**#1:** Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial^2 f}{\partial x \partial y}$ for the following function:

$$f(x,y) = \frac{e^x}{1+e^y}$$

**Solution:**

We have:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\frac{e^x}{1+e^y} = \frac{1}{1+e^y}\frac{\partial}{\partial x}e^x = \frac{e^x}{1+e^y} = f$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\frac{e^x}{1+e^y} = e^x\frac{\partial}{\partial y}(1+e^y)^{-1} = e^x\frac{-e^y}{(1+e^y)^2} = \frac{-e^{x+y}}{(1+e^y)^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial y}f = \frac{-e^{x+y}}{(1+e^y)^2}$$

**#2:** Let $R$ be the square given by $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1$ in the $xy$-plane. Calculate the volume of the solid lying over $R$, bounded above by the graph of $f(x,y) = 1 + e^{x+y}$.

**Solution:**

We can write this as a double integral:

$$\iint_R (1 + e^{x+y}) dx dy$$

By the "fundamental result" on page 384, we can evaluate this as an iterated integral:

$$\int_0^1 \left( \int_0^1 (1 + e^{x+y}) dy \right) dx$$

First, we calculate the inner integral as a function of $x$:

$$\int_0^1 (1 + e^{x+y}) dy = [y + e^{x+y}]_{y=0}^{y=1} = (1 + e^{x+1}) - (0 + e^x) = 1 + e^{x+1} - e^x$$

Next, we plug in to the outer integral:

$$\begin{aligned} \int_0^1 (1 + e^{x+1} - e^x) dx = [x + e^{x+1} - e^x]_{x=0}^{x=1} &= (1 + e^2 - e^1) - (0 + e^1 - e^0) = 1 + e^2 - e - e + 1 \\ &= 2 - 2e + e^2 \approx 3.95 \end{aligned}$$

**#3:** Find the value of the following integral:

$$\int_{\pi/8}^{\pi/6} \frac{3}{\cos^2 2x} dx$$

**Solution:**

Observe that $\frac{d}{dx}\frac{3}{2}\tan 2x = 3\sec^2(2x)$. Therefore:

$$\int_{\pi/8}^{\pi/6} \frac{3}{\cos^2 2x} dx = \int_{\pi/8}^{\pi/6} 3\sec^2(2x) dx = [\frac{3}{2}\tan 2x]_{\pi/8}^{\pi/6} = \frac{3}{2}(\tan \pi/3 - \tan \pi/4)$$

To further simplify this, we can observe that $\tan \pi/3 = (\sin \pi/3)/(\cos \pi/3) = (\sqrt{3}/2)/(1/2) = \sqrt{3}$, and $\tan \pi/4 = (\sin \pi/4)/(\cos \pi/4) = (\sqrt{2}/2)/(\sqrt{2}/2) = 1$, giving us a final answer of:

$$\frac{3}{2}(\sqrt{3} - 1)$$

**#4:** Evaluate the following indefinite integral:

$$\int \ln(\sin x) \cdot \cos x \ dx$$

**Solution:**

First, we make the substitution $u = \sin x$, $du = \cos x dx$, transforming the integral into:

$$\int \ln u du$$

We now apply integration by parts:

$$\begin{aligned}
\int \ln u du &= u \ln u - \int u d(\ln u) \\
&= u \ln u - \int u \cdot \frac{1}{u} du \\
&= u \ln u - \int du \\
&= u \ln u - u + C
\end{aligned}$$

Finally, we substitute back $u = \sin x$:

$$\sin x \cdot \ln(\sin x) - \sin x + C$$

**#5:** Solve the following initial value problem:

$$\frac{dy}{dt} = \left(\frac{1+t}{1+y}\right)^2, y(0) = 2$$

**Solution:**

First, we separate the variables to get:

$$(1+y)^2 dy = (1+t)^2 dt$$

Next, we integrate both sides:

$$\int (1+y)^2 dy = \int (1+t)^2 dt$$

$$\frac{1}{3}(1+y)^3 = \frac{1}{3}((1+t)^3 + C)$$

Multiplying by 3 and taking cube roots yields:

$$1 + y = \sqrt[3]{(1+t)^3 + C}$$
$$y = -1 + \sqrt[3]{(1+t)^3 + C}$$

To solve for $C$, we plug in the initial condition $t = 0$, $y = 2$:

$$2 = -1 + \sqrt[3]{1+C}$$

$$3^3 = 1 + C$$

$$C = 26$$

So our solution is:

$$y = -1 + \sqrt[3]{(1+t)^3 + 26}$$

**#6:** Find the second Taylor polynomials at $x = 0$ for the following functions: $e^{x^2}, \sin(x^2), \cos(x^2)$.

**Solution:**

All three functions are of the form $f(x) = g(x^2)$. Using the product and chain rules, we compute the first two derivatives:

$$f'(x) = 2xg'(x^2)$$

$$f''(x) = 2g'(x^2) + 4x^2g''(x^2)$$

This gives us the three values

$$f(0) = g(0), f'(0) = 0, f''(0) = 2g'(0)$$

So the second Taylor polynomial is:

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = g(0) + g'(0)x^2$$

In conclusion, the second Taylor polynomials for $e^{x^2}$, $\sin(x^2)$, and $\cos(x^2)$, are:

$$e^0 + e^0x^2 = 1 + x^2$$
$$\sin 0 + \cos 0x^2 = x^2$$
$$\cos 0 - \sin 0x^2 = 1$$

**#7:** Using the comparison test—and possibly additional methods—to determine whether the infinite series

$$\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^3}$$

converges or diverges. (Hint: $0 \leqslant \sin^2 x \leqslant 1$.)

**Solution:**

Dividing $0 \leqslant \sin^2 x \leqslant 1$ by $k^3$ gives us the inequality:

$$0 \leqslant \frac{\sin^2 k}{k^3} \leqslant \frac{1}{k^3}$$

This is the condition to apply the comparison test: if the sum $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges, the sum $\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^3}$ must converge as well.

To see that $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges, we can apply the integral test to the (continuous, non-negative, decreasing) function $f(x) = x^{-3}$:

$$\begin{aligned}
\int_1^{\infty} f(x)dx &= \lim_{b\to\infty} \int_1^b x^{-3} \\
&= \lim_{b\to\infty} -\frac{1}{2}x^{-2}|_1^b \\
&= \lim_{b\to\infty} (\frac{1}{2} - \frac{1}{2b^2}) \\
&= \frac{1}{2}
\end{aligned}$$

Since $\int_1^{\infty} f(x)dx$ converges, this gives our desired result, that $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges.