Problem 1. Perform the Gram-Schmidt algorithm on the set of vectors

\[ \vec{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad \vec{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \]

and deduce an orthogonal basis for the space

\[ V = \{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} : x + y + z + t = 0 \}. \]

We first perform the Gram-Schmidt algorithm on \{\vec{x}_1, \vec{x}_2, \vec{x}_3\}. We define \( \vec{v}_1 = \vec{x}_1 \) and

\[ \vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1, \quad \vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1. \]

We find:

\[ \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \\ 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \\ -1 \end{pmatrix}. \]

We next prove that \( V \) has dimension 3. For that we note that \( V \) is the kernel of the linear application \( T \) defined by

\[ T \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = x + y + z + t. \]

The matrix of \( T \) in the standard basis is \((1 1 1 1)\) thus \( T \) has rank 1. By the rank theorem, its kernel must have dimension 3. That is, \( V \) has dimension 3. In addition \( V \) contains the vectors \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) which are linearly independent and orthogonal as they result from the Gram-Schmidt process. Since the dimension of \( V \) is 3, \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) form an orthogonal basis of \( V \).

Problem 2. Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear transformation given by

\[ T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ 3x - y \end{pmatrix}. \]

Show that \( T \) is linear. Find a basis \( B \) in which the matrix of \( T \) with respect to the basis \( \{B, B\} \) is diagonal.
\( T \) is linear because if \( x, y, x', y', a \) are arbitrary numbers then

\[
T\left( \begin{array}{c} x \\ y \end{array} \right) + T\left( \begin{array}{c} x' \\ y' \end{array} \right) = \left( \begin{array}{c} x + y \\ 3x - y \end{array} \right) + \left( \begin{array}{c} x' + y' \\ 3x' - y' \end{array} \right) = \left( \begin{array}{c} (x + x') + (y + y') \\ 3(x + x') - (y + y') \end{array} \right) = T\left( \begin{array}{c} x + x' \\ y + y' \end{array} \right)
\]

\[
a T\left( \begin{array}{c} x \\ y \end{array} \right) = a \left( \begin{array}{c} x + y \\ 3x - y \end{array} \right) = \left( \begin{array}{c} ax + ay \\ 3ax - ay \end{array} \right) = T\left( \begin{array}{c} x \\ y \end{array} \right).
\]

The matrix of \( T \) in \( B \) is diagonal if \( B = \{\vec{u}_1, \vec{v}\} \) is a basis of eigenvectors of \( T \): indeed in this case we have \( T\vec{u} = \lambda_1 \vec{u}, T\vec{v} = \lambda_2 \vec{v} \) for some eigenvalues \( \lambda_1, \lambda_2 \) and eigenvectors \( \vec{u}_1, \vec{u}_2 \). In the basis \( B \), \( \vec{v}_1 \) has coordinates \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \vec{v}_2 \) has coordinates \( \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix} \). The matrix \( D \) of \( T \) in the basis \( B \) must satisfy

\[
D \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix}, \quad \text{corresponding to } T\vec{v}_1 = \lambda_1 \vec{v}_1
\]

\[
D \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix}, \quad \text{corresponding to } T\vec{v}_2 = \lambda_2 \vec{v}_2.
\]

Hence \( D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \) is diagonal.

Now we must explicitly find \( \vec{u}_1, \vec{u}_2 \). For that we write the matrix \( A \) of \( T \) in the standard basis and we look for its eigenvectors:

\[
A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}, \quad \text{det}(A - \lambda \text{Id}) = \text{det} \begin{bmatrix} 1 - \lambda & 1 \\ 3 & -1 - \lambda \end{bmatrix} = (1 - \lambda)(-1 - \lambda) - 3 = \lambda^2 - 4.
\]

There are two eigenvalues, solutions of \( \lambda^2 - 4 = 0 \): \( \lambda_1 = 2 \) and \( \lambda_2 = -2 \). We next find the corresponding eigenvectors by solving the augmented matrix system \([A - \lambda \text{Id}, 0] \):

\[
\lambda = 2 \Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 3 & -3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{cases} -x + y = 0 \\ y \text{ free} \end{cases} \quad \text{thus } \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

\[
\lambda = -2 \Rightarrow \begin{bmatrix} 3 & 1 & 0 \\ 3 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{cases} 3x + y = 0 \\ y \text{ free} \end{cases} \quad \text{thus } \vec{v}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.
\]

This gives the basis \( B = \{\vec{v}_1, \vec{v}_2\} \).

**Problem 3.** If possible, diagonalize the matrix

\[
A = \begin{bmatrix} -3 & 5 & -6 \\ 0 & 0 & 0 \\ 2 & -3 & 4 \end{bmatrix}.
\]

If \( A \) is not diagonalizable, explain why.
We first look for the eigenvalues of $A$. For that we write the characteristic equation \[ \det(A - \lambda I) = 0. \] It gives
\[
\begin{vmatrix}
-3 - \lambda & 5 & -6 \\
0 & -\lambda & 0 \\
2 & -3 & 4 - \lambda \\
\end{vmatrix} = 0
\]
Expand this with respect to the second line: we get
\[-\lambda \begin{vmatrix}
-3 - \lambda & -6 \\
2 & 4 - \lambda \\
\end{vmatrix} = 0, \quad -\lambda((-3 - \lambda)(4 - \lambda) + 12) = 0, \quad \lambda^2(1 - \lambda) = 0.
\]
There are two eigenvalues, 0 of multiplicity 2 and 1 of multiplicity 1. The matrix $A$ is diagonalizable if the multiplicity of each eigenvalue is equal to the dimension of the corresponding eigenspace. Since 0 has multiplicity 2 we try to look for the dimension of the eigenspace corresponding to the 0 eigenvalue. For that we try to solve the equation with augmented matrix system $[A - 0I, 0]$:
\[
\begin{bmatrix}
-3 & 5 & -6 & 0 \\
0 & 0 & 0 & 0 \\
2 & -3 & 4 & 0 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
-3 & 5 & -6 & 0 \\
0 & 0 & 0 & 0 \\
3 & -9/2 & 6 & 0 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
-3 & 5 & -6 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{bmatrix}.
\]
Hence there can be only one free variable in the set of solutions. This proves that the eigenspace corresponding to the zero eigenvalue has dimension 1, which is different from the multiplicity of the eigenvalue 0: $A$ is not diagonalizable.

**Problem 4.** Let $V$ be given by
\[ V = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad x + y + z = 0 \}. \]
Find a basis of $V$ and a basis of $V^\perp$. Find an orthonormal basis of $V$ and an orthonormal basis of $V^\perp$. Find the matrix $P$ of the orthogonal projection on $V$ in the standard basis. Explain without calculation why the rank of $P$ is 2 and why $P^2 = P$.

We write $V$ as a span:
\[ V = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad x + y + z = 0 \} = \{ \begin{pmatrix} x \\ y \\ -y - x \end{pmatrix} \} = \text{span}\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \}. \]
These two vectors are clearly linearly independent hence they form a basis of $V$. However they are not orthogonal. But for every $a$,
\[ V = \text{span}\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \} = \text{span}\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + a \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \}. \]
Now we look for $a$ so that these two vectors are orthogonal: it suffices to have
\[
\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \left( \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + a \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right) = 0, \text{ equivalent to } 1 + 2a = 0, \quad a = -1/2.
\]

Hence an orthogonal basis is given by
\[
\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \} = \{ \begin{pmatrix} 1 \\ 0 \\ -1/2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \}.
\]

To obtain an orthonormal basis of $V$ it suffices to renormalize the vectors. The first vector has norm $\sqrt{2}$ while the second has norm $\sqrt{6}/2$. Hence
\[
\{ \vec{u}_1, \vec{u}_2 \} = \{ \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{pmatrix} \}
\]
forms an orthonormal basis of $V$. In addition $V$ is a plane with equation $x + y + z = 0$, hence its normal vector is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. This vector has norm $\sqrt{3}$. It follows that an orthonormal basis for $V^\perp$ is
\[
\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}.
\]

To obtain the matrix of the orthogonal projection to $V$ we write $U = [\vec{u}_1, \vec{u}_2]$ and we recall that $P = UU^T$, hence
\[
P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 2/\sqrt{6} \\ -1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 2/\sqrt{6} & -1/\sqrt{6} \\ -1/\sqrt{2} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.
\]

The rank of $P$ must be equal to 2 because $P$ projects to $V$, which is two-dimensional. Hence the range of $P$ is $V$ thus has dimension 2. $P^2$ must be equal to $P$ because if one takes a vector $\vec{x}$ then project it to $V$, we obtain $P\vec{x}$ which is in $V$. If we project $P\vec{x}$ to $V$, we obtain $P^2\vec{x}$. But $P\vec{x}$ is already in $V$, thus $P^2\vec{x} = P\vec{x}$. Since $\vec{x}$ was arbitrary, we obtain $P^2 = P$.

**Problem 5.** Let $V$ be the subspace of $\mathbb{R}^2$ given by
\[
V = \{ \begin{pmatrix} x \\ y \end{pmatrix}, x + y = 0 \}
\]
and $P$ with coordinates $(0, 2)$ in the standard basis. Find the coordinates of the point in $V$ that is the closest to $P$.

The point that is closest to $P$ on $V$ is the orthogonal projection on $V$ of the point $P$. $V$ is the vector space spanned by the vector $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Hence if $\vec{y}$ has coordinates
\begin{pmatrix} 0 \\ 2 \end{pmatrix} \text{ then the orthogonal projection } T \vec{y} \text{ of } \vec{y} \text{ is given by }

T \vec{y} = \frac{\vec{y} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} = \frac{-2}{2} \vec{v} = -\vec{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.

One can easily double check this on a picture.