2} \xrightarrow{\sim} G \times_{\delta^{1/2}} \mathbb{C}.$$

We denote by the same letter the induced isomorphism at the level of sections

$$A : \Gamma^\infty(G/B, |T(G/B)|^{1/2}) \xrightarrow{\sim} \Gamma^\infty(G/B, G \times_{\delta^{1/2}} \mathbb{C}).$$

It is $G-$equivariant because, for all $g \in G$ and $g_0B \in G/B$, we have, using (2.6),

$$\begin{aligned}\left(A(\alpha_{g^{-1}}^*\mu)\right)(g_0B) &= A\left(\left(\frac{\rho(g^{-1}g_0)}{\rho(g_0)}\right)^{1/2} \mu(g_0B)\right)\\ &= \left[g_0, \rho(g_0)^{1/2}\left(\frac{\rho(g^{-1}g_0)}{\rho(g_0)}\right)^{1/2} \mu(g_0B)\right]\\ &= g\cdot\left[g^{-1}g_0, \rho(g^{-1}g_0)^{1/2}\mu(g_0B)\right]\\ &= g\cdot(A(\mu(g_0B))) = (g\cdot A\mu)(g_0B).\end{aligned}$$

□

*Remark* 2.2.9. The previous lemma allows to give a slightly different description of the vector bundle $E_\chi \otimes |T(G/B)|^{1/2}$. Let us define the character $\tilde{\chi}$ of $B$ by

$$\tilde{\chi} : B \to \mathbb{C} \; ; \; b \mapsto \chi(b) \,.\, \delta^{1/2}(b). \tag{2.8}$$

Then, by Lemma 2.2.8, we have the isomorphism of vector bundles

$$E_\chi \otimes |T(G/B)|^{1/2} \simeq G \times_{\tilde{\chi}} \mathbb{C} =: E_{\tilde{\chi}}, \tag{2.9}$$

and the natural left action of $G$ on $E_{\tilde{\chi}}$ given, for all $g_0, g \in G$ and $z \in \mathbb{C}$, by:[6]

$$g_0 \cdot [g, z] := [g_0g, z].$$

[6]We keep the same notation $[\cdot,\cdot]$ for the equivalence classes defining the elements of $E_\chi$ and $E_{\tilde{\chi}}$ since it should not introduce any confusion.

This induces an isomorphism $\Gamma_c^\infty\left(G/B, E_\chi \otimes |T(G/B)|^{1/2}\right) \xrightarrow{\sim} \Gamma_c^\infty\left(G/B, E_{\tilde{\chi}}\right)$ and, under this identification, the representation $U_\chi$ reads, for every $g_0, g \in G$ and $\varphi \in \Gamma_c^\infty\left(G/B, E_{\tilde{\chi}}\right)$:

$$(U_\chi(g_0)\varphi)(gB) := g_0 \cdot \varphi(g_0^{-1} gB). \tag{2.10}$$

If we fix a positive half-density $\mu$ and denote by $\rho$ the corresponding $\rho-$function, the transported Hermitian structure on $E_{\tilde{\chi}}$ is given, for $g \in G$ and $z, z' \in \mathbb{C}$, by

$$h_{gB}([g,z'],[g,z]) = \overline{z} z' \, \rho(g)^{-1} \tag{2.11}$$

and the inner product on $\Gamma_c^\infty\left(G/B, E_{\tilde{\chi}}\right)$ is, for all $\varphi, \psi \in \Gamma_c^\infty\left(G/B, E_{\tilde{\chi}}\right)$, given by

$$\langle \varphi, \psi \rangle = \int_{G/B} h_{gB}(\varphi(gB), \psi(gB)) \, \mu^2(gB). \tag{2.12}$$

$\triangleleft$

*Remark* 2.2.10. Recall that we can identify the space $\Gamma^\infty\left(G/B, E_{\tilde{\chi}}\right)$ of smooth sections of $E_{\tilde{\chi}}$ with the space of smooth $(B, \tilde{\chi})-$equivariant functions on $G$

$$\mathcal{C}^\infty(G)^{(B,\tilde{\chi})} := \left\{\hat{\varphi} \in \mathcal{C}^\infty(G) \mid \hat{\varphi}(gb) = \tilde{\chi}(b)^{-1}\hat{\varphi}(g) \;\forall g \in G, b \in B\right\}$$

through the isomorphism

$$\hat{\varphi} \in \mathcal{C}^\infty(G)^{(B,\tilde{\chi})} \mapsto \Gamma^\infty\left(G/B, E_{\tilde{\chi}}\right) \ni \varphi := [\;\; gB \mapsto [g, \hat{\varphi}(g)] \in E_{\tilde{\chi}} \;\;].$$

We denote by $\mathcal{C}^\infty(G)_c^{(B,\tilde{\chi})}$ the pre-image of the space of compactly supported smooth sections under this correspondence [7], which can be explicitly described as

$$\mathcal{C}^\infty(G)_c^{(B,\tilde{\chi})} = \left\{\hat{\varphi} \in \mathcal{C}^\infty(G)^{(B,\tilde{\chi})} \mid \pi(\mathrm{supp}(\hat{\varphi})) \text{ is compact}\right\}, \tag{2.13}$$

where $\pi : G \to G/B$ is the natural projection. Then, under this identification, the representation $U_\chi$ corresponds to the restriction to $\mathcal{C}^\infty(G)_c^{(B,\tilde{\chi})}$ of the left regular representation of $G$: for every $\hat{\varphi} \in \mathcal{C}^\infty(G)_c^{(B,\tilde{\chi})}$ and $g_0, g \in G$,

$$(U_\chi(g_0)\hat{\varphi})(g) = \hat{\varphi}(g_0^{-1} g). \tag{2.14}$$

Indeed, $U_\chi(g_0)\hat{\varphi}$ is defined by the identity $(U_\chi(g_0)\varphi)(gB) =: [g, (U_\chi(g_0)\hat{\varphi})(g)]$ and $\hat{\varphi}$ by $\varphi(gB) =: [g, \hat{\varphi}(g)]$, so we have:

$$\begin{aligned}(U_\chi(g_0)\varphi)(gB) &= g_0 \cdot \varphi(g_0^{-1} gB) = g_0 \cdot [g_0^{-1} g, \hat{\varphi}(g_0^{-1} g)] \\ &= [g, \hat{\varphi}(g_0^{-1} g)],\end{aligned}$$

[7] Notice that the subscript is on the right of the parenthesis, to distinguish it from compactly supported functions.

which implies (2.14) by identification. Regarding the inner product, if we fix a positive half-density $\mu$ and denote by $\rho$ the corresponding $\rho-$function, from (2.11) and (2.12), we have, for all $\hat{\varphi}, \hat{\psi} \in \mathcal{C}^\infty(G)_c^{(B,\tilde{\chi})}$,

$$\langle \hat{\varphi}, \hat{\psi} \rangle = \int_{G/B} \overline{\hat{\psi}(g)} \hat{\varphi}(g)\, \rho(g)^{-1}\, \mu^2(gB). \tag{2.15}$$

Notice that the integrand in (2.15) is a well-defined function on $G/B$ because of the $B-$equivariance of $\hat{\varphi}$, $\hat{\psi}$ and $\rho$ and the unitarity of the character $\chi$. $\lhd$

*Remark* 2.2.11. In the local case where $G = QB$, we have a natural half-density on $G/B \simeq Q$, the one corresponding to the $\rho-$function defined, for all $q \in Q$ and $b \in B$, by $\rho(qb) = \frac{\Delta_B(b)}{\Delta_G(b)}$. We denote it by $\mu$, and by $d_Q$ the measure corresponding to $\mu^2$. Furthermore, we have an isomorphism of vector spaces given by

$$\mathcal{C}^\infty(G)_c^{(B,\tilde{\chi})} \xrightarrow{\sim} \mathcal{C}_c^\infty(Q)\ ;\ \hat{\varphi} \mapsto \tilde{\varphi} := \hat{\varphi}_{|Q}.$$

Notice that it indeed maps $\mathcal{C}^\infty(G)_c^{(B,\tilde{\chi})}$ on *compactly* supported sections because of the characterization (2.13). Under this identification, for every $g \in G$ and $q_0 \in Q$, we have

$$\begin{aligned}(U(g)\tilde{\varphi})(q_0) = (U(g)\hat{\varphi})(q_0) &= \hat{\varphi}\left((g^{-1}q_0)^Q\,(g^{-1}q_0)^B\right) \qquad (2.16)\\ &= \tilde{\chi}\left((g^{-1}q_0)^B\right)^{-1} \tilde{\varphi}\left((g^{-1}q_0)^Q\right). \qquad \lhd\end{aligned}$$

From (2.15), we get that the inner product is given, for all $\tilde{\varphi}, \tilde{\psi} \in \mathcal{C}_c^\infty(Q)$, by

$$\langle \tilde{\varphi}, \tilde{\psi} \rangle = \int_Q \overline{\tilde{\psi}(g)} \tilde{\varphi}(g)\, d_Q(q),$$

which shows that $\mathcal{H}_\chi \simeq L^2(Q, d_Q)$.

*Remark* 2.2.12. We now have four different descriptions of the Hilbert space $\mathcal{H}_\chi$. In the following, we will use one or another depending on which one turns out to be the most convenient for a given purpose. Notice that we are a bit sloppy with the notation since we keep the same symbol for an operator when it acts on one space or another. However, it should not cause any confusion because it will be clear to which space the vector on which the operator acts belongs (for instance, the "hat" indicates that $\hat{\varphi} \in \mathcal{C}^\infty(G)_c^{(B,\tilde{\chi})}$ and the "tilde" indicates that $\tilde{\varphi} \in \mathcal{C}_c^\infty(G/B)$). For the sake of clarity, let us summarize the four descriptions of $\mathcal{H}_\chi$. It is defined as a completion of either:

1. the space of compactly supported smooth sections of the vector bundle

$$(E_\chi := G \times_\chi \mathbb{C}) \otimes |T(G/B)|^{1/2} \to G/B;$$

2. the space of compactly supported smooth sections of the vector bundle

$$E_{\tilde{\chi}} := G \times_{\tilde{\chi}} \mathbb{C} \to G/B;$$

3. the subspace $\mathcal{C}^\infty(G)_c^{(B,\tilde{\chi})}$ of $(B,\tilde{\chi})$−equivariant functions on $G$.
4. in the local case, the space of compactly supported functions on $G/B \simeq Q$. ◁

*Remark* 2.2.13. Let us also mention that the chosen positive half-density on $G/B$ appears in the expression of the inner products, so the completion of the space of sections will depend on that choice. However, all the obtained representations are unitarily equivalent and, since the half-density does not appear in the expressions of the representation of $G$, we often won't need to specify that choice. ◁

The second ingredient of the quantization map arises from the observation that the involution $\sigma$ – which encodes the symmetric structure of $G/B$ – allows to define an operator on $\mathcal{H}_\chi$ which commutes with $U_\chi(k)$ for all $k \in K$. It is based on the following lemmas.

**Lemma 2.2.14.** *Let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space. Then, the character $\tilde{\chi}$ is invariant under the restriction of $\sigma$ to the subgroup $B$, that is, $\tilde{\chi} \circ \sigma_{|B} = \tilde{\chi}$.*

*Proof.* The character $\chi$ is invariant under $\sigma$ by assumption. The invariance under $\sigma$ of $\delta^{1/2}$ follows from its definition and from the invariance of the modular function of a Lie group under any involutive homomorphism of the group. □

**Lemma 2.2.15.** *Let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space. The map*

$$\underline{\sigma} : G/B \to G/B \ ; \ gB \mapsto \sigma(g)B,$$

*is a well-defined involutive diffeomorphism of $G/B$. The map*

$$\tilde{\sigma} : E_{\tilde{\chi}} \to E_{\tilde{\chi}} \ ; \ [g, z] \mapsto [\sigma(g), z].$$

*is a well-defined involutive isomorphism of $E_{\tilde{\chi}}$ which lifts $\underline{\sigma}$.*

*Proof.* The fact that $\tilde{\sigma}$ and $\underline{\sigma}$ are well-defined follows from the stability of $B$ under $\sigma$ and Lemma 2.2.14. They both are involutive diffeomorphisms since $\sigma$ is. Finally, for all $g \in G$, $b \in B$ and $z \in \mathbb{C}$, $\tilde{\sigma}([gb, z]) = [\sigma(g)\sigma(b), z] \in (E_{\tilde{\chi}})_{\underline{\sigma}(gB)}$, so the fiber $(E_{\tilde{\chi}})_{gB}$ is mapped onto the fiber $(E_{\tilde{\chi}})_{\underline{\sigma}(gB)}$. □

In view of this Lemma, we can define a linear operator

$$\Sigma : \Gamma_c^\infty(G/B, E_{\tilde{\chi}}) \to \Gamma_c^\infty(G/B, E_{\tilde{\chi}})$$

by the formula, for all $\varphi \in \Gamma_c^\infty(G/B, E_{\tilde{\chi}})$ and $gB \in G/B$,

$$(\Sigma\varphi)(gB) := \tilde{\sigma}\left(\varphi\left(\underline{\sigma}(gB)\right)\right). \tag{2.17}$$

Notice that $\Sigma\varphi$ is compactly supported since $\underline{\sigma}$ is a diffeomorphism.

*Remark* 2.2.16. As in Remark 2.2.10, we can realize $\Sigma$ on the space of $(B,\tilde{\chi})-$ equivariant functions on $G$. For all $\hat{\varphi} \in \mathcal{C}^\infty(G)_c^{(B,\tilde{\chi})}$ and $g \in G$, we have

$$(\Sigma\hat{\varphi})(g) = \hat{\varphi}(\sigma(g)).$$

Indeed, $\Sigma\hat{\varphi}$ is defined by the identity $(\Sigma\varphi)(gB) =: [g, (\Sigma\hat{\varphi})(g)]$, and, $\hat{\varphi}$ being such that $\varphi(gB) =: [g, \hat{\varphi}(g)]$, we have

$$\begin{aligned}(\Sigma\varphi)(gB) &= \tilde{\sigma}(\varphi(\underline{\sigma}(gB))) = \tilde{\sigma}(\varphi(\sigma(g)B)) = \tilde{\sigma}([\sigma(g), \hat{\varphi}(\sigma(g))]) \\ &= [\sigma(\sigma((g)), \hat{\varphi}(\sigma(g))] = [g, \hat{\varphi}(\sigma(g))]. \qquad \lhd\end{aligned}$$

**Lemma 2.2.17.** *Let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space. The operator $\Sigma$ defined by (2.17) extends to an involutive unitary – hence self-adjoint – operator on $\mathcal{H}_\chi$. Furthermore, for all $k \in K$:*

$$U_\chi(k)\, \Sigma = \Sigma\, U_\chi(k). \tag{2.18}$$

*Proof.* We will realize $\Sigma$ on $(B,\tilde{\chi})-$equivariant functions on $G$ and first show that $\Sigma$ leaves the inner product (2.3) invariant. Let $\hat{\varphi}, \hat{\psi} \in \mathcal{C}^\infty(G)_c^{(B,\tilde{\chi})}$, and let $\mu$ be a half-density on $G/B$ and $\rho$ the associated $\rho-$function. We have, from the expression (2.15) for the inner product,

$$\begin{aligned}\langle\hat{\varphi}, \Sigma\hat{\psi}\rangle &= \int_{G/B} \hat{\varphi}(g)\, \overline{\Sigma\hat{\psi}(g)}\, \rho(g)^{-1}\, \mu^2(gB) \\ &= \int_{G/B} \hat{\varphi}(g)\, \overline{\hat{\psi}(\sigma(g))}\, \rho(g)^{-1}\, \mu^2(gB) \\ &= \int_{G/B} \hat{\varphi}(\sigma(g))\, \overline{\hat{\psi}(g)}\, \rho(\sigma(g))^{-1}\, \frac{\rho(\sigma(g))}{\rho(g)}\, \mu^2(\sigma(g)B) \\ &= \int_{G/B} \Sigma\hat{\varphi}(gB)\, \overline{\hat{\psi}(g)}\, \rho(\sigma(g))^{-1}\, \mu^2(gB) = \langle\Sigma\hat{\varphi}, \hat{\psi}\rangle,\end{aligned}$$

where, for the transformation of $\mu$ under the change of variable, we have used the fact that $\sigma$ is an involutive automorphism of $G$. This implies that $\Sigma$ extends to a unitary operator on $\mathcal{H}_\chi$, which is involutive because $\tilde{\sigma}$ and $\underline{\sigma}$ are. Let $k \in K$. In terms of equivariant functions, for all $\hat{\varphi} \in \mathcal{C}^\infty(G)_c^{(B,\tilde{\chi})}$ and $g \in G$, we have

$$\begin{aligned}(U_\chi(h)\Sigma\hat{\varphi})(g) &= \hat{\varphi}(\sigma(h^{-1}g)) = \hat{\varphi}(h^{-1}\sigma(g)) \\ &= (\Sigma U_\chi(h)\hat{\varphi})(g),\end{aligned}$$

which shows the last assertion. □

We now have everything we need to define a map that associates in a natural way an operator to every point of the symmetric space $G/K$. It is constructed by intertwining the operator $\Sigma$ by the representation $U_\chi$ of $G$.

**Proposition 2.2.18.** *Let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space. The map*

$$\Omega : G/K \to \mathcal{U}(\mathcal{H}_\chi) \; ; \; gK \mapsto U_\chi(g) \, \Sigma \, U_\chi(g)^{-1}$$

*is well-defined and defines a* unitary representation *of $G/K$ in the sense that, for every $x, y \in G/K$ and $g \in G$, we have:*

1. $\Omega(x)^2 = \mathrm{Id}_{\mathcal{H}_\chi}$,
2. $\Omega(x)\Omega(y)\Omega(x) = \Omega(s_x(y))$,
3. $U_\chi(g)\Omega(x)U_\chi(g)^{-1} = \Omega(g \cdot x)$.

*Proof.* Let us define

$$\Omega : G \to \mathcal{U}(\mathcal{H}_\chi) \; ; \; g \mapsto \Omega(g) := U_\chi(g) \, \Sigma \, U_\chi(g)^{-1},$$

which is indeed valued in $\mathcal{U}(\mathcal{H}_\chi)$ since the representation $U_\chi$ and $\Sigma$ are unitary. From (2.18), we get

$$\begin{aligned}\Omega(gk) &= U_\chi(g)U_\chi(k) \, \Sigma \, U_\chi(k)^{-1}U_\chi(g)^{-1} \\ &= U_\chi(g) \, \Sigma \, U_\chi(g)^{-1} = \Omega(g),\end{aligned}$$

so $\Omega$ induces a well defined map on $G/K$, which we still denote by $\Omega$. We then give the explicit formula for $\Omega$ on different realizations of $\mathcal{H}_\chi$. For all $x = g_x K \in G/K$ and $\varphi \in \Gamma_c^\infty(G/B, E_{\tilde{\chi}})$ and $g_0 B \in G/B$, we have:

$$\begin{aligned}(\Omega(x)\varphi)(g_0 B) &= (U_\chi(g_x)\Sigma U_\chi(g_x^{-1})\varphi)(g_0 B) \\ &= g_x \cdot \tilde{\sigma}\left(g_x^{-1} \cdot \varphi\left(g_x \sigma(g_x)^{-1}\sigma(g_0)B\right)\right).\end{aligned}$$

In terms of equivariant functions, we have, for all $\hat{\varphi} \in \mathcal{C}^\infty(G)^{(B,\tilde{\chi})}$ and $g_0 \in G$:

$$\begin{aligned}(\Omega(x)\hat{\varphi})(g_0) &= (U_\chi(g_x) \, \Sigma \, U_\chi(g_x^{-1})\hat{\varphi})(g_0) \\ &= \hat{\varphi}\left(g_x \sigma(g_x^{-1} g_0)\right).\end{aligned}$$

This last expression allows to verify the three properties of the claim by explicit computation. Let $x = g_x K, y = g_y K \in G/K$, $g, g_0 \in G$ and $\hat{\varphi} \in \mathcal{C}^\infty(G)_c^{(B,\tilde{\chi})}$. Then,

1. $\Omega(x)^2 = U_\chi(g) \, \Sigma \, U_\chi(g)^{-1} \, U_\chi(g) \, \Sigma \, U_\chi(g)^{-1} = \mathrm{Id}_{\mathcal{H}_\chi}$;
2. Since $s_x(y) = g_x \sigma(g_x^{-1} g_y)K$, we have

$$\begin{aligned}(\Omega(x)\Omega(y)\Omega(x)\hat{\varphi})(g_0) &= \hat{\varphi}\left(g_x\sigma(g_x^{-1}g_y\sigma(g_y^{-1}g_x\sigma(g_x^{-1}g_0)))\right) \\ &= \hat{\varphi}\left(g_x\sigma(g_x^{-1}g_y)\sigma(\sigma(g_y^{-1}g_x)g_x^{-1}g_0))\right) \\ &= (\Omega(s_x(y))\hat{\varphi})(g_0).\end{aligned}$$

3. Finally, $\Omega$ is $G-$equivariant since, from the definition, we have

$$\begin{aligned} U_\chi(g)\Omega(x)U_\chi(g)^{-1} &= U_\chi(g)U_\chi(g_x)\Sigma U_\chi(g_x)^{-1}U_\chi(g)^{-1} \\ &= U_\chi(gg_x)\Sigma U_\chi(gg_x)^{-1} = \Omega(g\cdot x). \end{aligned}$$ $\square$

*Remark* 2.2.19. Notice that for all $x \in G/K$, $\Omega(x)$ is not only unitary but also self-adjoint since it is an involution. $\triangleleft$

Following [BG15], we introduce the following definition.

**Definition 2.2.20.** *Let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space. The pair $(\mathcal{H}_\chi, \Omega)$ is called the* unitary representation of $(G/K, s)$ induced by the character $\chi$ of $B$.

We are now able to construct our first quantization map.

**Proposition 2.2.21.** *Let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space. Let us denote by $d_{G/K}$ the $G-$invariant measure on $G/K$ and by $L^1(G/K)$ the space of functions on $G/K$ integrable with respect to $d_{G/K}$. Then, the map*

$$\Omega : L^1(G/K) \to \mathcal{L}(\mathcal{H}_\chi)\ ;\ f \mapsto \Omega(f) \tag{2.19}$$

*where $\Omega(f)$ is the operator defined, for $\varphi \in \Gamma_c^\infty(G/B, E_{\tilde{\chi}})$ and $g_0B \in G/B$, by*

$$(\Omega(f)\varphi)(g_0B) := \int_{G/K} f(x)\,(\Omega(x)\varphi)(g_0B)\,d_{G/K}(x), \tag{2.20}$$

*is well-defined, continuous and $G-$equivariant in the sense that, for all $g \in G$:*

$$U_\chi(g)\Omega(f)U_\chi(g)^{-1} = \Omega({}^g f), \tag{2.21}$$

*where ${}^g f : G/K \to \mathbb{C}\ ;\ g_xK \mapsto f(g^{-1}g_xK)$.*

*Proof.* For all $g_x \in G$, from the unitarity of $U_\chi(g_x)$ and of $\Sigma$, we get the unitarity of $\Omega(g_x)$, which leads to

$$||\Omega(f)|| \leq ||f||_1$$

for all $f \in L^1(G/K)$, so $\Omega$ is well-defined and continuous. The $G-$ equivariance follows from Property 3 in Proposition 2.2.18 and from the $G-$ invariance of the measure. $\square$

**Definition 2.2.22.** *Let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space. The map $\Omega$ defined by (2.19) is called the* quantization map *of $G/K$ induced by $(B, \chi)$, or simply the quantization map of $G/K$ when the context is clear.*

*Remark* 2.2.23. Notice that, since $\Omega(x)$ is self-adjoint for all $x \in G/K$ (see Remark 2.2.19), we have, for all $f \in L^1(G/K)$,

$$\Omega(f)^* = \Omega(\overline{f}).$$

Indeed, this is easily seen from (2.20) for compactly supported $f$, and the property extends to the whole $L^1$ by continuity of $\Omega$. In particular, real-valued functions are mapped to self-adjoint operators. $\lhd$

### 2.2.3 Another quantization map

Although the quantization map $\Omega$ naturally encodes the symmetric space structure of $G/K$, it will turn out that a slight modification of it is more convenient in order to define a deformed (i.e. noncommutative) product on $G/K$. The modified quantization map arises from a very similar construction, which however involves a functional parameter. [8]

**Definition 2.2.24.** *Let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space. Let $\mathbf{m}$ be a smooth function on $G/B$ and denote also by $\mathbf{m}$ the operator of multiplication by $\mathbf{m}$ of sections of $E_{\tilde{\chi}}$. We define the operator $\Sigma_{\mathbf{m}}$ as the composition $\mathbf{m} \circ \Sigma$:*

$$\Sigma_{\mathbf{m}} : \Gamma_c^\infty (G/B, E_{\tilde{\chi}}) \to \Gamma_c^\infty (G/B, E_{\tilde{\chi}}) \ ; \ \varphi \mapsto (\mathbf{m} \circ \Sigma)(\varphi)$$

*that is, for all $\varphi \in \Gamma_c^\infty (G/B, E_{\tilde{\chi}})$ and $g_0B \in G/B$:*

$$\begin{aligned}(\Sigma_{\mathbf{m}}\varphi)(g_0B) &:= \mathbf{m}(g_0B)\ (\Sigma\varphi)(g_0B) = \mathbf{m}(g_0B)\ \tilde{\sigma}\left(\varphi(\sigma(g_0)B)\right) \\ &= \mathbf{m}(g_0B)\ \tilde{\sigma}\left(\varphi(\underline{\sigma}(g_0B))\right).\end{aligned} \tag{2.22}$$

*In terms of $(B, \tilde{\chi})-$equivariant functions, we have, for all $g_0 \in G$ and $\hat{\varphi} \in \mathcal{C}^\infty(G)_c^{(B,\tilde{\chi})}$,*

$$(\Sigma_{\mathbf{m}}\hat{\varphi})(g_0) = \mathbf{m}(g_0B)\ \hat{\varphi}(\sigma(g_0)). \tag{2.23}$$

It should be noted that $\Sigma_{\mathbf{m}}$ is only defined as a linear operator on the vector space of compactly supported smooth sections. In general, it does not extend to a bounded operator on $\mathcal{H}_\chi$, unless $\mathbf{m}$ is bounded. We will see that it might be needed to consider such unbounded $\mathbf{m}$. For a short time, we will therefore leave the realm of bounded operators on Hilbert spaces and consider our operators as linear operators on $\Gamma^\infty (G/B, E_{\tilde{\chi}})$. In this spirit, observe that for every $g \in G$, $U_\chi(g)$ gives an endomorphism of $\Gamma^\infty (G/B, E_{\tilde{\chi}})$. Notice also that $\Sigma_{\mathbf{m}}$ as well as $U(g)$ map compactly supported sections to compactly supported sections

[8] Later on, this function will then be chosen in such a way that the quantization map defines a unitary operator from the Hilbert space of square integrable functions on $G/K$ and the Hilbert space of Hilbert-Schmidt operators on $\mathcal{H}_\chi$.

since $\underline{\sigma}$ is a diffeomorphism and $G$ acts on $G/B$ by diffeomorphisms. As we did before with $\Sigma$, we can therefore intertwine $\Sigma_{\mathbf{m}}$ by the representation $U_\chi$ to define $\Omega_{\mathbf{m}}(g) = U_\chi(g)\,\Sigma_{\mathbf{m}}\,U_\chi(g)^{-1}$ for all $g \in G$. However, many properties of Proposition 2.2.18 do not hold anymore for $\Omega_{\mathbf{m}}$ for a generic $\mathbf{m}$ – for instance, $\Omega_{\mathbf{m}}$ is not constant on the left cosets of $K$ in $G$. Still, we can recover some of them by imposing some conditions on $\mathbf{m}$.

**Definition 2.2.25.** *Let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space. A smooth function $\mathbf{m}$ on $G/B$ is called* admissible *if the two following conditions are satisfied:*

1. *it is invariant for the natural left action of $K$ on $G/B$, that is, if for all $k \in K$ and $g_0B \in G/B$:*
$$\mathbf{m}(kg_0B) = \mathbf{m}(g_0B).$$
2. $\mathbf{m} \circ \underline{\sigma} = \overline{\mathbf{m}}$.

**Lemma 2.2.26.** *Let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space and let $\mathbf{m}$ be an admissible smooth function on $G/B$. Then, $\Sigma_{\mathbf{m}}$ is a symmetric operator on $\Gamma_c^\infty(G/B, E_{\tilde{\chi}})$ for the inner product (2.12) for any choice of positive half-density, that is, for all $\varphi, \psi \in \Gamma_c^\infty(G/B, E_{\tilde{\chi}})$,*
$$\langle \varphi, \Sigma_{\mathbf{m}}\psi \rangle = \langle \Sigma_{\mathbf{m}}\varphi, \psi \rangle\,.$$

*Proof.* We will use the expression (2.23) of $\Sigma_{\mathbf{m}}$ on $(B, \tilde{\chi})-$equivariant functions on $G$. Let $\hat{\varphi}, \hat{\psi} \in \mathcal{C}^\infty(G)_c^{(B,\tilde{\chi})}$, and let $\mu$ be a half-density on $G/B$ and $\rho$ the associated $\rho-$function. We have, from the expression (2.15) for the inner product,
$$\begin{aligned}
\langle \hat{\varphi}, \Sigma_{\mathbf{m}}\hat{\psi} \rangle &= \int_{G/B} \hat{\varphi}(g)\,\overline{\Sigma_{\mathbf{m}}\hat{\psi}(g)}\,\rho(g)^{-1}\,\mu^2(gB)\\
&= \int_{G/B} \hat{\varphi}(g)\,\overline{\mathbf{m}(gB)}\,\overline{\hat{\psi}(\sigma(g))}\,\rho(g)^{-1}\,\mu^2(gB)\\
&= \int_{G/B} \mathbf{m}(gB)\hat{\varphi}(\sigma(g))\,\overline{\hat{\psi}(g)}\,\rho(\sigma(g))^{-1}\,\frac{\rho(\sigma(g))}{\rho(g)}\,\mu^2(\sigma(g)B)\\
&= \int_{G/B} \Sigma_{\mathbf{m}}\hat{\varphi}(gB)\,\overline{\hat{\psi}(g)}\,\rho(\sigma(g))^{-1}\,\mu^2(gB) = \langle \Sigma_{\mathbf{m}}\hat{\varphi}, \hat{\psi} \rangle,
\end{aligned}$$
where we have used the fact that $\mathbf{m}$ is admissible, and the fact that $\sigma$ is an involutive automorphism of $G$ for the transformation of $\mu$ under the change of variable. □

**Lemma 2.2.27.** *Let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space and let $\mathbf{m}$ be an admissible smooth function on $G/B$. Then, the map*
$$\Omega_{\mathbf{m}} : G \to \operatorname{End}\left(\Gamma_c^\infty\left(G/B, E_{\tilde{\chi}}\right)\right)\;;\; g \mapsto \Omega_{\mathbf{m}}(g) := U_\chi(g)\,\Sigma_{\mathbf{m}}\,U_\chi(g)^{-1}$$
*satisfies the following properties:*

1. *$\Omega_{\mathbf{m}}$ is $G-$equivariant, that is, for all $g, g_0 \in G$,*

$$\Omega_{\mathbf{m}}(gg_0) = U_\chi(g)\,\Omega_{\mathbf{m}}(g_0)\,U_\chi(g)^{-1};$$

2. *for all $g \in G$, $\Omega_{\mathbf{m}}(g)$ is symmetric on $\Gamma_c^\infty(G/B, E_{\tilde{\chi}})$ for the inner product (2.12) for any choice of positive half-density;*

3. *for all $k \in K$:*

$$U_\chi(k)\Sigma_{\mathbf{m}} = \Sigma_{\mathbf{m}} U_\chi(k).$$

*Proof.* As in the case of $\Omega$, the first claim follows immediately from the definition. The last one is more easily proved by realizing the operators on $(B, \tilde{\chi})-$equivariant functions on $G$. Let $\hat{\varphi} \in \mathcal{C}^\infty(G)_c^{(B,\tilde{\chi})}$ and $g \in G$. For all $k \in K$, we have, by the $K-$invariance of $\mathbf{m}$ and the fact that $\sigma$ is the identity on $K$:

$$\begin{aligned}(U_\chi(k)\Sigma_{\mathbf{m}}\hat{\varphi})(g) &= \mathbf{m}(k^{-1}gB)\hat{\varphi}(\sigma(k^{-1}g))\\ &= \mathbf{m}(gB)\hat{\varphi}(k^{-1}\sigma(g)) = (\Sigma_{\mathbf{m}}U_\chi(k)\hat{\varphi})(g).\end{aligned}$$

□

In view of the those results, we get a weaker but similar statement to Proposition 2.2.18, which allows to attach an operator to each point of the symmetric space $G/K$.

**Proposition 2.2.28.** *Let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space and let $\mathbf{m}$ be an admissible smooth function on $G/B$. Then, the map*

$$\Omega_{\mathbf{m}} : G/K \to \mathrm{End}\left(\Gamma_c^\infty(G/B, E_{\tilde{\chi}})\right)\ ;\ gK \mapsto U_\chi(g)\,\Sigma_{\mathbf{m}}\,U_\chi(g)^{-1}$$

*is well-defined and $G-$equivariant, and we have, for all $\varphi \in \Gamma_c^\infty(G/B, E_{\tilde{\chi}})$, $gK \in G/K$ and $g_0B \in G/B$:*

$$(\Omega_{\mathbf{m}}(gK)\varphi)(g_0B) = \mathbf{m}(g^{-1}g_0B)\ \left(g\sigma(g^{-1})\right)\cdot\tilde{\sigma}\left(\varphi(g\sigma(g^{-1}g_0)B)\right). \tag{2.24}$$

*Proof.* Because of Property 3 in Lemma 2.2.27, we have $\Omega_{\mathbf{m}}(gh) = \Omega_{\mathbf{m}}(g)$ for all $g \in G$ and $k \in K$, so $\Omega_{\mathbf{m}}$ induces a map on $G/K$ which we still denote by $\Omega_{\mathbf{m}}$. The $G-$equivariance still holds after passing to the quotient since the left and right multiplications in $G$ commute.
The expression (2.24) follows by the explicit formulas (2.10) and (2.22) for $U_\chi(g)$ and $\Sigma_{\mathbf{m}}$, and from the fact that for all $g \in G$ and $[g_0, z] \in E_{\tilde{\chi}}$:

$$g\cdot\tilde{\sigma}\left(g^{-1}\cdot[g_0, z]\right) = \left[g\sigma(g^{-1}g_0), z\right] = \left(g\sigma(g^{-1})\right)\cdot\tilde{\sigma}([g_0, z]).$$

□

*Remark* 2.2.29. For a later use, let us give the expression of $\Omega_{\mathbf{m}}$ on $(B, \tilde{\chi})$-equivariant functions. For all $gK \in G/K$, $\hat{\varphi} \in \mathcal{C}^\infty(G)_c^{(B,\tilde{\chi})}$ and $g_0 \in G$, we have, from (2.14) and (2.23),

$$(\Omega_{\mathbf{m}}(gK)\hat{\varphi})(g_0) = \mathbf{m}(g^{-1}g_0B)\ \hat{\varphi}\left(g\sigma(g^{-1}g_0)\right).$$

From this expression, we also get that, in the local case $G = QB$, for all $gK \in G/K$, $\tilde{\varphi} \in \mathcal{C}_c^\infty(Q)$ and $q \in Q$,

$$(\Omega_{\mathbf{m}}(gK)\tilde{\varphi})(q) = \mathbf{m}\left((g^{-1}g_0)^Q\right)\ \tilde{\chi}\left((g\sigma(g^{-1}q))^B\right)^{-1}\tilde{\varphi}\left((g\sigma(g^{-1}q)^Q\right).$$

$\triangleleft$

In a similar way as we did for $\Omega$, we can use this family of operators to construct a quantization map for the functions on $G/K$. Let $f \in \mathcal{C}_c^\infty(G/K)$, we define a linear operator

$$\Omega_{\mathbf{m}}(f) : \Gamma_c^\infty\left(G/B, E_{\tilde{\chi}}\right) \to \Gamma^\infty\left(G/B, E_{\tilde{\chi}}\right) \tag{2.25}$$

by the formula given, for each $\varphi \in \Gamma_c^\infty\left(G/B, E_{\tilde{\chi}}\right)$ and $g_0B \in G/B$, by:

$$(\Omega_{\mathbf{m}}(f)\varphi)(g_0B) := \int_{G/K} f(x)\ (\Omega_{\mathbf{m}}(x)\varphi)(g_0B)\ d_{G/K}(x), \tag{2.26}$$

where $d_{G/K}(x)$ is the $G-$invariant measure on $G/K$.

**Lemma 2.2.30.** *Let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space and let $\mathbf{m}$ be an admissible smooth function on $G/B$. The map $f \mapsto \Omega_{\mathbf{m}}(f)$ is $G-$equivariant in the sense that, for all $f \in \mathcal{C}_c^\infty(G/K)$ and $g \in G$,*

$$\Omega_{\mathbf{m}}({}^g f) = U(g)\,\Omega(f)\,U(g)^{-1},$$

*where for all $g_0K \in G/K$, $({}^g f)(g_0K) := f(g^{-1}g_0K)$.*

*Proof.* This follows from the $G-$invariance of $d_{G/K}$ and Property (1) of Proposition 2.2.28. $\square$

The next property will be useful later on, to show that, when it makes sense, $\Omega_{\mathbf{m}}(f)^* = \Omega_{\mathbf{m}}(\overline{f})$.

**Lemma 2.2.31.** *Let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space and let $\mathbf{m}$ be an admissible smooth function on $G/B$. Then, for all $f \in \mathcal{C}_c^\infty(G/K)$ and $\varphi, \psi \in \Gamma_c^\infty(G/B, E_{\tilde{\chi}})$,*

$$\langle \varphi, \Omega_{\mathbf{m}}(f)\psi\rangle = \left\langle \Omega_{\mathbf{m}}(\overline{f})\varphi, \psi\right\rangle,$$

*where $\langle\cdot,\cdot\rangle$ denotes the inner product (2.12) for any choice of half-density on $G/B$.*

*Proof.* This follows from statement 2 in Lemma 2.2.27, and from the explicit expression (2.26). $\square$

By identifying the smooth section $\Omega_{\mathbf{m}}(f)\varphi$ with the corresponding generalized section (see Example 1.3.8) of $E_{\tilde{\chi}}$, $\Omega_{\mathbf{m}}(f)$ defines a general operator from $E_{\tilde{\chi}}$ to itself (see Section 1.4, Definition 1.4.1).

**Definition 2.2.32.** *Let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space and let $\mathbf{m}$ be an admissible smooth function on $G/B$. The* quantization map *of $G/K$ induced by $(B, \chi, \mathbf{m})$ is the linear map*

$$\Omega_{\mathbf{m}} : \mathcal{C}_c^\infty(G/K) \to \mathcal{L}_b\big(\mathcal{D}(G/B, E_{\tilde{\chi}}) \to \mathcal{D}'(G/B, E_{\tilde{\chi}})\big)\ ;\ f \mapsto \Omega_{\mathbf{m}}(f), \quad (2.27)$$

*where $\Omega_{\mathbf{m}}(f)$ is defined by (2.25) and (2.26).*[9] *We will call it simply the quantization map when the context is clear.*

*Remark* 2.2.33. Although $\Omega_{\mathbf{m}}(gK)$ might not be a bounded operator, it is a legitimate question to ask whether $\Omega_{\mathbf{m}}(f)$ extends to a honest bounded operator on $\mathcal{H}_\chi$. Also, we would like to know to which extent the domain of the quantization map $\Omega_{\mathbf{m}}$ can be enlarged to a larger space than compactly supported functions, such as the space of square-integrable functions. Let us however postpone that discussion to Section 2.3 and stick with general operators for the moment, in order to make a link with the previous chapter. ◁

### 2.2.4 The family of geometric morphisms of the quantization map

We will now see that the construction of the quantization map $\Omega_{\mathbf{m}}$ exactly fits into the setting of Section 1.8. Indeed, the operators $\Omega_{\mathbf{m}}(x)$ and $\Omega_{\mathbf{m}}(f)$ of our quantization map correspond to the pullback operators associated in Subsection 1.8.1 to a family of geometric morphisms. Later on, this identification will allow us to use the techniques that we developed in the previous chapter to compute the trace of (compositions of) $\Omega_{\mathbf{m}}(f)$.

More precisely, let $\mathbf{m}$ be an admissible smooth function on $G/B$ and let us consider the smooth map

$$\begin{aligned} \tau : G/K \times G/B \to G/B \\ ;\ (gK, g_0B) \mapsto \tau_{gK}(g_0B) := g\sigma\left(g^{-1}g_0\right)B \end{aligned} \quad (2.28)$$

and, for each $gK \in G/K$ and $g_0B \in G/B$, the linear map

$$\begin{aligned} r_{gK}(g_0B) : (E_{\tilde{\chi}})_{\tau_{gK}(g_0B)} \to (E_{\tilde{\chi}})_{g_0B} \\ ;\ [g_1, z] \mapsto \mathbf{m}(g^{-1}g_0B)\ \left(g\sigma(g^{-1})\right) \cdot [\sigma(g_1), z]. \end{aligned} \quad (2.29)$$

Notice that $r_{gK}(g_0B)$ is well-defined since any element of $(E_{\tilde{\chi}})_{\tau_{gK}(g_0B)}$ is of the form $[g\sigma\left(g^{-1}g_0\right), z]$ for some $z \in \mathbb{C}$ and that $\left(g\sigma(g^{-1})\right) \cdot [\sigma\left(g\sigma\left(g^{-1}g_0\right)\right), z] = [g_0, z] \in (E_{\tilde{\chi}})_{g_0B}$. Since $r$ depends smoothly on $gK$ and $g_0B$, the data $\underline{\tau} = (\tau, r)$ gives a smooth family of geometric morphisms of $E_{\tilde{\chi}}$ parametrized by $G/K$ (see Definition 1.8.1). For $gK \in G/K$, we see from (2.24) and (2.29) that

[9] Notice that in the case of $\mathbf{m} = 1$, we recover the quantization map $\Omega$ of Definition 2.2.22.

the pullback operator defined by (1.57) coincides with $\Omega_{\mathbf{m}}(gK)$. Now, denote by $|d_{G/K}(x)|$ the smooth density corresponding to the $G-$invariant measure on $G/K$. It is non-vanishing, so any compactly supported smooth density on $G/K$ is of the form $f|d_{G/K}(x)|$ for some $f \in \mathcal{C}_c^\infty(G/K)$. Then, the operator associated to $f|d_{G/K}(x)|$ by the expression (1.59) is equal to $\Omega_{\mathbf{m}}(f)$, given by (2.26).

This rephrasing in terms of geometric morphisms allows to highlight some of the geometric structure underlying the quantization map. In analogy with a group action, the map $\tau$ may be considered as an action of the symmetric space $G/K$ on $G/B$ in the sense of the following lemma.

**Lemma 2.2.34.** *Let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space. The smooth map*

$$\begin{aligned} \tau : G/K \times G/B &\to G/B \\ ;\ (gK, g_0B) &\mapsto \tau_{gK}(g_0B) := g\sigma\left(g^{-1}g_0\right)B \end{aligned} \tag{2.30}$$

*defines a $G-$equivariant action of the symmetric space $G/K$ on $G/B$ in the sense that, for all $x, y \in G/K$, we have:*

1. $(\tau_x)^2 = \mathrm{Id}_{G/B}$,
2. $\tau_x \circ \tau_y \circ \tau_x = \tau_{s_x(y)}$,
3. $\alpha_g \circ \tau_x \circ \alpha_{g^{-1}} = \tau_{g\cdot x}$ *for all* $g \in G$,

*where $\alpha$ is the natural left action (2.5) of $G$ on $G/B$.*

*Proof.* Let $g_xK, g_yK \in G/K$, $g_0B \in G/B$ and $g \in G$. The results follow from explicit computations and using the fact that $\sigma$ is an involutive automorphism of $G$.

1. $\tau_{g_xK}(\tau_{g_xK}(g_0B)) = g_x\sigma\left(g_x^{-1}g_x\sigma\left(g_x^{-1}g_0\right)\right)B = g_0B$;
2. Since $s_{g_xK}(g_yK) = g_x\sigma\left(g_x^{-1}g_y\right)K$, we have
$$\begin{aligned} (\tau_{g_xK} \circ \tau_{g_yK} \circ \tau_{g_xK})(g_0B) &= g_x\sigma\left(g_x^{-1}g_y\sigma\left(g_y^{-1}g_x\sigma\left(g_x^{-1}g_0\right)\right)\right)B \\ &= g_x\sigma\left(g_x^{-1}g_y\right)\sigma\left(\sigma\left(g_y^{-1}g_x\right)g_x^{-1}g_0\right)B \\ &= \tau_{s_{g_xK}(g_yK)}(g_0B); \end{aligned}$$
3. $(\alpha_g \circ \tau_x \circ \alpha_{g^{-1}})(g_0B) = g\,g_x\sigma\left(g_x^{-1}\,g^{-1}g_0\right)B = \tau_{gg_xK}(g_0B)$. □

**Definition 2.2.35.** *Let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space. The map (2.30) is called the* symmetric action of $G/K$ on $G/B$.

The map $r$ then corresponds to a lift to the vector bundle $E_{\tilde{\chi}}$ of the symmetric action $\tau$. Although, as noted in Lemma 2.2.27, it does not give a genuine representation of symmetric space for generic admissible $\mathbf{m}$, the $G-$equivariance is always satisfied.

*Remark* 2.2.36. In the local case $G = QB$, we identify $G/B$ with $Q$ and the bundle $E_{\tilde{\chi}}$ with the trivial bundle over $Q$. The morphisms $r_{gK}(q)$ are thus simply given by a complex number. Recall from Remark 2.2.29 that, for all $gK \in G/K$, $\tilde{\varphi} \in \mathcal{C}_c^\infty(Q)$ and $q \in Q$, we have

$$(\Omega_{\mathbf{m}}(gK)\tilde{\varphi})(q) = \mathbf{m}\left((g^{-1}q)^Q\right)\ \tilde{\chi}\left((g\sigma(g^{-1}q))^B\right)^{-1} \tilde{\varphi}\left((g\sigma(g^{-1}q)^Q\right). \tag{2.31}$$

Therefore, we have for all $gK \in G/K$, $q \in Q$:

$$\begin{aligned} \tau_{gK}(q) &= (g\sigma(g^{-1}q))^Q \\ r_{gK}(q) &= \mathbf{m}\left((g^{-1}q)^Q\right)\ \tilde{\chi}\left((g\sigma(g^{-1}q))^B\right)^{-1}. \end{aligned} \tag{2.32}$$

$\lhd$

Building on the results of the previous chapter, we get the following Proposition.

**Proposition 2.2.37.** *Let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space and let $\mathbf{m}$ be an admissible smooth function on $G/B$. Suppose that the action $\tau$ defined by (2.28) is locally transitive. Then, for all $f \in \mathcal{C}_c^\infty(G/K)$, the operator $\Omega_{\mathbf{m}}(f)$ has a smooth kernel.*

*Proof.* In view of the previous discussion identifying $\Omega_{\mathbf{m}}(f)$ with (1.58) for the family of geometric morphisms $(\tau, r)$ given by (2.28) and (2.29), this is an immediate consequence of Proposition 1.8.5. $\square$

Let us now comment on the requirement that the action $\tau$ is locally transitive, an essential feature to be able to apply the results of the previous chapter. As the following counter-example shows, local transitivity needs not hold in general.

*Example* 2.2.38. Let us consider the cylinder, as introduced in Example 2.2.4. Recall that, as a manifold, $M \simeq S^1 \times \mathbb{R}$ and the symmetric structure is given, for all $(e^{ia}, n), (e^{ia'}, n') \in M$, by

$$s_{(e^{ia},n)}(e^{ia'}, n') = (e^{i(2a-a')}, 2x\cos(a - a') - x').$$

For this example, we consider $B = K$, so $M$ acts on itself (i.e. $\tau = s$). From the latter expression, we can see that the action of $M$ on itself is not locally transitive. Indeed, identifying the tangent space to $M$ with $\mathbb{R}^2$, for every $(e^{ia}, n) \in M$ and

$$(X_a, X_n, 0, 0) \in T_{((e^{ia},n),(1,0))}(M \times M),$$

we have:

$$\tau_{*_{((e^{ia},n),(1,0))}}(X_a, X_n, 0, 0) = (2X_a, 2X_n\cos(a) - 2X_a\, x\, \sin(a)),$$

which is not surjective as soon as $e^{ia} = e^{i\frac{\pi}{2}}$. In order to suggest a more geometric intuition of what is going on, the situation is pictured in Figure 2.2.

$\diamondsuit$

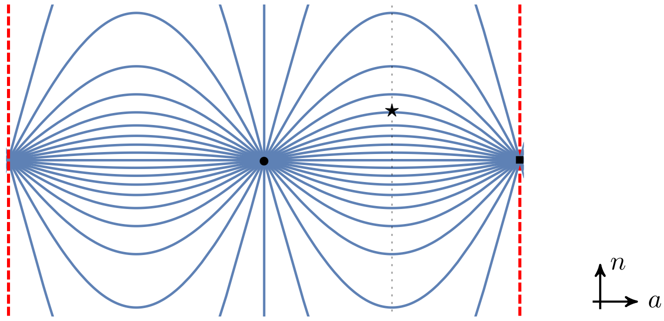



Figure 2.2: Picture of the symmetric space $M$ (the horizontal axis corresponds to the coordinate $a$, and the vertical one to $n$). The plain lines are the geodesics issued from the point $\bullet = (0,0)$ corresponding to the Loos connection. $\star = (e^{i\frac{\pi}{2}}, n)$, $\blacksquare = s_{(e^{i\frac{\pi}{2}},n)}(0,0)$ and the thin dotted line ($a = \frac{\pi}{2}$) shows the midpoints of the geodesics between the points $\bullet$ and $\blacksquare$. The dashed lines ($a = \pm\pi$) represent points $z \in M$ such that there is no $y \in M$ such that $s_y(0,0) = z$. In this situation, $s$ fails to be locally transitive because moving $\star$ vertically along the dotted line does not move $\blacksquare$. Thus, the differential of $s$ with respect to $\star$ cannot be surjective.

One possible way to ensure local transitivity is to require that any pair of points in $M$ admits a midpoint in the following sense.

**Definition 2.2.39.** *Let $(M, s)$ be a symmetric space. For $x, y \in M$, a point $z$ satisfying $s_z(x) = y$ is called a* midpoint *of $x$ and $y$. A* midpoint map *on $M$ is a smooth map*

$$M \times M \to M \; ; \; (x, y) \mapsto \mathrm{mid}(x, y)$$

*such that, for all $x, y \in M$, $\mathrm{mid}(x, y)$ is a midpoint of $x$ and $y$, that is:*

$$s_{\,\mathrm{mid}(x,y)}(x) = y.$$

Midpoints in the present context of symmetric spaces have first been studied by Qian [Qia97]. As can be seen in Example 2.2.38, they need not exist for generic pairs of points, neither should they be unique. Also, as it is the case on the circle, there might be topological obstructions to the smoothness of a midpoint map. The relation between the existence of a midpoint map and properties of the exponential map has been analyzed by Voglaire [Vog11]. Rephrasing [Vog14, Theorem 1.1] and [Vog11, Theorem 2.2.20], we get the following important characterization.

**Theorem 2.2.40.** *Let $(M, s)$ be a connected symmetric space. Then, the following conditions are equivalent:*

1. *there exists a midpoint map on $M$;*

2. *any two points in $M$ have at most one midpoint;*

3. *there exists $x \in M$ such that the exponential map at $x$ of the Loos connection is a global diffeomorphism;*

4. *the exponential map at any point of $M$ of the Loos connection is a global diffeomorphism.*

*Remark* 2.2.41. In particular, this implies that if every pair of points has at least one midpoint, then they have exactly one. It also implies that if a midpoint map exists, then it is unique, and every two points have a unique midpoint. $\lhd$

**Proposition 2.2.42.** *Let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space. Suppose that there exists a midpoint map on $G/K$. Then, the smooth map $\tau$ defined by (2.28) is locally transitive. Therefore, for any admissible smooth function $\mathbf{m}$ on $G/B$, the operator $\Omega_{\mathbf{m}}(f)$ is smooth for every $f \in \mathcal{C}_c^\infty(G/K)$.*

*Proof.* Let $g_0K \in G/K$, $g_1B \in G/B$ and define $g_2 := g_0\sigma(g_0^{-1}g_1)$. We have thus $\tau_{g_0K}(g_1B) = g_2B$. Let $X_0 \in T_{g_2B}(G/B)$ and $X \in \mathfrak{g}$ such that $X_0 = \frac{d}{dt}\big|_0 \exp(-tX)g_2B$ (such an $X$ exists since $G/B$ is a homogeneous space). For $\epsilon > 0$ sufficiently small, consider the path $x : \ I_\epsilon := ]-\epsilon, \epsilon[ \to M$ defined, for $t \in I_\epsilon$, by

$$x(t) := \operatorname{mid}\left(g_1K, \exp(-tX)g_2K\right).$$

It is smooth since mid is and, by uniqueness of the midpoints (Remark 2.2.41), $x(0) = g_0K$. Let $g : \ I_\epsilon \to G$ be a smooth lift of $x$ such that $g(0) = g_0$ – which exists since $G \to G/K$ is a $K-$principal bundle. Then, by definition of $x(t)$, we have:

$$\begin{aligned}\exp(-tX)g_2K &= s_{x(t)}(g_1K) = s_{g(t)K}(g_1K)\\ &= g(t)\,\sigma(g(t)^{-1}g_1)K,\end{aligned}$$

so for each $t \in I_\epsilon$, there exists $k(t) \in K$ such that

$$\exp(-tX)g_2k(t) = g(t)\,\sigma(g(t)^{-1}g_1).$$

Then, since $K \subset B$, we have:

$$\begin{aligned}\tau_{x(t)}(g_1B) &= \tau_{g(t)K}(g_1B) = g(t)\,\sigma(g(t)^{-1}g_1)B\\ &= \exp(-tX)g_2k(t)B = \exp(-tX)g_2B.\end{aligned}$$

Setting $Y := \frac{d}{dt}\big|_0 \ x(t) \in T_{g_0K}(G/K)$ the latter equation shows that

$$\tau_{*(g_0K,g_1B)}(Y, 0) = X_0,$$

which proves the local transitivity of $\tau$.

The last part of the claim follows from Proposition 2.2.37. $\square$

*Remark* 2.2.43. It is worth mentioning that the existence of a midpoint map is not a necessary condition for the local transitivity. Indeed, coming back to the Example 2.2.4 of the cylinder, but this time choosing $B = NK$, we have $G/B = Q \simeq S^1$ and, for all $(e^{ia}, n) \in M$ and $e^{ia_0} \in Q$,

$$\tau_{(e^{ia},n)}(e^{ia_0}) = e^{i(2a-a_0)},$$

which is locally transitive. ◁

## 2.3 Hilbert-Schmidt and trace-class operators

We now come back to the realm of Hilbert spaces and pause for a moment to review some fundamental facts about Hilbert-Schmidt and trace-class operators. The two notions are very much related, leading to a variety of ways to introduce them, depending for instance on which one is introduced first. For a complete exposition of the subject, we refer to Conway [Con00, Chapter 3, §18]. Although his approach is slightly different than ours, all the equivalences are stated and proved. Regarding the study of traces of operators, Pietsch [Pie14] gives an illuminating review of its history, which shows that it extends far beyond Hilbert spaces.

At the end of the section, we briefly discuss how our quantization map fits into the setting of Hilbert-Schmidt operators.

Throughout this section, let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and denote by $\mathcal{L}(\mathcal{H})$ the space of bounded linear operators on $\mathcal{H}$ and by $|| \cdot ||$ the operator norm on $\mathcal{L}(\mathcal{H})$.

**Proposition 2.3.1.** *Let $A \in \mathcal{L}(\mathcal{H})$ and $\{e_i\}_{i\in I}$ be an orthonormal basis of $\mathcal{H}$ such that $\sum_{i\in I} \langle Ae_i, Ae_i\rangle < +\infty$. Then, for all orthonormal basis $\{f_i\}_{i\in I}$ of $\mathcal{H}$:*

$$\sum_{i\in I} \langle Af_i, Af_i\rangle = \sum_{i\in I} \langle Ae_i, Ae_i\rangle .$$

This justifies the following definition.

**Definition 2.3.2.** *An operator $A \in \mathcal{L}(\mathcal{H})$ is called a* Hilbert-Schmidt operator *if there exists an orthonormal basis $\{e_i\}_{i\in I}$ of $\mathcal{H}$ such that*

$$\sum_{i\in I} \langle Ae_i, Ae_i\rangle < +\infty.$$

*The set of Hilbert-Schmidt operators on $\mathcal{H}$ is denoted by $\mathcal{L}^2(\mathcal{H})$ and we define*

$$|| \cdot ||_{\mathcal{L}^2} : \mathcal{L}^2(\mathcal{H}) \to \mathbb{R}\ ;\ A \mapsto ||A||_{\mathcal{L}^2} := \left(\sum_{i\in I} \langle Ae_i, Ae_i\rangle\right)^{1/2},$$

*where* $\{e_i\}_{i\in I}$ *is any basis of* $\mathcal{H}$.[10]

**Theorem 2.3.3.** *We have:*

1. $\mathcal{L}^2(\mathcal{H})$ *is a vector subspace of* $\mathcal{L}(\mathcal{H})$ *and* $||\cdot||_{\mathcal{L}^2}$ *is a norm on* $\mathcal{L}^2(\mathcal{H})$ *which turns it into a Banach space;*

2. *for all* $A \in \mathcal{L}^2(\mathcal{H})$, $||A|| \leq ||A||_{\mathcal{L}^2}$*;*

3. *for all* $A \in \mathcal{L}^2(\mathcal{H})$*, its adjoint* $A^*$ *belongs to* $\mathcal{L}^2(\mathcal{H})$ *and* $||A||_{\mathcal{L}^2} = ||A^*||_{\mathcal{L}^2}$*;*

4. $\mathcal{L}^2(\mathcal{H})$ *is a two-sided ideal of* $\mathcal{L}(\mathcal{H})$*, that is, for all* $A \in \mathcal{L}^2(\mathcal{H})$ *and* $T \in \mathcal{L}(\mathcal{H})$, $AT \in \mathcal{L}^2(\mathcal{H})$ *and* $TA \in \mathcal{L}^2(\mathcal{H})$*;*

5. *for all* $A \in \mathcal{L}^2(\mathcal{H})$ *and* $T \in \mathcal{L}(\mathcal{H})$, $||AT||_{\mathcal{L}^2} \leq ||T||\,||A||_{\mathcal{L}^2}$ *and* $||TA||_{\mathcal{L}^2} \leq ||T||\,||A||_{\mathcal{L}^2}$.

The subset of products of Hilbert-Schmidt operators turns out to be as important as Hilbert-Schmidt operators themselves. This definition was introduced by Schatten and von Neumann [SvN46].

**Definition 2.3.4.** *An operator* $A \in \mathcal{L}(\mathcal{H})$ *is called a* trace-class operator *if it is the product of two Hilbert-Schmidt operators. The set of trace-class operators on* $\mathcal{H}$ *is denoted by* $\mathcal{L}^1(\mathcal{H})$.

**Theorem 2.3.5.** *We have:*

1. $\mathcal{L}^1(\mathcal{H})$ *is a vector subspace of* $\mathcal{L}(\mathcal{H})$*;* [11]

2. *for all* $A \in \mathcal{L}^1(\mathcal{H})$*, its adjoint* $A^*$ *belongs to* $\mathcal{L}^1(\mathcal{H})$*;*

3. $\mathcal{L}^1(\mathcal{H})$ *is a two-sided ideal of* $\mathcal{L}(\mathcal{H})$*, that is, for all* $A \in \mathcal{L}^1(\mathcal{H})$ *and* $T \in \mathcal{L}(\mathcal{H})$, $AT \in \mathcal{L}^1(\mathcal{H})$ *and* $TA \in \mathcal{L}^1(\mathcal{H})$*;*

4. *for all* $A \in \mathcal{L}(\mathcal{H})$, $A \in \mathcal{L}^1(\mathcal{H})$ *if and only if for every orthonormal basis* $\{e_i\}_{i\in I}$ *of* $\mathcal{H}$,
$$\sum_{i\in I} |\langle Ae_i, e_i\rangle| < +\infty. \tag{2.33}$$
*In that case, the number* $\sum_{i\in I} \langle Ae_i, e_i\rangle$ *is independent on the basis.*

*Remark* 2.3.6. Some authors take (4) in Theorem 2.3.5 as a definition of trace-class operators. However, it should be emphasized that, unlike in Proposition 2.3.1, the condition (2.33) is not independent on the basis for a generic bounded operator. That is, there exists $A \in \mathcal{L}(\mathcal{H})$ which is not trace-class such that

[10] By Proposition 2.3.1, $||\cdot||_{\mathcal{L}^2}$ is independent of the choice of basis.

[11] As for Hilbert-Schmidt operators, it is possible to define a norm $||\cdot||_{\mathcal{L}^1}$ on $\mathcal{L}^1(\mathcal{H})$, which turns it into a Banach space, and which satisfies the same properties as $||\cdot||_{\mathcal{L}^2}$ in Theorem 2.3.3. The trace functional (to be defined in a moment) turns out to be continuous with respect to that norm.

(2.33) holds for one basis but not for all. This is one reason why we take the definition as a product of two Hilbert-Schmidt operators, which seems more elegant and more practical to use. ◁

We can now define the trace of a trace-class operator which, by analogy with the finite dimensional case, justifies the terminology and, by the previous Theorem, is independent on the basis in the definition.

**Definition 2.3.7.** *The* trace *of an operator* $A \in \mathcal{L}^1(\mathcal{H})$ *is the number*

$$\mathrm{Tr}\,(A) := \sum_{i \in I} \langle Ae_i, e_i \rangle ,$$

*where* $\{e_i\}_{i \in I}$ *is any orthonormal basis of* $\mathcal{H}$*.*

**Theorem 2.3.8.** *We have:*

1. $\mathrm{Tr} : \mathcal{L}^1(\mathcal{H}) \to \mathbb{C}$ *is a linear functional;*
2. *for all* $A \in \mathcal{L}^1(\mathcal{H})$ $\mathrm{Tr}\,(A^*) = \overline{\mathrm{Tr}\,(A)}$*.*
3. *for all* $A \in \mathcal{L}^1(\mathcal{H})$ *and* $T \in \mathcal{L}(\mathcal{H})$*,* $\mathrm{Tr}\,(AT) = \mathrm{Tr}\,(TA)$*.*

The trace allows to define an inner product on $\mathcal{L}^2(\mathcal{H})$ which turns it into a separable Hilbert space.

**Theorem 2.3.9.** *The space* $\mathcal{L}^2(\mathcal{H})$ *endowed with the inner product defined, for every* $A, B \in \mathcal{L}^2$*, by*

$$\langle A, B \rangle_{\mathcal{L}^2} := \mathrm{Tr}\,(B^* A)$$

*is a separable Hilbert space whose norm coincides with* $|| \cdot ||_{\mathcal{L}^2}$*.*

In the case where $\mathcal{H}$ is the space of square-integrable functions on a measurable space, we have an important characterization of Hilbert-Schmidt operators on $\mathcal{H}$. We refer to [RS81, Theorem VI.23] for a proof of this result.

**Theorem 2.3.10.** *Let* $M$ *be a manifold,* $\mu$ *a measure on* $M$ *and* $\mathcal{H} = L^2(M, \mu)$ *the Hilbert space of square-integrable functions on* $M$ *with respect to* $\mu$*. Then, an operator* $A \in \mathcal{L}(\mathcal{H})$ *is Hilbert-Schmidt if and only if there exists a function* $K \in L^2(M \times M, \mu \times \mu)$ *such that, for all* $f \in \mathcal{H}$*:*

$$(Af)(x) = \int_M K(x, y) f(y)\, d\mu(y).$$

*In that case, we have*

$$||A||^2_{\mathcal{L}^2} = \int_{M \times M} |K(x, y)|^2 \, d\mu(x)\, d\mu(y).$$

As we have already mentioned in subsection 1.4.4, there is no such nice characterization of a trace-class operator. However, when we know that an operator is trace-class and has an almost-everywhere continuous kernel, let us recall that we have the following trace formula. It follows from [Bri91, Corollary 3.2].

**Theorem 2.3.11.** *Let $\mu$ be a measure on a manifold $M$, and let $K$ be a trace-class operator on $L^2(M,\mu)$. If the kernel $K(x,y)$ is continuous at $(x,y)$ for almost every $x$, then*

$$\mathrm{Tr}\,(K) = \int_M K(x,x)d\mu(x).$$

We end up this section by discussing how our quantization procedure fits into the setting of Hilbert-Schmidt operators. Recall that, in Definition 2.2.32, the quantization map $\Omega_{\mathbf{m}}$ is defined as a general operator $\mathcal{L}_b\big(\mathcal{D}(G/B,E_{\tilde{\chi}}) \to \mathcal{D}'(G/B,E_{\tilde{\chi}})\big)$. However, in many interesting cases, it turns out that $\Omega_{\mathbf{m}}(f)$ can be extended to a Hilbert-Schmidt operator on $\mathcal{H}_\chi$. As we will see in the next section, it gives a powerful setting to compute the inverse of $\Omega_{\mathbf{m}}$. It is therefore useful to study whether the image of $\Omega_{\mathbf{m}}$ is contained in $\mathcal{L}^2(\mathcal{H}_\chi)$, and the following result gives a first sufficient condition.

**Proposition 2.3.12.** *Let $(G,K,\sigma,B,\chi)$ be a local nearly-quantum symmetric space and $\mathbf{m}$ an admissible smooth function on $G/B$. Suppose that $G/B$ is compact and that the action of $G/K$ on $G/B$ is locally transitive. Then, for every $f \in \mathcal{C}_c^\infty(G/K)$, the operator $\Omega_{\mathbf{m}}(f)$ extends to an Hilbert-Schmidt operator on $\mathcal{H}_\chi$.*

*Proof.* Recall from Definition 2.2.32 that for all $f \in \mathcal{C}_c^\infty(G/K)$, $\Omega_{\mathbf{m}}(f)$ is a general operator of $E_{\tilde{\chi}}$, that is, an element of $\mathcal{L}_b\big(\mathcal{D}(G/B,E_{\tilde{\chi}}) \to \mathcal{D}'(G/B,E_{\tilde{\chi}})\big)$. By Proposition 2.2.37, the kernel of $\Omega_{\mathbf{m}}(f)$ is smooth, and it is therefore square-integrable on $G/B \times G/B$ since $G/B$ is compact. Since we are in the local case, $G/B \simeq Q$ and $\mathcal{H}_\chi \simeq L^2(Q)$ by Remark 2.2.11. We can thus apply Theorem 2.3.10, which implies that $\Omega_{\mathbf{m}}(f)$ extends to a Hilbert-Schmidt operator on $\mathcal{H}_\chi$. □

## 2.4 Symbol map, deformed product and three-point kernel

Now we have built a quantization map, that associates operators to functions, we would like to define a *symbol map*, which goes the other way around and "dequantize" a quantized operator by assigning to it a function – its so-called *symbol*. Ideally, we would like those two maps to be inverse of each other, in the sense that the symbol of the operator quantizing a given function would be precisely that function. However, this won't be true in general and the defect of the inversion procedure will be encoded by the so-called *Berezin transform*.

Throughout this section, let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space, let $\mathbf{m}$ be an admissible smooth function on $G/B$ and denote by $d_{G/K}(x)$ the $G-$invariant measure on $G/K$. We further make the two following assumptions.

**Hypothesis.**

(H1) the symmetric action $\tau$ of $G/K$ on $G/B$ is locally transitive;

(H2) for every $f \in \mathcal{C}_c^\infty(G/K)$, the operator $\Omega_{\mathbf{m}}(f)$ extends to an Hilbert-Schmidt operator on $\mathcal{H}_\chi$, that is, the quantization map (2.27) induces a linear map
$$\Omega_{\mathbf{m}} : \mathcal{C}_c^\infty(G/K) \to \mathcal{L}^2(\mathcal{H}_\chi). \qquad \clubsuit$$

As we mentioned earlier, hypothesis (H2) is verified for many interesting examples. It allows to use the trace as a powerful computational tool in the quantization procedure. Hypothesis (H1) will then be used in to order to apply the fixed point formula for the trace that has been developed in the previous chapter. This will lead to an explicit geometric expression of a deformed product on $G/K$.

### 2.4.1 The symbol map and the deformed product

Let us begin with a heuristic explanation of how the symbol map arises. Let $A \in \mathcal{L}^2(\mathcal{H}_\chi)$. For all $\phi \in \mathcal{C}_c^\infty(G/K)$, since $\Omega_{\mathbf{m}}(\phi)$ is Hilbert-Schmidt, $\Omega_{\mathbf{m}}(\phi)^*\, A$ is trace-class and we can define the following map:

$$\varsigma_{\mathbf{m}}(A) : \mathcal{C}_c^\infty(G/K) \to \mathbb{C}\ ;\ \phi \mapsto \operatorname{Tr}(\Omega_{\mathbf{m}}(\phi)^*\, A). \tag{2.34}$$

In the good cases, we can hope that $\varsigma_{\mathbf{m}}(A)$ is in fact continuous and that, moreover, this antilinear distribution is represented by a (locally integrable) function $\sigma_{\mathbf{m}}(A)$. That is, for all $\phi \in \mathcal{C}_c^\infty(G/K)$,

$$\operatorname{Tr}(\Omega_{\mathbf{m}}(\phi)^*\, A) = \int_{G/K} \sigma_{\mathbf{m}}(A)(x)\, \overline{\phi(x)}\, d_{G/K}(x). \tag{2.35}$$

Applying this to $A := \Omega_{\mathbf{m}}(f)$ for $f \in \mathcal{C}_c^\infty(G/K)$ would give a dequantization procedure, the symbol of $\Omega_{\mathbf{m}}(f)$ being defined as the function $\sigma_{\mathbf{m}}(\Omega_{\mathbf{m}}(f))$. However, this function has no reason to be a smooth compactly supported function (and it won't be in general, even in the good cases). We therefore need to require that the quantization map $\Omega_{\mathbf{m}}$ can be extended to a larger domain $\mathcal{F} \supset \mathcal{C}_c^\infty(G/K)$ which is large enough for $\sigma_{\mathbf{m}}(\Omega_{\mathbf{m}}(f))$ to fall back in $\mathcal{F}$.

Notice that since $|\mathrm{Tr}\,(A\,\Omega_{\mathbf{m}}(\phi)^*)| \leq ||A||_{\mathcal{L}^2}\,||\Omega_{\mathbf{m}}(\phi)||_{\mathcal{L}^2}$, the continuity of (2.34) is guaranteed as soon as the quantization maps is continuous for the $L^2$ norm. It seems therefore natural to take $\mathcal{F}$ as the space of square-integrable functions, which leads us to strengthen (H2) with the following additional hypothesis.

**Hypothesis.**

(H3) the quantization map $\Omega_{\mathbf{m}}$ extends to a bounded linear operator [12]

$$\Omega_{\mathbf{m}} : L^2(G/K) \to \mathcal{L}^2(\mathcal{H}_\chi). \qquad \clubsuit$$

Now, observe that, by definition of the inner products of $\mathcal{L}^2(\mathcal{H}_\chi)$ and $L^2(G/K)$, the defining property (2.35) of $\sigma_{\mathbf{m}}(A)$ reads

$$\langle A, \Omega(\phi)\rangle_{\mathcal{L}^2} = \langle \sigma_{\mathbf{m}}(A), \Phi\rangle_{L^2}\,,$$

which is exactly the definition of the adjoint map of $\Omega_{\mathbf{m}}$. Assuming (H3) therefore makes the whole dequantization procedure well-defined, and leads naturally to the following definition.

**Definition 2.4.1.** *Let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space and let* $\mathbf{m}$ *be an admissible smooth function on $G/B$. Suppose that (H1), (H2) and (H3) hold. Then, the adjoint of the quantization map $\Omega_{\mathbf{m}}$,*

$$\sigma_{\mathbf{m}} : \mathcal{L}^2(\mathcal{H}_\chi) \to L^2(G/K),$$

*is called the* symbol map*.*

This is not the end of the story since the symbol map has no reason to be a left inverse of the quantization map. This would require the latter to be an isometry, which is not always the case. The obstruction is encoded by the notion of the so-called Berezin transform.

**Definition 2.4.2.** *Let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space and let* $\mathbf{m}$ *be an admissible smooth function on $G/B$. Suppose that (H1), (H2) and (H3) hold. The* Berezin transform *is the linear operator defined by*

$$B_{\mathbf{m}} : L^2(G/K) \to L^2(G/K)\ ;\ f \mapsto (\sigma_{\mathbf{m}} \circ \Omega_{\mathbf{m}})(f).$$

**Proposition 2.4.3.** *Let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space and let* $\mathbf{m}$ *be an admissible smooth function on $G/B$. Suppose that (H1), (H2) and (H3) hold. The Berezin transform is a positive bounded linear operator on $L^2(G/K)$, whose norm satisfies $||B_{\mathbf{m}}|| \leq ||\Omega_{\mathbf{m}}||^2$.*

*Proof.* This is immediate from the definition of $B_{\mathbf{m}}$ as a composition of a bounded operator and its adjoint, and from the fact that $\Omega_{\mathbf{m}}$ is bounded. □

[12] Later on, we will give examples for which it is indeed the case, and for which the functional parameter $\mathbf{m}$ can be chosen so that $\Omega_{\mathbf{m}}$ is even unitary.

Now we have a complete working quantization and dequantization procedure, we are ready to define a deformed product on $L^2(G/H)$. For this, we assume the quantization map to be unitary. It is therefore invertible and its inverse is precisely the symbol map.

**Hypothesis.**

(H4) the quantization map $\Omega_{\mathbf{m}} : L^2(G/K) \to \mathcal{L}^2(\mathcal{H}_\chi)$ is unitary.

♣

**Definition 2.4.4.** *Let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space and let $\mathbf{m}$ be an admissible smooth function on $G/B$. Suppose that (H1), (H2), (H3) and (H4) hold. We define a product $\star_{\mathbf{m}}$ on $L^2(G/K)$ by the formula, for all $f_1, f_2 \in L^2(G/K)$:*

$$f_1 \star_{\mathbf{m}} f_2 := \sigma_{\mathbf{m}} \left(\Omega_{\mathbf{m}}(f_1)\Omega_{\mathbf{m}}(f_2)\right). \tag{2.36}$$

**Theorem 2.4.5.** *Let $(G, K, \sigma, B, \chi)$ be a nearly-quantum symmetric space and let $\mathbf{m}$ be an admissible smooth function on $G/B$. Suppose that (H1), (H2), (H3) and (H4) hold. Then, $\star_{\mathbf{m}}$ satisfies the following properties:*

1. *it is bilinear, continuous and associative;*
2. *it is $G-$equivariant in the sense that, for all $g \in G$ and $f_1, f_2 \in L^2(G/K)$,*

   $$({}^g f_1) \star_{\mathbf{m}} ({}^g f_2) = {}^g(f_1 \star_{\mathbf{m}} f_2),$$

   *with $({}^g f)(g_0K) := f\left(g^{-1}g_0K\right)$ for all $f \in L^2(G/K)$ and $g_0K \in G/K$.*
3. *the complex conjugation is an involution of $\star_{\mathbf{m}}$, that is, for all $f_1, f_2 \in L^2(G/K)$,*

   $$\overline{f_1 \star_{\mathbf{m}} f_2} = \overline{f_2} \star_{\mathbf{m}} \overline{f_1}.$$

*Proof.* The bilinearity and the continuity follow from the linearity and continuity of the operators $\Omega_{\mathbf{m}}$ and $\sigma_{\mathbf{m}}$. Regarding the associativity, let $f_1, f_2, f_3 \in L^2(G/K)$. Since $\Omega_{\mathbf{m}}$ is unitary, we have $\Omega_{\mathbf{m}} \circ \sigma_{\mathbf{m}} = \mathrm{id}_{L^2(G/K)}$, so

$$\begin{aligned}
((f_1 \star_{\mathbf{m}} f_2) \star_{\mathbf{m}} f_3) &= \sigma_{\mathbf{m}} \left(\Omega(\sigma_{\mathbf{m}} \left(\Omega(f_1)\Omega(f_2)\right))\Omega(f_3)\right) \\
&= \sigma_{\mathbf{m}} \left(\Omega(f_1)\Omega(f_2)\Omega(f_3)\right) \\
&= \sigma_{\mathbf{m}} \left(\Omega(f_1)\Omega(\sigma_{\mathbf{m}}(\Omega(f_2)\Omega(f_3)))\right) \\
&= (f_1 \star_{\mathbf{m}} (f_2 \star_{\mathbf{m}} f_3)).
\end{aligned}$$

Next, recall from Lemma 2.2.30 that $\Omega_{\mathbf{m}}$ if $G-$equivariant on compactly supported functions. By continuity of $\Omega_{\mathbf{m}}$, it is also on $L^2(G/K)$. For every

$A \in \mathcal{L}^2(\mathcal{H}_\chi)$, $f \in L^2(G/K)$ and $g \in G$, we thus have, by definition of the adjoint,

$$\begin{aligned}\left\langle \sigma_{\mathbf{m}}\left(U(g)AU(g)^{-1}\right), f\right\rangle_{L^2} &= \left\langle U(g)AU(g)^{-1}, \Omega_{\mathbf{m}}(f)\right\rangle_{L^2} \\ &= \left\langle A, U(g)^{-1}\Omega_{\mathbf{m}}(f)U(g)\right\rangle_{L^2} \\ &= \left\langle A, \Omega_{\mathbf{m}}({}^{g^{-1}}f)\right\rangle_{L^2} \\ &= \left\langle \sigma_{\mathbf{m}}(A), {}^{g^{-1}}f\right\rangle_{L^2} \\ &= \langle {}^g(\sigma_{\mathbf{m}}(A)), f\rangle_{L^2},\end{aligned} \tag{2.37}$$

where the last line follows from the $G-$invariance of the $G-$invariant measure on $G/K$. This shows that $\sigma_{\mathbf{m}}\left(U(g)AU(g)^{-1}\right) = {}^g(\sigma_{\mathbf{m}}(A))$. This in turn gives the $G-$equivariance of $\star_{\mathbf{m}}$ since, for all $g \in G$, we have

$$\begin{aligned}({}^g f_1) \star_{\mathbf{m}} ({}^g f_2) &= \sigma_{\mathbf{m}}\left(\Omega_{\mathbf{m}}({}^g f_1)\,\Omega_{\mathbf{m}}({}^g f_2)\right) \\ &= \sigma_{\mathbf{m}}\left(U(g)\Omega_{\mathbf{m}}(f_1)\,U(g)^{-1}\,U(g)\Omega_{\mathbf{m}}(f_2)\,U(g)^{-1}\right) \\ &= \sigma_{\mathbf{m}}\left(U(g)\Omega_{\mathbf{m}}(f_1)\,\Omega_{\mathbf{m}}(f_2)\,U(g)^{-1}\right) \\ &= {}^g\left(\sigma_{\mathbf{m}}\left(\Omega_{\mathbf{m}}(f_1)\,\Omega_{\mathbf{m}}(f_2)\right)\right) = {}^g(f_1 \star_{\mathbf{m}} f_2).\end{aligned}$$

Finally, regarding the last assertion, notice that, because of Lemma 2.2.31, $\Omega_{\mathbf{m}}(\overline{f}) = \Omega_{\mathbf{m}}(f)^*$ for all $f \in \mathcal{C}_0^\infty(G/K)$. By continuity of $\Omega_{\mathbf{m}}$, the same holds for all $f \in L^2(G/K)$. Using the definition of the adjoint and a computation similar to (2.37), we get that $\sigma_{\mathbf{m}}(A^*) = \overline{\sigma(A)}$ for all $A \in \mathcal{L}^2(\mathcal{H}_\chi)$. Therefore,

$$\begin{aligned}\overline{f_1 \star_{\mathbf{m}} f_2} &= \overline{\sigma_{\mathbf{m}}\left(\Omega_{\mathbf{m}}(f_1)\,\Omega_{\mathbf{m}}(f_2)\right)} \\ &= \sigma_{\mathbf{m}}\left(\left(\Omega_{\mathbf{m}}(f_1)\,\Omega_{\mathbf{m}}(f_2)\right)^*\right) = \sigma_{\mathbf{m}}\left(\Omega_{\mathbf{m}}(f_2)^*\,\Omega_{\mathbf{m}}(f_1)^*\right) \\ &= \sigma_{\mathbf{m}}\left(\Omega_{\mathbf{m}}\left(\overline{f_2}\right)\Omega_{\mathbf{m}}\left(\overline{f_1}\right)\right) = \overline{f_2} \star_{\mathbf{m}} \overline{f_1},\end{aligned}$$

which ends the proof. □

*Remark* 2.4.6. The definition (2.36) of $\star_{\mathbf{m}}$ being rather abstract, let us give a way to a more explicit expression of the product. By density of $\mathcal{C}_0^\infty(G/K)$ in $L^2(G/K)$ and by continuity of $\star_{\mathbf{m}}$, it is enough to compute $f_1 \star_{\mathbf{m}} f_2$ for $f_1, f_2 \in \mathcal{C}_0^\infty(G/K)$. The latter is uniquely determined by the datum of its inner product with all $f_3 \in \mathcal{C}_0^\infty(G/K)$, that is (we rather take $\overline{f_3}$ instead of $f_3$ since it will be more convenient later on):

$$\begin{aligned}\left\langle f_1 \star_{\mathbf{m}} f_2, \overline{f_3}\right\rangle_{L^2} &= \left\langle \sigma_{\mathbf{m}}\left(\Omega_{\mathbf{m}}(f_1)\,\Omega_{\mathbf{m}}(f_2)\right), \overline{f_3}\right\rangle_{L^2} \\ &= \left\langle \Omega_{\mathbf{m}}(f_1)\,\Omega_{\mathbf{m}}(f_2), \Omega_{\mathbf{m}}(\overline{f_3})\right\rangle_{\mathcal{L}^2} \\ &= \operatorname{Tr}\left(\Omega_{\mathbf{m}}(\overline{f_3})^*\,\Omega_{\mathbf{m}}(f_1)\,\Omega_{\mathbf{m}}(f_2)\right) \\ &= \operatorname{Tr}\left(\Omega_{\mathbf{m}}(f_1)\,\Omega_{\mathbf{m}}(f_2)\,\Omega_{\mathbf{m}}(f_3)\right).\end{aligned} \tag{2.38}$$

The next section is dedicated to compute this last quantity using the fixed point formula of the previous chapter. ◁

### 2.4.2 The three point kernel

In this subsection, we take one step back regarding our hypotheses, and suppose solely (H1) and (H2). Indeed, it might be instructive to be able to compute $\mathrm{Tr}\,(\Omega_{\mathbf{m}}(f_1)\,\Omega_{\mathbf{m}}(f_2)\,\Omega_{\mathbf{m}}(f_3))$ even for examples which do not lead to a genuine deformed product. In order to simplify a bit the notations, let us denote $M := G/K$ and $Q := G/B$. We will also restrict ourself to the local case.

We start with the following straightforward lemma, which shows that a composition of a locally transitive action is also locally transitive.

**Lemma 2.4.7.** *Let $M$ and $Q$ be two manifolds, and $\tau : M \times Q \to Q$ a locally transitive smooth map. Then,*

$$\tilde{\tau} : (M \times M \times M) \times Q \to Q \;;\; \Big((x,y,z),q\Big) \mapsto \tau_z(\tau_y(\tau_x(q)))$$

*is also locally transitive.*

*Proof.* Let $x,y,z \in M$, $q \in Q$ and $X \in T_{\tilde{\tau}_{(x,y,z)}(q)}(Q)$. Denote $q' := \tau_y(\tau_x(q))$. Then, by definition of $\tilde{\tau}$, $X \in T_{\tau_z(q')}(Q)$. By local transitivity of $\tau$ at $(z,q')$, there exists $Y \in T_z(M)$ such that $\tau_{*_{(z,q')}}(Y,0) = X$. Therefore,

$$\tilde{\tau}_{*_{((x,y,z),q)}}(0,0,Y,0) = X,$$

which shows the local transitivity of $\tilde{\tau}$. □

We are now ready for our main theorems.

**Theorem 2.4.8.** *Let $(G,K,\sigma,B,\chi)$ be a local nearly-quantum symmetric space and let $\mathbf{m}$ be an admissible smooth function on $Q$. Suppose that*

1. *the symmetric action $\tau$ of $M$ on $Q$ is locally transitive;*
2. *for every $f \in \mathcal{C}^\infty_c(M)$, the operator $\Omega_{\mathbf{m}}(f)$ extends to an Hilbert-Schmidt operator on $\mathcal{H}_\chi$, that is, the quantization map (2.27) induces a linear map*
$$\Omega_{\mathbf{m}} : \mathcal{C}^\infty_c(M) \to \mathcal{L}^2(\mathcal{H}_\chi);$$
3. $\mathrm{pr}_{|Z}$ *is proper, where* $\mathrm{pr} : M^3 \times Q \to M^3$ *denotes the projection, and $Z$ is the fixed point bundle*
$$Z := \big\{((x,y,z),q) \in M^3 \times Q \quad|\quad (\tau_z \circ \tau_y \circ \tau_x)(q) = q\big\}\,;$$
4. *for all $x,y,z \in M$, all the fixed points $p$ of $\tau_z \circ \tau_y \circ \tau_x$ are simple, that is*
$$\det\big(\mathrm{id} - (\tau_z \circ \tau_y \circ \tau_x)_{*_p}\big) \neq 0.$$

*Then, for every $f_1, f_2, f_3 \in \mathcal{C}_c^\infty(M)$, we have*

$$\begin{aligned}
&\operatorname{Tr}\left(\Omega_{\mathbf{m}}(f_1)\, \Omega_{\mathbf{m}}(f_2)\, \Omega_{\mathbf{m}}(f_3)\right) \\
&= \int_{M^3} f_1(x) f_2(y) f_3(z)\, K(x,y,z)\, d_M(x) d_M(y) d_M(z),
\end{aligned} \tag{2.39}$$

*where $K : M^3 \to \mathbb{C}$ is smooth and given, for all $x, y, z \in M$, by*

$$K(x,y,z) = \sum_{p=\tau_x(p)} \frac{r_x(p)\; r_y\big(\tau_x(p)\big)\; r_z\big(\tau_y(\tau_x(p))\big)}{\big|\det\big(\mathrm{id} - (\tau_z \circ \tau_y \circ \tau_x)_{*_p}\big)\big|},$$

*where the sum is over the fixed points of $\tau_z \circ \tau_y \circ \tau_x$, and, for all $gK \in M$ and $q \in Q$,*

$$\begin{aligned}
r_{gK}(q) &= \mathbf{m}(g^{-1}qB)\; \tilde{\chi}\left((g\sigma(g^{-1}q))^B\right)^{-1} \\
&= \mathbf{m}(g^{-1}qB) \left(\frac{\Delta_B\left((g\sigma(g^{-1}q))^B\right)}{\Delta_G\left((g\sigma(g^{-1}q))^B\right)}\right)^{\frac{1}{2}} \chi\left((g\sigma(g^{-1}q))^B\right)^{-1}.
\end{aligned}$$

*Proof.* Let $f_1, f_2, f_3 \in \mathcal{C}_0^\infty(M)$. Since $\tau$ is locally transitive, each of the $\Omega(f_i)$ is smooth. They are also Hilbert-Schmidt by assumption, so their product $\Omega_{\mathbf{m}}(f_1)\, \Omega_{\mathbf{m}}(f_2)\, \Omega_{\mathbf{m}}(f_3)$ is a Hilbert-Schmidt operator with smooth kernel on $L^2(Q, d_Q)$.[13] By Theorem 2.3.11, its trace is given by the integral along the diagonal of its kernel, that is, its smooth trace $\operatorname{tr}\left(\Omega_{\mathbf{m}}(f_1)\, \Omega_{\mathbf{m}}(f_2)\, \Omega_{\mathbf{m}}(f_3)\right)$. To compute that smooth trace, we will express $\Omega_{\mathbf{m}}(f_1)\, \Omega_{\mathbf{m}}(f_2)\, \Omega_{\mathbf{m}}(f_3)$ as the pullback operator corresponding to a family of geometric morphisms, and use the results of the previous chapter. This is how it works.

Since $f_1, f_2$ and $f_3$ are compactly supported, we get from (2.26) that, for all $\varphi \in \Gamma_c^\infty(Q, E_{\tilde{\chi}})$,

$$\begin{aligned}
&\Omega_{\mathbf{m}}(f_1)\, \Omega_{\mathbf{m}}(f_2)\, \Omega_{\mathbf{m}}(f_3)\, \varphi \\
&= \int_{M\times M\times M} f_1(x) f_2(y) f_3(z)\, \left(\Omega_{\mathbf{m}}(x)\, \Omega_{\mathbf{m}}(y)\, \Omega_{\mathbf{m}}(z)\, \varphi\right) d_M(x) d_M(y) d_M(z).
\end{aligned} \tag{2.40}$$

Let us consider the family of geometric morphisms of $E_{\tilde{\chi}}$ parametrized by $M \times M \times M$ defined by

$$\tilde{\tau} : (M \times M \times M) \times Q \to Q \;;\; \Big((x,y,z), q\Big) \mapsto \tau_z(\tau_y(\tau_x(q)))$$

and, for all $(x,y,z) \in M \times M \times M$ and $q \in Q$,

$$\begin{aligned}
&\tilde{r}_{(x,y,z)}(q) : (E_{\tilde{\chi}})_{\tilde{\tau}_{(x,y,z)}(q)} \to (E_{\tilde{\chi}})_q, \\
&\tilde{r}_{(x,y,z)}(q) := r_x(q) \circ r_y(\tau_x(q)) \circ r_z(\tau_y(\tau_x(q))),
\end{aligned}$$

[13] $d_Q$ denotes the measure introduced in Remark 2.2.11.

where $\tau$ is the symmetric action (2.30) of $M$ on $Q$ and $r$ is its lift (2.29). Notice that, since we are in the local case, as mentioned in Remark 2.2.36, $E_{\tilde{\chi}}$ is identified with the trivial bundle over $Q$, and the morphisms $\tilde{r}_{(x,y,z)}(q)$ are simply given by the complex numbers

$$\tilde{r}_{(x,y,z)}(q) = r_x\big(q\big)\; r_y\big(\tau_x(q)\big)\; r_z\big(\tau_y(\tau_x(q))\big), \tag{2.41}$$

with $r$ given by (2.32). If we denote by $\tilde{\Omega}_{\mathbf{m}}$ the pullback operators associated to the family $(\tilde{\tau}, \tilde{r})$ as in subsection 1.8.1, we have, for all $x, y, z \in M$,

$$\Omega_{\mathbf{m}}(x)\; \Omega_{\mathbf{m}}(y)\; \Omega_{\mathbf{m}}(z) = \tilde{\Omega}_{\mathbf{m}}(x, y, z)\,.$$

From (2.40), we thus have (we identify smooth densities and smooth functions on $M$ through the $G-$invariant measure $d_M$)

$$\Omega_{\mathbf{m}}(f_1)\; \Omega_{\mathbf{m}}(f_2)\; \Omega_{\mathbf{m}}(f_3) = \tilde{\Omega}_{\mathbf{m}}(f_1 \otimes f_2 \otimes f_3)\,.$$

$\tau$ being locally transitive, $\tilde{\tau}$ is also locally transitive by Lemma 2.4.7. Together with hypotheses 3 and 4, it allows to apply Corollary 1.8.13 which, given the discussion at the beginning of the proof about the trace of $\Omega_{\mathbf{m}}(f_1)\; \Omega_{\mathbf{m}}(f_2)$ $\Omega_{\mathbf{m}}(f_3)$, leads to

$$\begin{aligned}
&\mathrm{Tr}\;(\Omega_{\mathbf{m}}(f_1)\; \Omega_{\mathbf{m}}(f_2)\; \Omega_{\mathbf{m}}(f_3)) = \mathrm{tr}\;(\Omega_{\mathbf{m}}(f_1)\; \Omega_{\mathbf{m}}(f_2)\; \Omega_{\mathbf{m}}(f_3))\\
&= \mathrm{tr}\;\Big(\tilde{\Omega}_{\mathbf{m}}(f_1 \otimes f_2 \otimes f_3)\Big)\\
&= \int_{M^3} f_1(x) f_2(y) f_3(z) \sum_{p=\tau_x(p)} \frac{\tilde{r}_{(x,y,z)}(p)}{\big|\det\big(\mathrm{id} - (\tilde{\tau}_{(x,y,z)})_{*_p}\big)\big|}\, d_M(x) d_M(y) d_M(z),
\end{aligned}$$

where the sum is over the fixed points of $\tilde{\tau}_x$, and is equal to 0 if $\tilde{\tau}_x$ has no fixed point. This shows (2.39) and the expression for $K(x, y, z)$ follows from the expressions (2.41) for $\tilde{r}$, (2.32) for $r$ and the definition (2.8) of $\tilde{\chi}$. The smoothness of $K(x, y, z)$ is also given by Corollary 1.8.13. □

*Remark* 2.4.9. Let us briefly comment on the hypotheses of Theorem 2.4.8. Regarding the first one, we have seen in Proposition 2.2.42 that $\tau$ is locally transitive as soon as there exists a midpoint map on $M$. Theorem 2.2.40 gives a characterization of such spaces. Notice however that it is not a necessary condition.

Regarding the hypothesis 3, we have seen in the proof of Theorem 1.8.12 that it is always verified if $Q$ is compact. If $Q$ is not compact, the following Lemma gives a sufficient condition. ◁

**Lemma 2.4.10.** *Let $(G, K, \sigma, B, \chi)$ be a local nearly-quantum symmetric space. If for all $(x, y, z) \in M^3$, $\tau_z \circ \tau_y \circ \tau_x$ admits a unique fixed point $p(x, y, z) \in Q$ and if the map*

$$M^3 \to Q\;;\; (x, y, z) \mapsto p(x, y, z)$$

*is smooth, then* $\mathrm{pr}_{|Z}$ *is proper, where* $\mathrm{pr} : M^3 \times Q \to M^3$ *denotes the projection, and* $Z := \big\{((x,y,z),q) \in M^3 \times Q \ \ | \ \ (\tau_z \circ \tau_y \circ \tau_x)(q) = q\big\}$.

*Proof.* The map $M^3 \ni (x,y,z) \mapsto ((x,y,z),p(x,y,z)) \in Z$ is the inverse map of $\mathrm{pr}_{|Z}$ by uniqueness of the fixed points, and it is smooth since $p$ is. $\mathrm{pr}_{|Z}$ is therefore a diffeomorphism, hence a proper map. □

If we further assume that the quantization map is unitary, then, we have an explicit formula for the associated deformed product (2.36).

**Theorem 2.4.11.** *Let* $(G,K,\sigma,B,\chi)$ *be a local nearly-quantum symmetric space, and let* $\mathbf{m}$ *be an admissible smooth function on* $Q$. *Suppose that*

1. *the symmetric action* $\tau$ *of* $M$ *on* $Q$ *is locally transitive;*
2. *the quantization map* $\Omega_{\mathbf{m}}$ *extends to a unitary operator*
$$\Omega_{\mathbf{m}} : L^2(M) \to \mathcal{L}^2(\mathcal{H}_\chi);$$
3. $\mathrm{pr}_{|Z}$ *is proper, where* $\mathrm{pr} : M^3 \times Q \to M^3$ *denotes the projection, and* $Z$ *is the fixed point bundle*
$$Z := \big\{((x,y,z),q) \in M^3 \times Q \ \ | \ \ (\tau_z \circ \tau_y \circ \tau_x)(q) = q\big\}\,;$$
4. *for all* $x,y,z \in M$, *all the fixed points* $p$ *of* $\tau_z \circ \tau_y \circ \tau_x$ *are simple, that is*
$$\det\big(\mathrm{id} - (\tau_z \circ \tau_y \circ \tau_x)_{*_p}\big) \neq 0.$$

*Then, the following formula defines an associative, bilinear and continuous product on* $L^2(M)$, *which is* $G-$*equivariant and admits the complex conjugation as a* $\star-$*involution in the sense of Theorem 2.4.5. It is given, for all* $f_1, f_2 \in \mathcal{C}_c^\infty(M)$ *and* $x \in M$, *by*

$$(f_1 \star_{\mathbf{m}} f_2)(x) = \int_{M\times M} f_1(y)\, f_2(z)\, K(x,y,z)\ d_M(y)\, d_M(z), \tag{2.42}$$

*where* $K : M^3 \to \mathbb{C}$ *is called the* three-point kernel of $\star_{\mathbf{m}}$ *and is a smooth map given, for all* $x,y,z \in M$, *by*

$$K(x,y,z) = \sum_{p\,=\,(\tau_z\circ\tau_y\circ\tau_x)(p)} \frac{r_x(p)\ r_y\big(\tau_x(p)\big)\ r_z\big(\tau_y(\tau_x(p))\big)}{\big|\det\big(\mathrm{id} - (\tau_z \circ \tau_y \circ \tau_x)_{*_p}\big)\big|},$$

*where the sum is over the fixed points of* $\tau_z \circ \tau_y \circ \tau_x$, *and, for all* $gK \in M$ *and* $q \in Q$,

$$\begin{aligned} r_{gK}(q) &= \mathbf{m}(g^{-1}qB)\ \tilde{\chi}\left((g\sigma(g^{-1}q))^B\right)^{-1} \\ &= \mathbf{m}(g^{-1}qB)\ \left(\frac{\Delta_B\left((g\sigma(g^{-1}q))^B\right)}{\Delta_G\left((g\sigma(g^{-1}q))^B\right)}\right)^{\frac{1}{2}}\ \chi\left((g\sigma(g^{-1}q))^B\right)^{-1}. \end{aligned}$$

*Proof.* Given our hypotheses, we can apply Theorem 2.4.5 to get a product $\star_{\mathbf{m}}$ on $L^2(M)$ with the desired properties. To get the explicit formula for $\star_{\mathbf{m}}$, let $f_1, f_2 \in \mathcal{C}^\infty(M)$. Repeating the arguments of Remark 2.4.6, we have to compute $\langle f_1 \star_{\mathbf{m}} f_2, \overline{f_3}\rangle_{L^2}$ for all $f_3 \in \mathcal{C}_c^\infty(M)$. On the one hand, by definition of the inner product, we have

$$\langle f_1 \star_{\mathbf{m}} f_2, \overline{f_3}\rangle_{L^2} = \int_M f_3(x)\,(f_1 \star_{\mathbf{m}} f_2)(x)\, d_M(x). \tag{2.43}$$

On the other hand, by Remark 2.4.6 and (2.38), we have

$$\begin{aligned} \langle f_1 \star_{\mathbf{m}} f_2, \overline{f_3}\rangle_{L^2} &= \mathrm{Tr}\,\left(\Omega_{\mathbf{m}}(f_1)\,\Omega_{\mathbf{m}}(f_2)\,\Omega_{\mathbf{m}}(f_3)\right) \\ &= \mathrm{Tr}\,\left(\Omega_{\mathbf{m}}(f_3)\,\Omega_{\mathbf{m}}(f_1)\,\Omega_{\mathbf{m}}(f_2)\right) \\ &= \int_{M^3} f_3(x) f_1(y) f_2(z)\, K(x,y,z)\, d_M(x) d_M(y) d_M(z), \end{aligned} \tag{2.44}$$

where the last line comes from the expression (2.39) for the trace in Theorem 2.4.8. Then, the expression (2.42) for $\star_{\mathbf{m}}$ follows by identification of (2.43) and (2.44). □

## 2.5 Elementary normal $\mathbf{j}$−groups

In this Section, we will apply the previous results to a particular class of symmetric spaces, the so called elementary normal $\mathbf{j}$−groups. [14] For these spaces $M$, the quantization map gives a unitary map from square-integrable functions $L^2(M)$ to Hilbert-Schmidt operators, which leads thus to a deformed equivariant product on $L^2(M)$. We will see that all the hypotheses of the previous section are satisfied, and will therefore be able to give an explicit fixed point formula for the product. The study of elementary normal $\mathbf{j}$−groups is motivated by the theory of Pyatetskii-Shapiro and collaborators on the classification of homogeneous bounded domains [GPSV64, PS69], where they appear as some kind of "building blocks". We refer to [Spi11] for some pedagogical details on that theory. We adopt here a more pragmatical approach to elementary normal $\mathbf{j}$−groups, and define them from their infinitesimal structure.

[14] To avoid any possible confusion, let us emphasize that, even if they are groups, we will consider them as a symmetric space $G/K$, not as the group $G$ acting on that symmetric space.

### 2.5.1 Definitions and properties

**Definition 2.5.1.** *Let $(V, \omega_0)$ be a symplectic vector space.*[15] *The* Heisenberg algebra *associated to $(V, \omega_0)$ is the Lie algebra $\mathfrak{h} := V \oplus \mathbb{R}E$, generated by the elements of $V$ and a generator $E$, which brackets are defined, for all $v, w \in V$, by*

$$[v, w] := \omega_0(v, w)E \quad \text{and} \quad [E, v] := 0.$$

*In particular, it is a central extension of the Abelian Lie algebra $V$.*

**Definition 2.5.2.** *Let $(V, \omega_0)$ be a symplectic vector space, and let $\mathfrak{h}$ be the associated Heisenberg algebra. The Lie algebra $\mathfrak{s} := \mathbb{R}H \oplus \mathfrak{h} = \mathbb{R}H \oplus V \oplus \mathbb{R}E$ with Lie brackets given, for all $v, w \in V$ and $a, t \in \mathbb{R}$, by*

$$[v, w] := \omega_0(v, w)E \quad , \quad [E, v] = 0 \quad \text{and} \quad [H, v + tE] := v + 2tE.$$

*is called an* elementary normal $\mathbf{j}-$algebra. *The connected simply connected Lie group whose Lie algebra is $\mathfrak{s}$ is called an* elementary normal $\mathbf{j}-$group.

*Remark* 2.5.3. The Lie algebra $\mathfrak{s}$ is a split extension of the Heisenberg algebra $\mathfrak{h}$:

$$0 \to \mathfrak{h} \to \mathbb{R}H \ltimes_{\rho_{\mathfrak{h}}} \mathfrak{h} \to \mathbb{R}H \to 0,$$

where the extension homomorpshism $\rho_{\mathfrak{h}} : \mathbb{R}H \to \mathrm{Der}(\mathfrak{h})$ is given, for all $v \in V$ and $t \in \mathbb{R}$, by

$$\rho_{\mathfrak{h}}(H)(v + tE) := [H, v + tE] := v + 2tE. \tag{2.45}$$

$\lhd$

*Remark* 2.5.4. Given an elementary $\mathbf{j}-$algebra $\mathfrak{s} = \mathbb{R}H \oplus V \oplus \mathbb{R}E$ associated to a symplectic vector space $(V, \omega_0)$ of dimension $2d$, we will always make the following identification

$$\mathbb{R}^{2d+2} \xrightarrow{\sim} \mathfrak{s} \ ; \ (a, v, t) \mapsto aH + v + tH,$$

where we identify $V \simeq \mathbb{R}^{2d}$. $\lhd$

Elementary normal $\mathbf{j}-$groups can be endowed with a natural symplectic structure, as well as a symmetric one. This is shown by the following Proposition, for which we refer to [BG15, Section 3.2] and references therein.

**Proposition 2.5.5.** *Let $\mathfrak{s}$ the elementary $\mathbf{j}-$algebra associated to a symplectic vector space $(V, \omega_0)$, and let $\mathbb{S}$ be the corresponding elementary normal $\mathbf{j}-$group. Then, $\mathbb{S}$ is an exponential (non-nilpotent) solvable Lie group. The map*

$$\mathfrak{s} \to \mathbb{S} \ ; \ (a, v, t) \mapsto \exp(aH)\exp(v + tE) = \exp(aH)\exp(v)\exp(tE) \tag{2.46}$$

*is a global coordinate chart. In this chart, we have*

[15] Recall that a *symplectic vector space* is a vector space $V$ endowed with a bilinear form $\omega_0$ which is antisymmetric ($\omega_0(u, v) = -\omega_0(v, u)$ for all $u, v \in V$) and non-degenerate (for all $u \in V$, if $\omega_0(u, v) = 0$ for all $v \in V$, then $u = 0$).

1. *The two form* $\omega^{\mathbb{S}} := 2da \wedge dt + \omega_0$ *is a symplectic form on* $\mathbb{S}$*;*
2. *The group law and inversion map on* $\mathbb{S}$ *are given, for every* $(a, v, t)$, $(a', v', t') \in \mathbb{S}$, *by*

$$(a, v, t)(a', v', t') = (a + a', e^{-a'}v + v', e^{-2a'}t + t' + \frac{1}{2}e^{-a'}\omega_0(v, v')),$$
$$(a, v, t)^{-1} = (-a, -e^{a}v, -e^{2a}t);$$

3. *The map* $s : \mathbb{S} \times \mathbb{S} \to \mathbb{S}$ *defined, for all* $(a, v, t)$, $(a', v', t') \in \mathbb{S}$, *by*

$$s_{(a,v,t)}(a', v', t') = \big(2a - a', 2v\cosh(a - a') - v', 2t\cosh(2a - 2a') - t' + \omega_0(v, v')\sinh(a - a')\big)$$

*defines a symmetric structure on* $\mathbb{S}$*;*

4. *there exists a midpoint map on* $(\mathbb{S}, s)$*, which is given, for all* $(a, v, t)$, $(a', v', t') \in \mathbb{S}$, *by*

$$\mathrm{mid}_{(a,v,t)}(a', v', t') = \left(\frac{a + a'}{2}, \frac{v + v'}{2\cosh\left(\frac{a-a'}{2}\right)}, \frac{t + t'}{2\cosh(a - a')} - \omega_0(v, v')\frac{\sinh\left(\frac{a-a'}{2}\right)}{4\cosh(a - a')\cosh\left(\frac{a-a'}{2}\right)}\right).$$

The following examples show that the reader might well have already encountered an elementary normal $\mathbf{j}$–group before.

*Example* 2.5.6. In the case $V = 0$, the elementary $\mathbf{j}$–algebra $\mathfrak{s}$ is generated by the two elements $H$ and $E$, with bracket $[H, E] = 2E$. The elementary normal $\mathbf{j}$– group $\mathbb{S}$ is the identity component of the group of affine transformations of the real line, the so-called $ax + b$ group. ♢

*Example* 2.5.7. More generally, consider the group $SU(1, n)$ and its Iwasawa decomposition $KAN$. Then, the factor $AN$ is an elementary normal $\mathbf{j}$– group. ♢

### 2.5.2 Associated nearly-quantum symmetric space

From now on, let us fix an elementary $\mathbf{j}$–algebra $\mathfrak{s} = \mathbb{R}H \oplus V \oplus \mathbb{R}E$ associated to a symplectic vector space $(V, \omega_0)$ of dimension $2d$. We denote by $\mathfrak{h}$ the Heisenberg algebra associated to $(V, \omega_0)$ and, as in Remark 2.5.4, we identify $\mathfrak{s} \simeq \mathbb{R}^{2d+2}$. We denote by $\mathbb{S}$ the corresponding elementary normal $\mathbf{j}$–group, and will always use the global coordinate chart $\mathfrak{s} \xrightarrow{\sim} \mathbb{S}$ given by (2.46).

In order to apply the quantization program developped before, the first step is to realize the symmetric space $(\mathbb{S}, s)$ as a symmetric triple $(G, K, \sigma)$. Following

[BG15, Chapter 7] [16], we first introduce the Lie algebra underlying $G$. Let us define $\mathfrak{g}_0$ as a one-dimensional split extension of two copies of the Heisenberg algebra in the following way. Let $\mathfrak{a} = \mathbb{R}H$ be the one-dimensional Lie algebra generated by $H$ and consider the extension homomorphism

$$\rho = \rho_{\mathfrak{h}} \oplus (-\rho_{\mathfrak{h}}) \in \mathrm{Der}(\mathfrak{h} \oplus \mathfrak{h}),$$

where $\rho_{\mathfrak{h}}$ is defined by (2.45). We define

$$\mathfrak{g}_0 := \mathfrak{a} \ltimes_{\rho} (\mathfrak{h} \oplus \mathfrak{h}).$$

Explicitely, the vector space underlying $\mathfrak{g}_0$ is $\mathbb{R}H \oplus (V \oplus \mathbb{R}E) \oplus (V \oplus \mathbb{R}E)$ and the brackets are given, for all $X_1 \oplus X_2, X_1' \oplus X_2' \in \mathfrak{h} \oplus \mathfrak{h}$, by

$$\begin{aligned} [H, X_1 \oplus X_2]_{\mathfrak{g}_0} &= \rho_{\mathfrak{h}}(H)(X_1) \oplus (-\rho_{\mathfrak{h}}(H)(X_2)), \\ [X_1 \oplus X_2, X_1' \oplus X_2']_{\mathfrak{g}_0} &= [X_1, X_1']_{\mathfrak{h}} \oplus [X_2, X_2']_{\mathfrak{h}}, \end{aligned}$$

where $\rho_{\mathfrak{h}}(H)(v + tE) = v + 2tE$ for all $v \in V$ and $t \in \mathbb{R}$. Then, let us consider the element $\Omega \in \Lambda^2\mathfrak{g}^*$ given, for all $v, v' \in V$, by

$$\begin{aligned} \Omega\big(H, E \oplus (-E)\big) &= 2, \\ \Omega\big(v \oplus (-v), v' \oplus (-v')\big) &= \omega_0(v, v'), \end{aligned}$$

and vanishing everywhere else on $\mathfrak{g} \times \mathfrak{g}$.

Finally, we define the Lie algebra $\mathfrak{g} := \mathfrak{g}_0 \oplus \mathbb{R}Z$ to be the one-dimensional central extension of $\mathfrak{g}_0$ with generator $Z$ whose brackets are given, for all $X, Y \in \mathfrak{g}$, by

$$[X, Y]_{\mathfrak{g}} = [X, Y]_{\mathfrak{g}_0} + \Omega(X, Y)Z$$

Notice that, as a vector space, $\mathfrak{g} \simeq \mathbb{R}H \oplus V \oplus V \oplus \mathbb{R}E \oplus \mathbb{R}E \oplus \mathbb{R}Z$.

Let $G$ be the connected simply connected Lie group whose Lie algebra is $\mathfrak{g}$. We have the global chart $\mathfrak{g} \xrightarrow{\sim} G$ given, for all $a, t_1, t_2, l \in \mathbb{R}$ and $v_1, v_2 \in V$, by

$$aH + v_1 \oplus v_2 + t_1E \oplus t_2E + lZ \mapsto \exp(aH)\exp(v_1 \oplus v_2 + t_1E \oplus t_2E + lZ).$$

In these global coordinates, the group law is given, for every $\big(a, v_1, v_2, t_1, t_2, l\big)$, $\big(a', v_1', v_2', t_1', t_2', l'\big) \in G$, by

$$\begin{aligned} &\big(a, v_1, v_2, t_1, t_2, l\big)\big(a', v_1', v_2', t_1', t_2', l'\big) = \\ &\Big(a + a'\ ,\ e^{-a'}v_1 + v_1'\ ,\ e^{a'}v_2 + v_2'\ , \\ &\quad e^{-2a'}t_1 + t_1' + \frac{1}{2}e^{-a'}\omega_0(v_1, v_1')\ ,\ e^{2a'}t_2 + t_2' + \frac{1}{2}e^{a'}\omega_0(v_2, v_2')\ , \\ &\quad l + l' + (e^{-2a'} - 1)t_1 + (e^{2a'} - 1)t_2 + \frac{1}{2}\omega_0(e^{-a'}v_1 - e^{a'}v_2, v_1' - v_2')\Big), \end{aligned} \tag{2.47}$$

[16] Beware that in [BG15], our group $G$ is denoted $\tilde{G}$, its Lie algebra $\tilde{\mathfrak{g}}$ and $K$ is denoted $\tilde{K}$. In this text, $G$ is not the transvection group of $\mathbb{S}$, but its central extension.

and the inversion map by

$$\begin{aligned}&\big(a, v_1, v_2, t_1, t_2, l\big)^{-1} = \\ &\quad\big( - a, -e^a v_1, -e^{-a} v_2, -e^{2a} t_1, -e^{-2a} t_2, -l - (e^{2a} - 1)t_1 - (e^{-2a} - 1)t_2\big).\end{aligned}$$

An involutive automorphism $\sigma : G \to G$ is given by the formula, for all $\big(a, v_1, v_2, t_1, t_2, l\big) \in G$,

$$\sigma(a, v_1, v_2, t_1, t_2, l) := (-a, v_2, v_1, t_2, t_1, l). \tag{2.48}$$

It is easy to see that the closed subgroup $K := G^\sigma = \{g \in G \mid \sigma(g) = g\}$ is given by

$$K = \{(0, v, v, t, t, l) \mid v \in V, t \in \mathbb{R}, l \in \mathbb{R}\}. \tag{2.49}$$

Now, to see that the symmetric triple $(G, K, \sigma)$ indeed realizes the symmetric space $(\mathbb{S}, s)$, notice that, for all $\big(a, v_1, v_2, t_1, t_2, l\big) \in G$, we have the decomposition

$$\big(a, v_1, v_2, t_1, t_2, l\big) = \big(a, v_1 - v_2, 0, t_1 - t_2 - \frac{1}{2}\omega_0(v_1, v_2), 0\big)\, \big(0, v_2, v_2, t_2, t_2, l\big).$$

Therefore, the map

$$\Phi : G/K \to \mathbb{R}^{2d+2}\ ;\ \big(a, v_1, v_2, t_1, t_2, l\big)K \mapsto \big(a, v_1 - v_2, t_1 - t_2 - \frac{1}{2}\omega_0(v_1, v_2)\big) \tag{2.50}$$

is a well-defined global chart on $G/K$, whose inverse is

$$\Phi^{-1} : \mathbb{R}^{2d+2} \to G/K\ ;\ (a, v, t) \mapsto (a, v, 0, t, 0, 0)K. \tag{2.51}$$

If we denote by $\tilde{s}$ the symmetric structure on $G/K$ coming from the symmetric triple $(G, K, \sigma)$, that is, for all $gK, g'K \in G/K$:

$$\tilde{s}_{gK}(g'K) = g\sigma\left(g^{-1}g'\right)K,$$

we compute

$$\begin{aligned}&\Phi\left(\tilde{s}_{(a,v,0,t,0,0)}(a', v', 0, t', 0)\right) \\ &\quad = \big(2a - a', 2v\cosh(a - a') - v', \\ &\qquad\qquad 2t\cosh(2a - 2a') - t' + \omega_0(v, v')\sinh(a - a')\big) \\ &\quad = s_{(a,v,t)}(a', v', t').\end{aligned}$$

This shows that under the identification $\mathbb{S} \simeq \mathbb{R}^{2d+2} \simeq G/K$ corresponding to the charts (2.50) and (2.46), the symmetric space $(\mathbb{S}, s)$ is isomorphic to the symmetric space $(G/K, \tilde{s})$ corresponding to the symmetric triple $(G, K, \sigma)$. From now on, we will always make the identification $\mathbb{S} \simeq \mathbb{R}^{2d+2} \simeq G/K$, and we will also denote by $s$ the symmetric structure on $G/K$.

We compute that the action of $G$ on $\mathbb{S}$ is given, for $(a, v_1, v_2, t_1, t_2, l) \in G$ and $(a', v', t') \in \mathbb{S}$, by

$$(a, v_1, v_2, t_1, t_2, l) \cdot (a', v', t') = \big(a' + a, v' + e^{-a'} v_1 - e^{a'} v_2,$$
$$t' + e^{-2a'} t_1 - e^{2a'} t_2 - \frac{1}{2}(\omega_0(v_1, v_2) - \omega_0(v_1, v') - \omega_0(v_2, v'))\big).$$

From that expression, we see that $d_{\mathbb{S}} := da\, dv\, dt$ is a $G-$invariant measure on $\mathbb{S} \simeq G/K$.

Our next step is to identify a subgroup $B$ of $G$ and a character $\chi$ of $B$ in order to define a nearly-quantum symmetric space $(G, K, \sigma, B, \chi)$. Let us consider a decomposition of $V$ as a direct sum of two complementary Lagrangian [17] subspaces $\mathfrak{l}$ and $\underline{\mathfrak{l}}$:

$$V = \underline{\mathfrak{l}} \oplus \mathfrak{l}. \tag{2.52}$$

For any vector $v \in V$, we will denote $v = v^{\underline{\mathfrak{l}}} + v^{\mathfrak{l}}$ its decomposition corresponding to (2.52). Then, we define the closed subgroup of $G$

$$B := \{(0, n \oplus m_1, n \oplus m_2, t_1, t_2, l) \mid m_1, m_2 \in \mathfrak{l},\ n \in \underline{\mathfrak{l}},\ t_1, t_2, l \in \mathbb{R}\}.$$

One explicitely see from the group law that $B$ is a subgroup of $G$. For any $\theta \in \mathbb{R}_0$, we also define the following character of $B$:

$$\chi_\theta(b) = e^{\frac{i}{\theta} l},$$

for all $b = (0, n \oplus m_1, n \oplus m_2, t_1, t_2, l) \in B$, which is clearly $\sigma-$invariant. We thus have defined a nearly-quantum symmetric space and we will now see that it is local. To this aim, let us consider the subgroup of $G$

$$Q := \{(a, n, 0, 0, 0, 0) \mid a \in \mathbb{R},\ n \in \underline{\mathfrak{l}}\}.$$

Notice that it is indeed a subgroup of $G$ because $\underline{\mathfrak{l}}$ is a Lagrangian subspace of $V$. Next, observe that, for all $q := (a, n, 0, 0, 0, 0) \in Q$ and $b := (0, n \oplus m_1, n \oplus m_2, t_1, t_2, l) \in B$, we have

$$qb = \big(a, (n + n') \oplus m_1', n' \oplus m_2', t_1' + \frac{1}{2}\omega_0(n, m_1'), t_2', l' + \frac{1}{2}\omega_0(n, m_1' - m_2')\big).$$

This shows that the map

$$Q \times B \to G\ ;\ (q, b) \mapsto qb$$

is a global diffeomorphism, so $(G, K, \sigma, B, \chi_\theta)$ is local. For a later use, we also compute the action of $G$ on $Q \simeq G/B$. For all $g := (a, n_1 \oplus m_1, n_2 \oplus m_2, t_1, t_2, l) \in G$ and $q := (a', n', 0, 0, 0, 0) \in Q$, we have

$$g \cdot q := (gq)^Q = (a + a', e^{-a'} n_1 - e^{a'} n_2 + n', 0, 0, 0, 0). \tag{2.53}$$

[17] Recall that a subspace $W$ of $V$ is called *Lagrangian* if $W = W^\perp$, where $W^\perp := \{v \in V \mid \omega_0(v, w) = 0\ \ \forall w \in W\}$.

From that expression, we notice that $d_Q := da\,dn$ is a $G-$invariant measure on $Q \simeq G/B$. In particular, this implies that the modular functions of $G$ and of $B$ coincides on $B$, a fact that we will use later on.

Putting all this together, we have:

**Proposition 2.5.8.** *Let $\mathbb{S}$ be an elementary normal $\mathbf{j}-$group, and let $\theta \in \mathbb{R}_0$. Then, the tuple $(G, K, \sigma, B, \chi_\theta)$ defined as above is a local nearly-quantum symmetric space. Moreover, the symmetric space $(\mathbb{S}, s)$ is isomorphic to the symmetric space $G/K$ corresponding to the symmetric triple $(G, K, \sigma)$.*

### 2.5.3 The deformed product and its three-point kernel

In the following, let $(V, \omega_0)$ be a symplectic vector space of dimension $2d$, and let $\mathbb{S}$ be the corresponding elementary normal $\mathbf{j}-$group. Let $\theta \in \mathbb{R}_0$ and let $(G, K, \sigma, B, \chi_\theta)$ be the local nearly-quantum symmetric space defined as above. As before, we will make the identification $\mathbb{S} \simeq \mathbb{R}^{2d+2} \simeq G/K$. Since we are in the local case $G = QB$, we will also make the identification $G/B \simeq Q$. Finally, let $\mathbf{m}$ be an admissible smooth function on $Q$.

Let us first describe the quantization map associated to the nearly-quantum symmetric space. Following what we have done in Section 2.2, for each point of $x \in \mathbb{S}$, we can define an operator $\Omega_{\theta,\mathbf{m}}(x)$ [18] acting on compactly supported smooth functions on $Q$. Recall from Remark 2.2.36 that it is given, for $gK \in \mathbb{S} \simeq G/K$, $\tilde{\varphi} \in \mathcal{C}_c^\infty(Q)$ and $q \in Q$, by

$$\Omega_{\theta,\mathbf{m}}(gK)\tilde{\varphi})(q) = r_{gK}(q)\,\tilde{\varphi}(\,\tau_{gK}(q)\,),$$

where

$$\begin{aligned} \tau_{gK}(q) &= (g\sigma(g^{-1}q))^Q \\ r_{gK}(q) &= \mathbf{m}\left((g^{-1}q)^Q\right)\ \tilde{\chi}_\theta\left((g\sigma(g^{-1}q))^B\right)^{-1}. \end{aligned}$$

Recall that from (2.53), we have seen that there is a $G-$invariant measure on $Q$, which implies that the modular functions of $G$ and $B$ coincide on $B$. Therefore, $\tilde{\chi}_\theta = \chi_\theta$. Let $x := (a, n \oplus m, t) \in \mathbb{S}$ and $q := (a', n') \in Q$. From (2.51), we get $x = gK$ with $g = (a, n \oplus m, 0, t, 0, 0)$. From the explicit expressions that we have given before, we compute the following identities:

$$\begin{aligned} (g^{-1}q)^Q &= \left(a' - a, n' - e^{a-a'}n\right), \\ (g\sigma(g^{-1}q))^Q &= \left(2a - a', 2\cosh(a - a')n - n'\right), \end{aligned}$$

[18] Notice that we now show the dependence on $\theta$, not only on $\mathbf{m}$.

$$
\begin{aligned}
&(g\sigma(g^{-1}q))^B = \\
&\Big(0,\ (n' - e^{a-a'}n) \oplus (e^{a'-a}m),\ (n' - e^{a-a'}n) \oplus (-e^{a-a'}m), \\
&\qquad e^{-2(a-a')}t - \frac{1}{2}e^{a'-a}\,\omega_0(m, 2\cosh(a-a')n - n'), \\
&\qquad - e^{2(a-a')}t - \frac{1}{2}e^{a-a'}\,\omega_0(m, n'), \\
&\qquad - 2\sinh\big(2(a-a')\big)t - 2\omega_0\big(\cosh(a-a')n - n', \cosh(a-a')m\big)\Big).
\end{aligned}
$$

This leads to the expressions:

$$
\tau_{(a,n\oplus m,t)}(a',n') = \big(2a - a', 2\cosh(a-a')n - n'\big), \tag{2.54}
$$

and

$$
\begin{aligned}
&r_{(a,n\oplus m,t)}(a',n') = \mathbf{m}\left(a' - a, n' - e^{a-a'}n\right) \\
&\times \exp\left(\frac{2i}{\theta}\Big(\sinh\big(2(a-a')\big)t + \omega_0\big(\cosh(a-a')n - n', \cosh(a-a')m\big)\Big)\right).
\end{aligned} \tag{2.55}
$$

We thus have a quantization map

$$
\Omega_{\theta,\mathbf{m}} : \mathcal{C}_c^\infty(\mathbb{S}) \to \mathcal{L}_b(\mathcal{D}(Q) \to \mathcal{D}'(Q))\ ;\ f \mapsto \Omega_{\theta,\mathbf{m}}(f), \tag{2.56}
$$

whith[19]

$$
(\Omega_{\theta,\mathbf{m}}(f)\tilde{\varphi})(q) = \int_{\mathbb{S}} f(x)\, r_x(q)\, \tilde{\varphi}(\tau_x(q))\, d_{\mathbb{S}}(x).
$$

We now come to the deformed product $\star_{\theta,\mathbf{m}}$ associated to the quantization map $\Omega_{\theta,\mathbf{m}}$ as in Section 2.4. Let us verify that the various needed hypotheses are fulfilled. The first question is whether the operators $\Omega_{\theta,\mathbf{m}}(f)$ can be extended to Hilbert-Schmidt operators on $L^2(Q)$, and defined for a larger class of functions than the compactly supported ones. To this aim, let us introduce the following smooth function on $Q$:

$$
\mathbf{m}_0 : Q \to \mathbb{R}\ ;\ (a,n) \mapsto 2^{d+2}\ \cosh^{1/2}(2a)\ \cosh^d(a). \tag{2.57}
$$

In [BG15, Theorem 6.43], it is shown that, if $||\mathbf{m}/\mathbf{m}_0||_\infty < +\infty$, the map (2.56) extends to a bounded operator:

$$
\Omega_{\theta,\mathbf{m}} : L^2(\mathbb{S}, d_{\mathbb{S}}) \to \mathcal{L}^2\left(L^2(Q, d_Q)\right).
$$

This is proved by explicitely computing the kernel of $\Omega_{\theta,\mathbf{m}}(f)$, and showing that it is square-integrable. Moreover, it is shown that if $\mathbf{m} = \mathbf{m}_0$, then the operator is unitary. Notice from (2.47), (2.49) and (2.48) that $\mathbf{m}_0$ is $K-$invariant and $\sigma-$invariant. Since it is a real function, it is admissible.

[19]Recall that we identify a smooth function with the corresponding generalized function.

Next, notice that, since $G/K$ admits a midpoint map by Proposition 2.5.5, Proposition 2.2.42 ensures that $\tau$ is locally transitive. This can also be checked directly from (2.54). We still need to settle the question of the fixed points. Let

$$\begin{aligned} x &= (a_x, v_x = n_x \oplus m_x, t_x) \in \mathbb{S}, \\ y &= (a_y, v_y = n_y \oplus m_y, t_y) \in \mathbb{S}, \\ z &= (a_z, v_z = n_z \oplus m_z, t_z) \in \mathbb{S}. \end{aligned}$$

From (2.54), we compute

$$\begin{aligned} (\tau_z \circ \tau_y \circ \tau_x)(q) = \big( &-a' + 2(a_x - a_y + a_z), \\ &- n' + 2n_x \cosh(a' - a_x) - 2n_y \cosh(a' - 2a_x + a_y) \\ &\quad + 2n_z \cosh(a' - 2a_x + 2a_y - a_z)\big), \end{aligned} \tag{2.58}$$

which allows to find that $\tau_z \circ \tau_y \circ \tau_x$ admits exactly one fixed point $p(x, y, z) \in Q$, and it is given by

$$\begin{aligned} p(x, y, z) = \big( &\, a_x - a_y + a_z, \\ &\cosh(a_x - a_y) n_z - \cosh(a_z - a_x) n_y + \cosh(a_y - a_z) n_x \,\big). \end{aligned} \tag{2.59}$$

Since the map $p : \mathbb{S}^3 \to Q$ ; $(x, y, z) \mapsto p(x, y, z)$ is smooth, Lemma 2.4.10 implies that Hypothesis 3 of Theorem 2.4.8 and Theorem 2.4.11 is verified. Finally, these fixed points are all simple. Indeed, from (2.58), we compute

$$(\tau_z \circ \tau_y \circ \tau_x)_{*_{p(x,y,z)}} = \begin{pmatrix} -1 & 0 \\ * & -\mathbf{1}_{d \times d} \end{pmatrix},$$

where $* = -2(\sinh(a_x - a_y) n_z + \sinh(a_y - a_z) n_x + \sinh(a_z - a_x) n_y)$. Therefore,

$$\det\big(\mathrm{Id} - (\tau_z \circ \tau_y \circ \tau_x)_{*_{p(x,y,z)}}\big) = 2^{d+1} \neq 0, \tag{2.60}$$

which is the condition for the fixed point $p(x, y, z)$ to be simple.

We are now finally able to get an bilinear, associative, continuous and $G$-equivariant deformed product on $L^2(\mathbb{S})$ which is compatible with the complex conjugation. Recall that if we choose $\mathbf{m} = \mathbf{m}_0$, then the quantization map is unitary, and the Berezin transform is trivial – that is, the symbol map is the inverse of the quantization map, see Section 2.4. Putting everything together, we have shown that the hypotheses of Theorem 2.4.5 are satisfied and we get thus a genuine associative deformed product $\star_{\theta,\mathbf{m}_0}$. Its kernel is given by Theorem 2.4.11, whose hypotheses are also satisfied by the preceding discussion. We therefore get:

**Theorem 2.5.9.** *Let $\mathbb{S}$ be an elementary normal $\mathbf{j}-$space. Then, the associative product $\star_{\theta,\mathbf{m}_0}$ on $L^2(\mathbb{S})$ given by Theorem 2.4.5 has the expression, for all $f_1, f_2 \in \mathcal{C}_c^\infty(\mathbb{S})$ and $x \in \mathbb{S}$,*

$$(f_1 \star_{\theta,\mathbf{m}_0} f_2)(x) = \int_{\mathbb{S}^2} f_1(y)\, f_2(z)\, K_{\theta,\mathbf{m}_0}(x,y,z)\; d_{\mathbb{S}}(y)\, d_{\mathbb{S}}(z),$$

*where the three-point kernel $K : \mathbb{S}^3 \to \mathbb{C}$ is given, for all $x, y, z \in \mathbb{S}$, by*

$$K_{\theta,\mathbf{m}_0}(x,y,z) = 2^{-(d+1)}\; r_x\big(p\big)\; r_y\big(\tau_x(p)\big)\; r_z\big(\tau_y(\tau_x(p))\big), \tag{2.61}$$

*where $p = p(x,y,z)$ is the unique fixed point of $\tau_z \circ \tau_y \circ \tau_x$. Explicitely, we have:*

$$K_{\theta,\mathbf{m}_0}(x,y,z) = A_{\mathbf{m}_0}(x,y,z)\; e^{\frac{2i}{\theta} S(x,y,z)},$$

*with, for all $x = (a_x, v_x, t_x)$, $y = (a_y, v_y, t_y)$, $z = (a_z, v_z, t_z) \in \mathbb{S}$,*

$$\begin{aligned} A_{\mathbf{m}_0}(x,y,z) = & \\ & 2^{2d+5} \cosh\big(a_x - a_y\big)^d \cosh\big(a_y - a_z\big)^d \cosh\big(a_z - a_x\big)^d \\ & \quad \times \cosh\big(2(a_x - a_y)\big)^{1/2} \cosh\big(2(a_y - a_z)\big)^{1/2} \cosh\big(2(a_z - a_x)\big)^{1/2} \end{aligned} \tag{2.62}$$

*and*

$$\begin{aligned} S(x,y,z) = & \sinh\big(2(a_x - a_y)\big)t_z + \sinh\big(2(a_y - a_z)\big)t_x + \sinh\big(2(a_z - a_x)\big)t_y \\ & + \cosh(a_x - a_y)\cosh(a_y - a_z)\,\omega_0(v_x, v_z) \\ & + \cosh(a_y - a_z)\cosh(a_z - a_x)\,\omega_0(v_y, v_x) \\ & + \cosh(a_z - a_x)\cosh(a_x - a_y)\,\omega_0(v_z, v_y). \end{aligned} \tag{2.63}$$

*Proof.* This follows from the previous discussion, which allows to apply Theorem 2.4.11. It implies that the kernel is given by

$$K_{\theta,\mathbf{m}}(x,y,z) = \frac{r_x\big(p\big)\; r_y\big(\tau_x(p)\big)\; r_z\big(\tau_y(\tau_x(p))\big)}{\big|\det\big(\mathrm{id} - (\tau_z \circ \tau_y \circ \tau_x)_{*_p}\big)\big|},$$

where $p = p(x,y,z)$ is the unique fixed point of $\tau_z \circ \tau_y \circ \tau_x$. From (2.60), we get the expression (2.61). Let

$$\begin{aligned} x &= (a_x, v_x = n_x \oplus m_x, t_x) \in \mathbb{S}, \\ y &= (a_y, v_y = n_y \oplus m_y, t_y) \in \mathbb{S}, \\ z &= (a_z, v_z = n_z \oplus m_z, t_z) \in \mathbb{S}. \end{aligned}$$

From the expressions (2.54) for $\tau$ and (2.59) for the fixed points, we get

$$\begin{aligned} p &= \big(\ a_x - a_y + a_z, \\ &\qquad \cosh(a_x - a_y)n_z - \cosh(a_z - a_x)n_y + \cosh(a_y - a_z)n_x\ \big), \\ \tau_x(p) &= \big(\ a_x + a_y - a_z, \\ &\qquad -\cosh(a_x - a_y)n_z + \cosh(a_z - a_x)n_y + \cosh(a_y - a_z)n_x\ \big), \\ \tau_y(\tau_x(p)) &= \big(\ -a_x + a_y + a_z, \\ &\qquad \cosh(a_x - a_y)n_z + \cosh(a_z - a_x)n_y - \cosh(a_y - a_z)n_x\ \big). \end{aligned}$$

From the formula (2.55) for $r$, we thus get (notice that $\mathbf{m}_0$ is a function of $a$ alone, not $n$):

$$A_{\mathbf{m}_0}(x, y, z) = \mathbf{m}_0(a_z - a_y)\mathbf{m}_0(a_x - a_z)\mathbf{m}_0(a_y - a_x),$$

which leads to (2.62) using the definition (2.57) of $\mathbf{m}_0$. For the phase – which does not depend on $\mathbf{m}$ –, a (long) computation leads to

$$\begin{aligned} S(x, y, z) &= \sinh\big(2(a_x - a_y)\big)t_z + \sinh\big(2(a_y - a_z)\big)t_x + \sinh\big(2(a_z - a_x)\big)t_y \\ &\quad + \cosh(a_x - a_y)\cosh(a_y - a_z)\left(\omega_0(n_x, m_z) + \omega_0(m_x, n_z)\right) \\ &\quad + \cosh(a_y - a_z)\cosh(a_z - a_x)\left(\omega_0(n_y, m_x) + \omega_0(m_y, n_x)\right) \\ &\quad + \cosh(a_z - a_x)\cosh(a_x - a_y)\left(\omega_0(n_z, m_y) + \omega_0(m_z, n_y)\right) \\ &= \sinh\big(2(a_x - a_y)\big)t_z + \sinh\big(2(a_y - a_z)\big)t_x + \sinh\big(2(a_z - a_x)\big)t_y \\ &\quad + \cosh(a_x - a_y)\cosh(a_y - a_z)\,\omega_0(v_x, v_z) \\ &\quad + \cosh(a_y - a_z)\cosh(a_z - a_x)\,\omega_0(v_y, v_x) \\ &\quad + \cosh(a_z - a_x)\cosh(a_x - a_y)\,\omega_0(v_z, v_y), \end{aligned}$$

where the second line is obtained by using the fact that $\mathfrak{l}$ and $\underline{\mathfrak{l}}$ are Lagrangian. □

*Remark* 2.5.10. The formula (2.62) and (2.63) for the three-point kernel of the deformed product has already been found in [BG15]. It has been computed in two different ways, first by intertwining the Moyal product, and second by using a quantization map as we do here. However, the computation of the trace of that quantization map is done explicitely from the kernel of the operators. The advantage of the approach we take here to the computation of the trace is that it makes more transparent why the fixed points appear in the kernel of the product. ◁

*Remark* 2.5.11. From [BG15, Equation 3.6], we extract that the area of the double triangle defined by $e = (0, \vec{0}, 0)$, $x_1 = (a_1, v_1, t_1)$, $x_2 = (a_2, v_2, t_2) \in \mathbb{S}$ is

$$S_{\mathrm{can}}(x_1, x_2) = \sinh(2a_1)t_2 - \sinh(2a_2)t_1 + \omega_0(v_1, v_2)\cosh(a_1)\cosh(a_2).$$

Using the fact that the area of double triangle is invariant under the group law on $\mathbb{S}$ – since the left translations are automorphisms of $(\mathbb{S}, s)$, and symplectomorphisms), a small computation shows that in the particular case of elementary normal **j**-groups, the phase $S(x, y, z)$ corresponds to minus the area of the double triangle determined by $x$, $y$ and $z$. $\triangleleft$

# Appendix A

# Locally convex vector spaces

We collect here some basic definitions and constructions related to locally convex topological vector spaces. We adopt a pragmatical approach, by defining their topology from a family of seminorms. We refer to [Trè06] for a complete treatment on the subject, as well as to [vdBC09] for a pedagogical exposition.

## Definitions and properties

**Definition A.1.** *A* topological vector space *is a vector space $V$ (over $\mathbb{C}$) endowed with a topology $\mathcal{T}$ such that the following maps are continuous:*

1. *addition:* $V \times V \to V$ ; $(v, w) \mapsto v + w$ ;
2. *scalar multiplication:* $\mathbb{C} \times V \to V$ ; $(\lambda, v) \mapsto \lambda v$.

**Proposition A.2.** *The topology of a topological vector space $(V, \mathcal{T})$ is completely determined by a basis of neighbourhoods of* $0$.

**Definition A.3.** *A* seminorm *$p$ on a vector space $V$ is a nonnegative function $p : V \to \mathbb{R}^+$ such that:*

1. $\forall v, w \in V \,:\, p(v + w) \leq p(v) + p(w)$ ;
2. $\forall \lambda \in \mathbb{C}, \forall v \in V \,:\, p(\lambda v) = |\lambda| p(v)$.

*Notice that condition 2 implies that $p(0) = 0$. A seminorm is called a* norm *if $\forall v \in V$, $p(v) = 0 \Rightarrow v = 0$. A family of seminorms $\{p_\alpha\}_{\alpha \in I}$ is called* total *if $\forall v \in V$:*

$$(\forall \alpha \in I \,,\, p_\alpha(v) = 0) \Rightarrow v = 0.$$

Let $\{p_\alpha\}_{\alpha\in I}$ be a family of seminorms on a vector space $V$. For $r > 0$, $n \in \mathbb{N}_0$ and $\{\alpha_1, \dots, \alpha_n\} \subset I$, we can define the ball

$$B^r_{\alpha_1,\dots,\alpha_n} := \{v \in V \mid p_{\alpha_i}(v) < r \ \ \forall 1 \leq i \leq n\} \subset V.$$

The collection of all those balls $\left\{B^r_{\alpha_1,\dots,\alpha_n}\right\}$ defines a family of neighbourhoods of 0 which gives a vector space topology on $V$.

**Definition A.4.** *A* locally convex topological vector space *(or l.c.v.s. in short) is a topological vector space $V$ such that there exists a family of seminorms on $V$ that induces the topology of $V$.*

*Remark* A.5. Locally convex vector spaces are so called because they admit enough "convex" neighbourhoods of 0. They are usually defined using this property but we chose the practical seminorm approach because this is how the topologies we deal with naturally arise. $\triangleleft$

*Remark* A.6. Different families of seminorms on a vector space $V$ can induce the same topology on $V$. As a topological vector space, $V$ should be considered as the same object, even though the seminorms are different. Often, the topology of $V$ is defined using a very large family of seminorms but this family can be restricted to a smaller one without changing the topology, as the following results show. $\triangleleft$

(▶ Restriction property of families of seminorms)

The following result is of first practical importance in order to verify the continuity of a map between locally convex spaces using their seminorms.

**Proposition A.7** ([Trè06], Proposition 7.7)**.** *Let $V$ and $W$ be two locally convex spaces. A linear map $f : V \to W$ is continuous if and only if for every continuous seminorm $q$ on $W$, there is a continuous seminorm $p$ on $V$ such that, for all $x \in V$, $q(f(x)) \leq p(x)$.*

## Continuous dual of a l.c.v.s.

**Definition A.8.** *Let $V$ be a locally convex vector space. The* continuous dual of $V$ *is the vector space of continuous linear functionals on $V$. It is denoted by $V'$.*

There are several topologies that we can consider on the continuous dual of a l.c.v.s. We will describe two of them that will be of importance for us.

**Definition A.9.** *Let $V$ be a locally convex vector space. The* weak* topology *on the continuous dual $V'$ is the locally convex topology induced by the family of seminorms*

$$\left\{p_v : V' \to \mathbb{R}^+ \, ; \; u \mapsto |u(v)| \quad \middle| \; v \in V\right\}.$$

*Remark* A.10. In the weak$^*$ topology, the convergence of a sequence is given by the pointwise convergence. That is, a sequence $(u_k)_{k\in\mathbb{N}}$ in $V'$ converges to $u \in V'$ if and only if, for all $v \in V$, $|u_n(v) - u(v)| \to 0$. $\lhd$

**Definition A.11.** *Let $V$ be a locally convex vector space whose topology is induced by the family of seminorms $\{p_\alpha\}_{\alpha\in I}$. A set $B \subset V$ is* bounded *is for all $\alpha \in I$, there exists $r_\alpha > 0$ such that $B \subset B_\alpha^{r_\alpha}$. Notice that any continuous linear functional $u$ is bounded on a bounded set $B$ in the sense that* $\sup_{v\in B} |u(v)| < \infty$.

**Definition A.12.** *Let $V$ be a locally convex vector space. The* strong topology *on the continuous dual $V'$ is the locally convex topology induced by the family of seminorms*

$$\left\{ p_B : V' \to \mathbb{R}^+ \; ; \; u \mapsto \sup_{v\in B} |u(v)| \quad | \; B \subset V \textit{ bounded} \right\}.$$

*Remark* A.13. A linear map on $V'$ is continuous in the weak$^*$ topology if it is continuous in the strong topology. However, since both topologies are in general different, the reverse is not true. $\lhd$

## Fréchet spaces and inductive limits of l.c.v.s.

**Definition A.14.** *A locally convex vector space $V$ is called* Fréchet *if its topology is induced by a countable total family of seminorms and if it is complete.*

In some cases, a vector space $V$ can be seen as the limit of an infinite strictly increasing family of vector subspaces, each of them carrying a locally convex topology. Among all the locally convex topology on $V$ such that all the inclusions are continuous, one of them is particularly interesting.

**Definition A.15.** *Let $V$ be a vector space and $V_1 \subset V_2 \subset \ldots$ an infinite srictly increasing sequence of vector subspaces of $V$ such that :*

1. *$V = \cup_{k=1}^{\infty} V_k$ ;*
2. *for each $k \geq 1$, $V_k$ is a locally convex vector space, its topology being denoted by $\mathcal{T}_k$ ;*
3. *for each $k \geq 1$, $V_k$ is closed in $V_{k+1}$ ;*
4. *for each $k \geq 1$, $\mathcal{T}_k = \mathcal{T}_{k+1|_{V_k}}$.*

*Then, we define the* inductive limit *topology on $V$ as the locally convex topology given by the following family of seminorms:*

$$\left\{ p \textit{ seminorm on } V \;\middle|\; p_{|_{V_k}} \textit{ is continuous } \forall k \geq 1 \right\}.$$

Although the previous definition is rather abstract, the following results describe in a more explicit way the continuity of a linear map as well as the convergence of a sequence in the inductive limit topology.

**Proposition A.16** ([Trè06], Proposition 13.1)**.** *Let $V = \cup_{k=1}^{\infty} V_k$ be a vector space endowed with the inductive limit as in (A.15), $W$ a locally convex vector space and $A : V \to W$ a linear map. Then, $A$ is continuous if and only if, for each $k \geq 1$, $A_{|V_k} : V_k \to W$ is continuous.*

**Proposition A.17** ([vdBC09], Theorem 2.1.11)**.** *Let $V = \cup_{k=1}^{\infty} V_k$ be a vector space endowed with the inductive limit as in (A.15). A sequence $(v_n)_{n\in\mathbb{N}}$ in $V$ converges to $v \in V$ if and only if the following two conditions are satisfied:*

1. *$\exists\, n_0 \in \mathbb{N}_0$ such that $v, v_n \in V_{n_0}$ for each $n \in \mathbb{N}$ ;*

2. *$(v_n)_{n\in\mathbb{N}}$ converges to $v$ in $V_{n_0}$.*

**Proposition A.18** ([vdBC09], Theorem 2.1.11)**.** *Let $V = \cup_{k=1}^{\infty} V_k$ be a vector space endowed with the inductive limit as in (A.15). Then $V$ cannot be metrizable.*

**Proposition A.19.** *Let $V$ and $W$ be two locally convex vector spaces. Suppose that $V$ is a Fréchet space, or an inductive limit of Fréchet spaces. Then a linear map $L : V \to W$ is continuous if it is sequentially continuous.*

*Proof.* If $V$ is a Fréchet space, it is metrizable and therefore, sequential continuity is equivalent to continuity. Suppose now that $V = \cup_{k=1}^{\infty} V_k$ is an inductive limit of Fréchet spaces. Let $L : V \to W$ be a linear map. By Proposition A.16, $L$ is continuous if and only if $L_{|V_k}$ is continuous for each $k$. Since each of the $V_k$ is a metrizable, $L_{|V_k}$ is continuous if it is sequentialy continuous. Since all the inclusions $V_k \hookrightarrow V$ are continuous, $L_{|V_k}$ is continuous if $L$ is sequentially continuous, so $L$ is continuous if it is sequentially continuous. $\square$

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