Eigenvalues for highly disordered potentials

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Waves and resonances

Waves scattered by a potential \( V \in C^\infty_0(\mathbb{R}^3, \mathbb{R}) \) are the solutions \( u \) of

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(\partial_t^2 - \Delta_{\mathbb{R}^3} + V)u = 0, \quad (u|_{t=0}, \partial_t u|_{t=0}) = (f_0, f_1).
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Suppose that \( u_0 \in L^2(\mathbb{R}^3) \) is an eigenvector of \(-\Delta_{\mathbb{R}^3} + V\), for an eigenvalue \( \lambda^2 \):

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Then we can construct a solution \( u(x, t) = e^{i\lambda t}u_0(x) \) to (1).
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**Problem:** since the domain is unbounded, we cannot obtain expansions for all solutions of (1) as linear combinations of functions of the above form.
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This is reflected in the spectrum of $-\Delta_{\mathbb{R}^3} + V$ on $L^2(\mathbb{R}^3)$: it is the union of a discrete set (eigenvalues) with the continuous spectrum $[0, \infty)$. 
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To overcome this difficulty, we use resonances, complex numbers $\{\lambda_j\}$ depending only on $V$, that quantize local decay of waves:
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$$\exists u_j, \ \forall A, L, \sup_{|x| \leq L} \left| u(x, t) - \sum_{\text{Im}\lambda_j > -A} u_j(x) e^{-i\lambda_j t} \right| = O(e^{-At}). \quad (2)$$
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Waves scattered by a potential \( V \in C_0^\infty(\mathbb{R}^3, \mathbb{R}) \) are the solutions \( u \) of

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R_V(\lambda) = (\Delta_{\mathbb{R}^3} + V - \lambda^2)^{-1} : C^\infty_0(\mathbb{R}^3) \to \mathcal{D}'(\mathbb{R}^3).
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In particular, when using resonances instead of eigenvalues, every solution of (1) can be locally expanded. Resonances are realized as the poles of the meromorphic continuation of

$$R_V(\lambda) = (-\Delta_{\mathbb{R}^3} + V - \lambda^2)^{-1} : C_0^{\infty}(\mathbb{R}^3) \rightarrow \mathcal{D}'(\mathbb{R}^3).$$

Eigenvalues $\mu$ are poles of $(-\Delta_{\mathbb{R}^3} + V - \mu)^{-1}$, hence (squares of) resonances. Conversely, resonances inducing eigenvalues are the one lying on the complex half-line $i[0, \infty)$. 
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The expansion (2) comes from a contour deformation in the representation of $u$ given by the spectral theorem:

$$u = \int_{\mathbb{R}} e^{-i t \lambda} \frac{R_V(\lambda) - R_V(-\lambda)}{2\pi} f_1 d\lambda - \int_{\mathbb{R}} \lambda e^{-i t \lambda} \frac{R_V(\lambda) - R_V(-\lambda)}{2\pi} f_0 d\lambda.$$
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The expansion (2) comes from a contour deformation in the representation of $u$ given by the spectral theorem:

$$u = \int_{\mathbb{R}} e^{-it\lambda} \frac{RV(\lambda) - RV(-\lambda)}{2\pi} f_1 d\lambda - \int_{\mathbb{R}} \lambda e^{-it\lambda} \frac{RV(\lambda) - RV(-\lambda)}{2\pi} f_0 d\lambda.$$

The poles $\lambda_j$ of $RV(\lambda)$ generate residues $u_j(x) e^{-i\lambda_j t}$ in (2).
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$$\exists u_j, \forall A, L, , \sup_{|x| \leq L} |u(x, t) - \sum_{\text{Im} \lambda_j > -A} u_j(x) e^{-i\lambda_j t}| = O(e^{-At}). \quad (2)$$

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The poles $\lambda_j$ of $R_V(\lambda)$ generate residues $u_j(x) e^{-i\lambda_j t}$ in (2). In particular, if $R_V(\lambda)$ has no poles above $\text{Im} \lambda \geq -A$ – resonance-free strip – waves scattered by $V$ decay locally like $e^{-At}$.
Resonances as poles of $R_V(\lambda) = (-\Delta + V - \lambda^2)^{-1}$
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$R_V(\lambda)$ holomorphic

$\mathbb{C}$

Re$\lambda$

Im$\lambda$

resonances of $V$
Resonances as poles of $R_V(\lambda) = (-\Delta + V - \lambda^2)^{-1}$

$R_V(\lambda)$ meromorphic
(spectral theorem)

$\star^2$ eigenvalues of $-\Delta_{\mathbb{R}^d} + V$
Resonances as poles of $R_V(\lambda) = (-\Delta + V - \lambda^2)^{-1}$

$R_V(\lambda)$ meromorphic (spectral theorem)

* 2 eigenvalues of $-\Delta_{\mathbb{R}^d} + V$
Resonances as poles of $R_V(\lambda) = (-\Delta + V - \lambda^2)^{-1}$

$R_V(\lambda)$ meromorphic (analytic Fredholm theory)

* resonances of $V$
Waves in heterogeneous media

Waves scattered by disordered media with scale of heterogeneity $N^{-1} \ll 1$ are modeled by $(\partial_t^2 - \Delta_{\mathbb{R}^3} + V_N)u = 0$, where
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\[
V_N(x) = q_0(x) + \sum_{j \in [-N,N]^3} u_j(\omega)q(Nx - j), \quad q, \ q_0 \in C_0^\infty(\mathbb{R}^3, \mathbb{R})
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\(u_j\) i.i.d, \(\mathbb{E}(u_j) = 0\), \(\mathbb{E}(u_j^2) = 1\), \(u_j \in L^\infty\).
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Model for disordered crystals plunged in a field $q_0$, whose sites $j/N$ come with a random charge $u_j$ and the potential $u_j q(Nx - j)$. $V_N$ is a typical function that varies randomly on a scale $N^{-1}$. 
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Example of potential $V_N$ with $N = 20$ in blue, with $q_0$ in red.
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$$\sum_{|j|_\infty \leq N} u_j \int q(Nx - j)\varphi(x)dx = \varepsilon^d \sum_{|j|_\infty \leq N} u_j \int q(x)\varphi(N(x + j))dx$$
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\[= N^{-3} \sum_{|j|_\infty \leq N} u_j \varphi(N^{-1}j) \cdot \int q(x)dx + O(N^{-4}) \sum_{|j|_\infty \leq N} |u_j| \xrightarrow{\mathbb{P}\text{-a.s.}} 0 \ (\text{K.S.L.L.N}).
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We observe a weak averaging effect on $V_N$.

Does this transfer to resonances of $V_N$, i.e. are resonances of $V_N$ well approximated by resonances of $q_0$?
Result 1: convergence of resonances

Recall that $V_N(x) = q_0(x) + \sum_j u_j q(Nx - j)$. Let $\text{Res}(V)$ denote the set of resonances of $V$. 
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**Theorem [Dr’17]**

$\mathbb{P}$-almost surely, the set of accumulation points of $\text{Res}(V_N)$ is equal to $\text{Res}(q_0)$.
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In other words, \( \mathbb{P} \)-a.s., *resonances of \( V_N \) converge to resonances of \( q_0 \); and there exists a sequence of resonances of \( V_N \) converging to each resonance of \( q_0 \).*
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**Remark:** If $q_0 \equiv 0$, then $q_0$ has no resonances. This implies that $\mathbb{P}$-a.s., $V_N$ has no resonances in any arbitrary large set, provided that $N$ is sufficiently large.
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In fact, after removing a set of probability \( O(e^{-cN^{3/2}}) \), for \( q_0 \equiv 0 \) resonances of \( V_N \) lie below the logarithmic line \( \Im \lambda = -A \ln(N) \); and waves scattered by \( V_N \) decay like \( N^{-At} \).
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Theorem [Dr’17]
If $-\lambda_2 < 0$ is a simple eigenvalue of $-\Delta R + q_0$, with normalized eigenvector $f$, then there exists a random variable such that

$$P(-\lambda_2 N \text{ is an eigenvalue of } V_N) \geq 1 - C e^{-cN^{1/4}},$$

and

$$\int_R q_0(x) dx \neq 0,$$

$$N_{d/2}(\lambda_N - \lambda_0) \int_R q_0(x) dx \xrightarrow{\text{law}} N(0, \sigma^2), \quad \sigma^2 \text{ def } = \int_{[-1,1]} |f(x)|^4 dx.$$

Remark: a similar, more complicated result holds for resonances. The convergence is faster when $\int_R q_0(x) dx = 0$, because $V_N$ is systematically highly oscillatory.
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If $-\lambda_0^2 < 0$ is a simple eigenvalue of $-\Delta_{\mathbb{R}^3} + q_0$, with normalized eigenvector $f$, then there exists $\lambda_N$ a random variable such that

$P(-\lambda_0^2 N \text{ is an eigenvalue of } V_N) \geq 1 - Ce^{-cN^{1/4}}$.

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- If $\int_{\mathbb{R}^3} q(x)dx \neq 0$,

$$\frac{N^{d/2} (\lambda_N - \lambda_0)}{\int_{\mathbb{R}^3} q(x)dx} \xrightarrow{\text{law}} \mathcal{N}(0, \sigma^2), \quad \sigma^2 \overset{\text{def}}{=} \int_{[-1,1]^3} |f(x)|^4 dx.$$
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and

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Remark: a similar, more complicated result holds for resonances. The convergence is faster when $\int_{\mathbb{R}^3} q(x)dx = 0$, because $V_N$ is systematically highly oscillatory.
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▶ Put the problem in a general framework due to [Golowich–Weinstein '05]. This allows to treat $V^# = V_N - q_0$ as a small perturbation of $q_0$, in the sense that $|V_N - q_0|_{H^{-2}} \to 0$ as $N \to \infty$.

▶ Show a local characteristic equation for resonances of $V_N$ near $\lambda_0 \in \text{Res}(q_0)$, of the form $\lambda - \lambda_0 = \infty \sum_{k=1}^{\infty} a_k(V^#, \lambda)$. The coefficients $a_k(V^#, \lambda)$ depend $k$-multilinearly on $V^#$, are holomorphic in $\lambda$; and the sum converges for $N$ sufficiently large and $\lambda$ near $\lambda_0$. Resonances/eigenvalues are thus the zeroes of a random holomorphic function.

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Conclusions

We show stability of resonances under highly oscillatory stochastic perturbations; we identify a stochastic and a deterministic regime for the speed of convergence of resonances, depending on the value of \[ \int_{\mathbb{R}^3} q(x) \, dx. \] Thanks for your attention!
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