Problem 1. Diagonalize the matrix

\[ A = \begin{bmatrix} -3 & -2 & -6 \\ 0 & 1 & 0 \\ 2 & 1 & 4 \end{bmatrix}. \]

We first look for the eigenvalues. For that it suffices to solve the characteristic equation \( \det(A - \lambda \text{Id}) = 0 \). It yields

\[ \det \begin{bmatrix} -3 - \lambda & -2 & -6 \\ 0 & 1 - \lambda & 0 \\ 2 & 1 & 4 - \lambda \end{bmatrix} \]

Expand this with respect to the second row to obtain

\[(1 - \lambda) \det \begin{bmatrix} -3 - \lambda & -6 \\ 2 & 4 - \lambda \end{bmatrix} = 0, \quad (1 - \lambda)((-3 - \lambda)(4 - \lambda) + 12) = 0\]

which is equivalent to \( \lambda(\lambda - 1)^2 = 0 \). There are two eigenvalues, 0 and 1. We first look for the eigenspace corresponding to the 0 eigenvalue. For that it suffices to solve the augmented matrix system \([A, 0]\\):

\[
\begin{bmatrix} -3 & -2 & -6 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

The corresponding system is

\[
\begin{cases}
x + 2z = 0 \\ y = 0
\end{cases}, \quad \text{an eigenvector is } \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}.
\]

Next we look for the eigenspace corresponding to the 1-eigenvalue. Since 1 has multiplicity 2 we need to find two linearly independent eigenvectors if we want to diagonalize \( A \) (if we cannot, then \( A \) is not diagonalizable). We solve the augmented matrix system \([A - \text{Id}, 0]\\):

\[
\begin{bmatrix} -4 & -2 & -6 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

There are two free variables so we will be able to find these two linearly independent eigenvectors. The corresponding system is \( 2x + y + 3z = 0 \), two linearly independent eigenvectors are for instance given by

\[
\begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}.
\]
Now we can conclude: we can write \( A = PDP^{-1} \) where \( P \) is the matrix of the eigenvectors in the order we found them and \( D \) is the diagonal matrix of eigenvalue:

\[
P = \begin{bmatrix}
2 & -5 & -3 \\
0 & 1 & 0 \\
-1 & 3 & 2
\end{bmatrix}, \quad D = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

If we want to push the calculation further we can compute \( P^{-1} \) and we will obtain:

\[
A = \begin{bmatrix}
2 & -5 & -3 \\
0 & 1 & 0 \\
-1 & 3 & 2
\end{bmatrix} \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
2 & 1 & 3 \\
0 & 1 & 0 \\
1 & -1 & 2
\end{bmatrix}.
\]

**Problem 2.** Let \( T \) be the transformation from \( \mathbb{P}_2 \) to \( \mathbb{P}_2 \) be defined by

\[ T(p(t)) = tp'(t) + p(t). \]

Show that \( T \) is linear. Find the matrix of \( T \) in the basis \( \{1, t, t^2\} \). Find the kernel and the range of \( T \).

We start by showing that \( T \) is linear. For that it suffices to show that \( T \) behaves well under addition and multiplication by a scalar. We have

\[
T(p(t) + q(t)) = t(p(t) + q(t))' + (p(t) + q(t)) = tp'(t) + p(t) + tq'(t) + q(t) = T(p(t)) + T(q(t)),
\]

\[
T(a \cdot p(t)) = t(a \cdot p(t))' + a \cdot p(t) = a(tp'(t) + p(t)) = aT(p(t)).
\]

Hence \( T \) is linear. To find the matrix of \( T \) in the basis \( \{1, t, t^2\} \) we take an arbitrary polynomial in \( \mathbb{P}_2 \), with coordinates \( (a, b, c) \) in the basis \( \{1, t, t^2\} \): \( p(t) = a + bt + ct^2 \).

Then we compute \( T(p(t)) \):

\[
T(p(t)) = T(a + bt + ct^2) = t(a + bt + ct^2)' + (a + bt + ct^2) = 3ct^2 + 2bt + a.
\]

Therefore the coordinates of \( T(p(t)) \) in the basis \( \{1, t, t^2\} \) are \( (a, 2b, 3c) \). It follows that \( T \) is represented by the matrix

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 3
\end{bmatrix}.
\]

This matrix has determinant 6, thus it is invertible. It follows that \( \text{null}(A) = \{0\} \) and \( \text{col}(A) = \mathbb{R}^3 \). Therefore the kernel of \( T \) is \( \{0\} \) and its range is \( \mathbb{P}_2 \).

**Problem 3** Find a basis of the null space and the column space of \( A \), where \( A \) is given by

\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 \\
-2 & 0 & -2 & -1 \\
-1 & 1 & -1 & 0
\end{bmatrix}.
\]
We first find the rank of $A$ by row reducing it. For that we note that $R_3 = R_1 + R_2$, thus
\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
-2 & 0 & -2 & -1 \\
-1 & 1 & -1 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 1 & 1 \\
-2 & 0 & -2 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 2 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\] (0.1)
Hence $A$ has rank 2. To find a basis of $\text{col}(A)$, it suffices to pick two linearly independent columns of $A$. For instance,
\[
\begin{pmatrix}
1 \\
-2 \\
-1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 \\
0 \\
1
\end{pmatrix}
\] forms a basis of $\text{col}(A)$. By the rank theorem, $\text{null}(A)$ has dimension $4 - 2 = 2$. To find a basis of $\text{null}(A)$, it suffices to find two linearly independent vectors in $\text{null}(A)$. By (0.1), the augmented matrix system $[A, 0]$ is equivalent to
\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 2 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\quad \begin{cases}
x + y + z + t = 0 \\
2y + t = 0
\end{cases}
\quad \begin{cases}
x = y - z \\
t = -2y.
\end{cases}
\]
For instance, the vectors
\[
\begin{pmatrix}
1 \\
1 \\
0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
-1 \\
0 \\
1
\end{pmatrix}
\] form a basis of $\text{null}(A)$.

**Problem 4.** Find $A^n$, where
\[
A = \begin{bmatrix}
-1 & -3 \\
2 & 4
\end{bmatrix}.
\]
(Hint: first diagonalize $A$.)

We first diagonalize the matrix $A$. Step 1: we solve the characteristic equation. We have
\[
\det \begin{bmatrix}
-1 - \lambda & -3 \\
2 & 4 - \lambda
\end{bmatrix} = \lambda^2 - 3\lambda + 2.
\]
The roots of this polynomial are 1, 2. Next we find the eigenspaces. For the eigenvalue 1, the augmented matrix system to solve is
\[
\begin{bmatrix}
-2 & -3 & 0 \\
2 & 3 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
2 & 3 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]
Therefore \( \begin{pmatrix} 3 \\ -2 \end{pmatrix} \) is an eigenvector. For the eigenvalue 2, the augmented matrix system to solve is
\[
\begin{bmatrix}
-3 & -3 & 0 \\
2 & 2 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]
Therefore \(
\begin{pmatrix}
1 \\
-1
\end{pmatrix}
\) is an eigenvector. It follows that \(A = PDP^{-1}\) where
\[
P = \begin{bmatrix}
3 & 1 \\
-2 & -1
\end{bmatrix}, \quad P^{-1} = \begin{bmatrix}
-1 & -1 \\
2 & 3
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
-2 & -3
\end{bmatrix}, \quad D = \begin{bmatrix}
1 & 0 \\
0 & 2
\end{bmatrix}.
\]

To find \(A^n\), it suffices to note that since \(A = PDP^{-1}\),
\[
A^n = (PDP^{-1}) \cdot (PDP^{-1}) \cdot \ldots \cdot (PDP^{-1}) = PD^nP^{-1}.
\]

Now, \(D\) is diagonal so \(D^n\) is just the \(n\)-th power of the diagonal elements. It follows
that
\[
A^n = PD^nP^{-1} = \begin{bmatrix}
3 & 1 \\
-2 & -1
\end{bmatrix} \begin{bmatrix}
1^n & 0 \\
0 & 2^n
\end{bmatrix} \begin{bmatrix}
1 & 1 \\
-2 & -3
\end{bmatrix} = \begin{bmatrix}
3 - 2^{n+1} & 3 - 3 \cdot 2^n \\
-2 + 2^{n+1} & -2 + 3 \cdot 2^n
\end{bmatrix}.
\]
We can double check with the cases \(n = 0, n = 1\): \(A^0 = \text{Id}\) which is correct and
\(A^1 = A\), which is correct too.

**Problem 9** Show that the matrix
\[
U = \frac{1}{3} \begin{bmatrix}
2 & -2 & 1 \\
1 & 2 & 2 \\
2 & 1 & -2
\end{bmatrix}
\]

is orthogonal.

Find \(U^{-1}\). Then find the coordinates of the vector
\[
\begin{pmatrix}
5 \\
1
\end{pmatrix}
\]
in the basis \{\(\begin{pmatrix}
2 \\
1
\end{pmatrix}, \begin{pmatrix}
-2 \\
2
\end{pmatrix}, \begin{pmatrix}
1 \\
-2
\end{pmatrix}\}\}.

To show that \(U\) is orthogonal we check that \(U^T U = \text{Id}\) and indeed,
\[
\frac{1}{3} \begin{bmatrix}
2 & 1 & 2 \\
-2 & 2 & 1 \\
1 & 2 & -2
\end{bmatrix} \cdot \frac{1}{3} \begin{bmatrix}
2 & -2 & 1 \\
1 & 2 & 2 \\
2 & 1 & -2
\end{bmatrix} = \text{Id}.
\]

Since \(U^T U = \text{Id}\), the inverse of \(U\) is \(U^T\), thus
\[
U^{-1} = U^T = \frac{1}{3} \begin{bmatrix}
2 & 1 & 2 \\
-2 & 2 & 1 \\
1 & 2 & -2
\end{bmatrix}
\]

To find the coordinates of that vector in the basis given by the columns of \(U\), it suffices to find coefficients \((a, b, c)\) such that
\[
\begin{pmatrix}
5 \\
1
\end{pmatrix} = a\begin{pmatrix}
2 \\
1
\end{pmatrix} + b\begin{pmatrix}
-2 \\
2
\end{pmatrix} + c\begin{pmatrix}
1 \\
-2
\end{pmatrix},
\]
that is, to find \((a, b, c)\) such that
\[
\begin{pmatrix}
5 \\
1
\end{pmatrix} = U \begin{pmatrix}
a \\
b \\
c
\end{pmatrix}, \text{ or equivalently } U^{-1} \begin{pmatrix}
5 \\
1
\end{pmatrix} = \begin{pmatrix}
a \\
b \\
c
\end{pmatrix}.
\]
Since we know the formula for $U^{-1}$ this is only a matrix-vector multiplication, hence
\[
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix} = U^{-1} \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{bmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & -2 \end{bmatrix} \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 13/3 \\ -7/3 \\ 5/3 \end{pmatrix}.
\]
The coordinates of this vector in this basis are then $(13/3, -7/3, 5/3)$. 