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Noncommutative Geometry and Supersymmetry in a Generalization of the Bigatti-Susskind System

by

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Chapter 1

Introduction

Many times throughout its history, Science has been the cause of deep changes in our perception of the world. The development of Quantum Mechanics at the beginning of the 20th century is certainly one of these revolutions of thought, and it still continues to confuse us. One of its most astonishing features is that, at quantum scales, physical observables should be described by operators, and that the position and momentum operators do not commute:

$$[\hat{x}, \hat{p}_x] \quad = \quad i\hbar \,. \tag{1.1}$$

This leads to the well known Heisenberg uncertainty principle which states that it is not possible to measure both the position and the momentum of a particle with an absolute precision.

A lot of work has been done since then, with the discovery of new fundamental interactions, the development of quantum field theory, etc., so that the Standard Model is now the theory that best fits our current understanding of particle physics. However, we have several reasons to think that it is not the end of the story and the last decades have given birth to many new theories aiming to address some of the still unresolved enigmas of Nature. Amongst them is the hypothesis that the fundamental structure of spacetime should be entirely revised, considering for instance that it is based on a noncommutative geometry.

In the framework of Quantum Mechanics, noncommutative geometry is generally studied through the following commutator: [1]

$$[\hat{x}_i, \hat{x}_j] \quad = \quad i\hbar\, \theta_{ij} \,, \tag{1.2}$$

where θ_{ij} is a constant matrix. One may also consider non vanishing commutators between the momentum operators. A classic example of how noncommutativity may emerge in a physical situation is given in the Landau problem [2] by projecting the whole system on the lowest Landau level. This idea has been introduced a long time ago by Snyder [3] and has been extensively studied by Connes [4] but it has gained a lot more popularity since it has been found that noncommutative geometry may emerge in string theories [5]. In a recent paper [6], a toy model – which we will refer to as the Bigatti-Susskind model – has been built in order to further understand how noncommutativity appears. Evidently, even if ordinary non relativistic quantum mechanics is now out of date in order to describe actual relativistic particle physics, it is still a very convenient framework to investigate new theories and to get a further insight into their physical consequences. The Bigatti-Susskind model consists in a system of two massive opposite charges constrained to move in a plane, subjected to a constant homogeneous and perpendicular magnetic field, and connected by a spring. By considering the limit of a strong magnetic field, these authors recover a situation similar to that in the Landau problem, namely, noncommutativity of the coordinates appear.

Another promising direction of actual research in physics is supersymmetry. Elementary particles are classified either as bosons or as fermions, depending on the value of their spin. Supersymmetry is a new internal symmetry of Nature that would relate these two kinds of particles by transforming a boson into a fermion and vice-versa. The following natural question then arises to ask whether it is possible to consider both noncommutative geometry and supersymmetry in one system. Once again, Quantum Mechanics turns to be a convenient framework to address this question. By extending the Landau problem to its $\mathcal{N}=1$ supersymmetric version [7], it has been shown that noncommutativity is indeed compatible with supersymmetry, leading to a deformation of the fermionic algebra consistent with the supersymmetry of the system. Turning back to the Bigatti-Susskind model, the same question may be considered. The answer is not straightforward since there is now a harmonic potential that seems difficult to handle in $\mathcal{N}=1$ Supersymmetric Quantum Mechanics. However, this question has been addressed in [8] and it was shown that a harmonic potential may indeed be considered together with a $\mathcal{N}=1$ supersymmetry. Following this last result, our work will therefore be devoted to the construction of a $\mathcal{N}=1$ supersymmetric extension of a generalization of the Bigatti-Susskind model.

Generalization of the Bigatti-Susskind Model

With the previous considerations in mind, the starting point of this work is a system of two particles of charges $\pm q$, mass m and respective positions $\mathbf{r}$ and $\mathbf{s}$. Their motion is constrained to a plane and is subjected to a constant, homogeneous magnetic field $\mathbf{B}_0 = B_0\,\hat{\mathbf{z}}$,[1] and the Coulomb interaction between the particles is neglected. Furthermore, the particles are connected by a spring, giving rise to a harmonic potential $\frac{1}{2}m\omega_0^2\left(\mathbf{s}-\mathbf{r}\right)^2$. Finally, the Bigatti-Susskind model is generalized by confining the center of mass of the system in a harmonic potential $\frac{1}{8}mk_0^2\left(\mathbf{r}+\mathbf{s}\right)^2$ – the numerical factor is chosen for later convenience.

The system may be described by the following Lagrangian :

$$\begin{aligned} L\left(\mathbf{r},\mathbf{s}\,;\dot{\mathbf{r}},\dot{\mathbf{s}}\right) &= \frac{1}{2}m\left(\dot{\mathbf{r}}^{\,2}+\dot{\mathbf{s}}^{\,2}\right)+q\,\dot{\mathbf{r}}\cdot\mathbf{A}\left(\mathbf{r}\,\right)-q\,\dot{\mathbf{s}}\cdot\mathbf{A}\left(\mathbf{s}\,\right) \\ &- \frac{1}{2}m\omega_0^2\left(\mathbf{s}-\mathbf{r}\,\right)^2-\frac{1}{8}mk_0^2\left(\mathbf{r}+\mathbf{s}\,\right)^2\,, \end{aligned} \tag{1.3}$$

where $\mathbf{A}\left(\mathbf{r}\right)$ is the magnetic vector potential. In order to keep the rotational covariance of the system explicit, the circular gauge will be used for the vector potential: $\mathbf{A}\left(\mathbf{r}\right)=\frac{1}{2}\mathbf{B}_0\times\mathbf{r}$. Introducing the variables

$$\begin{aligned} \mathbf{x} &= \frac{1}{2}\left(\mathbf{r}+\mathbf{s}\,\right), \\ \mathbf{u} &= \mathbf{s}-\mathbf{r}\,, \end{aligned} \tag{1.4}$$

the Lagrangian may be expressed as

$$\begin{aligned} L\left(\mathbf{x},\mathbf{u}\,;\dot{\mathbf{x}},\dot{\mathbf{u}}\right) &= m\dot{\mathbf{x}}^{\,2}+\frac{1}{4}m\dot{\mathbf{u}}^{\,2}+\frac{1}{2}q\,\mathbf{B}_0\cdot\left(\dot{\mathbf{x}}\times\mathbf{u}-\mathbf{x}\times\dot{\mathbf{u}}\,\right) \\ &- \frac{1}{2}m\omega_0^2\mathbf{u}^{\,2}-\frac{1}{2}mk_0^2\mathbf{x}^{\,2} \\ L\left(x_i,u_i\,;\dot{x}_i,\dot{x}_j\right) &= m\,x_i^2+\frac{1}{4}m\,u_i^2+\frac{1}{2}\,qB_0\,\epsilon_{ij}\left(\dot{x}_i\,u_j-x_i\,\dot{u}_j\right) \\ &- \frac{1}{2}m\omega_0^2\,x_i^2-\frac{1}{2}mk_0^2\,u_i^2 \end{aligned} \tag{1.5}$$

[1]Up to a global rotation of the system, there is no loss of generality in choosing a particular direction for the magnetic field perpendicular to the plane.

Outline

In the following Chapter, the question of constructing a supersymmetric extension of the previous system will be addressed. The appropriate formalism will first be presented, and then the actual extension will be constructed. In Chapter 3, the Hamiltonian formulation of the system will be described using Dirac's constraint analysis formalism. The symmetries of the system will also be discussed. This will lead to the quantization of the system in Chapter 4. The diagonalization of the Hamiltonian will be addressed, first for the bosonic sector, then for the fermionic one. The diagonalization of the angular momentum operator will be carried on, followed by an analysis of the symmetry transformations of the quantized system. In Chapter 5, a representation of the system will be constructed, and the energy spectrum will be discussed. The supercharge will finally be diagonalized. The last Chapter is devoted to the construction of $\mathcal{N} = 1$ supersymmetric non(anti)commutative superplanes through the projection of the system on two different subspaces of its Hilbert space of states. Some conclusions and perspectives will then end this work.

Mathematica©

The main part of the calculations of this work has been performed using Mathematica©. In particular, we had to implement Grassmannian calculus, and we have developed some convenient tools to perform Dirac's constraint analysis. The use of Mathematica© has turned to be very convenient to do many cross-checks during the calculations. We also would like to mentioned that, for the calculations of Chapter 5 – namely, those related to the representation of the system, the analysis of the energy spectrum and the supercharge diagonalization –, we have used the convenient *Quantum* package of José Luis Gómez-Muñoz and Francisco Delgado that may be found here: http://homepage.cem.itesm.mx/lgomez/quantum/.

Chapter 2

Supersymmetric Extension

2.1 Formalism

We will now build a $\mathcal{N} = 1$ supersymmetric extension of the generalization of the Bigatti-Susskind system we have just exposed. Introducing some new fermionic degrees of freedom in addition to the bosonic ones x_i and u_i, we will then define a supersymmetry transformation, that is, a transformation that maps the bosonic variables into the fermionic ones and conversely. We will then construct an extension of the previous Lagrangian such that the system action is invariant under that new symmetry. In order to achieve this programme, we will make use of the supertime formalism which allows to easily construct such supersymmetric action.

2.1.1 Grassmann variables

A fermionic degree of freedom is a feature specific to quantum mechanics. However, it is possible to introduce analogues to fermions in classical mechanics using Grassmann odd numbers. These are numbers that anticommute, so that if we have a set $\{\theta_1, \theta_2, ..., \theta_n\}$ of Grassmann odd numbers, we have:

$$\theta_i\theta_j = -\theta_j\theta_i, \qquad 1 \leq i,j \leq n \tag{2.1}$$

In particular, the square of any of those numbers gives zero: $\theta_i^2 = 0$ (no summation on i). In this context, usual commuting numbers are referred to as Grassmann even numbers. Even though Grassmann odd quantities have no direct geometrical interpretation at the classical level as Grassmann even variables do, they will give rise to anticommuting operators after quantization, as expected for physical fermions.

Here we will sketch only some useful rules of Grassmannian calculus but a complete review may be found in [9]. Any function of one Grassmann odd variable θ can be Taylor expanded and, due to the vanishing of θ^2, the expansion ends after the first order:

$$f(\theta) = a + \theta\, b \tag{2.2}$$

It is then possible to define (left) derivation and integration operations in term of Grassmann odd variables:

$$\begin{aligned} \frac{d}{d\theta}\theta = 1\ , \qquad & \frac{d}{d\theta}1 = 0 \\ \int d\theta\ \theta = 1\ , \qquad & \int d\theta\ 1 = 0 \end{aligned} \tag{2.3}$$

For Grassmann odd variables, integration is therefore the same operation as derivation. Note also that, due to the anticommutation of the θ_i's, we have to care about their order when deriving (resp. integrating) a product of the θ_i's: the derivation (resp. integration) variable has to be brought to the foremost left-handed position in any product:

$$\begin{aligned} \frac{d}{d\theta_j}(\theta_i\theta_j) &= -\frac{d}{d\theta_j}(\theta_j\theta_i) = -\theta_i \\ \int d\theta_j\ \theta_i\theta_j &= -\int d\theta_j\ \theta_j\theta_i = -\theta_i \end{aligned} \tag{2.4}$$

Finally, we define complex conjugation to include a reversal in the order of the variables in any product:

$$(\theta_i\theta_j)^* = \theta_j^*\,\theta_i^* \tag{2.5}$$

The reason of the reversed order of the variables is to keep the integration measure over complex Grassmann odd numbers consistent under complex conjugation (see [10]).

2.1.2 Supertime

As a convenient way to include supersymmetry in our system, let us extend the usual time coordinate t into a "supertime" (t, θ) where θ is a real Grassmann odd variable, that is, $\theta^* = \theta$ and $\theta^2 = 0$. This is sometimes referred to as "supermechanics". For some literature on the subject, see [11, 12], and [13, 14] for a more mathematical introduction. As noted before, we can Taylor expand any function f of the supertime to get

$$f(t, \theta) = a(t) + \theta\, b(t) \tag{2.6}$$

with $a(t)$ and $b(t)$ being some functions of the usual time t, of which the Grassmann parity is related to that of the function $f(t, \theta)$. We then promote the usual bosonic coordinates to the following Grassmann even supercoordinates[1]:

$$\begin{aligned} X_i(t,\theta) &= x_i(t) + i\theta\,\psi_i(t) \\ U_i(t,\theta) &= u_i(t) + i\theta\,\mu_i(t) \end{aligned} \qquad i = 1, 2, ..., d \tag{2.7}$$

where x_i and u_i are real Grassmann even functions of t while ψ_i and μ_i are real Grassmann odd functions of t. Notice that a factor i appears in the definitions of X_i and U_i in order to make them real functions of the supertime. For a reason that will be explained later, we also introduce the following Grassmann odd supercoordinates:

$$\begin{aligned} \Lambda_i(t,\theta) &= \lambda_i(t) + \theta\, y_i(t) \\ \Gamma_i(t,\theta) &= \gamma_i(t) + \theta\, z_i(t) \end{aligned} \qquad i = 1, 2, ..., d \tag{2.8}$$

where y_i and z_i are real Grassmann even functions of t while λ_i and γ_i are real Grassmann odd functions of t. Finally, any function $F(\mathbf{x}, \mathbf{u})$ of the usual coordinates can be extended to a function of the supercoordinates by Taylor expansion so that we get:

$$F(\mathbf{X}, \mathbf{U}) = F(\mathbf{x}, \mathbf{u}) + i\theta\,\psi_j \frac{\partial F}{\partial x_j}(\mathbf{x}, \mathbf{u}) + i\theta\,\mu_j \frac{\partial F}{\partial u_j}(\mathbf{x}, \mathbf{u}) \tag{2.9}$$

where the usual implicit summation over repeated indices is to be understood (this notation applies throughout hereafter, unless otherwise specified).

[1] For the moment, we consider a d-dimensional space and will later on restrict ourself to 2 dimensions.

2.1.3 Supersymmetry transformation

Time translations play a crucial role in physics since invariance under such transformations implies the conservation of the system energy. It therefore seems natural to discuss supertime translations which correspond to the following transformations:

$$\begin{array}{rcl} t & \rightarrow & t + i\epsilon\theta \\ \theta & \rightarrow & \theta + \epsilon \end{array} \tag{2.10}$$

where ϵ is a real Grassmann odd parameter. The operator that generates these translations is given by:

$$Q = \partial_\theta + i\theta\,\partial_t \tag{2.11}$$

and is such that:

$$Q^\dagger = Q\ , \qquad \{Q, Q\} = 2i\,\partial_t \tag{2.12}$$

The second property shows that Q can be viewed as sort of a "square root" of the Hamiltonian – namely the generator of time translations –, which is an expected feature for a supersymmetry generator. To this transformation can be associated a supercovariant derivative D:

$$D = \partial_\theta - i\theta\,\partial_t \tag{2.13}$$

$$\{Q, D\} = 0\ , \qquad \{D, D\} = -2i\,\partial_t \tag{2.14}$$

When acting with Q on the supercoordinates, we get:

$$\delta_\epsilon X_i(t,\theta) = -i\epsilon\, Q X_i(t,\theta) \tag{2.15}$$

so that the transformation acts on each component as follows:

$$\delta_\epsilon x_i(t) = \epsilon\lambda_i(t)\ , \qquad \delta_\epsilon\lambda_i(t) = i\epsilon\dot{x}_i(t) \tag{2.16}$$

We therefore see that Q indeed maps fermionic degrees of freedom onto bosonic ones and vice-versa. It thus displays all the feature required of the generator of $\mathcal{N} = 1$ supersymmetry transformations. In the following, a superfield refers to any function of specific Grassmann parity of the supertime that transforms under Q as in (2.15). Note that any usual combination of superfields (product, sum, covariant derivative, scalar product, etc.) is still a superfield.

2.1.4 Supersymmetric invariant action

We now end the presentation of the supertime formalism by showing how it allows to build in a straightforward way a system action which is invariant under supersymmetry. We consider an action of the form:

$$S = \int dt\, L = \int dt d\theta\, \mathbb{L} \tag{2.17}$$

where $\mathbb{L}$ is a Grassmann odd function of superfields, the superlagrangian, so that the action S is Grassmann even, as usual. The key point is that if we choose our superlagrangian to be any combination of superfields, the action S will be automatically invariant under a supersymmetry transformation. Indeed, since it is in that case itself a superfield which may be expressed as

$$\mathbb{L} = \mathbb{L}_1 + \theta\,\mathbb{L}_2 \tag{2.18}$$

with $\mathbb{L}_1$ being Grassmann odd and $\mathbb{L}_2$ Grassmann even, the transformation of the Lagrangian L will be equal to $\delta\mathbb{L}_2$ under a supersymmetry transformation. Because of the expressions (2.15), we have $\delta\mathbb{L}_2 = -\frac{d}{dt}\mathbb{L}_1$, that is, a total time derivative. The action of the system therefore changes by a surface term in time, which makes the system invariant under a supersymmetry transformation. Note that the supercovariant derivative can be used to build an odd superfield from an even one, and vice-versa.

2.1.5 Potential and $\mathcal{N} = 1$ supersymmetry

Because of the Grassmann odd character of the superlagrangian, any term built only from the Grassmann even supercoordinates X_i and U_i is forbidden. Therefore, it first seems that this formalism is not compatible with a potential $V(X_i, U_i)$. [2] However, it was shown in [8] that it is possible to introduce a harmonic potential for, say, x_i by using an auxiliary Grassmann odd superfield coupled to X_i. The auxiliary bosonic degree of freedom will turn out to be redundant and to be proportional to x_i, which upon reduction of that auxiliary variable induces a harmonic potential, as will be shown later. This is the reason why we have introduced the Grassmann odd superfields Λ_i and Γ_i in (2.8).

2.2 Supersymmetric Lagrangian

We now have all the tools necessary to build a supersymmetric extension of our generalization of the Bigatti-Susskind system. Let us for a moment introduce the superfields associated to the position of the superparticles[3]:

$$\begin{aligned} R_i(t,\theta) &= X_i(t,\theta) - \frac{1}{2}U_i(t,\theta) \\ S_i(t,\theta) &= X_i(t,\theta) + \frac{1}{2}U_i(t,\theta) \end{aligned} \tag{2.19}$$

We can verify that, after integrating over θ, the following term gives the right kinetic energy terms for the superparticles:

$$-\frac{1}{2}mD^2R_i\,DR_i - \frac{1}{2}mD^2S_i\,DS_i \tag{2.20}$$

Considering now the vector potential $A_i(x_j)$ associated to the magnetic field:

$$B_{ij}(x_k) = \frac{\partial}{\partial x_i}A_j(x_k) - \frac{\partial}{\partial x_j}A_i(x_k)\ , \tag{2.21}$$

we extend it to a function of the supercoordinates as in (2.9). The coupling of the superparticles to the magnetic field is then given by:

$$iq\Big(DR_i\,A_i(R_k) - DS_i\,A_i(S_k)\Big) \tag{2.22}$$

Reverting back to the superfields X_i and U_i and introducing the terms involving the auxiliary superfields, the full superlagrangian becomes:

$$\begin{aligned} \mathbb{L} &= -mD^2X_i\,DX_i - \frac{1}{4}mD^2U_i\,DU_i \\ &+ iq\,DX_i\left(A_i(X_k - \frac{1}{2}U_k) - A_i(X_k + \frac{1}{2}U_k)\right) \\ &- \frac{i}{2}q\,DU_i\left(A_i(X_k - \frac{1}{2}U_k) + A_i(X_k + \frac{1}{2}U_k)\right) \\ &+ \frac{1}{2}m\omega_0^2\,D\Lambda_i\,\Lambda_i + \frac{1}{2}mk_0^2\,D\Gamma_i\,\Gamma_i \\ &+ m\omega_0^2\beta_0\,U_i\,\Lambda_i + mk_0^2\kappa_0\,X_i\,\Gamma_i \end{aligned} \tag{2.23}$$

[2] This remark is only valid for a $\mathcal{N} = 1$ supersymmetry. Indeed, for $\mathcal{N} \geq 2$ for instance, the supertime is spanned by two Grassmann odd variables in addition to the usual time t so that the superlagrangian is Grassmann even.

[3] We recall that X_i is associated to the center of mass of the system, while U_i to the relative position of the particles.

with β_0 and κ_0 being dimensionless real scaling factors. Up to a redefinition of the sign of the superfields Λ and Γ, their signs may be chosen arbitrarily. In the following, we will choose to take $\mathrm{Sign}(\beta_0) = \mathrm{Sign}(\kappa_0) = 1$ without any loss of generality. The integration over θ then produces the Lagrange function:

$$
\begin{aligned}
L &= m\dot{x}_i^2 + \frac{1}{4}m\dot{u}_i^2 - im\dot{\psi}_i\psi_i - \frac{i}{4}m\dot{\mu}_i\mu_i \\
&+ q\,\dot{x}_i\left(A_i(x_k - \frac{1}{2}u_k) - A_i(x_k + \frac{1}{2}u_k)\right) \\
&- \frac{q}{2}\,\dot{u}_i\left(A_i(x_k - \frac{1}{2}u_k) + A_i(x_k + \frac{1}{2}u_k)\right) \\
&+ \frac{i}{2}q\left(\psi_i\mu_j\right)\left(B_{ij}(x_k - \frac{1}{2}u_k) + B_{ij}(x_k + \frac{1}{2}u_k)\right) \\
&+ iq\left(\psi_i\psi_j + \frac{iq}{4}\mu_i\mu_j\right)\left(\partial_j A_i(x_k - \frac{1}{2}u_k) - \partial_j A_i(x_k + \frac{1}{2}u_k)\right) \\
&+ \frac{1}{2}m\omega_0^2\left(y_i^2 - i\dot{\lambda}_i\lambda_i\right) + \frac{1}{2}mk_0^2\left(z_i^2 - i\dot{\gamma}_i\gamma_i\right) \\
&+ m\omega_0^2\beta_0\left(u_iy_i + i\mu_i\lambda_i\right) + mk_0^2\kappa_0\left(x_iz_i + i\psi_i\gamma_i\right)
\end{aligned}
\tag{2.24}
$$

In order to be able to further study this Lagrangian, let us restrict ourself to a two dimensional space, and consider a constant, homogeneous and perpendicular magnetic field $B_{12} = B_0$. To keep the rotational covariance explicit, we also use the circular gauge for the vector potential:

$$
A_i(r_j) = -\frac{1}{2}B_0\,\epsilon_{ij}r_j \tag{2.25}
$$

The Lagrangian then simplifies to the following form:

$$
\begin{aligned}
L &= m\dot{x}_i^2 + \frac{1}{4}m\dot{u}_i^2 - im\dot{\psi}_i\psi_i - \frac{i}{4}m\dot{\mu}_i\mu_i \\
&+ \frac{q}{2}B_0\epsilon_{ij}\left(\dot{x}_iu_j - \dot{u}_jx_i + 2i\psi_i\mu_j\right) \\
&+ \frac{1}{2}m\omega_0^2\left(y_i^2 - i\dot{\lambda}_i\lambda_i\right) + \frac{1}{2}mk_0^2\left(z_i^2 - i\dot{\gamma}_i\gamma_i\right) \\
&+ m\omega_0^2\beta_0\left(u_iy_i + i\mu_i\lambda_i\right) + mk_0^2\kappa_0\left(x_iz_i + i\psi_i\gamma_i\right)
\end{aligned}
\tag{2.26}
$$

Note how this expression does not involve any time derivative of the bosonic auxiliary variables y_i and z_i, and how the latter are coupled to dynamical bosonic coordinates u_i and x_i, respectively.

2.3 Equations of Motion

Using the above Lagrange function the Euler-Lagrange equations of the system are:

$$\begin{aligned}
2m\,\epsilon_{ij}\,\ddot{x}_j - m\kappa_0 k_0^2\,\epsilon_{ij}\,z_j - qB\,\dot{u}_i &= 0\\
\frac{1}{2}m\,\ddot{u}_i - m\beta_0\omega_0^2\,y_i + qB\,\epsilon_{ij}\,\dot{x}_j &= 0\\
-m\beta_0\omega_0^2\,u_i - m\omega_0^2\,y_i &= 0\\
-m\kappa_0 k_0^2\,x_i - mk_0^2\,z_i &= 0\\
-im\kappa_0 k_0^2\,\epsilon_{ij}\,\gamma_j - 2im\,\epsilon_{ij}\,\dot{\psi}_j + iqB\,\mu_i &= 0\\
-im\beta_0\omega_0^2\,\lambda_i - \frac{1}{2}im\,\dot{\mu}_i - iqB\,\epsilon_{ij}\,\psi_j &= 0\\
im\beta_0\omega_0^2\,\mu_i - im\omega_0^2\,\dot{\lambda}_i &= 0\\
im\kappa_0 k_0^2\,\psi_i - imk_0^2\,\dot{\gamma}_i &= 0
\end{aligned} \tag{2.27}$$

As was pointed out earlier, the auxiliary bosonic degrees of freedom z_i and y_i are redundant being proportional to x_i and u_i respectively:

$$y_i = -\beta_0\,u_i\;, \qquad z_i = -\kappa_0\,x_i \tag{2.28}$$

This is how harmonic potential contributions arise for x_i and u_i. The last two equations of motion also show that the time evolution of the auxiliary fermionic degrees of freedom is specified by that of their associated fermionic non-auxiliary variables. Given this observation, we find the following independent set of equations of motion:

$$\begin{aligned}
2m\,\epsilon_{ij}\,\ddot{x}_j + m\kappa_0^2 k_0^2\,\epsilon_{ij}\,x_j - qB\,\dot{u}_i &= 0\\
\frac{m}{2}\,\ddot{u}_i + m\beta_0^2\omega_0^2\,u_i + qB\,\epsilon_{ij}\,\dot{x}_j &= 0\\
2m\,\epsilon_{ij}\,\ddot{\psi}_j + m\kappa_0^2 k_0^2\,\epsilon_{ij}\,\psi_j - qB\,\dot{\mu}_i &= 0\\
\frac{m}{2}\,\ddot{\mu}_i + m\beta_0^2\omega_0^2\,\mu_i + qB\,\epsilon_{ij}\,\dot{\psi}_j &= 0
\end{aligned} \tag{2.29}$$

Note the symmetry between the bosonic and fermionic sets of equations. These may be solved, leading to the following solutions for the bosonic degrees of freedom:

$$\begin{aligned}
x_i(t) &= i\frac{m}{2qB\omega_1}\left(2\beta_0^2\omega_0^2 - \omega_1^2\right)\,\epsilon_{ij}\left(A_j\,e^{-i\omega_1 t} - A_j^*\,e^{+i\omega_1 t}\right) + \left(B_i\,e^{-i\omega_2 t} + B_i^*\,e^{+i\omega_2 t}\right)\\
u_i(t) &= \left(A_i\,e^{-i\omega_1 t} + A_i^*\,e^{+i\omega_1 t}\right) + i\frac{2m}{qB\omega_2}\left(\frac{1}{2}\kappa_0^2 k_0^2 - \omega_2^2\right)\,\epsilon_{ij}\left(B_j\,e^{-i\omega_2 t} - B_j^*\,e^{+i\omega_2 t}\right)
\end{aligned} \tag{2.30}$$

with the following definitions of angular frequencies:

$$\begin{aligned}
\omega_1 &= \frac{1}{2}(\omega_+ - \omega_-)\\
\omega_2 &= \frac{1}{2}(\omega_+ + \omega_-)\\
\omega_+ &= \sqrt{\left(\frac{Bq}{m}\right)^2 + \frac{1}{2}\left(\kappa_0\,k_0\; + \;2\beta_0\,\omega_0\right)^2}\\
\omega_- &= \sqrt{\left(\frac{Bq}{m}\right)^2 + \frac{1}{2}\left(\kappa_0\,k_0\; - \;2\beta_0\,\omega_0\right)^2}
\end{aligned} \tag{2.31}$$

while A_i and B_i are arbitrary complex Grassmann even integration constants. It is useful to note the following identity (no summation on i):

$$\frac{m}{2qB\omega_i}\left(2\omega^2-\omega_i^2\right)=\frac{qB\omega_i}{2m}\left(\frac{k^2}{2}-\omega_i^2\right)^{-1} \tag{2.32}$$

The fermionic solutions have the same form as the bosonic ones, but the corresponding complex integration constants G_i and H_i are of course rather now Grassmann odd quantities:

$$\begin{aligned}\psi_i(t) &= i\frac{m}{2qB\omega_1}\left(2\beta_0^2\omega_0^2-\omega_1^2\right)\epsilon_{ij}\left(G_j\,e^{-i\omega_1 t}-G_j^*\,e^{+i\omega_1 t}\right)+\left(H_i\,e^{-i\omega_2 t}+H_i^*\,e^{+i\omega_2 t}\right)\\ \mu_i(t) &= \left(G_i\,e^{-i\omega_1 t}+G_i^*\,e^{+i\omega_1 t}\right)+i\frac{2m}{qB\omega_2}\left(\frac{1}{2}\kappa_0^2k_0^2-\omega_2^2\right)\epsilon_{ij}\left(H_j\,e^{-i\omega_2 t}-H_j^*\,e^{+i\omega_2 t}\right)\end{aligned} \tag{2.33}$$

Chapter 3

Hamiltonian Formulation

On the road towards the quantization of the system, we now have to move from the Lagrangian formalism to the Hamiltonian one. The classical description of the system will not be anymore in terms of positions and velocities, but rather in terms of the momentum phase space variables, that is the configuration space coordinates and their conjugate momenta, treated as independent variables. This is nothing new, but here we are in actual fact also dealing with a constrained system, that is, the momentum phase space variables are not all independent (think for instance to the bosonic auxiliary degrees of freedom). We have chosen to handle this particular feature using Dirac's constraints analysis to be described briefly. Then, we will study the rotational and supersymmetrical invariances of the system to compute the associated Noether charges.

3.1 Conjugate Momenta

To each degree of freedom of the supercoordinates, we associate a conjugate momentum:

$$\begin{aligned} x_i: \quad p_i &= \frac{\partial L}{\partial \dot{x}_i} = 2m\dot{x}_i + \frac{q}{2}B_0\epsilon_{ij}u_j \\ u_i: \quad \pi_i &= \frac{\partial L}{\partial \dot{u}_i} = \frac{1}{2}m\dot{u}_i + \frac{q}{2}B_0\epsilon_{ij}x_j \end{aligned}$$

$$\begin{aligned} z_i: \quad \zeta_i &= \frac{\partial L}{\partial \dot{z}_i} = 0 & y_i: \quad \varphi_i &= \frac{\partial L}{\partial \dot{y}_i} = 0 \\ \psi_i: \quad \chi_i &= \frac{\partial L}{\partial \dot{\psi}_i} = -im\psi_i & \mu_i: \quad \tau_i &= \frac{\partial L}{\partial \dot{\mu}_i} = -\frac{i}{4}m\mu_i \\ \gamma_i: \quad \xi_i &= \frac{\partial L}{\partial \dot{\gamma}_i} = -\frac{i}{2}mk_0^2\gamma_i & \lambda_i: \quad \eta_i &= \frac{\partial L}{\partial \dot{\lambda}_i} = -\frac{i}{2}m\omega_0^2\lambda_i \end{aligned}$$

From this, we see that the system is singular, in the sense that there is no one-to-one correspondence between the velocity phase space and the momentum phase space. In mathematical terms, this is encoded in the fact that the Hessian of the Lagrangian has a vanishing determinant:

$$\det\left[\frac{\partial^2 L}{\partial \dot{q}_i\,\partial \dot{q}_j}\right] = \det\left[\frac{\partial p_j}{\partial \dot{q}_i}\right] = 0 \tag{3.1}$$

where q_i denotes the coordinates of the system, and p_i their associated conjugate momenta. In such cases, the Lagrangian is said to be irregular. This means that the velocities of the system cannot all be uniquely determined from the momenta and vice-versa, so that we need to take into account the existence of the constraints originating from (3.1):

$$\begin{aligned}
\mathcal{M}_i &= \zeta_i\ , & \mathcal{M}'_i &= \varphi_i \\
\mathcal{F}_i &= \chi_i + im\psi_i\ , & \mathcal{F}'_i &= \tau_i + \frac{i}{4}m\mu_i \\
\mathcal{T}_i &= \xi_i + \frac{i}{2}mk_0^2\gamma_i\ , & \mathcal{T}'_i &= \eta_i + \frac{i}{2}m\omega_0^2\lambda_i
\end{aligned} \tag{3.2}$$

These restrictions on the phase space dynamics are referred to as *primary* constraints in the sense that they follow from the definitions of the conjugate momenta. Let us delay the discussion of the constraints until the next Section, and define the canonical Hamiltonian:

$$H_0 \equiv \dot{x}_i p_i + \dot{u}_i \pi_i + \dot{\psi}_i \chi_i + \dot{\mu}_i \tau_i + \dot{y}_i \varphi_i + \dot{z}_i \zeta_i + \dot{\lambda}_i \eta_i + \dot{\gamma}_i \xi_i - L \tag{3.3}$$

Using the relations (3.1), we get:

$$\begin{aligned}
H_0 &= \frac{1}{4m}\left(p_i - \frac{q}{2}B_0\epsilon_{ij}u_j\right)^2 + \frac{1}{m}\left(\pi_i - \frac{q}{2}B_0\epsilon_{ij}x_j\right)^2 \\
&- iqB_0\epsilon_{ij}\psi_i\mu_j - \frac{1}{2}m\omega_0^2 y_i^2 - \frac{1}{2}mk_0^2 z_i^2 \\
&- m\omega_0^2\beta_0\left(u_i y_i + i\mu_i\lambda_i\right) - mk_0^2\kappa_0\left(x_i z_i + i\psi_i\gamma_i\right)
\end{aligned} \tag{3.4}$$

From this Hamiltonian, the dynamic of the system is specified through the use of the Grassmann graded Poisson brackets. For two arbitrary momentum phase space functions F and G of respective Grassmann parity ϵ_F and ϵ_G, their Poisson bracket is denoted as $\{F, G\}$ and is defined by:

$$\{F, G\} = \sum_a (-1)^{\epsilon_a \epsilon_F}\left(\frac{\partial F}{\partial q_a}\frac{\partial G}{\partial p_a} - (-1)^{\epsilon_a}\frac{\partial F}{\partial p_a}\frac{\partial G}{\partial q_a}\right) \tag{3.5}$$

where the index a labels all momentum phase space variables (q_a, p_a) while ϵ_a denotes their Grassmann parity. The time evolution of any momentum phase space quantity $f(t, q_a, p_a)$ is then given by:

$$\frac{d}{dt}f = \frac{\partial}{\partial t}f + \{f, H_0\} \tag{3.6}$$

3.2 Dirac's Constraints Analysis

As pointed to before, since we work with an irregular Lagrangian, the complete momentum phase space has to be restricted to a surface specified by a collection of constraints

$$\phi_r(q_a, p_a) = 0\ , \qquad r = 1, ..., m \tag{3.7}$$

A possible approach to handle such constraints is to solve them explicitly and then work with independent degrees of freedom. However, this way of proceeding leads in general to intricate calculations and often turns to hide the covariance of the system under some of its symmetries. Therefore, here we will follow Dirac's analysis of constraints that is exposed for instance in [15] or [16]. Here we will only sketch the main ideas of this formalism without addressing all the general details, by restricting the discussion to the situation which is encountered in our system. The following considerations require the constraints to be locally independent and regular, that is, the matrix $\frac{\partial \phi^r}{\partial z^i}$ has to be finite and of maximal rank, even when the constraints are imposed.

The starting point is to generalize the time evolution generator of the system by extending the Hamiltonian to include a linear combination of the primary constraints ϕ_r:

$$H = H_0 + \sum_r u_r(t, q_a, p_a)\,\phi_r(q_a, p_a) \tag{3.8}$$

where the u_r's are would-be Lagrange multiplier functions of the momentum phase space and of time, and of the same Grassmann parity ϵ_r as their corresponding constraints. Note that the properties of the u_r's under complex conjugation have to be chosen such that H is real. This generalization is motivated by the fact that, on the constraint surface of the momentum phase space, H coincides with the canonical Hamiltonian. H and H_0 are said to be weakly equal:

$$H \approx H_0 \tag{3.9}$$

which means that the equality holds *after* the constraints have been imposed. Therefore, there is no obvious reason why we should not consider H as generating the time evolution of the system:

$$\begin{aligned} \frac{d}{dt}f &= \frac{\partial}{\partial t}f + \{f, H\} \\ &\approx \frac{\partial}{\partial t}f + \{f, H_0\} + \sum_r (-1)^{\epsilon_f \epsilon_r}\, u_r\, \{f, \phi_r\} \end{aligned} \tag{3.10}$$

However, for the description of the system to be consistent, the time evolution of the constraints themselves has to vanish so that they hold at any time:

$$0 = \dot{\phi}_r \approx \{\phi_r, H_0\} + \sum_s (-1)^{\epsilon_f \epsilon_s}\, u_s\, \{\phi_r, \phi_s\} \tag{3.11}$$

If these equations are not trivial, they either give some conditions on the Lagrange multipliers, or give rise to new constraints, called *secondary* constraints. The latter should then also be introduced in the generalized Hamiltonian, and the consistency check has to be reiterated. Repeating the operation until no new constraints appear, one ends up with a possibly larger set of constraints, still denoted as ϕ_r. A constraint is said to be *second-class* if its Poisson bracket with at least one other constraint does not vanish on the constraint surface, and *first-class* otherwise. Second-class constraints correspond to redundant degrees of freedom in the system description. The case which implies to the system considered in this work is when all constraints we end up with are second-class. All would-be Lagrange multipliers are then completely determined by the consistency equations. This being done, the Hamiltonian H generates a consistent time evolution of the system inside the constraint surface. However, from a practical point of view, we still have to deal with constraints. A convenient way to avoid this has been introduced by Dirac through the definition of the Dirac brackets:

$$\{f, g\}_D \equiv \{f, g\} - \sum_{r,r'} \{f, \phi_r\}\, C^{rr'}\, \{\phi_{r'}, g\} \tag{3.12}$$

with

$$C^{rr'} \equiv \Big(\{\phi_r, \phi_{r'}\} \Big)^{-1} \tag{3.13}$$

It can be shown that by using Dirac brackets instead of Poisson brackets allows to solve the second-class constraints to extract the independent degrees of freedom of the system and work with a restricted momentum phase space inside the constraint surface, thus "forgetting" about the constraints altogether.

Let us now apply the previous discussion to our system. We can first check that the set of primary constraints (3.2) is irreducible and that they all are regular. Then, we define the generalized Hamiltonian as follows:

$$\begin{aligned} H &= \frac{1}{4m}\left(p_i - \frac{q}{2}B_0\epsilon_{ij}u_j\right)^2 + \frac{1}{m}\left(\pi_i - \frac{q}{2}B_0\epsilon_{ij}x_j\right)^2 \\ &- iqB_0\epsilon_{ij}\psi_i\mu_j - \frac{1}{2}m\omega_0^2 y_i^2 - \frac{1}{2}mk_0^2 z_i^2 \\ &- m\omega_0^2\beta_0\left(u_i y_i + i\mu_i\lambda_i\right) - mk_0^2\kappa_0\left(x_i z_i + i\psi_i\gamma_i\right) \\ &+ \mathcal{U}_i\mathcal{M}_i + \mathcal{U}'_i\mathcal{M}'_i + \Phi_i\mathcal{F}_i + \Phi'_i\mathcal{F}'_i + \Xi_i\mathcal{T}_i + \Xi'_i\mathcal{T}'_i \end{aligned} \tag{3.14}$$

where $\mathcal{U}_i$ and $\mathcal{U}'_i$ are Grassmann even would-be Lagrange multipliers and Φ_i, Φ'_i, Ξ_i and Ξ'_i Grassmann odd would-be Lagrange multipliers, according to the Grassmann parity of their associated constraints. Requiring the consistency conditions under time evolution allows to determine some of the would-be Lagrange multipliers:

$$\begin{aligned}\Phi_i &= -\epsilon_{ij}\frac{qB}{2m}\mu_j - i\frac{\kappa_0}{m}\xi_i \\ \Phi'_i &= -\epsilon_{ij}\frac{2qB}{m}\psi_j - i4\frac{\beta_0}{m}\eta_i \\ \Xi_i &= i\frac{\kappa_0}{m}\chi_i \\ \Xi'_i &= i\frac{4\beta_0}{m}\tau_i\end{aligned} \tag{3.15}$$

while also leading to the following additional or secondary constraints:

$$\mathcal{N}_i = y_i + \beta_0 u_i\ , \qquad \mathcal{N}'_i = z_i + \kappa_0 x_i \tag{3.16}$$

Introducing two new Grassmann even would-be Lagrange multipliers $\mathcal{V}_i$ and $\mathcal{V}'_i$, these secondary constraints are included in the generalized Hamiltonian through the following term:

$$\mathcal{V}_i\mathcal{N}_i + \mathcal{V}'_i\mathcal{N}'_i \tag{3.17}$$

Reiterating the requirement of the consistency conditions, one finds:

$$\begin{aligned}\mathcal{U}_i &= \epsilon_{ij}\frac{qB\kappa_0}{4m}u_j - \frac{\kappa_0}{2m}p_i \\ \mathcal{U}'_i &= \epsilon_{ij}\frac{qB\beta_0}{m}x_j - \frac{2\beta_0}{m}\pi_i \\ \mathcal{V}_i &= 0 \\ \mathcal{V}'_i &= 0\end{aligned} \tag{3.18}$$

The reiteration process thus terminates since we do not have any new constraint; all would-be Lagrange multipliers have been determined. Again, we can check that the set of all primary and secondary constraints is irreducible and that they are regular. The constraints are also all second-class, since otherwise some Lagrange multipliers would have left arbitrary. We can therefore reduce the redundant degrees of freedom, and compute the relevant Dirac brackets to carry on the analysis of our system taking only the independent degrees of freedom into account. We end up with 8 bosonic and 8 fermionic degrees of freedom. The non vanishing Dirac brackets of the variables spanning the restricted phase space are given by:

$$\begin{aligned}\{x_i, p_j\}_D &= \delta_{ij}\ , & \{u_i, \pi_j\}_D &= \delta_{ij} \\ \{\psi_i, \psi_j\}_D &= -\frac{i}{2m}\delta_{ij}\ , & \{\mu_i, \mu_j\}_D &= -\frac{2i}{m}\delta_{ij} \\ \{\lambda_i, \lambda_j\}_D &= -\frac{i}{m\omega_0^2}\delta_{ij}\ , & \{\gamma_i, \gamma_j\}_D &= -\frac{i}{mk_0^2}\delta_{ij}\end{aligned} \tag{3.19}$$

which, for the quantized system, leads to a Heisenberg algebra for the bosonic degrees of freedom, and a (non normalized) Clifford algebra for the fermionic ones. In the following, since we will only use Dirac brackets, we will omit the D subscript. The Hamiltonian of the system reduced to the constraint surface is given by:

$$\begin{aligned}H &= \frac{1}{4m}\left(p_i - \frac{qB}{2}\epsilon_{ij}u_j\right)^2 + \frac{1}{m}\left(\pi_i - \frac{qB}{2}\epsilon_{ij}x_j\right)^2 - iqB\epsilon_{ij}\psi_i\mu_j \\ &+ \frac{1}{2}m\kappa_0^2k_0^2\,x_i^2 + \frac{1}{2}m\beta_0^2\omega_0^2\,u_i^2 - im\kappa_0k_0^2\,\psi_i\gamma_i - im\beta_0\omega_0^2\,\mu_i\lambda_i\end{aligned} \tag{3.20}$$

As expected, we now see explicitly the positive definite harmonic potentials for the bosonic variables x_i and u_i.

3.3 Symmetries

We will now identify the Noether charges associated to the continuous symmetries of the system. We consider a symmetry transformation parametrized by Grassmann even or odd parameters ϵ_a. The fact it is a symmetry of the system means that the action has to transform at most by a surface term (in time), that is, the Lagrangian is changed by a total time derivative. In infinitesimal form, the transformation is then expressed as:

$$\begin{aligned} t &\to t + \epsilon_a \chi^a \\ q_n &\to q_n + \epsilon_a \phi_n^a \\ L &\to L + \epsilon_a \frac{d}{dt} G^a \end{aligned} \qquad (3.21)$$

Because it is a symmetry transformation, it is associated to conserved quantities, the Noether charges, given by: [16]

$$Q^a = G^a - \chi^a L - \sum_n (\phi_n^a - \chi^a \dot{q}_n) \frac{\partial L}{\partial \dot{q}_n} \qquad (3.22)$$

These Noether charges are the generators of the symmetry transformation since, acting on a phase space variable through the Dirac brackets, the Q^a's precisely generate the previous transformation. Furthermore under Dirac brackets, the Noether charges also obey the Lie algebra of their symmetry group, possibly including a central extension. Notice that, in this Section, the study of each symmetry is initiated from the Lagrangian point of view – inclusived of the redundant degrees of freedom – to finally identify the Noether charges on the reduced momentum phase space.

3.3.1 Rotational invariance

By construction, the system is invariant under $SO(2)$ rotations in the plane. There is a unique Grassmann even parameter ϵ_R for these transformations which, in infinitesimal form, are given by:

$$\begin{aligned} \delta x_i &= \epsilon_R \epsilon_{ij} x_j \,, & \delta u_i &= \epsilon_R \epsilon_{ij} u_j \\ \delta \psi_i &= \epsilon_R \epsilon_{ij} \psi_j \,, & \delta \mu_i &= \epsilon_R \epsilon_{ij} \mu_j \\ \delta z_i &= \epsilon_R \epsilon_{ij} z_j \,, & \delta y_i &= \epsilon_R \epsilon_{ij} y_j \\ \delta \gamma_i &= \epsilon_R \epsilon_{ij} \gamma_j \,, & \delta \lambda_i &= \epsilon_R \epsilon_{ij} \lambda_j \end{aligned} \qquad (3.23)$$

The time coordinate is not affected, and the Lagrangian remains unmodified. Given the general formula (3.22), and using the constraints of the previous Sections, the angular momentum, namely the generator of rotations, acquires the following expression in terms of the reduced momentum phase space variables:

$$L_{Noether} = \epsilon_{ij}\, x_i\, p_j + \epsilon_{ij}\, u_i\, \pi_j - im\epsilon_{ij}\, \psi_i\, \psi_j - \frac{i}{4} m\epsilon_{ij}\, \mu_i\, \mu_j - \frac{i}{2} m\omega_0^2 \epsilon_{ij}\, \lambda_i\, \lambda_j - \frac{i}{2} mk_0^2 \epsilon_{ij}\, \gamma_i\, \gamma_j \qquad (3.24)$$

As a cross check, one may explicitly verify that $L_{Noether}$ indeed generates the transformations (3.23) while it is also a conserved quantity as it should:

$$\{H,\, L_{Noether}\} = 0 \qquad (3.25)$$

3.3.2 Supersymmetry invariance

According to (2.15), under a supersymmetry transformation a superfield F_i transforms as:

$$F_i \to F_i - i\,\epsilon_Q\,Q\,F_i \tag{3.26}$$

where ϵ_Q is a Grassmann odd parameter. This gives, for the bosonic and fermionic coordinates of the system:

$$\begin{aligned}
\delta x_i &= \epsilon_Q\,\psi_i\,, & \delta u_i &= \epsilon_Q\,\mu_i \\
\delta \psi_i &= \epsilon_Q\,(i\,\dot{x}_i)\,, & \delta \mu_i &= \epsilon_Q\,(i\,\dot{u}_i) \\
\delta z_i &= \epsilon_Q\,(-\dot{\gamma}_i)\,, & \delta y_i &= \epsilon_Q\,(-\dot{\lambda}_i) \\
\delta \gamma_i &= \epsilon_Q\,(-i\,z_i)\,, & \delta \lambda_i &= \epsilon_Q\,(-i\,y_i)
\end{aligned} \tag{3.27}$$

As noted in Subsection (2.1.4), by construction the action is invariant under a supersymmetry transformation, however up to a surface term. As was explained, if we define $\mathbb{L} = \mathbb{L}_1 + \theta\,\mathbb{L}_2$, the transformation of the Lagrangian is given by:

$$\delta_{\epsilon_Q} L = \epsilon_Q \left(-\frac{d}{dt}\mathbb{L}_1 \right) \tag{3.28}$$

Given (2.23), one finds:

$$\begin{aligned}
\mathbb{L}_1 \;=\; & -m\,\dot{x}_i\,\psi_i - \frac{1}{4}m\,\dot{u}_i\,\mu_i + \frac{qB}{2}\epsilon_{ij}\,\left(x_i\,\mu_j + u_i\,\psi_j\right) \\
& + \frac{1}{2}m\omega_0^2\,y_i\,\lambda_i + \frac{1}{2}mk_0^2\,z_i\,\gamma_i + m\omega_0^2\beta_0\,u_i\,\lambda_i + mk_0^2\kappa_0\,x_i\,\gamma_i
\end{aligned} \tag{3.29}$$

Using (3.22), the Noether charge associated to the invariance of the system under supersymmetry transformations is then given by:

$$Q_{Noether} = -\mathbb{L}_1 - \psi_i\,\frac{\partial L}{\partial \dot{x}_i} - \mu_i\,\frac{\partial L}{\partial \dot{u}_i} - m\,\dot{x}_i\,\psi_i - \frac{1}{4}m\,\dot{u}_i\,\mu_i + \frac{1}{2}m\omega_0^2\,y_i\,\lambda_i + \frac{1}{2}mk_0^2\,z_i\,\gamma_i \tag{3.30}$$

Substituting the expression of $\mathbb{L}_1$ and using the Hamiltonian constraints, we finally establish:

$$Q_{Noether} = -p_i\,\psi_i - \pi_i\,\mu_i - m\beta_0\omega_0^2\,u_i\,\lambda_i - m\kappa_0 k_0^2\,x_i\,\gamma_i - \frac{qB}{2}\epsilon_{ij}\,u_i\,\psi_j - \frac{qB}{2}\epsilon_{ij}\,x_i\,\mu_j \tag{3.31}$$

Again, one may verify that $Q_{Noether}$ generates the supersymmetric transformations (3.27). We also have the following Dirac brackets:

$$\begin{aligned}
\{H,\,Q_{Noether}\} &= 0 \\
\{L,\,Q_{Noether}\} &= 0 \\
\{Q_{Noether},\,Q_{Noether}\} &= -2i\,H
\end{aligned} \tag{3.32}$$

Notice that, for "historical reasons", we will use $-Q_{Noether}$ in the following, instead of the previous definition (3.31).

Chapter 4

Quantization

We now promote each degree of freedom to an operator acting on a Hilbert space to be defined. Following the canonical quantization procedure, we define the (anti-)commutation relations as ($i\hbar$) times the Dirac bracket of the classical quantities:

$$\begin{array}{rclcrcl} [\hat{x}_i, \hat{p}_j] &=& i\hbar\,\delta_{ij}\,\mathbb{I}\,, & & [\hat{u}_i, \hat{\pi}_j] &=& i\hbar\,\delta_{ij}\,\mathbb{I} \\ \{\hat{\psi}_i, \hat{\psi}_j\} &=& \frac{\hbar}{2m}\,\delta_{ij}\,\mathbb{I}\,, & & \{\hat{\mu}_i, \hat{\mu}_j\} &=& \frac{2\hbar}{m}\,\delta_{ij}\,\mathbb{I} \\ \{\hat{\lambda}_i, \hat{\lambda}_j\} &=& \frac{\hbar}{m\omega_0^2}\,\delta_{ij}\,\mathbb{I}\,, & & \{\hat{\gamma}_i, \hat{\gamma}_j\} &=& \frac{\hbar}{mk_0^2}\,\delta_{ij}\,\mathbb{I} \end{array} \tag{4.1}$$

where $i, j = 1, 2$ and all the operators are their own hermitian conjugate. Let us recall the expression of the Hamiltonian, the angular momentum and the supersymmetric Noether charge operators:

$$\begin{aligned} \hat{H} &= \frac{1}{4m}\left(\hat{p}_i - \frac{qB}{2}\epsilon_{ij}\hat{u}_j\right)^2 + \frac{1}{m}\left(\hat{\pi}_i - \frac{qB}{2}\epsilon_{ij}\hat{x}_j\right)^2 - iqB\epsilon_{ij}\hat{\psi}_i\hat{\mu}_j \\ &+ \frac{1}{2}m\kappa_0^2k_0^2\,\hat{x}_i^2 + \frac{1}{2}m\beta_0^2\omega_0^2\,\hat{u}_i^2 - im\kappa_0k_0^2\,\hat{\psi}_i\hat{\gamma}_i - im\beta_0\omega_0^2\,\hat{\mu}_i\hat{\lambda}_i \\ \hat{L} &= \epsilon_{ij}\,\hat{x}_i\,\hat{p}_j + \epsilon_{ij}\,\hat{u}_i\,\hat{\pi}_j - im\epsilon_{ij}\,\hat{\psi}_i\,\hat{\psi}_j - \frac{i}{4}m\epsilon_{ij}\,\hat{\mu}_i\,\hat{\mu}_j - \frac{i}{2}m\omega_0^2\epsilon_{ij}\,\hat{\lambda}_i\,\hat{\lambda}_j - \frac{i}{2}mk_0^2\epsilon_{ij}\,\hat{\gamma}_i\,\hat{\gamma}_j \\ \hat{Q} &= \hat{p}_i\,\hat{\psi}_i + \hat{\pi}_i\,\hat{\mu}_i + m\beta_0\omega_0^2\,\hat{u}_i\,\hat{\lambda}_i + m\kappa_0k_0^2\,\hat{x}_i\,\hat{\gamma}_i + \frac{qB}{2}\epsilon_{ij}\,\hat{u}_i\,\hat{\psi}_j + \frac{qB}{2}\epsilon_{ij}\,\hat{x}_i\,\hat{\mu}_j \end{aligned} \tag{4.2}$$

In the following, we will omit the hat used to distinguish an operator from the corresponding classical quantity.

4.1 Hamiltonian

First, we should note that the bosonic and fermionic sectors of the Hamiltonian are decoupled from each other. Furthermore, using the fact that $\epsilon_{ij} = -\epsilon_{ji}$ and that $(\epsilon_{ij})^2 = 1$ (no summation on i and j), we can write the Hamiltonian as follows:

$$\begin{aligned} H &= \tfrac{1}{m}\left(\pi_i - \tfrac{qB}{2}\epsilon_{ij}x_j\right)^2 + \tfrac{1}{4m}\left(\epsilon_{ij}\,p_j + \tfrac{qB}{2}u_i\right)^2 - iqB\,\mu_i\,(\epsilon_{ij}\psi_j) \\ &+ \tfrac{1}{2}m\beta_0^2\omega_0^2\,u_i^2 + \tfrac{1}{2}m\kappa_0^2k_0^2\,(\epsilon_{ij}x_j)^2 - im\kappa_0k_0^2\,(\epsilon_{ij}\psi_j)\,(\epsilon_{ik}\gamma_k) - im\beta_0\omega_0^2\,\mu_i\lambda_i \\ &\equiv H_1 + H_2 \end{aligned} \tag{4.3}$$

with H_1 involving only the the bosonic operators u_1, π_1, x_2, p_2 and the fermionic operators μ_1, λ_1, ψ_2, γ_2 ; and H_2 involving only u_2, π_2, x_1, p_1 and μ_2, λ_2, ψ_1, γ_1. This splitting corresponds to the two chiral sectors and will prove to be useful in the following.

4.1.1 Bosonic sector

Let us first focus on the bosonic sector of the Hamiltonian. Since we are dealing with a quadratic Hamiltonian, we would like to follow the usual procedure of constructing annihilation and creation Fock algebra operators. We could use the solutions to the equations of motion in order to guess the form of those operators, which has been done, but it appears not to be a useful technique. The expressions become quickly highly intricate, especially when expressing and diagonalizing the supersymmetry Noether charge operator. Furthermore, that way of proceeding tends to hide the reason why such complicated expressions arise in this system in contrast with the Landau problems with or without a harmonic potential that have been discussed in [7] and [8]. As we will see, the subtlety is due to the two different frequencies ω_0 and k_0 appearing in the two harmonic potentials.

Before rushing into the calculation, it is useful and inspiring to take some time to motivate the general idea behind our following construction of Fock algebra operators. Suppose we have found a convenient set of such operators $a_i, a_i^\dagger$ for $i = 1, ..., 4$. They have to fulfil the following conditions:

$$\begin{aligned} [a_i, a_j^\dagger] &= \delta_{ij}\,\mathbb{I} \\ [a_i, a_j] &= [a_i^\dagger, a_j^\dagger] = 0 \\ H_b &= \sum_i \Omega_i a_i^\dagger a_i + C \end{aligned} \tag{4.4}$$

where C is a constant, and H_b is the bosonic part of the Hamiltonian. In order to avoid to deal with too complicated equations, we will not diagonalize directly the Hamiltonian, but rather relax the third condition. We will thus proceed in two steps, corresponding to two successive linear combinations of the bosonic degrees of freedom:

1. impose the commutation relations and a Hamiltonian expression of the form

$$H_b = \sum_{ij} M_{ij}\,\alpha_i^\dagger \alpha_j + C \tag{4.5}$$

2. finalize the diagonalization of the Hamiltonian.

First change of variables

To get further insight into the form of such operators, let us write them in the following form:

$$\begin{aligned} \alpha_i &= A_i + iB_i \\ \alpha_i^\dagger &= A_i - iB_i \end{aligned} \tag{4.6}$$

with A_i and B_i being hermitian operators. We then have (no summation over i):

$$[\alpha_i, \alpha_i^\dagger] = -2i[A_i, B_i] \tag{4.7}$$

Now, let us see what these basic considerations imply for the construction of the Fock operators. From those concerning the commutator of α_i and $\alpha_i^\dagger$, we get that (no summation on i)

$$[A_i, B_i] = \frac{i}{2}\,\mathbb{I}, \tag{4.8}$$

which in turn implies that $\alpha_i^\dagger \alpha_i = A_i^2 + B_i^2$ (no summation on i). From the third condition in (4.4), because the kinetic energy terms of our Hamiltonian couple u_i to $\epsilon_{ij}\, p_j$ (respectively, π_i to $\epsilon_{ij}\, x_j$), we get that they both need to appear in the expression of at least one of the A_i's or B_i's, say in A_k for the sake of the argument. In order for (4.8) to hold, B_k has to involve π_i or

$\epsilon_{ij}\, x_j$ (respectively, u_i or $\epsilon_{ij}\, p_j$). This leads us to the conclusion that we will need the following more general form for the Fock algebra operators:

$$\begin{aligned}
\alpha_i &= A u_i + F\,(\epsilon_{ij}\, p_j) + i\,(C\pi_i + D\,(\epsilon_{ij}\, x_j)) \\
\alpha_i^\dagger &= A u_i + F\,(\epsilon_{ij}\, p_j) - i\,(C\pi_i + D\,(\epsilon_{ij}\, x_j)) \\
\beta_i &= A' u_i + F'\,(\epsilon_{ij}\, p_j) + i\,(C'\pi_i + D'\,(\epsilon_{ij}\, x_j)) \\
\beta_i^\dagger &= A' u_i + F'\,(\epsilon_{ij}\, p_j) - i\,(C'\pi_i + D'\,(\epsilon_{ij}\, x_j))
\end{aligned} \tag{4.9}$$

with A, F, C and D being real constant parameters. We should notice that the fact that H_1 and H_2 have the same form justifies the equality of the coefficients for $i = 1, 2$. We have the following inverse relations:

$$\begin{aligned}
u_i &= \frac{F'\left(\alpha_i^\dagger + \alpha_i\right) - F\left(\beta_i^\dagger + \beta_i\right)}{2AF' - 2FA'} \\
\pi_i &= \frac{i\left(D'\left(\alpha_i^\dagger - \alpha_i\right) + D\left(\beta_i - \beta_i^\dagger\right)\right)}{2CD' - 2DC'} \\
x_i &= -i\,\epsilon_{ij}\frac{\left(C'\left(\alpha_j - \alpha_j^\dagger\right) + C\left(\beta_j^\dagger - \beta_j\right)\right)}{2CD' - 2DC'} \\
p_i &= -\epsilon_{ij}\frac{A\left(\beta_j^\dagger + \beta_j\right) - A'\left(\alpha_j^\dagger + \alpha_j\right)}{2AF' - 2FA'}
\end{aligned} \tag{4.10}$$

Imposing the commutation relations of (4.4), we get the following system of six equations:

$$\begin{aligned}
2\hbar(AC + FD) &= 1 = 2\hbar\,(A'C' - F'D') \\
\hbar\,(CA' - AC' - DF' + FD') &= 0 = \hbar\,(CA' + AC' - DF' - FD') \\
\hbar\,(-CA' - AC' + DF' + FD') &= 0 = \hbar\,(-CA' + AC' + DF' - FD')
\end{aligned} \tag{4.11}$$

Amongst the solutions of that system, we will pick out the ones corresponding to F and C' equal to zero and $A = \frac{1}{2C\hbar}$, $A' = -\frac{D}{2C\hbar D'}$ and $F' = -\frac{1}{2\hbar D'}$, leaving three free parameters. Substituting these expressions in the Hamiltonian, and imposing the coefficients of the terms $\alpha_i\alpha_j$ and $\alpha_i^\dagger\alpha_j^\dagger$ to vanish according to (4.5), we get the following system:

$$\begin{aligned}
\hbar^2 q^2 B^2 C^4 - 4\hbar^2 qB\, C^3 D\hbar^2 + 4\hbar^2 C^2 D^2 + 8m^2\beta_0^2\omega_0^2\hbar^2 C^4 - 4 = 0 \\
-q^2 B^2 C^2 - 4\,qB\,CD + 4\hbar^2 C^2 D'^4 - 2m^2\kappa_0^2 k_0^2\, C^2 - 4D^2 = 0 \\
-\hbar^2\, qB\, C^3 D'^2 + qB\, C + 2\hbar^2 C^2 D D'^2 + 2D = 0
\end{aligned} \tag{4.12}$$

Solving it gives the following unique solution for the values of the parameters, up to some global signs:

$$A = \frac{1}{\sqrt{2}}\sqrt{\frac{m\beta_0\omega_0\omega_+}{\hbar\,(k_0\kappa_0 + 2\beta_0\omega_0)}}, \qquad A' = -\frac{Bq}{4\sqrt{k_0 m\hbar\,\kappa_0\omega_+}}\frac{(k_0\kappa_0 - 2\beta_0\omega_0)}{\sqrt{(k_0\kappa_0 + 2\beta_0\omega_0)}}$$

$$C = \frac{1}{\sqrt{2}}\sqrt{\frac{k_0\kappa_0 + 2\beta_0\omega_0}{m\hbar\beta_0\omega_0\omega_+}}, \qquad F' = -\frac{1}{2}\sqrt{\frac{k_0\kappa_0 + 2\beta_0\omega_0}{k_0 m\hbar\kappa_0\omega_+}}$$

$$D = \frac{Bq}{2\sqrt{2}\sqrt{m\hbar\,\omega_+}}\frac{(k_0\kappa_0 - 2\beta_0\omega_0)}{\sqrt{\beta_0\omega_0\,(k_0\kappa_0 + 2\beta_0\omega_0)}}, \qquad D' = \sqrt{\frac{k_0 m\kappa_0\omega_+}{\hbar\,(k_0\kappa_0 + 2\beta_0\omega_0)}}$$

$$\tag{4.13}$$

where ω_+ is defined as in (2.31). Substituting these values and the combinations (4.10) in the Hamiltonian, we obtain the expression:

$$\begin{aligned} H_b &= \left(\frac{2\hbar\beta_0\omega_0\omega_+}{k_0\kappa_0 + 2\beta_0\omega_0}\right) \alpha_i^\dagger \alpha_i + \left(\frac{\hbar k_0\kappa_0\omega_+}{k_0\kappa_0 + 2\beta_0\omega_0}\right) \beta_i^\dagger \beta_i \\ &- \left(\frac{\sqrt{2}Bq\hbar\sqrt{k_0\beta_0\kappa_0\omega_0}}{m\,(k_0\kappa_0 + 2\beta_0\omega_0)}\right) \left(\alpha_i^\dagger \beta_i + \beta_i^\dagger \alpha_i\right) + \hbar\omega_+ \end{aligned} \tag{4.14}$$

As expected, we have the right commutators:

$$[\alpha_i, \alpha_j^\dagger] = \delta_{ij}\,\mathbb{I} = [\beta_i, \beta_j^\dagger], \tag{4.15}$$

the remaining ones all being equal to zero.

Final diagonalization

We may now apply a second change of variables to remove the remaining non-diagonal terms in the Hamiltonian. It is straightforward to check that the previous commutation relations do not change under the following change of variables:

$$\begin{aligned} a_i &= \frac{1}{\sqrt{2}}\left(\sqrt{1-\xi}\,\alpha_i - \sqrt{1+\xi}\,\beta_i\right) \\ b_i &= \frac{1}{\sqrt{2}}\left(\sqrt{1+\xi}\,\alpha_i + \sqrt{1-\xi}\,\beta_i\right) \\ a_i^\dagger &= \frac{1}{\sqrt{2}}\left(\sqrt{1-\xi}\,\alpha_i^\dagger - \sqrt{1+\xi}\,\beta_i^\dagger\right) \\ b_i^\dagger &= \frac{1}{\sqrt{2}}\left(\sqrt{1+\xi}\,\alpha_i^\dagger + \sqrt{1-\xi}\,\beta_i^\dagger\right) \end{aligned} \tag{4.16}$$

where ξ is a real parameter such that $\xi^2 < 1$. We then have:

$$[a_i, a_j^\dagger] = \delta_{ij}\,\mathbb{I} = [b_i, b_j^\dagger] \tag{4.17}$$

with the other commutators all being zero. The inverse transformations are given by:

$$\begin{aligned} \alpha_i &= \frac{1}{\sqrt{2}}\left(\sqrt{1-\xi}\,a_i + \sqrt{\xi+1}\,b_i\right) \\ \beta_i &= \frac{1}{\sqrt{2}}\left(\sqrt{1-\xi}\,b_i - \sqrt{\xi+1}\,a_i\right) \\ \alpha_i^\dagger &= \frac{1}{\sqrt{2}}\left(\sqrt{1-\xi}\,a_i^\dagger + \sqrt{\xi+1}\,b_i^\dagger\right) \\ \beta_i^\dagger &= \frac{1}{\sqrt{2}}\left(\sqrt{1-\xi}\,b_i^\dagger - \sqrt{\xi+1}\,a_i^\dagger\right) \end{aligned} \tag{4.18}$$

If we now substitute these expressions in the Hamiltonian and require the non-diagonal terms to vanish, we get a specific value for ξ given by:

$$\xi = s\,\frac{(\kappa_0\,k_0 - 2\beta_0\,\omega_0)\ \omega_+}{(\kappa_0\,k_0 + 2\beta_0\,\omega_0)\ \omega_-} \tag{4.19}$$

with $s = \mathrm{Sign}(qB)$. Note that it is readily checked that this solution for ξ does indeed meet the condition $\xi^2 < 1$. The expressions (4.18) finally lead to the following diagonal bosonic Hamiltonian:

$$H_b = \hbar\,\Omega_+\,a_i^\dagger\,a_i + \hbar\,\Omega_-\,b_i^\dagger\,b_i + \hbar\,\omega_+ \tag{4.20}$$

with

$$\begin{aligned}\Omega_+ &= \frac{1}{2}\,(\omega_+ + s\,\omega_-) \\ \Omega_- &= \frac{1}{2}\,(\omega_+ - s\,\omega_-)\end{aligned} \tag{4.21}$$

It is interesting to compare our system to the one studied in [8], that is, a supersymmetric extension of a system of one charged particle constrained to move in a plane, subjected to a constant, homogeneous and perpendicular magnetic field, and confined in a harmonic potential. Naively, one could have expected that our system would be equivalent to two copies of that simpler one. However, this is only true in the specific case where the two frequencies ω_0 and k_0 of the harmonic potentials are related as follows:

$$\kappa_0\,k_0 = 2\,\beta_0\,\omega_0 \tag{4.22}$$

In this case, many of the previous expressions simplify so that we recover the results found in [8]. The reason why our general system exhibits a richer structure is that the center of mass x_i of the system and the relative position u_i of the particles are coupled by the magnetic field while they are subjected to different harmonic potentials.

4.1.2 Fermionic sector

We will now focus on the fermionic sector of the Hamiltonian. Again, rushing into the diagonalization without any further considerations leads to complicated expressions hiding some characteristic properties of the system. Instead, we will use the supersymmetry of the system to guess the convenient linear combination that diagonalizes the fermionic Hamiltonian. Let us first rescale the fermionic degrees of freedom:

$$\begin{aligned}\psi'_i &= \sqrt{\tfrac{4m}{\hbar}}\,\psi_i\,, & \mu'_i &= \sqrt{\tfrac{m}{\hbar}}\,\mu_i \\ \lambda'_i &= \sqrt{\tfrac{2m\omega_0^2}{\hbar}}\,\lambda_i\,, & \gamma'_i &= \sqrt{\tfrac{2mk_0^2}{\hbar}}\,\gamma_i\end{aligned} \tag{4.23}$$

$$\psi_i'^\dagger = \psi'_i\,, \qquad \mu_i'^\dagger = \mu'_i\,, \qquad \lambda_i'^\dagger = \lambda'_i\,, \qquad \gamma_i'^\dagger = \gamma_i \tag{4.24}$$

so that the anti-commutation relations give a $SO(8)$ Clifford algebra:

$$\begin{aligned}\{\psi'_i, \psi'_j\} &= 2\,\delta_{ij}\,\mathbb{I} = \{\mu'_i, \mu'_j\} \\ \{\lambda'_i, \lambda'_j\} &= 2\,\delta_{ij}\,\mathbb{I} = \{\gamma'_i, \gamma'_j\}\end{aligned} \tag{4.25}$$

The fermionic part of the Hamiltonian now reads as:

$$H_f = -\frac{i\hbar}{4m}\Big[\sqrt{2}\,m\Big(\kappa_0\,k_0\,\Big(\epsilon_{ij}\,\psi'_j\Big)\,\Big(\epsilon_{ij}\,\gamma'_j\Big) + 2\,\beta_0\,\omega_0\;\mu'_i\;\lambda'_i\Big) - 2\,qB\,\Big(\epsilon_{ij}\,\psi'_j\Big)\;\mu'_i\Big] \tag{4.26}$$

Since we will need it in this Section, it is useful to compute the charge Q in terms of the new bosonic operators:

$$
\begin{aligned}
Q \;=\;& \frac{s\,qB\hbar\,\beta_0\omega_0\,\Omega_+}{2m\sqrt{s\,\beta_0\,\omega_0\omega_-\omega_+\,(2\beta_0\,\omega_0\,\Omega_+\,-\,\kappa_0\,k_0\,\Omega_-)}}\;\left(a_i\left(\epsilon_{ij}\,\psi'_j\right)\,+\,a_i^\dagger\left(\epsilon_{ij}\,\psi'_j\right)\right)\\
-\;& \frac{s\,qB\hbar\,\beta_0\omega_0\,\Omega_-}{2m\sqrt{s\,\beta_0\,\omega_0\omega_-\omega_+\,(\kappa_0\,k_0\,\Omega_+\,-\,2\beta_0\,\omega_0\,\Omega_-)}}\;\left(b_i\left(\epsilon_{ij}\,\psi'_j\right)\,+\,b_i^\dagger\left(\epsilon_{ij}\,\psi'_j\right)\right)\\
-\;& \frac{i\,s\,qB\,\hbar\,\kappa_0k_0\,\Omega_+}{2\sqrt{2}m\sqrt{s\,\kappa_0\,k_0\omega_-\omega_+\,(\kappa_0\,k_0\,\Omega_+\,-\,2\beta_0\,\omega_0\,\Omega_-)}}\;\left(a_i\,\mu'_i\,-\,a_i^\dagger\,\mu'_i\right)\\
-\;& \frac{i\,s\,qB\,\hbar\,\kappa_0k_0\,\Omega_-}{2\sqrt{2}m\sqrt{s\,\kappa_0\,k_0\omega_-\omega_+\,(2\beta_0\,\omega_0\,\Omega_+\,-\,\kappa_0\,k_0\,\Omega_-)}}\;\left(b_i\,\mu'_i\,-\,b_i^\dagger\,\mu'_i\right)\\
+\;& \frac{\hbar}{2\sqrt{2}}\sqrt{\frac{s\,\beta_0\,\omega_0}{\omega_+\,\omega_-}\,(2\beta_0\,\omega_0\,\Omega_+\,-\,\kappa_0\,k_0\,\Omega_-)}\;\left(a_i\,\lambda'_i+a_i^\dagger\,\lambda'_i\right)\\
+\;& \frac{\hbar}{2\sqrt{2}}\sqrt{\frac{s\,\beta_0\,\omega_0}{\omega_+\,\omega_-}\,(\kappa_0\,k_0\,\Omega_+\,-\,2\beta_0\,\omega_0\,\Omega_-)}\;\left(b_i\,\lambda'_i+b_i^\dagger\,\lambda'_i\right)\\
+\;& \frac{i\hbar}{4}\sqrt{\frac{s\,\kappa_0\,k_0}{\omega_+\,\omega_-}\,(\kappa_0\,k_0\,\Omega_+\,-\,2\beta_0\,\omega_0\,\Omega_-)}\;\left(a_i\left(\epsilon_{ij}\,\gamma'_j\right)\,-\,a_i^\dagger\left(\epsilon_{ij}\,\gamma'_j\right)\right)\\
-\;& \frac{i\hbar}{4}\sqrt{\frac{s\,\kappa_0\,k_0}{\omega_+\,\omega_-}\,(2\beta_0\,\omega_0\,\Omega_+\,-\,\kappa_0\,k_0\,\Omega_-)}\;\left(b_i\left(\epsilon_{ij}\,\gamma'_j\right)\,-\,b_i^\dagger\left(\epsilon_{ij}\,\gamma'_j\right)\right)
\end{aligned}
\tag{4.27}
$$

The point that should be stressed here is that each term involves the product of one bosonic and one fermionic operator which, in particular, commute. This form suggests that the commutator of Q with a bosonic operator is a purely fermionic operator, made of a linear combination of the fermionic degrees of freedom. This will indeed turn to be the right way to diagonalize the fermionic Hamiltonian. We therefore introduce the following operators:

$$
\begin{aligned}
\Gamma_i \;=\;& [Q,a_i]\\
=\;& -\,\frac{s\,qB\hbar\,\beta_0\omega_0\,\Omega_+}{2m\sqrt{s\,\beta_0\,\omega_0\omega_-\omega_+\,(2\beta_0\,\omega_0\,\Omega_+\,-\,\kappa_0\,k_0\,\Omega_-)}}\;\left(\epsilon_{ij}\,\psi'_j\right)\\
-\;& \frac{i\,s\,qB\,\hbar\,\kappa_0k_0\,\Omega_+}{2\sqrt{2}m\sqrt{s\,\kappa_0\,k_0\omega_-\omega_+\,(\kappa_0\,k_0\,\Omega_+\,-\,2\beta_0\,\omega_0\,\Omega_-)}}\;\mu'_i\\
-\;& \frac{\hbar}{2\sqrt{2}}\sqrt{\frac{s\,\beta_0\,\omega_0}{\omega_+\,\omega_-}\,(2\beta_0\,\omega_0\,\Omega_+\,-\,\kappa_0\,k_0\,\Omega_-)}\;\lambda'_i\\
+\;& \frac{i\hbar}{4}\sqrt{\frac{s\,\kappa_0\,k_0}{\omega_+\,\omega_-}\,(\kappa_0\,k_0\,\Omega_+\,-\,2\beta_0\,\omega_0\,\Omega_-)}\;\left(\epsilon_{ij}\,\gamma'_j\right)
\end{aligned}
\tag{4.28}
$$

$$
\begin{aligned}
\Gamma_i^\dagger \;=\;& -[Q,a_i^\dagger]\\
=\;& -\,\frac{s\,qB\hbar\,\beta_0\omega_0\,\Omega_+}{2m\sqrt{s\,\beta_0\,\omega_0\omega_-\omega_+\,(2\beta_0\,\omega_0\,\Omega_+\,-\,\kappa_0\,k_0\,\Omega_-)}}\;\left(\epsilon_{ij}\,\psi'_j\right)\\
+\;& \frac{i\,s\,qB\,\hbar\,\kappa_0k_0\,\Omega_+}{2\sqrt{2}m\sqrt{s\,\kappa_0\,k_0\omega_-\omega_+\,(\kappa_0\,k_0\,\Omega_+\,-\,2\beta_0\,\omega_0\,\Omega_-)}}\;\mu'_i\\
-\;& \frac{\hbar}{2\sqrt{2}}\sqrt{\frac{s\,\beta_0\,\omega_0}{\omega_+\,\omega_-}\,(2\beta_0\,\omega_0\,\Omega_+\,-\,\kappa_0\,k_0\,\Omega_-)}\;\lambda'_i\\
-\;& \frac{i\hbar}{4}\sqrt{\frac{s\,\kappa_0\,k_0}{\omega_+\,\omega_-}\,(\kappa_0\,k_0\,\Omega_+\,-\,2\beta_0\,\omega_0\,\Omega_-)}\;\left(\epsilon_{ij}\,\gamma'_j\right)
\end{aligned}
\tag{4.29}
$$

$$
\begin{aligned}
\Lambda_i &= [Q, b_i] \\
&= -\frac{s\, qB\hbar\, \beta_0\omega_0\, \Omega_-}{2m\sqrt{s\, \beta_0\, \omega_0\omega_-\omega_+\, (\kappa_0\, k_0\, \Omega_+ \,-\, 2\beta_0\, \omega_0\, \Omega_-)}}\, \left(\epsilon_{ij}\, \psi'_j\right) \\
&- \frac{i\, s\, qB\, \hbar\, \kappa_0 k_0\, \Omega_-}{2\sqrt{2}m\sqrt{s\, \kappa_0\, k_0\omega_-\omega_+\, (2\beta_0\, \omega_0\, \Omega_+ \,-\, \kappa_0\, k_0\, \Omega_-)}}\, \mu'_i \\
&- \frac{\hbar}{2\sqrt{2}}\sqrt{\frac{s\, \beta_0\, \omega_0}{\omega_+\, \omega_-}\, (\kappa_0\, k_0\, \Omega_+ \,-\, 2\beta_0\, \omega_0\, \Omega_-)}\;\, \lambda'_i \\
&- \frac{i\hbar}{4}\sqrt{\frac{s\, \kappa_0\, k_0}{\omega_+\, \omega_-}\, (2\beta_0\, \omega_0\, \Omega_+ \,-\, \kappa_0\, k_0\, \Omega_-)}\;\, \left(\epsilon_{ij}\, \gamma'_j\right)
\end{aligned}
\tag{4.30}
$$

$$
\begin{aligned}
\Lambda_i^\dagger &= -[Q, b_i^\dagger] \\
&= -\frac{s\, qB\hbar\, \beta_0\omega_0\, \Omega_-}{2m\sqrt{s\, \beta_0\, \omega_0\omega_-\omega_+\, (\kappa_0\, k_0\, \Omega_+ \,-\, 2\beta_0\, \omega_0\, \Omega_-)}}\, \left(\epsilon_{ij}\, \psi'_j\right) \\
&+ \frac{i\, s\, qB\, \hbar\, \kappa_0 k_0\, \Omega_-}{2\sqrt{2}m\sqrt{s\, \kappa_0\, k_0\omega_-\omega_+\, (2\beta_0\, \omega_0\, \Omega_+ \,-\, \kappa_0\, k_0\, \Omega_-)}}\, \mu'_i \\
&- \frac{\hbar}{2\sqrt{2}}\sqrt{\frac{s\, \beta_0\, \omega_0}{\omega_+\, \omega_-}\, (\kappa_0\, k_0\, \Omega_+ \,-\, 2\beta_0\, \omega_0\, \Omega_-)}\;\, \lambda'_i \\
&+ \frac{i\hbar}{4}\sqrt{\frac{s\, \kappa_0\, k_0}{\omega_+\, \omega_-}\, (2\beta_0\, \omega_0\, \Omega_+ \,-\, \kappa_0\, k_0\, \Omega_-)}\;\, \left(\epsilon_{ij}\, \gamma'_j\right)
\end{aligned}
\tag{4.31}
$$

The key point that will save us a fair amount of work is that these fermionic operators still commute with the bosonic degrees of freedom:

$$
\begin{aligned}
[\Gamma_i^{(\dagger)}, a_j] &= 0 = [\Gamma_i^{(\dagger)}, b_j] \\
[\Gamma_i^{(\dagger)}, a_j^\dagger] &= 0 = [\Gamma_i^{(\dagger)}, b_j^\dagger] \\
[\Lambda_i^{(\dagger)}, a_j] &= 0 = [\Lambda_i^{(\dagger)}, b_j] \\
[\Lambda_i^{(\dagger)}, a_j^\dagger] &= 0 = [\Lambda_i^{(\dagger)}, b_j^\dagger]
\end{aligned}
\tag{4.32}
$$

These relations allow us to compute the anticommutation relations of the Γ's and Λ's in an abstract way. Using the Jacobi identity, we have that:

$$
\begin{aligned}
\{\Gamma_i, \Gamma_j\} &= \Big\{\Gamma_i, [Q, a_j]\Big\} \\
&= \Big\{Q, [a_j, \Gamma_i]\Big\} - \Big[a_j, \{\Gamma_i, Q\}\Big] \\
&= -\Big[a_j, \{\Gamma_i, Q\}\Big]
\end{aligned}
\tag{4.33}
$$

We also have that:

$$
\begin{aligned}
\{Q, \Gamma_i\} &= \Big\{Q, [Q, a_i]\Big\} \\
&= Q^2 a_i - Q a_i Q + Q a_i Q - a_i Q^2 \\
&= \hbar\, [H, a_i] \,=\, -\hbar^2\, \Omega_+\, a_i
\end{aligned}
\tag{4.34}
$$

where we used the fact that $Q^2 = \hbar H$. This implies that $\{\Gamma_i, \Gamma_j\} = 0$. In a similar way, we have that:

$$\begin{aligned}
\{\Gamma_i^\dagger, \Gamma_j^\dagger\} &= \Big\{\Gamma_i, -[Q, a_j^\dagger]\Big\} \\
&= \Big\{Q, [a_j^\dagger, \Gamma_i]\Big\} - \Big[a_j^\dagger, \{\Gamma_i, Q\}\Big] \\
&= -\Big[a_j^\dagger, \{\Gamma_i, Q\}\Big] \\
\{Q, \Gamma_i^\dagger\} &= \Big\{Q, -[Q, a_i^\dagger]\Big\} \\
&= -Q^2 a_i^\dagger + Q a_i^\dagger Q - Q a_i^\dagger Q + a_i^\dagger Q^2 \\
&= -\hbar\,[H, a_i^\dagger] = -\hbar^2\,\Omega_+\,a_i^\dagger
\end{aligned} \tag{4.35}$$

so that, again, $\{\Gamma_i^\dagger, \Gamma_j^\dagger\} = 0$. Finally, in a similar way, we have:

$$\begin{aligned}
\{\Gamma_i, \Gamma_j^\dagger\} &= -\Big\{\Gamma_i, [Q, a_j^\dagger]\Big\} \\
&= \Big[a_j^\dagger, \{\Gamma_i, Q\}\Big] - \Big\{Q, [a_j^\dagger, \Gamma_i]\Big\} \\
&= \Big[a_j^\dagger, -\hbar^2 \Omega_+ a_i\Big] = \hbar^2 \Omega_+\,\delta_{ij}\,\mathbb{I}
\end{aligned} \tag{4.36}$$

The same argument applies to the Λ's to get the same anticommutation relations provided Ω_+ is replaced by Ω_-. It is now convenient to normalize the fermionic operators:

$$\begin{aligned}
\Gamma_i' &= \tfrac{1}{\hbar\sqrt{\Omega_+}}\,\Gamma_i\,, & \Gamma_i^{\dagger\,\prime} &= \tfrac{1}{\hbar\sqrt{\Omega_+}}\,\Gamma_i^\dagger \\
\Lambda_i' &= \tfrac{1}{\hbar\sqrt{\Omega_-}}\,\Lambda_i\,, & \Lambda_i^{\dagger\,\prime} &= \tfrac{1}{\hbar\sqrt{\Omega_-}}\,\Lambda_i^\dagger
\end{aligned} \tag{4.37}$$

so that we get:

$$\{\Gamma_i', \Gamma_j^{\dagger\,\prime}\} = \delta_{ij}\,\mathbb{I} = \{\Lambda_i', \Lambda_j^{\dagger\,\prime}\} \tag{4.38}$$

while all other anticommutation relations not being given here are understood to be identically vanishing. It is now rather straightforward to check that:

$$Q = -\hbar\sqrt{\Omega_+}\,a_i\,\Gamma_i^{\dagger\,\prime} - \hbar\sqrt{\Omega_+}\,a_i^\dagger\,\Gamma_i' - \hbar\sqrt{\Omega_-}\,b_i\,\Lambda_i^{\dagger\,\prime} - \hbar\sqrt{\Omega_-}\,b_i^\dagger\,\Lambda_i' \tag{4.39}$$

In order to establish how the Hamiltonian is expressed in terms of these new operators, we will use the fact that $Q^2 = \hbar\,H$ instead of directly substituting the expressions (4.28-4.31) in the Hamiltonian. Using the anticommutation of the non conjugated fermionic variables, we have:

$$\begin{aligned}
H &= \frac{1}{\hbar} Q^2 \\
&= \sum_i \Big[\hbar\Omega_+\,\Big(a_i a_i^\dagger\,\Gamma_i^{\dagger\,\prime}\Gamma_i' + a_i^\dagger a_i\,\Gamma_i'\Gamma_i^{\dagger\,\prime}\Big) + \hbar\Omega_-\,\Big(b_i b_i^\dagger\,\Lambda_i^{\dagger\,\prime}\Lambda_i' + b_i^\dagger b_i\,\Lambda_i'\Lambda_i^{\dagger\,\prime}\Big)\Big] \\
&= \hbar\,\Omega_+\,\Big(a_i^\dagger a_i + \Gamma_i^{\dagger\,\prime}\,\Gamma_i'\Big) + \hbar\,\Omega_-\,\Big(b_i^\dagger b_i + \Lambda_i^{\dagger\,\prime}\,\Lambda_i'\Big)
\end{aligned} \tag{4.40}$$

It is worth noticing that, as may be expected for a supersymmetric Hamiltonian, the latter does not contain any constant term. Indeed, the ones that come from the (anti)commutation relations of the fermionic and bosonic Fock operators in the bosonic and fermionic parts of the Hamiltonian cancel each other.

4.2 Angular Momentum

As may be seen from (4.2), the angular momentum operator mixes the two chiral sectors that were held decoupled in the Hamiltonian. It is therefore natural that a last change of variable that mixes them is needed to diagonalize the operator L. Let us introduce the following chiral Fock algebra operators:

$$a_\pm = \frac{1}{\sqrt{2}}\left(a_1 \mp i\, a_2\right)\,, \qquad a_\pm^\dagger = \frac{1}{\sqrt{2}}\left(a_1^\dagger \pm i\, a_2^\dagger\right)$$
$$b_\pm = \frac{1}{\sqrt{2}}\left(b_1 \mp i\, b_2\right)\,, \qquad b_\pm^\dagger = \frac{1}{\sqrt{2}}\left(b_1^\dagger \pm i\, b_2^\dagger\right) \tag{4.41}$$

$$\Gamma_\pm = \frac{1}{\sqrt{2}}\left(\Gamma_1' \mp i\,\Gamma_2'\right)\,, \qquad \Gamma_\pm^\dagger = \frac{1}{\sqrt{2}}\left(\Gamma_1^{\dagger\prime} \pm i\,\Gamma_2^{\dagger\prime}\right)$$
$$\Lambda_\pm = \frac{1}{\sqrt{2}}\left(\Lambda_1' \mp i\,\Lambda_2'\right)\,, \qquad \Lambda_\pm^\dagger = \frac{1}{\sqrt{2}}\left(\Lambda_1^{\dagger\prime} \pm i\,\Lambda_2^{\dagger\prime}\right) \tag{4.42}$$

such that:

$$\begin{aligned}
[a_\pm, a_\pm^\dagger] &= \mathbb{I} = [b_\pm, b_\pm^\dagger] \\
[a_\pm, a_\mp^\dagger] &= 0 = [b_\pm, b_\mp^\dagger] \\
\{\Gamma_\pm, \Gamma_\pm^\dagger\} &= \mathbb{I} = \{\Lambda_\pm, \Lambda_\pm^\dagger\} \\
\{\Gamma_\pm, \Gamma_\mp^\dagger\} &= 0 = \{\Lambda_\pm, \Lambda_\mp^\dagger\}
\end{aligned} \tag{4.43}$$

We have the following inverse transformations:

$$\begin{aligned}
a_1 &= \frac{1}{\sqrt{2}}\left(a_- + a_+\right)\,, & a_1^\dagger &= \frac{1}{\sqrt{2}}\left(a_-^\dagger + a_+^\dagger\right) \\
a_2 &= \frac{i}{\sqrt{2}}\left(a_+ - a_-\right)\,, & a_2^\dagger &= -\frac{i}{\sqrt{2}}\left(a_+^\dagger - a_-^\dagger\right) \\
b_1 &= \frac{1}{\sqrt{2}}\left(b_- + b_+\right)\,, & b_1^\dagger &= \frac{1}{\sqrt{2}}\left(b_-^\dagger + b_+^\dagger\right) \\
b_2 &= \frac{i}{\sqrt{2}}\left(b_+ - b_-\right)\,, & b_2^\dagger &= -\frac{i}{\sqrt{2}}\left(b_+^\dagger - b_-^\dagger\right)
\end{aligned} \tag{4.44}$$

$$\begin{aligned}
\Gamma_1' &= \frac{1}{\sqrt{2}}\left(\Gamma_- + \Gamma_+\right)\,, & \Gamma_1^{\dagger\prime} &= \frac{1}{\sqrt{2}}\left(\Gamma_-^\dagger + \Gamma_+^\dagger\right) \\
\Gamma_2' &= \frac{i}{\sqrt{2}}\left(\Gamma_+ - \Gamma_-\right)\,, & \Gamma_2^{\dagger\prime} &= -\frac{i}{\sqrt{2}}\left(\Gamma_+^\dagger - \Gamma_-^\dagger\right) \\
\Lambda_1' &= \frac{1}{\sqrt{2}}\left(\Lambda_- + \Lambda_+\right)\,, & \Lambda_1^{\dagger\prime} &= \frac{1}{\sqrt{2}}\left(\Lambda_-^\dagger + \Lambda_+^\dagger\right) \\
\Lambda_2' &= \frac{i}{\sqrt{2}}\left(\Lambda_+ - \Lambda_-\right)\,, & \Lambda_2^{\dagger\prime} &= -\frac{i}{\sqrt{2}}\left(\Lambda_+^\dagger - \Lambda_-^\dagger\right)
\end{aligned} \tag{4.45}$$

In terms of these transformations, on may check that the angular momentum operator L is diagonalized:

$$L = \hbar\left(a_+^\dagger a_+ - a_-^\dagger a_- + b_+^\dagger b_+ - b_-^\dagger b_- + \Gamma_+^\dagger \Gamma_+ - \Gamma_-^\dagger \Gamma_- + \Lambda_+^\dagger \Lambda_+ - \Lambda_-^\dagger \Lambda_-\right) \tag{4.46}$$

while the form of the operators H and Q remains unchanged:

$$\begin{aligned} H &= \hbar\,\Omega_+ \left(a_+^\dagger\, a_+ + a_-^\dagger\, a_-\right) + \hbar\,\Omega_- \left(b_+^\dagger\, b_+ + b_-^\dagger\, b_-\right) \\ &+ \hbar\,\Omega_+ \left(\Gamma_+^\dagger\, \Gamma_+ + \Gamma_-^\dagger\, \Gamma_-\right) + \hbar\,\Omega_- \left(\Lambda_+^\dagger\, \Lambda_+ + \Lambda_-^\dagger\, \Lambda_-\right) \\ Q &= -\,\hbar\,\sqrt{\Omega_+}\left(a_+\,\Gamma_+^\dagger + a_+^\dagger\,\Gamma_+ + a_-\,\Gamma_-^\dagger + a_-^\dagger\,\Gamma_-\right) \\ &\quad -\,\hbar\,\sqrt{\Omega_-}\left(b_+\,\Lambda_+^\dagger + b_+^\dagger\,\Lambda_+ + b_-\,\Lambda_-^\dagger + b_-^\dagger\,\Lambda_-\right) \end{aligned} \tag{4.47}$$

4.3 Symmetry Transformations

To close this chapter, it is instructive to look at the symmetry transformations of the bosonic and fermionic chiral Fock algebra operators under H, Q, and L. The first thing to note is that the classical invariance of the system under the associated symmetries still holds at the quantum level. The three operators indeed commute with each other:

$$[H,Q]=0, \qquad [H,L]=0, \qquad [L,Q]=0 \tag{4.48}$$

4.3.1 Time invariance

For the Hamiltonian, which generates time translations, we have:

$$\begin{aligned} [H,a_+] &= -\hbar\,\Omega_+\,a_+\,, & [H,a_+^\dagger] &= \hbar\,\Omega_+\,a_+^\dagger \\ [H,a_-] &= -\hbar\,\Omega_+\,a_-\,, & [H,a_-^\dagger] &= \hbar\,\Omega_+\,a_-^\dagger \\ [H,b_+] &= -\hbar\,\Omega_-\,b_+\,, & [H,b_+^\dagger] &= \hbar\,\Omega_-\,b_+^\dagger \\ [H,b_-] &= -\hbar\,\Omega_-\,b_-\,, & [H,b_-^\dagger] &= \hbar\,\Omega_-\,b_-^\dagger \\ [H,\Gamma_+] &= -\hbar\,\Omega_+\,\Gamma_+\,, & [H,\Gamma_+^\dagger] &= \hbar\,\Omega_+\,\Gamma_+^\dagger \\ [H,\Gamma_-] &= -\hbar\,\Omega_+\,\Gamma_-\,, & [H,\Gamma_-^\dagger] &= \hbar\,\Omega_+\,\Gamma_-^\dagger \\ [H,\Lambda_+] &= -\hbar\,\Omega_-\,\Lambda_+\,, & [H,\Lambda_+^\dagger] &= \hbar\,\Omega_-\,\Lambda_+^\dagger \\ [H,\Lambda_-] &= -\hbar\,\Omega_-\,\Lambda_-\,, & [H,\Lambda_-^\dagger] &= \hbar\,\Omega_-\,\Lambda_-^\dagger \end{aligned} \tag{4.49}$$

which shows that the operators $a_+^\dagger$, $a_-^\dagger$, $\Gamma_+^\dagger$ and $\Gamma_-^\dagger$ create a quantum of energy $\hbar\,\Omega_+$ while their hermitian conjugates annihilate one. For the b's and Λ's operators, the associated quantum of energy is obviously $\hbar\,\Omega_-$. It is worth noticing the equality between the bosonic and fermionic quanta of energy, which is a feature implied by the supersymmetry of the system.

4.3.2 Supersymmetry invariance

A supersymmetry transformation transforms a bosonic degree of freedom into a fermionic one and vice-versa. The following relations show that the $a_\pm$'s and the $\Gamma_\pm$'s, and the $b_\pm$'s and the $\Lambda_\pm$'s form respective doublet representations of the supersymmetry algebra since they are transformed into each other:

$$
\begin{aligned}
[Q,a_+] &= \hbar\sqrt{\Omega_+}\,\Gamma_+\,, & [Q,a_+^\dagger] &= -\hbar\sqrt{\Omega_+}\,\Gamma_+^\dagger \\
\{Q,\Gamma_+\} &= -\hbar\sqrt{\Omega_+}\,a_+\,, & \{Q,\Gamma_+^\dagger\} &= -\hbar\sqrt{\Omega_+}\,a_+^\dagger \\
[Q,a_-] &= \hbar\sqrt{\Omega_+}\,\Gamma_-\,, & [Q,a_-^\dagger] &= -\hbar\sqrt{\Omega_+}\,\Gamma_-^\dagger \\
\{Q,\Gamma_-\} &= -\hbar\sqrt{\Omega_+}\,a_-\,, & \{Q,\Gamma_-^\dagger\} &= -\hbar\sqrt{\Omega_+}\,a_-^\dagger \\
[Q,b_+] &= \hbar\sqrt{\Omega_-}\,\Lambda_+\,, & [Q,b_+^\dagger] &= -\hbar\sqrt{\Omega_-}\,\Lambda_+^\dagger \\
\{Q,\Lambda_+\} &= -\hbar\sqrt{\Omega_-}\,b_+\,, & \{Q,\Lambda_+^\dagger\} &= -\hbar\sqrt{\Omega_-}\,b_+^\dagger \\
[Q,b_-] &= \hbar\sqrt{\Omega_-}\,\Lambda_-\,, & [Q,b_-^\dagger] &= -\hbar\sqrt{\Omega_-}\,\Lambda_-^\dagger \\
\{Q,\Lambda_-\} &= -\hbar\sqrt{\Omega_-}\,b_-\,, & \{Q,\Lambda_-^\dagger\} &= -\hbar\sqrt{\Omega_-}\,b_-^\dagger
\end{aligned}
\tag{4.50}
$$

4.3.3 Rotational invariance

Finally, here is the action of the operator L on the degrees of freedom:

$$
\begin{aligned}
[L,a_+] &= -\hbar\,a_+\,, & [L,a_+^\dagger] &= \hbar\,a_+^\dagger \\
[L,a_-] &= \hbar\,a_-\,, & [L,a_-^\dagger] &= -\hbar\,a_-^\dagger \\
[L,b_+] &= -\hbar\,b_+\,, & [L,b_+^\dagger] &= \hbar\,b_+^\dagger \\
[L,b_-] &= \hbar\,b_-\,, & [L,b_-^\dagger] &= -\hbar\,b_-^\dagger \\
[L,\Gamma_+] &= -\hbar\,\Gamma_+\,, & [L,\Gamma_+^\dagger] &= \hbar\,\Gamma_+^\dagger \\
[L,\Gamma_-] &= \hbar\,\Gamma_-\,, & [L,\Gamma_-^\dagger] &= -\hbar\,\Gamma_-^\dagger \\
[L,\Lambda_+] &= -\hbar\,\Lambda_+\,, & [L,\Lambda_+^\dagger] &= \hbar\,\Lambda_+^\dagger \\
[L,\Lambda_-] &= \hbar\,\Lambda_-\,, & [L,\Lambda_-^\dagger] &= -\hbar\,\Lambda_-^\dagger
\end{aligned}
\tag{4.51}
$$

This shows that each creation operator subscripted with a "$+$" (respectively, "$-$") creates a unit $+\hbar$ (respectively, "$-\hbar$") of angular momentum. In order to prevent any confusion, it should be emphasised that the $\pm$ subscript of the operators refers to their angular momentum eigenvalues $\pm\hbar$, *not* to their energy eigenvalues $\hbar\,\Omega_\pm$.

By an appropriate use of the supersymmetry of the system, we have so far introduced some convenient bosonic and fermionic Fock algebra operators so that the Hamiltonian and the angular momentum operators are both diagonal. We are now prepared to work out the representation of the quantum system, and proceed to the diagonalization of the supercharge Q, which is the aim of the next Chapter.

Chapter 5

Representation and Supercharge Diagonalization

In the previous Chapter, we have constructed convenient creation and annihilation operators which span bosonic or fermionic Fock algebras, and diagonalize the Hamiltonian of the system. To complete the description of the quantum system, we now have to find a representation of these operators. We therefore have to construct the Hilbert space of the physical states, and associate to each operator a linear transformation on that space, such that the (anti)commutation relations hold. We will then be able to determine the energy spectrum, as well as the spectrum of the operators L and Q.

5.1 Bosonic Fock Space

A usual representation of the bosonic Fock algebra is given by considering the space of physical bosonic states $\mathcal{H}_b$ as being the bosonic Fock space that will be defined in the following. That space is constructed from the so called *vacuum state*, denoted by $|\,\Omega_b\,\rangle$, having the following properties:

$$\begin{aligned} \langle\,\Omega_b\,|\,\Omega_b\,\rangle &= 1\,, \\ a_\pm\,|\,\Omega_b\,\rangle &= 0 = b_\pm\,|\,\Omega_b\,\rangle. \end{aligned} \tag{5.1}$$

The other (orthonormalized) bosonic Fock states are constructed from that vacuum as follows:

$$|\,N_{a_+}, N_{a_-}, N_{b_+}, N_{b_-}\,\rangle = \frac{1}{\sqrt{N_{a_+}!N_{a_-}!N_{b_+}!N_{b_-}!}}\left(a_+^\dagger\right)^{N_{a_+}}\left(a_-^\dagger\right)^{N_{a_-}}\left(b_+^\dagger\right)^{N_{b_+}}\left(b_-^\dagger\right)^{N_{b_-}}|\,\Omega_b\,\rangle\,, \tag{5.2}$$

$$\langle\,N_{a_+}, N_{a_-}, N_{b_+}, N_{b_-}\,|\,M_{a_+}, M_{a_-}, M_{b_+}, M_{b_-}\,\rangle = \delta_{N_{a_+},M_{a_+}}\,\delta_{N_{a_-},M_{a_-}}\,\delta_{N_{b_+},M_{b_+}}\,\delta_{N_{b_-},M_{b_-}}\,. \tag{5.3}$$

These states form a complete orthonormal basis of the infinite dimensional bosonic Fock space, so that we have:

$$\begin{aligned} \mathcal{H}_b &= \mathrm{Span}\left\langle\ \{|\,N_{a_+}, N_{a_-}, N_{b_+}, N_{b_-}\,\rangle\}_{N_{a_+},N_{a_-},N_{b_+},N_{b_-}=0,1,2,\ldots}\ \right\rangle, \\ \mathbb{I}_b &= \sum_{N_{a_+},N_{a_-},N_{b_+},N_{b_-}=0}^{\infty} |\,N_{a_+}, N_{a_-}, N_{b_+}, N_{b_-}\,\rangle\langle\,N_{a_+}, N_{a_-}, N_{b_+}, N_{b_-}\,|\,, \end{aligned} \tag{5.4}$$

where $\mathbb{I}_b$ is the identity operator on $\mathcal{H}_b$. It is then straightforward to determine how the bosonic components of the operators H and L act on the bosonic Fock states. As mentioned before, the latter are eigenstates of the Hamiltonian:

$$\begin{aligned} H_b \,|\, N_{a_+}, N_{a_-}, N_{b_+}, N_{b_-} \,\rangle &= E_b\left(N_{a_+}, N_{a_-}, N_{b_+}, N_{b_-}\right) |\, N_{a_+}, N_{a_-}, N_{b_+}, N_{b_-} \,\rangle \,, \\ E_b\left(N_{a_+}, N_{a_-}, N_{b_+}, N_{b_-}\right) &= \left(N_{a_+} + N_{a_-}\right) \hbar\,\Omega_+ + \left(N_{b_+} + N_{b_-}\right) \hbar\,\Omega_- \,, \end{aligned} \tag{5.5}$$

as well as of the angular momentum operator:

$$L_b \,|\, N_{a_+}, N_{a_-}, N_{b_+}, N_{b_-} \,\rangle = \hbar\left(N_{a_+} - N_{a_-} + N_{b_+} - N_{b_-}\right) |\, N_{a_+}, N_{a_-}, N_{b_+}, N_{b_-} \,\rangle \,. \tag{5.6}$$

5.2 Fermionic Fock Space

Because fermionic operators obey anticommutation relations in contradistinction to commutation relations, their representation is easily constructed from the Pauli matrices σ_i. Let us define

$$\sigma_\pm = \frac{1}{2}\left(\sigma_1 \pm \sigma_2\right) \tag{5.7}$$

that act on $\mathbb{R}^2$. Denoting $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively by $|+\rangle$ and $|-\rangle$, we have:

$$\begin{aligned} \sigma_+ \,|+\rangle &= 0 \qquad & \sigma_+ \,|-\rangle &= |+\rangle \\ \sigma_- \,|-\rangle &= 0 \qquad & \sigma_- \,|+\rangle &= |-\rangle \end{aligned} \tag{5.8}$$

We will then consider the Hilbert space $\mathcal{H}_f$ of physical fermionic states as being the $2^4 = 16$ dimensional tensor product of four copies of the Hilbert space spanned by $|+\rangle$ and $|-\rangle$:

$$\begin{aligned} \mathcal{H}_f &= \mathrm{Span}\left\langle \left\{ |\, s_{\Gamma_+}, s_{\Gamma_-}, s_{\Lambda_+}, s_{\Lambda_-} \,\rangle \right\}_{s_{\Gamma_+}, s_{\Gamma_-}, s_{\Lambda_+}, s_{\Lambda_-} \in \{+,-\}} \right\rangle \\ \mathbb{I}_f &= \sum_{s_{\Gamma_+}, s_{\Gamma_-}, s_{\Lambda_+}, s_{\Lambda_-} \in \{+,-\}} |\, s_{\Gamma_+}, s_{\Gamma_-}, s_{\Lambda_+}, s_{\Lambda_-} \,\rangle \langle\, s_{\Gamma_+}, s_{\Gamma_-}, s_{\Lambda_+}, s_{\Lambda_-} \,| \end{aligned} \tag{5.9}$$

where $\mathbb{I}_f$ is the identity operator on $\mathcal{H}_f$. The states are orthonormal $|\, s_{\Gamma_+}, s_{\Gamma_-}, s_{\Lambda_+}, s_{\Lambda_-} \,\rangle$, so that:

$$\langle\, s_{\Gamma_+}, s_{\Gamma_-}, s_{\Lambda_+}, s_{\Lambda_-} \,|\, t_{\Gamma_+}, t_{\Gamma_-}, t_{\Lambda_+}, t_{\Lambda_-} \,\rangle = \delta_{s_{\Gamma_+}, t_{\Gamma_+}} \, \delta_{s_{\Gamma_-}, t_{\Gamma_-}} \, \delta_{s_{\Lambda_+}, t_{\Lambda_+}} \, \delta_{s_{\Lambda_-}, t_{\Lambda_-}} \tag{5.10}$$

We finally associate to the fermionic operators the following tensor products of the $\sigma_\pm$ and σ_3 matrices:

$$\begin{aligned} \Gamma_+ &= \sigma_- \otimes \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \qquad & \Gamma_+^\dagger &= \sigma_+ \otimes \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \\ \Gamma_- &= \sigma_3 \otimes \sigma_- \otimes \mathbb{I} \otimes \mathbb{I} \qquad & \Gamma_-^\dagger &= \sigma_3 \otimes \sigma_+ \otimes \mathbb{I} \otimes \mathbb{I} \\ \Lambda_+ &= \sigma_3 \otimes \sigma_3 \otimes \sigma_- \otimes \mathbb{I} \qquad & \Lambda_+^\dagger &= \sigma_3 \otimes \sigma_3 \otimes \sigma_+ \otimes \mathbb{I} \\ \Lambda_- &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_- \qquad & \Lambda_-^\dagger &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_+ \end{aligned} \tag{5.11}$$

This provides a representation of the (complex) $SO(8)$ Clifford algebra. Note that the presence of the σ_3 matrices ensure the anticommutativity of the fermionic operators. Once again, the fermionic states are eigenstates of the fermionic Hamiltonian:

$$
\begin{aligned}
&H_f\,|-,-,-,-\rangle = 0 && \\
&H_f\,|+,-,-,-\rangle = \hbar\Omega_+|+,-,-,-\rangle && H_f\,|-,+,-,-\rangle = \hbar\Omega_+|-,+,-,-\rangle \\
&H_f\,|-,-,+,-\rangle = \hbar\Omega_-|-,-,+,-\rangle && H_f\,|-,-,-,+\rangle = \hbar\Omega_-|-,-,-,+\rangle \\
&H_f\,|+,+,-,-\rangle = 2\hbar\Omega_+|+,+,-,-\rangle && H_f\,|-,-,+,+\rangle = 2\hbar\Omega_-|-,-,+,+\rangle \\
&H_f\,|-,+,+,-\rangle = \hbar\,(\Omega_+ + \Omega_-)\,|-,+,+,-\rangle && H_f\,|+,-,-,+\rangle = \hbar\,(\Omega_+ + \Omega_-)\,|+,-,-,+\rangle \\
&H_f\,|+,-,+,-\rangle = \hbar\,(\Omega_+ + \Omega_-)\,|+,-,+,-\rangle && H_f\,|-,+,-,+\rangle = \hbar\,(\Omega_+ + \Omega_-)\,|-,+,-,+\rangle \\
&H_f\,|+,+,+,-\rangle = \hbar\,(2\Omega_+ + \Omega_-)\,|+,+,+,-\rangle && H_f\,|+,+,-,+\rangle = \hbar\,(2\Omega_+ + \Omega_-)\,|+,+,-,+\rangle \\
&H_f\,|-,+,+,+\rangle = \hbar\,(\Omega_+ + 2\Omega_-)\,|-,+,+,+\rangle && H_f\,|+,-,+,+\rangle = \hbar\,(\Omega_+ + 2\Omega_-)\,|+,-,+,+\rangle \\
&H_f\,|+,+,+,+\rangle = 2\hbar\,(\Omega_+ + \Omega_-)\,|+,+,+,+\rangle &&
\end{aligned}
\tag{5.12}
$$

This is also the case for the fermionic part L_f of the angular momentum:

$$
\begin{aligned}
&L_f\,|-,+,-,+\rangle = -2\hbar|-,+,-,+\rangle && \\
&L_f\,|-,+,-,-\rangle = -\hbar|-,+,-,-\rangle && L_f\,|-,-,-,+\rangle = -\hbar|-,-,-,+\rangle \\
&L_f\,|+,+,-,+\rangle = -\hbar|+,+,-,+\rangle && L_f\,|-,+,+,+\rangle = -\hbar|-,+,+,+\rangle \\
&L_f\,|-,-,-,-\rangle = 0 && L_f\,|+,+,-,-\rangle = 0 \\
&L_f\,|-,+,+,-\rangle = 0 && L_f\,|+,-,-,+\rangle = 0 \\
&L_f\,|-,-,+,+\rangle = 0 && L_f\,|+,+,+,+\rangle = 0 \\
&L_f\,|+,-,-,-\rangle = \hbar|+,-,-,-\rangle && L_f\,|-,-,+,-\rangle = \hbar|-,-,+,-\rangle \\
&L_f\,|+,+,+,-\rangle = \hbar|+,+,+,-\rangle && L_f\,|+,-,+,+\rangle = \hbar|+,-,+,+\rangle \\
&L_f\,|+,-,+,-\rangle = 2\hbar|+,-,+,-\rangle &&
\end{aligned}
\tag{5.13}
$$

5.3 Energy Spectrum and Degeneracies

The construction of the representation of the whole system is now straightforward if we consider the tensor product of the representations of the bosonic and fermionic sectors. We therefore get:

$$
\begin{aligned}
\mathcal{H} &= \mathcal{H}_b \otimes \mathcal{H}_f \\
\mathbb{I}_b &= \sum_{N_{a_+},N_{a_-},N_{b_+},N_{b_-}=0}^{\infty} \quad \sum_{s_{\Gamma_+},s_{\Gamma_-},s_{\Lambda_+},s_{\Lambda_-}\in\{+,-\}} * \\
&\quad |\,N_{a_+},N_{a_-},N_{b_+},N_{b_-}\,;\,s_{\Gamma_+},s_{\Gamma_-},s_{\Lambda_+},s_{\Lambda_-}\,\rangle\langle\,N_{a_+},N_{a_-},N_{b_+},N_{b_-}\,;\,s_{\Gamma_+},s_{\Gamma_-},s_{\Lambda_+},s_{\Lambda_-}\,|
\end{aligned}
\tag{5.14}
$$

Using the eigenvalues listed in the two previous Sections, we can group these basis states according to their energy, and determine the degeneracy of the Hamiltonian eigenvalues. The latter are specified by two natural numbers, N_+, $N_- \geq 0$, corresponding to the number of quanta of energy $\hbar\,\Omega_+$ and $\hbar\,\Omega_-$ respectively:

$$E\,(N_+,N_-) = N_+\,\hbar\,\Omega_+ + N_-\,\hbar\,\Omega_- \tag{5.15}$$

In order to classify the energy eigenstates, it is convenient to divide them into different sectors, according to their values of N_+ and N_-.

5.3.1 Ground state

As mentioned before, the manifest supersymmetry of the system leads to a zero energy for the ground state:

$$|\,\Omega\,\rangle = |\,0,0,0,0\,;\ -,-,-,-\,\rangle \tag{5.16}$$

Note that this state is not degenerate, as it should since supersymmetry is not spontaneously broken in this system.

5.3.2 $N_+ \geq 1$ and $N_- = 0$ energy states

The eigenstates of the Hamiltonian with an eigenvalue equal to $N_+\,\hbar\Omega_+$ with $N_+ \geq 1$ can be specified in terms of two parameters. The first one, a natural number n_+, is related to the occupancy number of the operator $a_-^\dagger$. The second one, S, is related to the spin configuration of the state. Let us define:

$$\begin{aligned}
|\,N_+\,,\,0\,;\,1\,\rangle_+ &\equiv |\,N_+\,,\,0\,,\,0\,,\,0\,;\ -,-,-,-\,\rangle\\
|\,N_+\,,\,0\,;\,2\,\rangle_+ &\equiv |\,N_+-1\,,\,0\,,\,0\,,\,0\,;\ +,-,-,-\,\rangle\\
|\,N_+\,,\,N_+\,;\,1\,\rangle_+ &\equiv |\,0\,,\,N_+\,,\,0\,,\,0\,;\ -,-,-,-\,\rangle\\
|\,N_+\,,\,N_+\,;\,2\,\rangle_+ &\equiv |\,0\,,\,N_+-1\,,\,0\,,\,0\,;\ -,+,-,-\,\rangle
\end{aligned} \tag{5.17}$$

and, for $1 \leq n_+ \leq N_+ - 1$:

$$\begin{aligned}
|\,N_+\,,\,n_+\,;\,1\,\rangle_+ &\equiv |\,N_+-n_+\,,\,n_+\,,\,0\,,\,0\,;\ -,-,-,-\,\rangle\\
|\,N_+\,,\,n_+\,;\,2\,\rangle_+ &\equiv |\,N_+-n_+\,,\,n_+-1\,,\,0\,,\,0\,;\ -,+,-,-\,\rangle\\
|\,N_+\,,\,n_+\,;\,3\,\rangle_+ &\equiv |\,N_+-n_+-1\,,\,n_+\,,\,0\,,\,0\,;\ +,-,-,-\,\rangle\\
|\,N_+\,,\,n_+\,;\,4\,\rangle_+ &\equiv |\,N_+-n_+-1\,,\,n_+-1\,,\,0\,,\,0\,;\ +,+,-,-\,\rangle
\end{aligned} \tag{5.18}$$

For the sake of clarity, we define:

$$\begin{aligned}
|\,N_+\,,\,0\,;\,3\,\rangle_+ &\equiv |\,N_+\,,\,N_+\,;\,1\,\rangle_+\\
|\,N_+\,,\,0\,;\,4\,\rangle_+ &\equiv |\,N_+\,,\,N_+\,;\,2\,\rangle_+
\end{aligned} \tag{5.19}$$

so that an eigenstate of eigenvalue $N_+\,\hbar\Omega_+$ can be generically written as:

$$|\,N_+\,,\,n_+\,;\,S\,\rangle_+\,, \qquad 0 \leq n_+ \leq N_+ - 1\,,\ s = 1,2,3,4 \tag{5.20}$$

The degeneracy of this energy level is therefore equal to $4N_+$. As angular momentum operator eigenstates, the eigenvalues of the states $|\,N_+\,,\,n_+\,,\,s\,\rangle$ are given by:

$$\begin{aligned}
L\,|\,N_+\,,\,0\,;\,1\,\rangle_+ &= \hbar N_+\ |\,N_+\,,\,0\,;\,1\,\rangle_+\\
L\,|\,N_+\,,\,0\,;\,2\,\rangle_+ &= \hbar N_+\ |\,N_+\,,\,0\,;\,2\,\rangle_+\\
L\,|\,N_+\,,\,0\,;\,3\,\rangle_+ &= -\hbar N_+\ |\,N_+\,,\,0\,;\,3\,\rangle_+\\
L\,|\,N_+\,,\,0\,;\,4\,\rangle_+ &= -\hbar N_+\ |\,N_+\,,\,0\,;\,4\,\rangle_+\\
L\,|\,N_+\,,\,n_+\,;\,1\,\rangle_+ &= \hbar(N_+-2n_+)\ |\,N_+\,,\,n_+\,;\,1\,\rangle_+\\
L\,|\,N_+\,,\,n_+\,;\,2\,\rangle_+ &= \hbar(N_+-2n_+)\ |\,N_+\,,\,n_+\,;\,2\,\rangle_+\\
L\,|\,N_+\,,\,n_+\,;\,3\,\rangle_+ &= \hbar(N_+-2n_+)\ |\,N_+\,,\,n_+\,;\,3\,\rangle_+\\
L\,|\,N_+\,,\,n_+\,;\,4\,\rangle_+ &= \hbar(N_+-2n_+)\ |\,N_+\,,\,n_+\,;\,4\,\rangle_+
\end{aligned} \tag{5.21}$$

5.3.3 $N_- \geq 1$ and $N_+ = 0$ energy states

The results of this Section are very similar to those of the previous one. The eigenstates of energy $N_- \hbar\Omega_-$ with $N_- \geq 1$ are specified by a natural number n_-, related to the occupancy number associated to the operator $b_-^\dagger$, and by a number S related to the spinor degrees of freedom:

$$|\, N_- \,,\, n_- \,;\, S\,\rangle_- \,, \qquad 0 \leq n_- \leq N_- - 1 \,,\; s = 1,2,3,4 \tag{5.22}$$

with, similarly to the previous section:

$$\begin{aligned}
|\, N_- \,,\, 0 \,;\, 1\,\rangle_- &= |\, 0 \,,\, 0 \,,\, N_- \,,\, 0 \,;\, -,-,-,-\,\rangle \\
|\, N_- \,,\, 0 \,;\, 2\,\rangle_- &= |\, 0 \,,\, 0 \,,\, N_- - 1 \,,\, 0 \,;\, -,-,+,-\,\rangle \\
|\, N_- \,,\, 0 \,;\, 3\,\rangle_- &= |\, 0 \,,\, 0 \,,\, 0 \,,\, N_- \,;\, -,-,-,-\,\rangle \\
|\, N_- \,,\, 0 \,;\, 4\,\rangle_- &= |\, 0 \,,\, 0 \,,\, 0 \,,\, N_- - 1 \,;\, -,-,-,+\,\rangle
\end{aligned} \tag{5.23}$$

and, for $1 \leq n_- \leq N_- - 1$:

$$\begin{aligned}
|\, N_- \,,\, n_- \,;\, 1\,\rangle_- &= |\, 0 \,,\, 0 \,,\, N_- - n_- \,,\, n_- \,;\, -,-,-,-\,\rangle \\
|\, N_- \,,\, n_- \,;\, 2\,\rangle_- &= |\, 0 \,,\, 0 \,,\, N_- - n_- \,,\, n_- - 1 \,;\, -,-,-,+\,\rangle \\
|\, N_- \,,\, n_- \,;\, 3\,\rangle_- &= |\, 0 \,,\, 0 \,,\, N_- - n_- - 1 \,,\, n_- \,;\, -,-,+,-\,\rangle \\
|\, N_- \,,\, n_- \,;\, 4\,\rangle_- &= |\, 0 \,,\, 0 \,,\, N_- - n_- - 1 \,,\, n_- - 1 \,;\, -,-,+,+\,\rangle
\end{aligned} \tag{5.24}$$

The degeneracy of the $N_- \hbar\Omega_-$ energy level is therefore equal to $4N_-$. Again, we can list the eigenvalues of the states corresponding to the angular momentum operator:

$$\begin{aligned}
L\,|\, N_- \,,\, 0 \,;\, 1\,\rangle_- &= \hbar N_- \; |\, N_- \,,\, 0 \,;\, 1\,\rangle_- \\
L\,|\, N_- \,,\, 0 \,;\, 2\,\rangle_- &= \hbar N_- \; |\, N_- \,,\, 0 \,;\, 2\,\rangle_- \\
L\,|\, N_- \,,\, 0 \,;\, 3\,\rangle_- &= -\hbar N_- \; |\, N_- \,,\, 0 \,;\, 3\,\rangle_- \\
L\,|\, N_- \,,\, 0 \,;\, 4\,\rangle_- &= -\hbar N_- \; |\, N_- \,,\, 0 \,;\, 4\,\rangle_- \\
L\,|\, N_- \,,\, n_- \,;\, 1\,\rangle_- &= \hbar(N_- - 2n_-) \; |\, N_- \,,\, n_- \,;\, 1\,\rangle_- \\
L\,|\, N_- \,,\, n_- \,;\, 2\,\rangle_- &= \hbar(N_- - 2n_-) \; |\, N_- \,,\, n_- \,;\, 2\,\rangle_- \\
L\,|\, N_- \,,\, n_- \,;\, 3\,\rangle_- &= \hbar(N_- - 2n_-) \; |\, N_- \,,\, n_- \,;\, 3\,\rangle_- \\
L\,|\, N_- \,,\, n_- \,;\, 4\,\rangle_- &= \hbar(N_- - 2n_-) \; |\, N_- \,,\, n_- \,;\, 4\,\rangle_-
\end{aligned} \tag{5.25}$$

5.3.4 $N_+ \geq 1$ and $N_- \geq 1$ energy states

It finally remains to describe the states of energy $(N_+ \hbar\Omega_+ + N_- \hbar\Omega_-)$ with $N_+, N_- \geq 1$. Each of these corresponds to the combination of one state of each of the two sectors described in the two previous Sections. Therefore, they are specified by two natural numbers n_+ and n_- and by a number S with the same physical interpretation as before:

$$|\, N_+ \,,\, n_+ \,,\, N_- \,,\, n_- \,;\, S\,\rangle \,, \qquad 0 \leq n_+, n_- \leq N_- - 1 \,,\; s = 1,2,...,16 \tag{5.26}$$

The list of the exact definition of these states can be found in Appendix A. The degeneracy of this energy level is equal to $16\,(N_+ N_-)$. Finally, we can again list the angular momentum operator eigenvalues:

$$
\begin{aligned}
L\,|\,N_+\,,\,0\,,\,N_-\,,\,0\,;\,S\,\rangle &= \hbar(N_+ + N_-)\;\;|\,N_+\,,\,0\,,\,N_-\,,\,0\,;\,S\,\rangle \quad s=1,2,3,4\\
L\,|\,N_+\,,\,0\,,\,N_-\,,\,0\,;\,S\,\rangle &= \hbar(N_+ - N_-)\;\;|\,N_+\,,\,0\,,\,N_-\,,\,0\,;\,S\,\rangle \quad s=5,6,7,8\\
L\,|\,N_+\,,\,0\,,\,N_-\,,\,0\,;\,S\,\rangle &= \hbar(N_- - N_+)\;\;|\,N_+\,,\,0\,,\,N_-\,,\,0\,;\,S\,\rangle \quad s=9,10,11,12\\
L\,|\,N_+\,,\,0\,,\,N_-\,,\,0\,;\,S\,\rangle &= -\hbar(N_+ + N_-)\;\;|\,N_+\,,\,0\,,\,N_-\,,\,0\,;\,S\,\rangle \quad s=13,14,15,16
\end{aligned}
$$

$$
\begin{aligned}
L\,|\,N_+\,,\,n_+\,,\,N_-\,,\,0\,;\,S\,\rangle &= \hbar(N_+ + N_- - 2n_+)\;\;|\,N_+\,,\,n_+\,,\,N_-\,,\,0\,;\,S\,\rangle\\
&\qquad\qquad \text{for } s=1,...,8\\
L\,|\,N_+\,,\,n_+\,,\,N_-\,,\,0\,;\,S\,\rangle &= \hbar(N_+ - N_- - 2n_+)\;\;|\,N_+\,,\,n_+\,,\,N_-\,,\,0\,;\,S\,\rangle\\
&\qquad\qquad \text{for } s=9,...,16\\
L\,|\,N_+\,,\,0\,,\,N_-\,,\,n_-\,;\,S\,\rangle &= \hbar(N_+ + N_- - 2n_-)\;\;|\,N_+\,,\,0\,,\,N_-\,,\,n_-\,;\,S\,\rangle\\
&\qquad\qquad \text{for } s=1,...,8\\
L\,|\,N_+\,,\,0\,,\,N_-\,,\,n_-\,;\,S\,\rangle &= \hbar(N_- - N_+ - 2n_-)\;\;|\,N_+\,,\,0\,,\,N_-\,,\,n_-\,;\,S\,\rangle\\
&\qquad\qquad \text{for } s=9,...,16\\
L\,|\,N_+\,,\,n_+\,,\,N_-\,,\,n_-\,;\,S\,\rangle &= \hbar(N_+ + N_- - 2(n_- + n_+))\;\;|\,N_+\,,\,n_+\,,\,N_-\,,\,n_-\,;\,S\,\rangle\\
&\qquad\qquad \text{for } s=1,...,16
\end{aligned}
\tag{5.27}
$$

5.4 Supercharge Diagonalization

As was noted earlier, the three operators H, L and Q commute amongst themselves, so that it is possible to find a basis of the space of states which diagonalizes the three operators at the same time. For now on, we have found how to diagonalize two of them, it remains to work on the diagonalization of the supercharge Q. In order to distinguish the new states from the previous ones, we will use the superscript ^{Q}: $|\,...\,\rangle^Q$. The Tables in this Section aim to sum up the different states and their eigenvalues corresponding to H, L and Q. As in the previous Sections, we will work sector by sector.

5.4.1 Ground state

The ground state is annihilated by the supercharge, so we have for its eigenvalues:

	H	L	Q
$\|\,\Omega\,\rangle$	0	0	0

(5.28)

It therefore forms a trivial representation of the $\mathcal{N}=1$ supersymmetry.

5.4.2 $N_+ \geq 1$ and $N_- = 0$ energy states

Let us recall that, since H, L and Q commute, the Q eigenstates will be built from linear combinations of states having the same energy and angular momentum. As shown by (5.21), we therefore have to diagonalize two 2-by-2 matrices, and one 4-by-4 matrix, so that we obtain at the end:

$$
\begin{aligned}
|\,N_+\,,\,0\,;\,1\,\rangle_+^Q &\equiv \frac{1}{\sqrt{2}}\Big(|\,N_+\,,\,0\,;\,1\,\rangle_+ + |\,N_+\,,\,0\,;\,2\,\rangle_+\Big) \\
|\,N_+\,,\,0\,;\,2\,\rangle_+^Q &\equiv \frac{1}{\sqrt{2}}\Big(|\,N_+\,,\,0\,;\,1\,\rangle_+ - |\,N_+\,,\,0\,;\,2\,\rangle_+\Big) \\
|\,N_+\,,\,0\,;\,3\,\rangle_+^Q &\equiv \frac{1}{\sqrt{2}}\Big(|\,N_+\,,\,0\,;\,3\,\rangle_+ - |\,N_+\,,\,0\,;\,4\,\rangle_+\Big) \\
|\,N_+\,,\,0\,;\,4\,\rangle_+^Q &\equiv \frac{1}{\sqrt{2}}\Big(|\,N_+\,,\,0\,;\,3\,\rangle_+ + |\,N_+\,,\,0\,;\,4\,\rangle_+\Big)
\end{aligned}
\tag{5.29}
$$

$$
\begin{aligned}
|\,N_+\,,\,n_+\,;\,1\,\rangle_+^Q &\equiv -\sqrt{\frac{n_+}{2N_+}}|\,N_+\,,\,n_+\,;\,1\,\rangle_+ + \frac{1}{\sqrt{2}}|\,N_+\,,\,n_+\,;\,2\,\rangle_+ + \sqrt{\frac{1}{2}\left(1-\frac{n_+}{N_+}\right)}|\,N_+\,,\,n_+\,;\,4\,\rangle_+ \\
|\,N_+\,,\,n_+\,;\,2\,\rangle_+^Q &\equiv -\sqrt{\frac{n_+}{2N_+}}|\,N_+\,,\,n_+\,;\,1\,\rangle_+ - \frac{1}{\sqrt{2}}|\,N_+\,,\,n_+\,;\,2\,\rangle_+ + \sqrt{\frac{1}{2}\left(1-\frac{n_+}{N_+}\right)}|\,N_+\,,\,n_+\,;\,4\,\rangle_+ \\
|\,N_+\,,\,n_+\,;\,3\,\rangle_+^Q &\equiv \sqrt{\frac{1}{2}\left(1-\frac{n_+}{N_+}\right)}|\,N_+\,,\,n_+\,;\,1\,\rangle_+ + \frac{1}{\sqrt{2}}|\,N_+\,,\,n_+\,;\,3\,\rangle_+ + \sqrt{\frac{n_+}{2N_+}}|\,N_+\,,\,n_+\,;\,4\,\rangle_+ \\
|\,N_+\,,\,n_+\,;\,4\,\rangle_+^Q &\equiv \sqrt{\frac{1}{2}\left(1-\frac{n_+}{N_+}\right)}|\,N_+\,,\,n_+\,;\,1\,\rangle_+ + \frac{1}{\sqrt{2}}|\,N_+\,,\,n_+\,;\,3\,\rangle_+ - \sqrt{\frac{n_+}{2N_+}}|\,N_+\,,\,n_+\,;\,4\,\rangle_+
\end{aligned}
\tag{5.30}
$$

These states are normalized and orthogonal:

$$
{}_+^Q\langle\,N_+\,,\,0\,;\,S\,|\,M_+\,,\,0\,;\,T\,\rangle_+^Q = \delta_{N_+,M_+}\delta_{S,T} \tag{5.31}
$$

They have the following eigenvalues:

(5.32)

	H	L	Q
$\|\,N_+\,,\,0\,;\,1\,\rangle_+^Q$	$\hbar N_+\Omega_+$	$\hbar N_+$	$-\hbar\sqrt{N_+\Omega_+}$
$\|\,N_+\,,\,0\,;\,2\,\rangle_+^Q$	$\hbar N_+\Omega_+$	$\hbar N_+$	$\hbar\sqrt{N_+\Omega_+}$
$\|\,N_+\,,\,0\,;\,3\,\rangle_+^Q$	$\hbar N_+\Omega_+$	$-\hbar N_+$	$-\hbar\sqrt{N_+\Omega_+}$
$\|\,N_+\,,\,0\,;\,4\,\rangle_+^Q$	$\hbar N_+\Omega_+$	$-\hbar N_+$	$\hbar\sqrt{N_+\Omega_+}$
$\|\,N_+\,,\,n_+\,;\,1\,\rangle_+^Q$	$\hbar N_+\Omega_+$	$\hbar(N_+-2n_+)$	$-\hbar\sqrt{N_+\Omega_+}$
$\|\,N_+\,,\,n_+\,;\,2\,\rangle_+^Q$	$\hbar N_+\Omega_+$	$\hbar(N_+-2n_+)$	$\hbar\sqrt{N_+\Omega_+}$
$\|\,N_+\,,\,n_+\,;\,3\,\rangle_+^Q$	$\hbar N_+\Omega_+$	$\hbar(N_+-2n_+)$	$-\hbar\sqrt{N_+\Omega_+}$
$\|\,N_+\,,\,n_+\,;\,4\,\rangle_+^Q$	$\hbar N_+\Omega_+$	$\hbar(N_+-2n_+)$	$\hbar\sqrt{N_+\Omega_+}$

It is worth noting that, in agreement with the fact that $Q^2 = \hbar H$, the supercharge of an eigenstate is equal to $\hbar$ times a square root of its energy. Note also that a state is in general not completely determined by its energy, angular momentum and supercharge. This point will be discussed later on.

5.4.3 $N_- \geq 1$ and $N_+ = 0$ energy states

On this subspace, Q may be diagonalized following exactly the same approach as in the previous Subsection. We therefore obtain the following states:

$$\begin{aligned}
|\,N_-\,,\,0\,;\,1\,\rangle_-^Q &\equiv \frac{1}{\sqrt{2}}\Big(|\,N_-\,,\,0\,;\,1\,\rangle_- + |\,N_-\,,\,0\,;\,2\,\rangle_-\Big)\\
|\,N_-\,,\,0\,;\,2\,\rangle_-^Q &\equiv \frac{1}{\sqrt{2}}\Big(|\,N_-\,,\,0\,;\,1\,\rangle_- - |\,N_-\,,\,0\,;\,2\,\rangle_-\Big)\\
|\,N_-\,,\,0\,;\,3\,\rangle_-^Q &\equiv \frac{1}{\sqrt{2}}\Big(|\,N_-\,,\,0\,;\,3\,\rangle_- - |\,N_-\,,\,0\,;\,4\,\rangle_-\Big)\\
|\,N_-\,,\,0\,;\,4\,\rangle_-^Q &\equiv \frac{1}{\sqrt{2}}\Big(|\,N_-\,,\,0\,;\,3\,\rangle_- + |\,N_-\,,\,0\,;\,4\,\rangle_-\Big)
\end{aligned} \tag{5.33}$$

$$\begin{aligned}
|\,N_-\,,\,n_-\,;\,1\,\rangle_-^Q &\equiv -\sqrt{\frac{n_-}{2N_-}}|\,N_-\,,\,n_-\,;\,1\,\rangle_- + \frac{1}{\sqrt{2}}|\,N_-\,,\,n_-\,;\,2\,\rangle_- + \sqrt{\frac{1}{2}\left(1-\frac{n_-}{N_-}\right)}|\,N_-\,,\,n_-\,;\,4\,\rangle_-\\
|\,N_-\,,\,n_-\,;\,2\,\rangle_-^Q &\equiv -\sqrt{\frac{n_-}{2N_-}}|\,N_-\,,\,n_-\,;\,1\,\rangle_- - \frac{1}{\sqrt{2}}|\,N_-\,,\,n_-\,;\,2\,\rangle_- + \sqrt{\frac{1}{2}\left(1-\frac{n_-}{N_-}\right)}|\,N_-\,,\,n_-\,;\,4\,\rangle_-\\
|\,N_-\,,\,n_-\,;\,3\,\rangle_-^Q &\equiv \sqrt{\frac{1}{2}\left(1-\frac{n_-}{N_-}\right)}|\,N_-\,,\,n_-\,;\,1\,\rangle_- + \frac{1}{\sqrt{2}}|\,N_-\,,\,n_-\,;\,3\,\rangle_- + \sqrt{\frac{n_-}{2N_-}}|\,N_-\,,\,n_-\,;\,4\,\rangle_-\\
|\,N_-\,,\,n_-\,;\,4\,\rangle_-^Q &\equiv \sqrt{\frac{1}{2}\left(1-\frac{n_-}{N_-}\right)}|\,N_-\,,\,n_-\,;\,1\,\rangle_- + \frac{1}{\sqrt{2}}|\,N_-\,,\,n_-\,;\,3\,\rangle_- - \sqrt{\frac{n_-}{2N_-}}|\,N_-\,,\,n_-\,;\,4\,\rangle_-
\end{aligned} \tag{5.34}$$

We also have the following eigenvalues:

	H	L	Q
$\|\,N_-\,,\,0\,;\,1\,\rangle_-^Q$	$\hbar N_-\Omega_-$	$\hbar N_-$	$-\hbar\sqrt{N_-\Omega_-}$
$\|\,N_-\,,\,0\,;\,2\,\rangle_-^Q$	$\hbar N_-\Omega_-$	$\hbar N_-$	$\hbar\sqrt{N_-\Omega_-}$
$\|\,N_-\,,\,0\,;\,3\,\rangle_-^Q$	$\hbar N_-\Omega_-$	$-\hbar N_-$	$-\hbar\sqrt{N_-\Omega_-}$
$\|\,N_-\,,\,0\,;\,4\,\rangle_-^Q$	$\hbar N_-\Omega_-$	$-\hbar N_-$	$\hbar\sqrt{N_-\Omega_-}$
$\|\,N_-\,,\,n_-\,;\,1\,\rangle_-^Q$	$\hbar N_-\Omega_-$	$\hbar(N_- - 2n_-)$	$-\hbar\sqrt{N_-\Omega_-}$
$\|\,N_-\,,\,n_-\,;\,2\,\rangle_-^Q$	$\hbar N_-\Omega_-$	$\hbar(N_- - 2n_-)$	$\hbar\sqrt{N_-\Omega_-}$
$\|\,N_-\,,\,n_-\,;\,3\,\rangle_-^Q$	$\hbar N_-\Omega_-$	$\hbar(N_- - 2n_-)$	$-\hbar\sqrt{N_-\Omega_-}$
$\|\,N_-\,,\,n_-\,;\,4\,\rangle_-^Q$	$\hbar N_-\Omega_-$	$\hbar(N_- - 2n_-)$	$\hbar\sqrt{N_-\Omega_-}$

(5.35)

5.4.4 $N_+ \geq 1$ and $N_- \geq 1$ energy states

This last subspace is a bit more tricky to diagonalize. As can be seen from (5.27), we indeed get up to 16-by-16 matrices that lead to cumbersome expressions. Furthermore, we will not require these expressions in the sequel of this work since we will only be interested in the previous subspaces. In order to save some paper, we therefore do not give the (known) complete and exact expressions of $|\,N_+\,,\,n_+\,,\,N_-\,,\,n_-\,;\,S\,\rangle^Q$, but their eigenvalues associated to the operators H, L and Q may be found in Appendix A. Let us however make the following remark. As shown by (5.27), for fixed $N_+, N_- \geq 1$, $1 \leq n_+ \leq N_+ - 1$ and $1 \leq n_- \leq N_- - 1$, the 16 states $|\, N_+ \;,\; n_+ \;,\; N_- \;,\; n_- \;;\; S\,\rangle$ all share the same energy and angular momentum. This will thus also be true for the alternative basis of states $|\, N_+ \;,\; n_+ \;,\; N_- \;,\; n_- \;;\; S\,\rangle^Q$. Since there are only two possible supercharges eigenvalues of opposite signs associated to a given energy, there will be 8 states sharing the same three eigenvalues. This observation hints at one additional global symmetry of the system at least that would distinguish these states.

5.4.5 Identity operator resolution

Summarizing the results of the previous Sections, we can write the identity resolution in terms of the considered eigenstates of the operators H, L and Q:

$$\mathbb{I} = |\,\Omega\,\rangle\langle\,\Omega\,| \tag{5.36}$$

$$+ \sum_{N_+=1}^{\infty} \sum_{n_+=0}^{N_+-1} \sum_{S=1}^{4} |\, N_+, n_+, 0, 0\,; S\,\rangle^Q_+ {}^Q_+\langle\, N_+, n_+, 0, 0\,; S\,| \tag{5.37}$$

$$+ \sum_{N_-=1}^{\infty} \sum_{n_-=0}^{N_+-1} \sum_{S=1}^{4} |\, 0, 0, N_-, n_-\,; S\,\rangle^Q_- {}^Q_-\langle\, 0, 0, N_-, n_-\,; S\,| \tag{5.38}$$

$$+ \sum_{N_+,N_-=1}^{\infty} \sum_{n_+=0}^{N_+-1} \sum_{n_-=0}^{N_--1} \sum_{S=1}^{16} |\, N_+, n_+, N_-, n_-\,; S\,\rangle^Q {}^Q\langle\, N_+, n_+, N_-, n_-\,; S\,| \tag{5.39}$$

Chapter 6

The $\mathcal{N} = 1$ Supersymmetric Non(anti)commutative Superplane

This Chapter aims to show how noncommutativity may emerge in our system. The general idea is the same as the one used in [7] or [8] and consists in projecting the whole system onto a well chosen subspace of its Hilbert space of states. This projection will lead to a deformation of the operators algebra that, in our case, implies that the system bosonic coordinates do not commute anymore. In the following Sections, we will consider two such possible projections, requiring them to be consistent with the supersymmetry of the system. That way, these will give examples of $\mathcal{N} = 1$ supersymmetric non(anti)commutative superplanes.

Let us first begin by a general consideration about projections that will help us to simplify the calculations of this Chapter. Let us consider a projection operator $\mathbb{P}$, that is, an operator such that $\mathbb{P}^2 = \mathbb{P}$ (and $\mathbb{P}^\dagger = \mathbb{P}$). For any operator A, we consider its projection

$$\overline{A} = \mathbb{P}A\mathbb{P}. \tag{6.1}$$

Since our aim is to determine the algebra that the projected operators obey, we will usually not be interested in the exact expression of the projected operators, but rather in the expression of their (anti)commutators. Therefore, it is useful to note that, if an operator A_1 commutes with $\mathbb{P}$, we have, for any other operator A_2:

$$\begin{aligned} [\,\overline{A}_1, \overline{A}_2] &= \mathbb{P}A_1\mathbb{P}A_2\mathbb{P} - \mathbb{P}A_2\mathbb{P}A_1\mathbb{P} \\ &= \mathbb{P}A_1A_2\mathbb{P} - \mathbb{P}A_2A_1\mathbb{P} = \mathbb{P}[A_1, A_2]\mathbb{P} \end{aligned} \tag{6.2}$$

Note that this argument holds also for an anticommutator instead of a commutator. We will shortly see how this result can save us valuable work.

In order to clearly identify the subspaces on which we will project the system, let us recall the identity operator resolution both in terms of the $|\,N_+, n_+, N_-, n_-\,; S\,\rangle$ and the $|\,N_+, n_+, N_-, n_-\,; S\,\rangle^Q$

eigenstates:

$$\begin{aligned}
\mathbb{I} \;=\;& \;|\,\Omega\,\rangle\langle\,\Omega\,| \\
&+ \sum_{N_+=1}^{\infty} \sum_{n_+=0}^{N_+-1} \sum_{S=1}^{4} |\,N_+, n_+\,;S\,\rangle_+ \;{}_+\langle\,N_+, n_+\,;S\,| \\
&+ \sum_{N_-=1}^{\infty} \sum_{n_-=0}^{N_+-1} \sum_{S=1}^{4} |\,N_-, n_-\,;S\,\rangle_- \;{}_-\langle\,N_-, n_-\,;S\,| \\
&+ \sum_{N_+,N_-=1}^{\infty} \sum_{n_+=0}^{N_+-1} \sum_{n_-=0}^{N_--1} \sum_{S=1}^{16} |\,N_+, n_+, N_-, n_-\,;S\,\rangle\langle\,N_+, n_+, N_-, n_-\,;S\,|
\end{aligned} \tag{6.3}$$

$$\begin{aligned}
\mathbb{I} \;=\;& \;|\,\Omega\,\rangle\langle\,\Omega\,| \\
&+ \sum_{N_+=1}^{\infty} \sum_{n_+=0}^{N_+-1} \sum_{S=1}^{4} |\,N_+, n_+\,;S\,\rangle_+^Q \;{}_+^Q\langle\,N_+, n_+\,;S\,| \\
&+ \sum_{N_-=1}^{\infty} \sum_{n_-=0}^{N_+-1} \sum_{S=1}^{4} |\,N_-, n_-\,;S\,\rangle_-^Q \;{}_-^Q\langle\,N_-, n_-\,;S\,| \\
&+ \sum_{N_+,N_-=1}^{\infty} \sum_{n_+=0}^{N_+-1} \sum_{n_-=0}^{N_--1} \sum_{S=1}^{16} |\,N_+, n_+, N_-, n_-\,;S\,\rangle^Q \;{}^Q\langle\,N_+, n_+, N_-, n_-\,;S\,|
\end{aligned} \tag{6.4}$$

6.1 Projection on the N_+ Sector

We will first consider the projection operator on the subspace spanned by the eigenstates corresponding to $N_- = 0$ and $N_+ \geq 0$:

$$\begin{aligned}
\mathbb{P}_0 \;=\;& \;|\,\Omega\,\rangle\langle\,\Omega\,| \\
&+ \sum_{N_+=1}^{\infty} \sum_{n_+=0}^{N_+-1} \sum_{S=1}^{4} |\,N_+, n_+\,;S\,\rangle_+^Q \;{}_+^Q\langle\,N_+, n_+\,;S\,|
\end{aligned} \tag{6.5}$$

In this Section, for any operator A, we will define its projection by $\mathbb{P}_0$ as:

$$\overline{A} \;=\; \mathbb{P}_0 A \mathbb{P}_0. \tag{6.6}$$

Let us note by $\mathcal{H}_0$ the subspace of all projected states. The projections of the operators H, Q and L are easily obtained since we deal with eigenstates:

$$\overline{H} \;=\; \sum_{N_+=1}^{\infty} \sum_{n_+=0}^{N_+-1} \sum_{S=1}^{4} (\hbar\, N_+\, \Omega_+)\; |\,N_+, n_+\,;S\,\rangle_+^Q \;{}_+^Q\langle\,N_+, n_+\,;S\,| \tag{6.7}$$

$$\overline{Q} \;=\; \sum_{N_+=1}^{\infty} \sum_{n_+=0}^{N_+-1} \sum_{S=1}^{4} (-1)^S\; \hbar\sqrt{N_+\,\Omega_+}\; |\,N_+, n_+\,;S\,\rangle_+^Q \;{}_+^Q\langle\,N_+, n_+\,;S\,| \tag{6.8}$$

$$\begin{aligned}
\overline{L} \;=\;& \sum_{N_+=1}^{\infty} \sum_{S=1}^{4} (-1)^{(S-1 \bmod 2)}\; \hbar\, N_+ |\,N_+\,;S\,\rangle_+^Q \;{}_+^Q\langle\,N_+\,;S\,| \\
+\;& \sum_{N_+=1}^{\infty} \sum_{n_+=0}^{N_+-1} \sum_{S=1}^{4} \hbar\;(N_+ - 2n_+)\; |\,N_+, n_+\,;S\,\rangle_+^Q \;{}_+^Q\langle\,N_+, n_+\,;S\,|
\end{aligned} \tag{6.9}$$

These expressions imply that $\mathcal{H}_0$ is consistent with the $\mathcal{N} = 1$ supersymmetry of the system since the supercharge is still the "square root" of the Hamiltonian on the projected subspace:

$$\overline{Q}^2 = \hbar \overline{H} \tag{6.10}$$

This could also have been deduced by observing that H, Q and L clearly commute with $\mathbb{P}_0$, so that (6.2) implies (6.10). In the same way, we get:

$$[\overline{H}, \overline{Q}] = 0, \qquad [\overline{H}, \overline{L}] = 0, \qquad [\overline{L}, \overline{Q}] = 0 \tag{6.11}$$

For operators that do not admit the states in (6.5) as eigenstates, the calculation of their projection is a bit more involved. As it may be seen, the projection (6.6) involves many sums, two of them being infinite. In Mathematica©, such sums are difficult to deal with. Therefore, for some operators, it turns out to be more efficient to compute their commutators with $\mathbb{P}_0$ and use the trick mentioned at the beginning of this Chapter instead of calculating the exact expression of their projection. Furthermore, since the eigenstates $|\, N_+, n_+, 0, 0\,; S\,\rangle^Q$ are orthonormal, one may verify that the projector $\mathbb{P}_0$ is equivalent to the following one:

$$\begin{aligned} \mathbb{P}'_0 = &\ |\,\Omega\,\rangle\langle\,\Omega\,| \\ &+ \sum_{N_+=1}^{\infty} \sum_{n_+=0}^{N_+-1} \sum_{S=1}^{4} |\, N_+, n_+\,; S\,\rangle_+ \, {}_+\langle\, N_+, n_+\,; S\,| \end{aligned} \tag{6.12}$$

Again, the use of $\mathbb{P}'_0$ instead of $\mathbb{P}_0$ leads to easier computations. Note however that in actual fact, even though the notation used is different for convenience, $\mathbb{P}'_0$ and $\mathbb{P}_0$ are identical projections.

6.1.1 Algebra of the projected chiral Fock algebra operators and invariance properties

We will now work on the algebra of the projected bosonic and fermionic chiral Fock algebra operators. Having the previous remarks in mind, we compute the commutators of the operators $a_\pm^{(\dagger)}$ and $\Gamma_\pm^{(\dagger)}$ with the projector, and we find that they identically vanish:

$$\begin{aligned} [\mathbb{P}_0, a_\pm] &= 0 = [\mathbb{P}_0, \Gamma_\pm] \\ [\mathbb{P}_0, a_\pm^\dagger] &= 0 = [\mathbb{P}_0, \Gamma_\pm^\dagger] \end{aligned} \tag{6.13}$$

This result can be easily understood since these operators create or annihilate a quantum of energy $\hbar\,\Omega_+$ and therefore leave $\mathcal{H}_0$ invariant, hence the commutation of these operators with $\mathbb{P}_0$. On the contrary, the operators $b_\pm^{(\dagger)}$ and $\Lambda_\pm^{(\dagger)}$ create/annihilate a quantum of energy $\hbar\,\Omega_-$ and therefore map a state of $\mathcal{H}_0$ outside of $\mathcal{H}_0$. As may therefore be expected, we find:

$$\begin{aligned} b_+\,\mathbb{P}_0 = \mathbb{P}_0\, b_+^\dagger &= 0 = b_-\,\mathbb{P}_0 = \mathbb{P}_0\, b_-^\dagger\,, \\ \Lambda_+\,\mathbb{P}_0 = \mathbb{P}_0\,\Lambda_+^\dagger &= 0 = \Lambda_-\,\mathbb{P}_0 = \mathbb{P}_0\,\Lambda_-^\dagger\,, \end{aligned} \tag{6.14}$$

so that

$$\begin{aligned} \overline{b}_+ = \overline{b}_+^\dagger &= 0 = \overline{\Lambda}_+ = \overline{\Lambda}_+^\dagger\,, \\ \overline{b}_- = \overline{b}_-^\dagger &= 0 = \overline{\Lambda}_- = \overline{\Lambda}_-^\dagger\,. \end{aligned} \tag{6.15}$$

Therefore, using (6.2), the algebra of the projected operators is given by the following non vanishing commutators:

$$\begin{aligned} [\overline{a}_+, \overline{a}_+^\dagger] &= \mathbb{P}_0 = \{\overline{\Gamma}_+, \overline{\Gamma}_+^\dagger\}\,, \\ [\overline{a}_-, \overline{a}_-^\dagger] &= \mathbb{P}_0 = \{\overline{\Gamma}_-, \overline{\Gamma}_-^\dagger\}\,. \end{aligned} \tag{6.16}$$

Using the commutation of H, Q and L with $\mathbb{P}_0$, it is straightforward to compute the transformations induced by $\overline{H}$, $\overline{Q}$ and $\overline{L}$ on the non vanishing chiral Fock algebra operators: they are given by the expressions of Section 4.3, the operators being replaced by the projected ones:

$$\begin{aligned}
[\overline{H}, \overline{a}_+] &= -\hbar\Omega_+ \overline{a}_+ , & [\overline{H}, \overline{a}_+^\dagger] &= \hbar\Omega_+ \overline{a}_+^\dagger \\
[\overline{H}, \overline{a}_-] &= -\hbar\Omega_+ \overline{a}_- , & [\overline{H}, \overline{a}_-^\dagger] &= \hbar\Omega_+ \overline{a}_-^\dagger \\
[\overline{H}, \overline{\Gamma}_+] &= -\hbar\Omega_+ \overline{\Gamma}_+ , & [\overline{H}, \overline{\Gamma}_+^\dagger] &= \hbar\Omega_+ \overline{\Gamma}_+^\dagger \\
[\overline{H}, \overline{\Gamma}_-] &= -\hbar\Omega_+ \overline{\Gamma}_- , & [\overline{H}, \overline{\Gamma}_-^\dagger] &= \hbar\Omega_+ \overline{\Gamma}_-^\dagger
\end{aligned} \tag{6.17}$$

$$\begin{aligned}
[\overline{Q}, \overline{a}_+] &= \hbar\sqrt{\Omega_+}\, \overline{\Gamma}_+ , & [\overline{Q}, \overline{a}_+^\dagger] &= -\hbar\sqrt{\Omega_+}\, \overline{\Gamma}_+^\dagger \\
\{\overline{Q}, \overline{\Gamma}_+\} &= -\hbar\sqrt{\Omega_+}\, \overline{a}_+ , & \{\overline{Q}, \overline{\Gamma}_+^\dagger\} &= -\hbar\sqrt{\Omega_+}\, \overline{a}_+^\dagger \\
[\overline{Q}, \overline{a}_-] &= \hbar\sqrt{\Omega_+}\, \overline{\Gamma}_- , & [\overline{Q}, \overline{a}_-^\dagger] &= -\hbar\sqrt{\Omega_+}\, \overline{\Gamma}_-^\dagger \\
\{\overline{Q}, \overline{\Gamma}_-\} &= -\hbar\sqrt{\Omega_+}\, \overline{a}_- , & \{\overline{Q}, \overline{\Gamma}_-^\dagger\} &= -\hbar\sqrt{\Omega_+}\, \overline{a}_-^\dagger
\end{aligned} \tag{6.18}$$

$$\begin{aligned}
[\overline{L}, \overline{a}_+] &= -\hbar\overline{a}_+ , & [\overline{L}, \overline{a}_+^\dagger] &= \hbar\overline{a}_+^\dagger \\
[\overline{L}, \overline{a}_-] &= \hbar\overline{a}_- , & [\overline{L}, \overline{a}_-^\dagger] &= -\hbar\overline{a}_-^\dagger \\
[\overline{L}, \overline{\Gamma}_+] &= -\hbar\overline{\Gamma}_+ , & [\overline{L}, \overline{\Gamma}_+^\dagger] &= \hbar\overline{\Gamma}_+^\dagger \\
[\overline{L}, \overline{\Gamma}_-] &= \hbar\overline{\Gamma}_- , & [\overline{L}, \overline{\Gamma}_-^\dagger] &= -\hbar\overline{\Gamma}_-^\dagger
\end{aligned} \tag{6.19}$$

This shows that, like the whole system, the projected system is still invariant under the $\mathcal{N} = 1$ supersymmetry transformations and the $SO(2)$ rotations in the plane.

6.1.2 Coordinates and momenta commutation relations

It is now time to turn back to the Cartesian bosonic coordinates x_i, u_i and their conjugated momenta p_i, π_i and the fermionic degrees of freedom ψ_i, μ_i, λ_i and γ_i. Their quantum algebra (4.1) is supposed to be deformed after the projection of the system by $\mathbb{P}_0$. In order to verify this, we will as before not consider the exact expression of the projection of the coordinates. Instead, we will directly compute their commutators from the expressions (6.15) and (6.16). Using the inverses of the linear transformations introduced in Chapter 4, the calculation will be implemented in Mathematica© using the following obvious identity:

$$[\sum_i c_i A_i , \sum_j d_j A_j] = \sum_{ij} c_i d_j [A_i, A_j] \tag{6.20}$$

where the A_i's label the bosonic and fermionic chiral Fock algebra operators. This leads us to the following expressions:

$$\begin{aligned}
[\epsilon_{ik}\,\overline{x}_k, \epsilon_{jl}\,\overline{p}_l] &= \frac{1}{2}\left(1 + \frac{k_0^2\,\kappa_0^2\ -\ 4\beta_0^2\,\omega_0^2}{2s\,\omega_+\omega_-}\right)\ i\hbar\,\delta_{ij}\,\mathbb{P}_0 \\
[\overline{u}_i, \overline{\pi}_j] &= \frac{1}{2}\left(1 - \frac{k_0^2\,\kappa_0^2\ -\ 4\beta_0^2\,\omega_0^2}{2s\,\omega_+\omega_-}\right)\ i\hbar\,\delta_{ij}\,\mathbb{P}_0 \\
[\epsilon_{ik}\,\overline{x}_k, \overline{u}_j] &= \frac{s\,qB}{m^2\omega_+\omega_-}\ i\hbar\,\delta_{ij}\,\mathbb{P}_0 \\
[\epsilon_{ik}\,\overline{p}_k, \overline{\pi}_j] &= \frac{s\,qB}{4m^2\omega_+\omega_-}(q^2B^2\ +\ m^2\left(k_0^2\,\kappa_0^2\ +\ 4\beta_0^2\,\omega_0^2\right))\ i\hbar\,\delta_{ij}\,\mathbb{P}_0 \\
\{\overline{\psi}_1, \overline{\psi}_2\} &= \left(1 + \frac{s}{\omega_+\omega_-}\left(\frac{q^2B^2}{m^2} + \frac{1}{2}\left(k_0^2\,\kappa_0^2\ -\ 4\beta_0^2\,\omega_0^2\right)\right)\right)\ \frac{\hbar}{4m}\,\delta_{ij}\,\mathbb{P}_0 \\
\{\overline{\mu}_1, \overline{\mu}_2\} &= \left(1 + \frac{s}{\omega_+\omega_-}\left(\frac{q^2B^2}{m^2} - \frac{1}{2}\left(k_0^2\,\kappa_0^2\ -\ 4\beta_0^2\,\omega_0^2\right)\right)\right)\ \frac{\hbar}{m}\,\delta_{ij}\,\mathbb{P}_0 \\
\{\overline{\lambda}_1, \overline{\lambda}_2\} &= \left(1 - \frac{s}{\omega_+\omega_-}\left(\frac{q^2B^2}{m^2} + \frac{1}{2}\left(k_0^2\,\kappa_0^2\ -\ 4\beta_0^2\,\omega_0^2\right)\right)\right)\ \frac{\hbar}{2m\omega_0^2}\,\delta_{ij}\,\mathbb{P}_0 \\
\{\overline{\gamma}_1, \overline{\gamma}_2\} &= \left(1 - \frac{s}{\omega_+\omega_-}\left(\frac{q^2B^2}{m^2} - \frac{1}{2}\left(k_0^2\,\kappa_0^2\ -\ 4\beta_0^2\,\omega_0^2\right)\right)\right)\ \frac{\hbar}{2mk_0^2}\,\delta_{ij}\,\mathbb{P}_0 \\
\{\epsilon_{ik}\,\overline{\psi}_k, \overline{\lambda}_j\} &= \frac{s\,qB\,\beta_0}{m^2\omega_+\omega_-}\ \hbar\,\delta_{ij}\,\mathbb{P}_0 \\
\{\overline{\mu}_i, \epsilon_{ik}\,\overline{\gamma}_j\} &= -\frac{s\,qB\,\kappa_0}{m^2\omega_+\omega_-}\ \hbar\,\delta_{ij}\,\mathbb{P}_0
\end{aligned} \tag{6.21}$$

As expected, the projected plane becomes noncommutative since the coordinates $\epsilon_{ij}\,\overline{x}_j$ and $\overline{u}_i$ do not commute anymore. This result can be understood by noticing that, since the $(b_\pm^{(\dagger)}, \Lambda_\pm^{(\dagger)})$ sector is reduced to the null operator by the projection, a dependence appears between $\epsilon_{ij}\overline{x}_j$ and $\overline{\pi}_i$ on the one hand, and between $\overline{u}_i$ and $\epsilon_{ij}\overline{p}_j$ on the other hand:

$$\begin{aligned}
\overline{\pi}_i &= \frac{m^2}{4qB}(\kappa_0^2k_0^2 - 4\beta_0^2\omega_0^2 - 2s\,\omega_+\omega_-)\ \epsilon_{ij}\,\overline{x}_j \\
\epsilon_{ij}\,\overline{p}_j &= \frac{m^2}{4qB}(\kappa_0^2k_0^2 - 4\beta_0^2\omega_0^2 + 2s\,\omega_+\omega_-)\ \overline{u}_i
\end{aligned} \tag{6.22}$$

Note that this projection gives an example of noncommutativity where not only the coordinates do not commute, but the momentum operators also. The fermionic algebra is also deformed, in a consistent way with the $\mathcal{N} = 1$ supersymmetry since the latter still holds on the projected subspace. Note that the anticommutator between $\epsilon_{ij}\,\overline{\psi}_j$ and $\overline{\lambda}_j$ does not vanish anymore, showing that these projected fermionic degrees of freedom are not independent. The same remark holds for $\overline{\mu}_i$ and $\epsilon_{ij}\,\overline{\gamma}_j$.

One may check that for the specific case where $\kappa_0\,k_0 = 2\,\beta_0\,\omega_0$, the expressions (6.21) are consistent with the results which have been calculated in [8].

Finally, it is interesting to notice that, if we turn off the harmonic potential confining the center of mass of the system, and if we take the massless limit, we get, for the bosonic coordinates:

$$[\epsilon_{ik}\,\overline{x}_k, \overline{u}_j] \underset{m,k_0\to 0}{=} s\frac{i\hbar}{qB}\,\delta_{ij}\,\mathbb{P}_0 \tag{6.23}$$

which is consistent with the discussion in [6]. However, this remark should be taken with great care, since this limit would require a more careful treatment, as is seen for instance in the fermionic anticommutation relations which become infinite naively in that limit.

6.2 Projection on the N_- Sector

In a very similar way as in the previous Section, it is also possible to consider the projection of the system on the subspace spanned by the eigenstates corresponding to $N_+ = 0$ and $N_- \geq 0$. The corresponding projector is given by

$$\begin{aligned}\mathbb{P}_1 \;=\;& |\,\Omega\,\rangle\langle\,\Omega\,| \\ &+\sum_{N_-=1}^{\infty}\sum_{n_-=0}^{N_--1}\sum_{S=1}^{4}|\,N_-,n_-\,;S\,\rangle_-^Q\;{}_-^Q\langle\,N_-,n_-\,;S\,|\end{aligned} \tag{6.24}$$

and, given an operator A, we will consider its projection

$$\overline{A} \;=\; \mathbb{P}_1 A \mathbb{P}_1. \tag{6.25}$$

As before, we have the following expressions for the projection of the operators H, L and Q:

$$\overline{H} \;=\; \sum_{N_-=1}^{\infty}\sum_{n_-=0}^{N_--1}\sum_{S=1}^{4}(\hbar N_-\,\Omega_-)\,|\,N_-,n_-\,;S\,\rangle_-^Q\;{}_-^Q\langle\,N_-,n_-\,;S\,| \tag{6.26}$$

$$\overline{Q} \;=\; \sum_{N_-=1}^{\infty}\sum_{n_-=0}^{N_--1}\sum_{S=1}^{4}(-1)^S\,\hbar\sqrt{N_-\,\Omega_-}\,|\,N_-,n_-\,;S\,\rangle_-^Q\;{}_-^Q\langle\,N_-,n_-\,;S\,| \tag{6.27}$$

$$\begin{aligned}\overline{L} \;=\;& \sum_{N_-=1}^{\infty}\sum_{S=1}^{4}(-1)^{(S-1 \bmod 2)}\,\hbar N_-|\,N_-\,;S\,\rangle_-^Q\;{}_-^Q\langle\,N_-\,;S\,| \\ &+\sum_{N_-=1}^{\infty}\sum_{n_-=0}^{N_--1}\sum_{S=1}^{4}\hbar\,(N_- - 2n_-)\;|\,N_-,n_-\,;S\,\rangle_-^Q\;{}_-^Q\langle\,N_-,n_-\,;S\,|\end{aligned} \tag{6.28}$$

This shows that, again, this subspace is consistent with the $\mathcal{N}=1$ supersymmetry since

$$\overline{Q}^2 \;=\; \hbar\overline{H}. \tag{6.29}$$

As for the N_+ sector, the three projected operators still commute. The following Subsections follow exactly the same arguments as in the previous Section, so we will not repeat them.

6.2.1 Algebra of the projected chiral Fock algebra operators and invariance properties

The results are the same as previously, except that we have to replace the $a_\pm^{(\dagger)}$'s by the $b_\pm^{(\dagger)}$'s and the $\Gamma_\pm^{(\dagger)}$'s by the $\Lambda_\pm^{(\dagger)}$'s and conversely. This is easily understood since this time, the considered subspace corresponds to states carrying $\hbar\,\Omega_-$ energy quanta. We therefore have:

$$\begin{aligned}[\mathbb{P}_1, b_\pm] \;&=\; 0 \;=\; [\mathbb{P}_1, \Lambda_\pm] \\ [\mathbb{P}_1, b_\pm^\dagger] \;&=\; 0 \;=\; [\mathbb{P}_1, \Lambda_\pm^\dagger]\end{aligned} \tag{6.30}$$

and

$$\begin{aligned} a_+\,\mathbb{P}_1 = \mathbb{P}_1\,a_+^\dagger &= 0 = a_-\,\mathbb{P}_1 = \mathbb{P}_1\,a_-^\dagger \\ \Gamma_+\,\mathbb{P}_1 = \mathbb{P}_1\,\Gamma_+^\dagger &= 0 = \Gamma_-\,\mathbb{P}_1 = \mathbb{P}_1\,\Gamma_-^\dagger \end{aligned} \tag{6.31}$$

so that

$$\begin{aligned} \overline{a}_+ = \overline{a}_+^\dagger &= 0 = \overline{\Gamma}_+ = \overline{\Gamma}_+^\dagger \\ \overline{a}_- = \overline{a}_-^\dagger &= 0 = \overline{\Gamma}_- = \overline{\Gamma}_-^\dagger \end{aligned} \tag{6.32}$$

This leads us to the following algebra for the non vanishing projected bosonic and fermionic chiral Fock algebra operators:

$$\begin{aligned} [\overline{b}_+, \overline{b}_+^\dagger] &= \mathbb{P}_1 = \{\overline{\Lambda}_+, \overline{\Lambda}_+^\dagger\} \\ [\overline{b}_-, \overline{b}_-^\dagger] &= \mathbb{P}_1 = \{\overline{\Lambda}_-, \overline{\Lambda}_-^\dagger\}, \end{aligned} \tag{6.33}$$

where it is understood that all the other (anti)commutators are equal to zero. Again, the expressions for the transformations induced by $\overline{H}$, $\overline{L}$ and $\overline{Q}$ still hold for the projected operators:

$$\begin{aligned} [\,\overline{H}, \overline{b}_+\,] &= -\hbar\,\Omega_-\,\overline{b}_+\,, & [\,\overline{H}, \overline{b}_+^\dagger\,] &= \hbar\,\Omega_-\,\overline{b}_+^\dagger \\ [\,\overline{H}, \overline{b}_-\,] &= -\hbar\,\Omega_-\,\overline{b}_-\,, & [\,\overline{H}, \overline{b}_-^\dagger\,] &= \hbar\,\Omega_-\,\overline{b}_-^\dagger \\ [\,\overline{H}, \overline{\Lambda}_+\,] &= -\hbar\,\Omega_-\,\overline{\Lambda}_+\,, & [\,\overline{H}, \overline{\Lambda}_+^\dagger\,] &= \hbar\,\Omega_-\,\overline{\Lambda}_+^\dagger \\ [\,\overline{H}, \overline{\Lambda}_-\,] &= -\hbar\,\Omega_-\,\overline{\Lambda}_-\,, & [\,\overline{H}, \overline{\Lambda}_-^\dagger\,] &= \hbar\,\Omega_-\,\overline{\Lambda}_-^\dagger \end{aligned} \tag{6.34}$$

$$\begin{aligned} [\,\overline{Q}, \overline{b}_+\,] &= \hbar\,\sqrt{\Omega_-}\;\overline{\Lambda}_+\,, & [\,\overline{Q}, \overline{b}_+^\dagger\,] &= -\hbar\,\sqrt{\Omega_-}\;\overline{\Lambda}_+^\dagger \\ \{\,\overline{Q}, \overline{\Lambda}_+\,\} &= -\hbar\,\sqrt{\Omega_-}\;\overline{b}_+\,, & \{\,\overline{Q}, \overline{\Lambda}_+^\dagger\,\} &= -\hbar\,\sqrt{\Omega_-}\;\overline{b}_+^\dagger \\ [\,\overline{Q}, \overline{b}_-\,] &= \hbar\,\sqrt{\Omega_-}\;\overline{\Lambda}_-\,, & [\,\overline{Q}, \overline{b}_-^\dagger\,] &= -\hbar\,\sqrt{\Omega_-}\;\overline{\Lambda}_-^\dagger \\ \{\,\overline{Q}, \overline{\Lambda}_-\,\} &= -\hbar\,\sqrt{\Omega_-}\;\overline{b}_-\,, & \{\,\overline{Q}, \overline{\Lambda}_-^\dagger\,\} &= -\hbar\,\sqrt{\Omega_-}\;\overline{b}_-^\dagger \end{aligned} \tag{6.35}$$

$$\begin{aligned} [\,\overline{L}, \overline{b}_+\,] &= -\hbar\,\overline{b}_+\,, & [\,\overline{L}, \overline{b}_+^\dagger\,] &= \hbar\,\overline{b}_+^\dagger \\ [\,\overline{L}, \overline{b}_-\,] &= \hbar\,\overline{b}_-\,, & [\,\overline{L}, \overline{b}_-^\dagger\,] &= -\hbar\,\overline{b}_-^\dagger \\ [\,\overline{L}, \overline{\Lambda}_+\,] &= -\hbar\,\overline{\Lambda}_+\,, & [\,\overline{L}, \overline{\Lambda}_+^\dagger\,] &= \hbar\,\overline{\Lambda}_+^\dagger \\ [\,\overline{L}, \overline{\Lambda}_-\,] &= \hbar\,\overline{\Lambda}_-\,, & [\,\overline{L}, \overline{\Lambda}_-^\dagger\,] &= -\hbar\,\overline{\Lambda}_-^\dagger \end{aligned} \tag{6.36}$$

The projected system is therefore still invariant under the $\mathcal{N}=1$ supersymmetry and the $SO(2)$ rotations.

6.2.2 Coordinates and momenta commutation relations

Proceeding as in the previous Section, we can compute the commutation relations of the projected Cartesian bosonic coordinates $\overline{x}_i$, $\overline{u}_i$ and their conjugated momenta $\overline{p}_i$, $\overline{\pi}_i$ and the projected fermionic degrees of freedom $\overline{\psi}_i$, $\overline{\mu}_i$, $\overline{\lambda}_i$ and $\overline{\gamma}_i$, and find out that their initial quantum algebra is deformed as follows:

$$
\begin{aligned}
[\epsilon_{ik}\,\overline{x}_k, \epsilon_{jl}\,\overline{p}_l] &= \frac{1}{2}\left(1 - \frac{k_0^2\,\kappa_0^2\; - 4\beta_0^2\,\omega_0^2}{2s\,\omega_+\omega_-}\right) i\hbar\,\delta_{ij}\,\mathbb{P}_1 \\
[\overline{u}_i, \overline{\pi}_j] &= \frac{1}{2}\left(1 + \frac{k_0^2\,\kappa_0^2\; - 4\beta_0^2\,\omega_0^2}{2s\,\omega_+\omega_-}\right) i\hbar\,\delta_{ij}\,\mathbb{P}_1 \\
[\epsilon_{ik}\,\overline{x}_k, \overline{u}_j] &= -\frac{s\,qB}{m^2\omega_+\omega_-}\; i\hbar\,\delta_{ij}\,\mathbb{P}_1 \\
[\epsilon_{ik}\,\overline{p}_k, \overline{\pi}_j] &= -\frac{s\,qB}{4m^2\omega_+\omega_-}\left(q^2B^2\; + m^2\left(k_0^2\,\kappa_0^2\; + 4\beta_0^2\,\omega_0^2\right)\right) i\hbar\,\delta_{ij}\,\mathbb{P}_1 \\
\{\overline{\psi}_1, \overline{\psi}_2\} &= \left(1 - \frac{s}{\omega_+\omega_-}\left(\frac{q^2B^2}{m^2} + \frac{1}{2}\left(k_0^2\,\kappa_0^2\; - 4\beta_0^2\,\omega_0^2\right)\right)\right)\frac{\hbar}{4m}\,\delta_{ij}\,\mathbb{P}_1 \\
\{\overline{\mu}_1, \overline{\mu}_2\} &= \left(1 - \frac{s}{\omega_+\omega_-}\left(\frac{q^2B^2}{m^2} - \frac{1}{2}\left(k_0^2\,\kappa_0^2\; - 4\beta_0^2\,\omega_0^2\right)\right)\right)\frac{\hbar}{m}\,\delta_{ij}\,\mathbb{P}_1 \\
\{\overline{\lambda}_1, \overline{\lambda}_2\} &= \left(1 + \frac{s}{\omega_+\omega_-}\left(\frac{q^2B^2}{m^2} + \frac{1}{2}\left(k_0^2\,\kappa_0^2\; - 4\beta_0^2\,\omega_0^2\right)\right)\right)\frac{\hbar}{2m\omega_0^2}\,\delta_{ij}\,\mathbb{P}_1 \\
\{\overline{\gamma}_1, \overline{\gamma}_2\} &= \left(1 + \frac{s}{\omega_+\omega_-}\left(\frac{q^2B^2}{m^2} - \frac{1}{2}\left(k_0^2\,\kappa_0^2\; - 4\beta_0^2\,\omega_0^2\right)\right)\right)\frac{\hbar}{2mk_0^2}\,\delta_{ij}\,\mathbb{P}_1 \\
\{\epsilon_{ik}\,\overline{\psi}_k, \overline{\lambda}_j\} &= -\frac{s\,qB\,\beta_0}{m^2\omega_+\omega_-}\;\hbar\,\delta_{ij}\,\mathbb{P}_1 \\
\{\overline{\mu}_i, \epsilon_{ik}\,\overline{\gamma}_j\} &= \frac{s\,qB\,\kappa_0}{m^2\omega_+\omega_-}\;\hbar\,\delta_{ij}\,\mathbb{P}_1
\end{aligned}
\tag{6.37}
$$

These commutation relations are very similar to those obtained for the N_+ sector, each differing only by a (not necessary global) sign from its analogue in the N_+ sector. As in the latter case, noncommutativity in the bosonic configuration space coordinates can be understood to result from a dependence between $\overline{u}_i$ and $\epsilon_{ij}\,\overline{p}_j$, and between $\epsilon_{ij}\,\overline{x}_j$ and $\overline{\pi}_i$ that is a consequence of the vanishing projection of the operators $a_\pm^{(\dagger)}$ and $\Gamma_\pm^{(\dagger)}$:

$$
\begin{aligned}
\overline{\pi}_i &= \frac{m^2}{4qB}\left(\kappa_0^2 k_0^2 - 4\beta_0^2\omega_0^2 + 2s\,\omega_+\omega_-\right)\epsilon_{ij}\,\overline{x}_j \\
\epsilon_{ij}\,\overline{p}_j &= \frac{m^2}{4qB}\left(\kappa_0^2 k_0^2 - 4\beta_0^2\omega_0^2 - 2s\,\omega_+\omega_-\right)\overline{u}_i
\end{aligned}
\tag{6.38}
$$

In the fermionic sector also, the algebra is rescaled, and there appear new non vanishing anticommutators coming from the dependence between $\epsilon_{ij}\,\overline{\psi}_j$ and $\overline{\lambda}_j$ and between $\overline{\mu}_i$ and $\epsilon_{ij}\,\overline{\gamma}_j$. Again, this algebra deformation must be consistent with the $\mathcal{N} = 1$ supersymmetry that still holds in the considered subspace.

This strong similarity between the results of the N_+ and N_- sectors suggests that it could be interesting to consider the projection on the subspace spanned by the eigenstates having either

N_+ or N_- equal to zero, or both of them. Due to the opposite signs appearing between the expressions in (6.21) and (6.37), such projection should lead to a more general quantum algebra deformation. However, in this case, there is no chiral Fock operators that are reduced to the null operator by the projection, and none of them commute with the projector. Therefore, the commutators of the projected chiral Fock algebra operators are not simply given by the sum of the commutators corresponding to the N_+ and N_- sectors, as may be seen by calculating the projected commutator of a_+ and $b_+^\dagger$. A complete analysis has thus to be carried on.

Chapter 7

Conclusion and Perspectives

The aim of this work was to construct a $\mathcal{N} = 1$ supersymmetric extension of a generalization of the Bigatti-Susskind model presented in [6], and to see whether, in a similar fashion as in [7] and [8], noncommutativity may arise in a consistent way with the supersymmetry of the system.

The first step towards the achievement of this programme was to build a supersymmetric action for the system. We used the formalism of supermechanics which, by extending the usual time with by a real Grassmann odd parameter, offers a convenient and rather straightforward way to construct a $\mathcal{N} = 1$ supersymmetric action. Note however that although it may first appear to be impossible to include a harmonic potential that way, this issue may be addressed by the introduction of Grassmann odd auxiliary fields [8]. This has led to add some fermionic degrees of freedom to the system, as well as redundant bosonic variables. Note also that main calculations of this work have been carried on using Mathematica©. This has required the development of some packages in order to deal with Grassmann odd variables but it also turned to be very convenient to perform many cross-checks during the calculations.

The Hamiltonian formulation of the system has been described using Dirac's constraint analysis formalism. Through the definition and use of Dirac brackets instead of the usual Poisson brackets, the dynamics of the system could be described in terms of the relevant degrees of freedom only, "forgetting" about the constraints resulting from the Lagrangian formulation. We then have identified the Noether charges corresponding to the rotational and supersymmetrical continuous symmetries of the system and we have found out that the Hamiltonian, the angular momentum and the supercharge all commute amongst themselves.

In order to proceed to the quantization of the system, the operators have been promoted to quantum operators such that their (anti)commutation relations are given by the Dirac brackets of the corresponding classical quantities. It then turned out that the diagonalization of the Hamiltonian was not as straightforward as it was thought. However, a careful and systematic approach have led to the introduction of the right Fock algebra operators in the bosonic sector, enlightening that the complications arise because of the non equality of the harmonic potential frequencies. The diagonalization of the fermionic part of the Hamiltonian has been conducted using the supersymmetry of the system. This way of proceeding appeared to be very convenient to avoid unnecessary calculations. Since it depends only on a few properties of the system – namely its supersymmetry and the form of the supercharge –, it may be asked to what extent this technique may be generalized.

The representation of the quantized system was constructed using the usual Fock space in the bosonic sector, and the Pauli matrices in the fermionic sector. We could then analyse the energy spectrum of the Hamiltonian and the eigenvalues of the angular momentum and the supercharge.

Note that all the results were consistent with the $\mathcal{N}=1$ supersymmetry of the system. We have found that the Hamiltonian, the supercharge and the angular momentum operator do not form a complete set of commuting observables since the specification of their three eigenvalues do not specify uniquely a state of the basis of states. This observation hints at one additional global symmetry of the system at least that would distinguish these states. Though this is out of the scope of this work, there are some reasons to think this question may be linked to the triality of the $SO(8)$ Clifford algebra of the fermionic operators.

Finally, we have addressed the question of whether noncommutativity may appear in this supersymmetric system. We have found that indeed, through appropriate projections on subspaces of the Hilbert space of states, the operators algebra is deformed in such a way that some coordinates become noncommutative. We have worked out two such projections, that have led to two different examples of $\mathcal{N}=1$ supersymmetric non(anti)commutative superplanes. It should be emphasized that in both cases, the projected system is still consistent with supersymmetry, and is invariant under $SO(2)$ rotations. The algebra of the bosonic sector is deformed in such a way that $\overline{u}_i$ do not commute anymore with $\epsilon_{ij}\,\overline{x}_j$. For the first projection $\mathbb{P}_0$, we get:

$$[\epsilon_{ik}\,\overline{x}_k, \overline{u}_j] \quad = \quad \frac{s\,qB}{m^2\omega_+\omega_-}\, i\hbar\,\delta_{ij}\,\mathbb{P}_0 \tag{7.1}$$

and a similar result holds for the second projection. It is instructive to note that this noncommutativity results from the fact that, since some of the Fock algebra operators are reduced to the null operator by the projection, a dependence appears between $\overline{u}_i$ and $\epsilon_{ij}\,\overline{p}_j$, and between $\epsilon_{ij}\,\overline{x}_j$ and $\overline{\pi}_i$, leading to the non vanishing commutator (7.1). The algebra of the fermionic sector is also deformed, in a way which must be consistent with supersymmetry. Note that when turning off the center of mass potential and when considering the massless limit of the system, the results of the bosonic sector indeed reduce to the ones of the Bigatti-Susskind system [6]. However, such limit should be studied in a more careful way for the fermionic sector.

Inspired by the strong similarities between the results of the two projections, we have tried to consider the projection on both subspaces at the same time. Some work has been done towards the answer to this question, but it appeared that the results cannot be obtained in an easy way from the results of the two previous projections, so that a complete and careful treatment has to be performed.

At the end of this work, we have achieved our initial objective to build a $\mathcal{N}=1$ supersymmetric extension of a generalization of the Bigatti-Susskind model. Recalling that the latter was introduced in order to get a further insight of how noncommutativity appears in bosonic string theories, the supersymmetric extension that has been developed in this work could be a first – although tiny – step towards the question of noncommutativity in superstring theories, and the way the fermionic algebra of such theories may be deformed while still being consistent with supersymmetry.

Appendix A

Eigenstates

A.1 Energy Eigenstates : $N_+ \geq 1$ and $N_- \geq 1$

Here we list the Hamiltonian eigenstates corresponding to the energy level $(N_+ \, \hbar\Omega_+ + N_- \, \hbar\Omega_-)$ with $N_+, N_- \geq 1$. They are denoted by :

$$| \, N_+ \, , \, n_+ \, , \, N_- \, , \, n_- \, ; \, S \, \rangle \, , \qquad 0 \leq n_+, n_- \leq N_- - 1 \, , \; s = 1, 2, ..., 16 \tag{A.1}$$

with, for $n_+ = n_- = 0$,

$$\begin{aligned}
| \, N_+ \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, 1 \, \rangle &= | \, N_+ \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, -,-,-,- \, \rangle \\
| \, N_+ \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, 2 \, \rangle &= | \, N_+ \, , \, 0 \, , \, N_- - 1 \, , \, 0 \, ; \, -,-,+,- \, \rangle \\
| \, N_+ \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, 3 \, \rangle &= | \, N_+ - 1 \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, +,-,-,- \, \rangle \\
| \, N_+ \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, 4 \, \rangle &= | \, N_+ - 1 \, , \, 0 \, , \, N_- - 1 \, , \, 0 \, ; \, +,-,+,- \, \rangle \\
| \, N_+ \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, 5 \, \rangle &= | \, N_+ \, , \, 0 \, , \, 0 \, , \, N_- \, ; \, -,-,-,- \, \rangle \\
| \, N_+ \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, 6 \, \rangle &= | \, N_+ \, , \, 0 \, , \, 0 \, , \, N_- - 1 \, ; \, -,-,-,+ \, \rangle \\
| \, N_+ \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, 7 \, \rangle &= | \, N_+ - 1 \, , \, 0 \, , \, 0 \, , \, N_- \, ; \, +,-,-,- \, \rangle \\
| \, N_+ \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, 8 \, \rangle &= | \, N_+ - 1 \, , \, 0 \, , \, 0 \, , \, N_- - 1 \, ; \, +,-,-,+ \, \rangle \\
| \, N_+ \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, 9 \, \rangle &= | \, 0 \, , \, N_+ \, , \, N_- \, , \, 0 \, ; \, -,-,-,- \, \rangle \\
| \, N_+ \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, 10 \, \rangle &= | \, 0 \, , \, N_+ \, , \, N_- - 1 \, , \, 0 \, ; \, -,-,+,- \, \rangle \\
| \, N_+ \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, 11 \, \rangle &= | \, 0 \, , \, N_+ - 1 \, , \, N_- \, , \, 0 \, ; \, -,+,-,- \, \rangle \\
| \, N_+ \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, 12 \, \rangle &= | \, 0 \, , \, N_+ - 1 \, , \, N_- - 1 \, , \, 0 \, ; \, -,+,+,- \, \rangle \\
| \, N_+ \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, 13 \, \rangle &= | \, 0 \, , \, N_+ \, , \, 0 \, , \, N_- \, ; \, -,-,-,- \, \rangle \\
| \, N_+ \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, 14 \, \rangle &= | \, 0 \, , \, N_+ \, , \, 0 \, , \, N_- - 1 \, ; \, -,-,-,+ \, \rangle \\
| \, N_+ \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, 15 \, \rangle &= | \, 0 \, , \, N_+ - 1 \, , \, 0 \, , \, N_- \, ; \, -,+,-,- \, \rangle \\
| \, N_+ \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, 16 \, \rangle &= | \, 0 \, , \, N_+ - 1 \, , \, 0 \, , \, N_- - 1 \, ; \, -,+,-,+ \, \rangle
\end{aligned} \tag{A.2}$$

for $1 \leq n_+ \leq N_+ - 1$ and $n_- = 0$,

$$
\begin{aligned}
|N_+\,,\,n_+\,,\,N_-\,,\,0\,;\,1\rangle &= |N_+\,,\,0\,,\,N_-\,,\,0\,;\,-,-,-,-\rangle\\
|N_+\,,\,n_+\,,\,N_-\,,\,0\,;\,2\rangle &= |N_+\,,\,0\,,\,N_- - 1\,,\,0\,;\,-,-,+,-\rangle\\
|N_+\,,\,n_+\,,\,N_-\,,\,0\,;\,3\rangle &= |N_+ - 1\,,\,0\,,\,N_-\,,\,0\,;\,+,-,-,-\rangle\\
|N_+\,,\,n_+\,,\,N_-\,,\,0\,;\,4\rangle &= |N_+ - 1\,,\,0\,,\,N_- - 1\,,\,0\,;\,+,-,+,-\rangle\\
|N_+\,,\,n_+\,,\,N_-\,,\,0\,;\,5\rangle &= |N_+\,,\,0\,,\,0\,,\,N_-\,;\,-,-,-,-\rangle\\
|N_+\,,\,n_+\,,\,N_-\,,\,0\,;\,6\rangle &= |N_+\,,\,0\,,\,0\,,\,N_- - 1\,;\,-,-,-,+\rangle\\
|N_+\,,\,n_+\,,\,N_-\,,\,0\,;\,7\rangle &= |N_+ - 1\,,\,0\,,\,0\,,\,N_-\,;\,+,-,-,-\rangle\\
|N_+\,,\,n_+\,,\,N_-\,,\,0\,;\,8\rangle &= |N_+ - 1\,,\,0\,,\,0\,,\,N_- - 1\,;\,+,-,-,+\rangle\\
|N_+\,,\,n_+\,,\,N_-\,,\,0\,;\,9\rangle &= |0\,,\,N_+\,,\,N_-\,,\,0\,;\,-,-,-,-\rangle\\
|N_+\,,\,n_+\,,\,N_-\,,\,0\,;\,10\rangle &= |0\,,\,N_+\,,\,N_- - 1\,,\,0\,;\,-,-,+,-\rangle\\
|N_+\,,\,n_+\,,\,N_-\,,\,0\,;\,11\rangle &= |0\,,\,N_+ - 1\,,\,N_-\,,\,0\,;\,-,+,-,-\rangle\\
|N_+\,,\,n_+\,,\,N_-\,,\,0\,;\,12\rangle &= |0\,,\,N_+ - 1\,,\,N_- - 1\,,\,0\,;\,-,+,+,-\rangle\\
|N_+\,,\,n_+\,,\,N_-\,,\,0\,;\,13\rangle &= |0\,,\,N_+\,,\,0\,,\,N_-\,;\,-,-,-,-\rangle\\
|N_+\,,\,n_+\,,\,N_-\,,\,0\,;\,14\rangle &= |0\,,\,N_+\,,\,0\,,\,N_- - 1\,;\,-,-,-,+\rangle\\
|N_+\,,\,n_+\,,\,N_-\,,\,0\,;\,15\rangle &= |0\,,\,N_+ - 1\,,\,0\,,\,N_-\,;\,-,+,-,-\rangle\\
|N_+\,,\,n_+\,,\,N_-\,,\,0\,;\,16\rangle &= |0\,,\,N_+ - 1\,,\,0\,,\,N_- - 1\,;\,-,+,-,+\rangle
\end{aligned}
\tag{A.3}
$$

for $1 \leq n_- \leq N_- - 1$ and $n_+ = 0$,

$$
\begin{aligned}
|N_+\,,\,0\,,\,N_-\,,\,n_-\,;\,1\rangle &= |N_+\,,\,0\,,\,N_-\,,\,0\,;\,-,-,-,-\rangle\\
|N_+\,,\,0\,,\,N_-\,,\,n_-\,;\,2\rangle &= |N_+\,,\,0\,,\,N_- - 1\,,\,0\,;\,-,-,+,-\rangle\\
|N_+\,,\,0\,,\,N_-\,,\,n_-\,;\,3\rangle &= |N_+ - 1\,,\,0\,,\,N_-\,,\,0\,;\,+,-,-,-\rangle\\
|N_+\,,\,0\,,\,N_-\,,\,n_-\,;\,4\rangle &= |N_+ - 1\,,\,0\,,\,N_- - 1\,,\,0\,;\,+,-,+,-\rangle\\
|N_+\,,\,0\,,\,N_-\,,\,n_-\,;\,5\rangle &= |N_+\,,\,0\,,\,0\,,\,N_-\,;\,-,-,-,-\rangle\\
|N_+\,,\,0\,,\,N_-\,,\,n_-\,;\,6\rangle &= |N_+\,,\,0\,,\,0\,,\,N_- - 1\,;\,-,-,-,+\rangle\\
|N_+\,,\,0\,,\,N_-\,,\,n_-\,;\,7\rangle &= |N_+ - 1\,,\,0\,,\,0\,,\,N_-\,;\,+,-,-,-\rangle\\
|N_+\,,\,0\,,\,N_-\,,\,n_-\,;\,8\rangle &= |N_+ - 1\,,\,0\,,\,0\,,\,N_- - 1\,;\,+,-,-,+\rangle\\
|N_+\,,\,0\,,\,N_-\,,\,n_-\,;\,9\rangle &= |0\,,\,N_+\,,\,N_-\,,\,0\,;\,-,-,-,-\rangle\\
|N_+\,,\,0\,,\,N_-\,,\,n_-\,;\,10\rangle &= |0\,,\,N_+\,,\,N_- - 1\,,\,0\,;\,-,-,+,-\rangle\\
|N_+\,,\,0\,,\,N_-\,,\,n_-\,;\,11\rangle &= |0\,,\,N_+ - 1\,,\,N_-\,,\,0\,;\,-,+,-,-\rangle\\
|N_+\,,\,0\,,\,N_-\,,\,n_-\,;\,12\rangle &= |0\,,\,N_+ - 1\,,\,N_- - 1\,,\,0\,;\,-,+,+,-\rangle\\
|N_+\,,\,0\,,\,N_-\,,\,n_-\,;\,13\rangle &= |0\,,\,N_+\,,\,0\,,\,N_-\,;\,-,-,-,-\rangle\\
|N_+\,,\,0\,,\,N_-\,,\,n_-\,;\,14\rangle &= |0\,,\,N_+\,,\,0\,,\,N_- - 1\,;\,-,-,-,+\rangle\\
|N_+\,,\,0\,,\,N_-\,,\,n_-\,;\,15\rangle &= |0\,,\,N_+ - 1\,,\,0\,,\,N_-\,;\,-,+,-,-\rangle\\
|N_+\,,\,0\,,\,N_-\,,\,n_-\,;\,16\rangle &= |0\,,\,N_+ - 1\,,\,0\,,\,N_- - 1\,;\,-,+,-,+\rangle
\end{aligned}
\tag{A.4}
$$

and, for $1 \leq n_+ \leq N_+ - 1$ and $1 \leq n_- \leq N_- - 1$,

$$
\begin{aligned}
|\, N_+ \, , \, n_+ \, , \, N_- \, , \, n_- \, ; \, 1 \,\rangle &= |\, N_+ \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, -,-,-,- \,\rangle \\
|\, N_+ \, , \, n_+ \, , \, N_- \, , \, n_- \, ; \, 2 \,\rangle &= |\, N_+ \, , \, 0 \, , \, N_- - 1 \, , \, 0 \, ; \, -,-,+,- \,\rangle \\
|\, N_+ \, , \, n_+ \, , \, N_- \, , \, n_- \, ; \, 3 \,\rangle &= |\, N_+ - 1 \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, +,-,-,- \,\rangle \\
|\, N_+ \, , \, n_+ \, , \, N_- \, , \, n_- \, ; \, 4 \,\rangle &= |\, N_+ - 1 \, , \, 0 \, , \, N_- - 1 \, , \, 0 \, ; \, +,-,+,- \,\rangle \\
|\, N_+ \, , \, n_+ \, , \, N_- \, , \, n_- \, ; \, 5 \,\rangle &= |\, N_+ \, , \, 0 \, , \, 0 \, , \, N_- \, ; \, -,-,-,- \,\rangle \\
|\, N_+ \, , \, n_+ \, , \, N_- \, , \, n_- \, ; \, 6 \,\rangle &= |\, N_+ \, , \, 0 \, , \, 0 \, , \, N_- - 1 \, ; \, -,-,-,+ \,\rangle \\
|\, N_+ \, , \, n_+ \, , \, N_- \, , \, n_- \, ; \, 7 \,\rangle &= |\, N_+ - 1 \, , \, 0 \, , \, 0 \, , \, N_- \, ; \, +,-,-,- \,\rangle \\
|\, N_+ \, , \, n_+ \, , \, N_- \, , \, n_- \, ; \, 8 \,\rangle &= |\, N_+ - 1 \, , \, 0 \, , \, 0 \, , \, N_- - 1 \, ; \, +,-,-,+ \,\rangle \\
|\, N_+ \, , \, n_+ \, , \, N_- \, , \, n_- \, ; \, 9 \,\rangle &= |\, 0 \, , \, N_+ \, , \, N_- \, , \, 0 \, ; \, -,-,-,- \,\rangle \\
|\, N_+ \, , \, n_+ \, , \, N_- \, , \, n_- \, ; \, 10 \,\rangle &= |\, 0 \, , \, N_+ \, , \, N_- - 1 \, , \, 0 \, ; \, -,-,+,- \,\rangle \\
|\, N_+ \, , \, n_+ \, , \, N_- \, , \, n_- \, ; \, 11 \,\rangle &= |\, 0 \, , \, N_+ - 1 \, , \, N_- \, , \, 0 \, ; \, -,+,-,- \,\rangle \\
|\, N_+ \, , \, n_+ \, , \, N_- \, , \, n_- \, ; \, 12 \,\rangle &= |\, 0 \, , \, N_+ - 1 \, , \, N_- - 1 \, , \, 0 \, ; \, -,+,+,- \,\rangle \\
|\, N_+ \, , \, n_+ \, , \, N_- \, , \, n_- \, ; \, 13 \,\rangle &= |\, 0 \, , \, N_+ \, , \, 0 \, , \, N_- \, ; \, -,-,-,- \,\rangle \\
|\, N_+ \, , \, n_+ \, , \, N_- \, , \, n_- \, ; \, 14 \,\rangle &= |\, 0 \, , \, N_+ \, , \, 0 \, , \, N_- - 1 \, ; \, -,-,-,+ \,\rangle \\
|\, N_+ \, , \, n_+ \, , \, N_- \, , \, n_- \, ; \, 15 \,\rangle &= |\, 0 \, , \, N_+ - 1 \, , \, 0 \, , \, N_- \, ; \, -,+,-,- \,\rangle \\
|\, N_+ \, , \, n_+ \, , \, N_- \, , \, n_- \, ; \, 16 \,\rangle &= |\, 0 \, , \, N_+ - 1 \, , \, 0 \, , \, N_- - 1 \, ; \, -,+,-,+ \,\rangle
\end{aligned}
\tag{A.5}
$$

A.2 Q Eigenstates : $N_+ \geq 1$ and $N_- \geq 1$

Here is the complete list of the energy, angular momentum and supercharge eigenvalues of the Q eigenstates $|\, N_+ \, , \, n_+ \, , \, N_- \, , \, n_- \, ; \, S \,\rangle^Q$ for $N_+ \geq 1$ and $N_- \geq 1$. This allows us to display the degeneracy associated to the specification of these three eigenvalues.

	H	L	Q	
$\|\, N_+ \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, 1 \,\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$-\hbar(N_- + N_+)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$	
$\|\, N_+ \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, 2 \,\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$-\hbar(N_- + N_+)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$	
$\|\, N_+ \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, 3 \,\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- - N_+)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$	
$\|\, N_+ \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, 4 \,\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- - N_+)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$	(A.6)
$\|\, N_+ \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, 5 \,\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_+ - N_-)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$	
$\|\, N_+ \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, 6 \,\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_+ - N_-)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$	
$\|\, N_+ \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, 7 \,\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$	
$\|\, N_+ \, , \, 0 \, , \, N_- \, , \, 0 \, ; \, 8 \,\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$	

	H	L	Q
$\vert N_+ , 0 , N_- , 0 ; 9\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$-\hbar(N_- + N_+)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ , 0 , N_- , 0 ; 10\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$-\hbar(N_- + N_+)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ , 0 , N_- , 0 ; 11\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- - N_+)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ , 0 , N_- , 0 ; 12\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- - N_+)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ , 0 , N_- , 0 ; 13\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_+ - N_-)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ , 0 , N_- , 0 ; 14\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_+ - N_-)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ , 0 , N_- , 0 ; 15\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ , 0 , N_- , 0 ; 16\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$

(A.7)

For $1 \leq n_+ \leq N_+ - 1$, we have:

	H	L	Q
$\vert N_+ , n_+ , N_- , 0 ; 1\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$-\hbar(N_- - N_+ + 2n_+)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ , n_+ , N_- , 0 ; 2\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$-\hbar(N_- - N_+ + 2n_+)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ , n_+ , N_- , 0 ; 3\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$-\hbar(N_- - N_+ + 2n_+)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ , n_+ , N_- , 0 ; 4\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$-\hbar(N_- - N_+ + 2n_+)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ , n_+ , N_- , 0 ; 5\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_+)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ , n_+ , N_- , 0 ; 6\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_+)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ , n_+ , N_- , 0 ; 7\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_+)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ , n_+ , N_- , 0 ; 8\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_+)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$

(A.8)

	H	L	Q
$\vert N_+ , n_+ , N_- , 0 ; 9\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$-\hbar(N_- - N_+ + 2n_+)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ , n_+ , N_- , 0 ; 10\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$-\hbar(N_- - N_+ + 2n_+)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ , n_+ , N_- , 0 ; 11\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$-\hbar(N_- - N_+ + 2n_+)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ , n_+ , N_- , 0 ; 12\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$-\hbar(N_- - N_+ + 2n_+)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ , n_+ , N_- , 0 ; 13\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_+)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ , n_+ , N_- , 0 ; 14\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_+)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ , n_+ , N_- , 0 ; 15\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_+)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ , n_+ , N_- , 0 ; 16\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_+)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$

(A.9)

For $1 \leq n_- \leq N_- - 1$, we have:

	H	L	Q
$\vert N_+ \,,\, 0 \,,\, N_- \,,\, n_- \,;\, 1 \rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$-\hbar(-N_- + N_+ + 2n_-)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ \,,\, 0 \,,\, N_- \,,\, n_- \,;\, 2 \rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$-\hbar(-N_- + N_+ + 2n_-)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ \,,\, 0 \,,\, N_- \,,\, n_- \,;\, 3 \rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$-\hbar(-N_- + N_+ + 2n_-)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ \,,\, 0 \,,\, N_- \,,\, n_- \,;\, 4 \rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- - N_+ - 2n_-)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ \,,\, 0 \,,\, N_- \,,\, n_- \,;\, 5 \rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_-)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ \,,\, 0 \,,\, N_- \,,\, n_- \,;\, 6 \rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_-)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ \,,\, 0 \,,\, N_- \,,\, n_- \,;\, 7 \rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_-)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ \,,\, 0 \,,\, N_- \,,\, n_- \,;\, 8 \rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_-)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$

(A.10)

	H	L	Q
$\vert N_+ \,,\, 0 \,,\, N_- \,,\, n_- \,;\, 9 \rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$-\hbar(-N_- + N_+ + 2n_-)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ \,,\, 0 \,,\, N_- \,,\, n_- \,;\, 10 \rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$-\hbar(-N_- + N_+ + 2n_-)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ \,,\, 0 \,,\, N_- \,,\, n_- \,;\, 11 \rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$-\hbar(-N_- + N_+ + 2n_-)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ \,,\, 0 \,,\, N_- \,,\, n_- \,;\, 12 \rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$-\hbar(-N_- + N_+ + 2n_-)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ \,,\, 0 \,,\, N_- \,,\, n_- \,;\, 13 \rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_-)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ \,,\, 0 \,,\, N_- \,,\, n_- \,;\, 14 \rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_-)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ \,,\, 0 \,,\, N_- \,,\, n_- \,;\, 15 \rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_-)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ \,,\, 0 \,,\, N_- \,,\, n_- \,;\, 16 \rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_-)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$

(A.11)

Finally, for $1 \leq n_+ \leq N_+ - 1$ and $1 \leq n_- \leq N_- - 1$, we have:

	H	L	Q
$\vert N_+ \,,\, n_+ \,,\, N_- \,,\, n_- \,;\, 1 \rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_- - 2n_+)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ \,,\, n_+ \,,\, N_- \,,\, n_- \,;\, 2 \rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_- - 2n_+)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ \,,\, n_+ \,,\, N_- \,,\, n_- \,;\, 3 \rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_- - 2n_+)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ \,,\, n_+ \,,\, N_- \,,\, n_- \,;\, 4 \rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_- - 2n_+)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ \,,\, n_+ \,,\, N_- \,,\, n_- \,;\, 5 \rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_- - 2n_+)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ \,,\, n_+ \,,\, N_- \,,\, n_- \,;\, 6 \rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_- - 2n_+)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ \,,\, n_+ \,,\, N_- \,,\, n_- \,;\, 7 \rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_- - 2n_+)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+ \,,\, n_+ \,,\, N_- \,,\, n_- \,;\, 8 \rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_- - 2n_+)$	$-\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$

(A.12)

	H	L	Q
$\vert N_+\,,\; n_+\,,\; N_-\,,\; n_-\,;\; 9\,\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_- - 2n_+)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+\,,\; n_+\,,\; N_-\,,\; n_-\,;\; 10\,\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_- - 2n_+)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+\,,\; n_+\,,\; N_-\,,\; n_-\,;\; 11\,\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_- - 2n_+)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+\,,\; n_+\,,\; N_-\,,\; n_-\,;\; 12\,\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_- - 2n_+)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+\,,\; n_+\,,\; N_-\,,\; n_-\,;\; 13\,\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_- - 2n_+)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+\,,\; n_+\,,\; N_-\,,\; n_-\,;\; 14\,\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_- - 2n_+)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+\,,\; n_+\,,\; N_-\,,\; n_-\,;\; 15\,\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_- - 2n_+)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$
$\vert N_+\,,\; n_+\,,\; N_-\,,\; n_-\,;\; 16\,\rangle^Q$	$\hbar(N_-\Omega_- + N_+\Omega_+)$	$\hbar(N_- + N_+ - 2n_- - 2n_+)$	$\hbar\sqrt{N_-\Omega_- + N_+\Omega_+}$

(A.13)

Bibliography

[1] F. G. Scholtz, L. Gouba, A. Hafver, and C. M. Rohwer. Formulation, interpretation and application of non-commutative quantum mechanics. *J.Phys.*, A42:175303, 2009.

[2] R. Jackiw. Physical instances of noncommuting coordinates. *Nucl. Phys. Proc. Suppl.*, 108:30–36, 2002.

[3] H. S. Snyder. Quantized space-time. *Phys. Rev.*, 71:38, 1947.

[4] A. Connes. *Noncommutative Geometry.* Academic Press, San Diego, 1994.

[5] N. Seiberg and E. Witten. String theory and noncommutative geometry. *JHEP*, 9909:032, 1999.

[6] D. Bigatti and L. Susskind. Magnetic fields, branes and noncommutative geometry. *Phys. Rev.*, D62:066004, 2000.

[7] J. Ben Geloun, J. Govaerts, and F. G. Scholtz. The N=1 Supersymmetric Landau Problem and its Supersymmetric Landau Level Projections: the N=1 Supersymmetric Moyal-Voros Superplane. *J. Phys.*, A42:495203, 2009.

[8] M. D. Yangoufolo Malessabo. Problème de Landau, supersymétrie et géométrie non-commutative. Master's thesis, Université catholique de Louvain, 2010.

[9] F. A. Berezin. The method of second quantization. *Pure Appl. Phys.*, 24:1–228, 1966.

[10] I. J. R. Aitchison and A. J. G. Hey. Gauge theories in particle physics: A practical introduction. Vol. 2: Non-Abelian gauge theories: QCD and the electroweak theory. Bristol, UK: IOP (2004).

[11] J. Govaerts. On the road towards the quantum geometer's universe: An introduction to four-dimensional supersymmetric quantum field theories. In *Proceedings of the Third International Workshop on Contemporary Problems in Mathematical Physics*, pages 94–150, Institut de Mathématiques et de Sciences Physiques (IMSP), Université d'Abomey-Calavi, Cotonou (Republic of Benin), November 1^{st} - 7^{th}, 2003. Eds. J. Govaerts, M.N. Hounkonnou and A.Z. Msezane. (World Scientific, Singapore, 2004).

[12] F. Ravndal. Elementary supersymmetry. *Proc. CERN School of Physics (CERN-85-11)*, pages 300–306, 1984.

[13] D. S. Freed. *Five Lectures on Supersymmetry.* American Mathematical Society, 1999.

[14] F. Hélein. SuPeyrsymmetry: An introduction to supersymmetry. *Talk given in Peyresq Summer School on Integrable Differential Equations and Quantum Field Theory*, June 2002.

[15] P.A.M. Dirac. *Lectures on Quantum Mechanics.* Belfer Graduate School of Science, Yeshiva University, New York, 1964.

[16] J. Govaerts. Hamiltonian Quantisation and Constrained Dynamics. In *Leuven Notes in Mathematical and Theoretical Physics*, volume 4. Leuven University Press, 1991.