Topologically protected edge states via highly oscillatory potentials.

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Floquet–Bloch theory

We consider periodic operators:

\[ \mathcal{P} = f(D_x) + V(x), \quad x \in \mathbb{R}, \quad f, V \in C^\infty(\mathbb{R}, \mathbb{R}), \]

\[ V(x + 1) = V(x), \quad f(\xi) \geq |\xi|^2 \quad \text{for } |\xi| \gg 1. \]

By periodicity, spaces of quasi-periodic functions

\[ L^2_\xi(\mathbb{R}) \overset{\text{def}}{=} \{ u \in L^2_{\text{loc}}(\mathbb{R}) : u(x + 1) = e^{i\xi} u(x) \} \]

are invariant. Hence \( L^2_\xi(\mathbb{R}) \) admits a basis of eigenvectors of \( \mathcal{P} \) with eigenvalues

\[ \lambda_0(\xi) \leq \lambda_1(\xi) \leq ... \]

The spectrum of \( \mathcal{P} \) on \( L^2(\mathbb{R}) \) is absolutely continuous, equal to

\[ \{ \lambda_j(\xi) : \xi \in [0, 2\pi), \ j \in \mathbb{N} \}. \]
Example 1: $V \equiv 0$, $\mathcal{P} = f(D_x)$

In the case $V \equiv 0$, the eigenvalues of $\mathcal{P}$ on $L^2_\xi(\mathbb{R})$ are

$$\{ f(\xi + 2\pi \ell) : \ell \in \mathbb{Z} \}.$$ 

We can then plot dispersion curves of $\mathcal{P}$ using the multi-valued function $\xi \mapsto f(\xi \mod 2\pi)$. 

![Graphs showing dispersion curves](diagrams)
Example 2: dimer models

Assume that $f$ and $V$ admit additional symmetries:

$$f(\xi) = f(-\xi), \quad V(x + 1/2) = V(x).$$

Basic example: $f(\xi) = \xi^2$, $V \equiv 0$.

There is a linear crossing: a Dirac point appears. Dirac points correspond to conical intersections of dispersion surfaces.
Dirac points

Mathematically, Dirac points are pairs \((\xi_\star, E_\star)\) such that there exists \(j, \nu\) with
\[
\lambda_j(\xi) = E_\star + \nu|\xi - \xi_\star| + O(\xi - \xi_\star)^2 \\
\lambda_{j+1}(\xi) = E_\star - \nu|\xi - \xi_\star| + O(\xi - \xi_\star)^2.
\]
(1)

Their theoretical existence was postulated by Hamilton. Solutions of
\(D_t u = \mathcal{P} u\) supported at \(t = 0\) near \(\xi_\star\) are expected to approximately evolve according to
\[D_t u = (E_\star + \nu|D_x|)u.\]
(2)

Tremendous amount of work in the physics litterature.

Mathematical work: [Berry ’80s], [Gérard ’90], [Colin de Verdière ’91], [Fefferman–Weinstein ’12] (genericity of Dirac points, rigorous formulation of (1) as a matrix Dirac equation), [Lee ’14] (point scatterers), [Fefferman–Lee-Thorp–Weinstein ’16, ’17] (perturbative results, tight binding regimes), [Berkolaiko–Comech ’16] (symmetry-theoretic approach), [Kuchment ’16] (survey),...
Physical motivation

Mathematically speaking a material is an insulator at energy \( \leq E \) if the corresponding operator has a spectral gap around \( E \).

In dimer models Dirac points come from the existence of two symmetries:

\[
x \mapsto -x, \ x \mapsto x + 1/2
\]

Break the second symmetry by adding \( \delta \cos(2\pi x) \): \( \mathcal{P} = f(D_x) \) becomes

\[
f(D_x) + \delta \cos(2\pi x).
\]

An energy gap of size \( \delta \) opens near the Dirac energy. The material becomes an insulator at energy \( \leq E_\star \).
Physical motivation

Mathematically speaking a material is an insulator at energy $\leq E$ if the corresponding operator has a spectral gap around $E$. In dimer models Dirac points come from the existence of two symmetries:

$$x \mapsto -x, \ x \mapsto x + 1/2$$

Break the second symmetry by adding $\delta \cos(2\pi x)$: $\mathcal{P} = f(D_x)$ becomes

$$f(D_x) + \delta \cos(2\pi x).$$

An energy gap of size $\delta$ opens near the Dirac energy. The material becomes an insulator at energy $\leq E_*$. 

![Graph](image.png)
Introducing phase defects

We study periodic structures with a phase defect. The typical potential is

\[
\delta \kappa(\delta x) \cos(2\pi x) : \quad \kappa(x) = \pm 1 \text{ for } x \text{ near } \pm \infty.
\]

The potential ”behaves” like \(\cos(2\pi x)\) at both ends but acquires a phase defect when going from \(-\infty\) to \(+\infty\). The periodic structure is ”stretched” in the middle.

We set \(P = f(D_x) + \delta \kappa(\delta x) \cos(2\pi x)\). The essential spectrum is characterized by the asymptotic operators:

\[
P_{\pm\delta} = f(D_x) \pm \delta \cos(2\pi x), \quad x \text{ near } \pm \infty
\]

Hence \(P\) has spectral gaps near Dirac energies of \(f(D_x)\).
Existing results

Recall that $D_x^2$ has a Dirac point at $(\pi, \pi^2)$.

**Theorem [Fefferman–Lee-Thorp–Weinstein ’14]**

For $\delta$ sufficiently small, the operator $D_x^2 + \delta \kappa(\delta x) \cos(2\pi x)$ has an eigenvalue of energy $\pi^2 + O(\delta^2)$.

The corresponding eigenstate takes the form

$$u(x) = \alpha_+(\delta x)e^{i\pi x} + \alpha_-(\delta x)e^{-i\pi x} + ...$$

where the vector $\alpha = (\alpha_-, \alpha_+)$ solves the Dirac equation

$$\mathcal{D}\alpha = 0, \quad \mathcal{D} \overset{\text{def}}{=} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} D_y + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \kappa(y)$$

**Comments:**

- $u \in L^2$ because $(-1, 1)$ is an essential spectrum gap of $\mathcal{D}$.
- This mode is topologically protected: it persists under arbitrarily large perturbations of $\kappa$ on compact sets.
- This supports the bulk/edge correspondence.
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$$D\alpha = 0, \quad D \overset{\text{def}}{=} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} D_y + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \kappa(y)$$

**Comments:**

- The theorem still holds when $D_x^2$ is replaced by $D_x^2$ plus an even cosine series and $\cos(2\pi x)$ is replaced by an odd cosine series.
- This theorem is the basis for deeper results on topologically protected modes in honeycomb lattices.
The multiscale analysis of [F–L–T–W ’14, ’16]

We derive formally this result with multiscale analysis. We look for an "linear" combination of Dirac eigenstates with slowly varying coefficients:

\[ u(x, y) = \alpha_+(y)e^{i\pi x} + \alpha_-(y)e^{-i\pi x} + \delta v(x, y) + \ldots, \quad y = \delta x. \]

In the variables \((x, y) \in S^1 \times \mathbb{R}:

\[ D_x^2 + \delta \kappa(\delta x) \cos(2\pi x) \mapsto (D_x + \delta D_y)^2 + \delta \kappa(y) \cos(2\pi x). \]

Plug \(u\) in RHS and group terms of order 1, \(\delta, \ldots\):

\[ \sum_{\pm} \alpha_\pm(y)(D_x^2 - \pi^2)e^{\pm i\pi x} = 0 \]

\[ (D_x^2 - \pi^2)v + \sum_{\pm} 2D_y \alpha_\pm(y) \cdot D_x e^{\pm i\pi x} + \alpha_\pm(y)\kappa(y) \cos(2\pi x)e^{\pm i\pi x} = 0. \]

The second equation has a solution iff the second term is \(L^2_x(S^1)\)-orthogonal to \(e^{\pm i\pi x}\). Thus we must have

\[ \left\langle \sum_{\pm} 2D_y \alpha_\pm(y) \cdot D_x e^{\pm i\pi x} + \alpha_\pm(y)\kappa(y) \cos(2\pi x)e^{\pm i\pi x}, e^{\pm i\pi x} \right\rangle_{L^2_x(S^1)} = 0 \]
The multiscale analysis of [F–L–T–W ’14, ’16]

This yields the Dirac equation

\[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
D_y \alpha_+ \\
D_y \alpha_-
\end{bmatrix}
+ \kappa(y)
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
\alpha_+ \\
\alpha_-
\end{bmatrix}
= 0.
\]

This way we construct a quasimode with energy in a spectral gap, hence there is an eigenvector with energy nearby. It is quite hard to show that

\[
\alpha_+(\delta x)e^{i\pi x} + \alpha_-(\delta x)e^{-i\pi x} + ...
\]

is indeed an eigenstate. Selfadjoint principles only show that (??) is near a linear combination of eigenstates of $P$ with energy near $\pi^2$. 
Honeycomb lattices

The model is a potential well at each vertex of a hexagonal lattice.

Such structures generically admit Dirac points ([Fefferman–Weinstein ’12], [Lee ’14], [Berkolaiko–Comech ’16], [Fefferman–Lee–Thorp–Weinstein ’17]).
Perturbation along an edge

An edge perturbation of honeycomb lattices is obtained by fixing (say) a rational edge and stretching adiabatically the system along this edge.

[Fefferman–Lee–Thorp–Weinstein ’16] studies the existence of states located along the edge with energy Dirac energies.
Perturbation along an edge

An edge perturbation of honeycomb lattices is obtained by fixing (say) a rational edge and stretching adiabatically the system along this edge.

Such states accounts for the insulator/conductor characteristics of the material, depending on the direction of propagation.
The no-fold condition of [F–L–T–W ’16]

The existence of an eigenstate depends whether ”stretching” the periodic structure along the edge opens a spectral gap ”in the edge direction”. 
The no-fold condition of [F–L-T–W ’16]

The existence of an eigenstate depends whether ”stretching” the periodic structure along the edge opens a spectral gap ”in the edge direction”.

![Diagram showing the no-fold condition](image_url)
The no-fold condition of [F–L–T–W ’16]

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\[ E_\ast \]

\[ \xi_\ast \]
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In the right picture, dispersion surfaces ”fold over” the energy $E_\star$. [Fefferman–Lee-Thorp–Weinstein ’16] conjectured that if the no-fold condition fails topologically protected resonances appear.

We will study 1D operators modeling the picture on the right. We first define resonances.
Resonances of periodic systems [Gérard ’90]

Resonances are poles of the meromorphic continuation of the resolvent. If $T$ is periodic, we write

$$T(\xi) = T : L^2_\xi(\mathbb{R}) \to L^2_\xi(\mathbb{R}).$$

$T(\xi)$ has compact resolvent. By Floquet–Bloch theory,

$$T = \int_0^{+2\pi} T(\xi) d\xi, \quad \Im \lambda > 0 \Rightarrow (T - \lambda)^{-1} = \int_0^{+2\pi} (T(\xi) - \lambda)^{-1} d\xi.$$

Since $(T(\xi) - \lambda)^{-1}$ is periodic we can change the contour $[0, 2\pi]$ to the unit circle:

$$\Im \lambda > 0 \Rightarrow (T - \lambda)^{-1} = \oint_{S^1} (T(z) - \lambda)^{-1} \frac{dz}{iz}, \quad T(e^{i\xi}) = T(\xi).$$
Resonances of periodic systems [Gérard ’90]

\[ \Im \lambda > 0 \Rightarrow (T - \lambda)^{-1} = \oint_{\mathbb{S}^1} (T(z) - \lambda)^{-1} \frac{dz}{iz}, \quad T(e^{i\xi}) = T(\xi). \]

\( z \mapsto (T(z) - \lambda)^{-1} \) has complex poles; as \( \lambda \) approaches \( \mathbb{R} \), these poles converge to points on \( \mathbb{S}^1 \). Except when poles end up pinching \( \mathbb{S}^1 \), we can deform \( \mathbb{S}^1 \) to avoid them. Pinching points correspond to extrema of dispersion hypersurfaces and induce resonances.

\[ z \in \mathbb{C} \quad \Im \lambda \gg 1 \]

\( \ast: \) pole of \( z \mapsto (T(z) - \lambda)^{-1} \)
Resonances of periodic systems [Gérard ’90]

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$*$: pole of $z \mapsto (T(z) - \lambda)^{-1}$
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\(z \in \mathbb{C}\)

\(\lambda \rightarrow \lambda_0 \in \mathbb{R}\)

\(*: \text{pole of } z \mapsto (T(z) - \lambda)^{-1}\)
Resonances of periodic systems [Gérard ’90]

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\( z \mapsto (T(z) - \lambda)^{-1} \) has complex poles; as \( \lambda \) approaches \( \mathbb{R} \), these poles converge to points on \( S^1 \). Except when poles end up pinching \( S^1 \), we can deform \( S^1 \) to avoid them. Pinching points correspond to extrema of dispersion hypersurfaces and induce resonances.

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\( z \in \mathbb{C} \)

\( \lambda = \lambda_0 \in \mathbb{R} \)

\( S^1 \)
Edge-perturbed periodic systems

The black-box approach of Sjöstrand–Zworski provides a meromorphic continuation of the resolvent of \( P = f(D_x) + \delta \kappa (\delta x) \cos(2\pi x) \).

Asymptotic operators: \( P_{\pm \delta} = f(D_x) \pm \delta \cos(2\pi x) \) for \( x \) near \( \pm \infty \).

This motivates an ad-hoc parametrix for \( P - \lambda \):

\[
Q(\lambda) = \frac{1 - \kappa}{2} (P_{-\delta} - \lambda)^{-1} + \frac{1 + \kappa}{2} (P_{\delta} - \lambda)^{-1}.
\]

We observe that \( (P - \lambda)Q(\lambda) = \text{Id} + K(\lambda) \) with

\[
K(\lambda) \overset{\text{def}}{=} \delta A_\delta \left( (P_{-\delta} - \lambda)^{-1} - (P_{\delta} - \lambda)^{-1} \right), \quad A_\delta \text{ of lower order}.
\]

This provides the meromorphic continuation of \( (P - \lambda)^{-1} \):

\[
(P - \lambda)^{-1} = Q(\lambda)(\text{Id} + K(\lambda))^{-1}.
\]

At distance \( \sim \delta \) from Dirac energies, resonances are poles of \((\text{Id} + K(\lambda))^{-1}\); the key operator is the resolvent difference \((P_{-\delta} - \lambda)^{-1} - (P_{\delta} - \lambda)^{-1}\).
Assumptions

Let $P$ be of the form

$$P = f(D_x) + \delta \kappa (\delta x) \cos(2\pi x), \quad f \text{ analytic and even}.$$ 

Set $f'(\pi) = E_\star$; WLOG $f'(\pi) = 1$. We assume that

1. $f(\xi) = E_\star$ implies $f'(\xi) \neq 0$;
2. The only Dirac point of $P$ at energy $E_\star$ is $(\pi, E_\star)$.

![Graph of $f$ and $P$ showing assumptions](image)
Assumptions

Let $P$ be of the form

\[ P = f(D_x) + \delta \kappa(\delta x) \cos(2\pi x), \quad f \text{ analytic and even.} \]

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2. The only Dirac point of $P$ at energy $E_\star$ is $(\pi, E_\star)$. 

\[ f \]

\[ \xi \]

\[ E_\star \]

\[ \pi \]
Result \( P = f(D_x) + \delta \kappa(\delta x) \cos(2\pi x) \)

The spectrum of the Dirac operator \( \mathcal{D} \) is \((-\infty, -1] \cup [1, \infty) \cup \{\mu_j\} \)
where \(-1 < -\mu_n \leq ... \leq -\mu_1 < \mu_0 = 0 < \mu_1 \leq ... \leq \mu_n < 1\).

**Theorem [Drouot–Fefferman–Weinstein, in progress]**

Fix \( \mu_n < \mu < 1 \). For \( \delta \) sufficiently small, \( P \) continues meromorphically to \( \mathcal{D}(E_\ast, \mu \delta) \) and has exactly \( 2n + 1 \) resonances in this disk, given by

\[ \lambda_j = E_\ast + \delta \mu_j + o(\delta). \]

If in addition the no-fold condition is satisfied (\( f(\xi) = 0 \iff \xi = \pm \pi \)) then these resonances are eigenvalues and the corresponding eigenstates are

\[ \alpha_{+, j}(\delta x)e^{i\pi x} + \alpha_{-, j}(\delta x)e^{-i\pi x} + ... \]

where \((\alpha_{+, j}, \alpha_{-, j})\) are the eigenvectors of \( \mathcal{D} \) at energy \( \mu_j \).

**Comments**

- This is some progress towards the F–L–T–W conjecture.
\textbf{Result} \quad (P = f(D_x) + \delta \kappa(\delta x) \cos(2\pi x))

The spectrum of the Dirac operator \(\mathcal{D}\) is \((-\infty, -1] \cup [1, \infty) \cup \{\mu_j\}\) where \(-1 < -\mu_n \leq \ldots \leq -\mu_1 < \mu_0 = 0 < \mu_1 \leq \ldots \leq \mu_n < 1\).

\textbf{Theorem [Drouot–Fefferman–Weinstein, in progress]} 

Fix \(\mu_n < \mu < 1\). For \(\delta\) sufficiently small, \(P\) continues meromorphically to \(\mathcal{D}(E_*, \mu \delta)\) and has exactly \(2n + 1\) resonances in this disk, given by

\[\lambda_j = E_* + \delta \mu_j + o(\delta).\]

If in addition the no-fold condition is satisfied \((f(\xi) = 0 \text{ iff } \xi = \pm \pi)\) then these resonances are eigenvalues and the corresponding eigenstates are

\[\alpha_{+,j}(\delta x)e^{i\pi x} + \alpha_{-,j}(\delta x)e^{-i\pi x} + \ldots\]

where \((\alpha_{+,j}, \alpha_{-,j})\) are the eigenvectors of \(\mathcal{D}\) at energy \(\mu_j\).

\textbf{Comments}

- When the no-fold condition is satisfied, it characterizes all the eigenstates of \(P\) in the gap. When \(f(D_x) = D_x^2\) this improves the F–L–T–W theorem; and it proves the bulk-edge correspondence.
Result \( P = f(D_x) + \delta \kappa(\delta x) \cos(2\pi x) \)

The spectrum of the Dirac operator \( \mathcal{D} \) is \((-\infty, -1] \cup [1, \infty) \cup \{\mu_j\}\) where \(-1 < -\mu_n \leq ... \leq -\mu_1 < \mu_0 = 0 < \mu_1 \leq ... \leq \mu_n < 1\).

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\[
\lambda_j = E_* + \delta \mu_j + o(\delta).
\]

If in addition the no-fold condition is satisfied (\( f(\xi) = 0 \) iff \( \xi = \pm \pi \)) then these resonances are eigenvalues and the corresponding eigenstates are

\[
\alpha_{+, j}(\delta x)e^{i\pi x} + \alpha_{-, j}(\delta x)e^{-i\pi x} + ...
\]

where \( (\alpha_{+, j}, \alpha_{-, j}) \) are the eigenvectors of \( \mathcal{D} \) at energy \( \mu_j \).

**Comments**

- However when the no-fold condition fails we cannot show that the resonances in \( \mathbb{D}(E_*, \mu \delta) \) are ”true” resonances (\( \Im \lambda_j < 0 \)). Classical perturbation theory seems to give only \( \Im \lambda_j = O(\delta \infty) \)!
Pictorial representation

Spectrum of the Dirac operator $\mathcal{D}$

Spectrum of $P$ under the no-fold condition

$E_\star - \delta$  $E_\star + \delta$
Pictorial representation

Spectrum of the Dirac operator $\mathcal{D}$

Spectrum of $P$ without the no-fold condition

$E_* - \delta$  $E_* + \delta$
Highly oscillatory potentials

**Theorem [Drouot–Fefferman–Weinstein, in progress]**

$E_\star + \mu \delta + o(\delta)$ is a resonance of $P$ if and only if $\mu^2 - 1$ is an eigenvalue of

$$D_x^2 + V \left( x, \frac{x}{\delta} \right)$$

where $V \in C_0^\infty(\mathbb{R} \times S^1)$ is a $2 \times 2$ matrix potential with

$$V \left( x, \frac{x}{\delta} \right) \rightarrow V \overset{\text{def}}{=} \begin{bmatrix} \kappa^2 - 1 & -i\kappa' \\ i\kappa' & \kappa^2 - 1 \end{bmatrix}.$$

The resonances of $V(x, x/\delta)$ were completely described in [Drouot ’15] (full expansion, derivation of effective potentials, ...) following work of [Duchêne–Vukićević–Weinstein ’14]. They converge to those of $V$.

The Dirac operator comes from

$$D_x^2 + V(x) - (\mu^2 - 1) = (D - \mu)(D + \mu).$$
Principle of proof (WLOG $E_\star = 0$)

Goal: study the resolvent difference $(P_\delta(\xi) - \lambda)^{-1} - (P_{-\delta}(\xi) - \lambda)^{-1}$ after projection in energy/momenta in three cases:

(whole dispersion curves)
Principle of proof (WLOG $E_\ast = 0$)

Goal: study the resolvent difference $(P_\delta(\xi) - \lambda)^{-1} - (P_{-\delta}(\xi) - \lambda)^{-1}$ after projection in energy/momenta in three cases:

Case I: Away from problems.
Principle of proof (WLOG $E_\star = 0$)

Goal: study the resolvent difference $(P_\delta(\xi) - \lambda)^{-1} - (P_{-\delta}(\xi) - \lambda)^{-1}$ after projection in energy/momenta in three cases:

Case II: Near resonant momenta.
Principle of proof (WLOG $E_\star = 0$)

Goal: study the resolvent difference $(P_\delta(\xi) - \lambda)^{-1} - (P_{-\delta}(\xi) - \lambda)^{-1}$ after projection in energy/momenta in three cases:

Case III: Near Dirac momenta.
I. Momenta with energies away from 0

Look at the resolvent difference:

\[
(P_\delta(\xi) - \lambda)^{-1} - (P_{-\delta}(\xi) - \lambda)^{-1}
\]

\[
= -2 \sum_{k=0}^{\infty} (P_0(\xi) - \lambda)^{-1} (\delta \cos(2\pi x)(P_0(\xi) - \lambda)^{-1})^{2k+1} = O(\delta)
\]

because \((P_0(\xi) - \lambda)^{-1}\) is sufficiently small when \(\lambda \in \mathbb{D}(0, \delta)\) and momenta/energy are in that range. Integrate over such \(\xi\) to deduce that this terms have negligible contributions.
II. Near resonant momenta/energy

Use a Cauchy formula, then the resolvent difference formula:

\[
\frac{\pi_\delta(\xi)}{\lambda_{j,\delta}(\xi) - \lambda} - \frac{\pi_{-\delta}(\xi)}{\lambda_{j,-\delta}(\xi) - \lambda}
= \oint_0 \left( (P_\delta(\xi) - z)^{-1} - (P_{-\delta}(\xi) - z)^{-1} \right) \frac{dz}{2\pi i(z - \lambda)}
\]

\[
= -2 \sum_{k=0}^{\infty} \oint_0 (P_0(\xi) - \lambda)^{-1} \left( \delta \cos(2\pi x)(P_0(\xi) - \lambda)^{-1} \right)^{2k+1} \frac{dz}{2\pi i(z - \lambda)}.
\]

Absence of resonances implies good bounds for complex-valued \( \xi \). Integrate over such \( \xi \) to obtain negligible (complex) terms.
III. Near Dirac momentum/energy

These terms contribute. In fact look at

$$\pi_\delta(\xi) P_\delta(\xi) \pi_\delta(\xi) \sim \pi_0(\xi) P_\delta(\xi) \pi_0(\xi).$$

In the above, $\pi_\delta(\xi)$ project onto the 2D vector space of Bloch modes and $\pi_0(\xi)$ projects on $\mathbb{C}e^{i(\xi-2\pi)x} \oplus \mathbb{C}e^{i\xi x}$. The matrix of $\pi_0(\xi) P_\delta(\xi) \pi_0(\xi)$ is

$$\begin{bmatrix} f(\xi - 2\pi) & \delta \\ \delta & f(\xi) \end{bmatrix} \sim \begin{bmatrix} -(\xi - \pi) & \delta \\ \delta & \xi - \pi \end{bmatrix}$$ (WLOG $f'(\pi) = 1$)
III. Near Dirac momentum/energy

Hence \((P_{\delta}(\xi) - \lambda)^{-1} - (P_{-\delta}(\xi) - \lambda)^{-1}\)

\[
\sim \begin{bmatrix}
-(\xi - \pi) - \lambda & \delta \\
\delta & \xi - \pi - \lambda
\end{bmatrix}^{-1} - \begin{bmatrix}
-(\xi - \pi) - \lambda & -\delta \\
\delta & \xi - \pi - \lambda
\end{bmatrix}^{-1}
\]

\[
= \frac{2\delta}{(\xi - \pi)^2 - \lambda^2 + \delta^2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]
We deduce

\[
(P_\delta(\xi) - \lambda)^{-1} - (P_{-\delta}(\xi) - \lambda)^{-1} \sim \frac{2\delta e^{i(\xi-2\pi)x} \otimes e^{i\xi x}}{(\xi - \pi)^2 - \lambda^2 + \delta^2} + \text{s.t.}
\]
Connection to the Laplacian resolvent

We showed: \((P_\delta(\xi) - \lambda)^{-1} - (P_{-\delta}(\xi) - \lambda)^{-1} \sim \frac{2\delta e^{i(\xi-2\pi)x} \otimes e^{ix}}{(\xi - \pi)^2 - \lambda^2 + \delta^2} + \text{s.t.}\)

Hence using \(-\lambda^2 + \delta^2 = -z^2\delta^2\) we get

\((P_\delta - \lambda)^{-1} - (P_{-\delta} - \lambda)^{-1} \sim \int_{|\xi - \pi| \leq \delta^{1/3}} \frac{2\delta e^{i(\xi-2\pi)x} \otimes e^{ix}}{(\xi - \pi)^2 - \delta^2 z^2} + \text{s.t.}\)

The RHS has kernel

\((x, x') \mapsto \int_{|\xi - \pi| \leq \delta^{1/3}} \frac{2\delta e^{i(\xi-2\pi)x - i\xi x'}}{(\xi - \pi)^2 - \delta^2 z^2} d\xi.\)

Rescale using \(\xi - \pi \mapsto \delta \xi, x \mapsto x/\delta, x' \mapsto x'/\delta\) to get

\(e^{i\pi x/\delta} \cdot \int_{|\xi| \leq \delta^{-2/3}} \frac{2e^{i\xi \cdot (x-x')}}{\xi^2 - z^2} d\xi \cdot e^{i\pi x'/\delta}.\)

This is how the resolvent of the Laplacian appears. Further algebraic manipulations show that the problem reduces to the analysis of \(D_x^2 + V(x, x/\delta)\). [Drouot '15] yields the Theorem.
Remaining problems/projects

- Show that these resonances have generically non-zero imaginary part.
- Study the dynamics of near-resonant states.
- Extend the analysis to the hexagonal lattice, Lieb lattice,...

Thanks for your attention!