Problem 1. Solve the linear system
\[
\begin{align*}
    x + 3y - 2z &= 0 \\
    2x - y + z &= 0 \\
    4x + 4y - 3z &= 0
\end{align*}
\]
(10 points). Describe the set of solution on a picture (5 points). What happens to this picture if you change the zeros on the right to any number (5 points)?

We solve the augmented matrix system
\[
\begin{pmatrix}
    1 & 3 & -2 & 0 \\
    2 & -1 & 1 & 0 \\
    4 & 4 & -3 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
    1 & 3 & -2 & 0 \\
    0 & -7 & 5 & 0 \\
    0 & -8 & 5 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
    1 & 3 & -2 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 5 & 0
\end{pmatrix}
\]

The equivalent system is now
\[
\begin{align*}
    x + 3y - 2z &= 0 \\
    y &= 0 \\
    5z &= 0
\end{align*}
\]

The last line gives \( z = 0 \) and the second \( y = 0 \). This forces \( x \) to be equal to 0 in the first line. Therefore the system has only one solution: \((0, 0, 0)\). The picture representing what’s happening is the intersection of three planes. If we change the zeros on the right to any number, we translate the planes to parallel planes. Therefore the picture remains the same, just translated. And there is still a unique intersection point, thus a single solution.

Problem 2. True or false?

- The product of two non-zero matrices is always non-zero.
- The vectors
  \[
  \begin{pmatrix}
      1 \\
      2 \\
      3
  \end{pmatrix},
  \begin{pmatrix}
      2 \\
      -1 \\
      3
  \end{pmatrix},
  \begin{pmatrix}
      7 \\
      4 \\
      -1
  \end{pmatrix}
  \]
  are not linearly independent.
- The linear span of
  \[
  \begin{pmatrix}
      1 \\
      2 \\
      0
  \end{pmatrix}
  \]
  and
  \[
  \begin{pmatrix}
      2 \\
      -1 \\
      3
  \end{pmatrix}
  \]
  is the plane of equation \( 7x - 6y - 5z = 0 \).
- If \( x, y, z \) are three numbers all non-zero and \( \vec{u}, \vec{v}, \vec{w} \) are three vectors all non-zero then \( x\vec{u} + y\vec{v} + z\vec{w} \) is non-zero.
False: counter example:

\[ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A \cdot B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \]

False. We next show that they are linearly independent. For that we solve the augmented matrix system:

\[
\begin{bmatrix}
1 & 2 & 7 & 0 \\
2 & -1 & 4 & 0 \\
3 & 3 & -1 & 0
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & 2 & 7 & 0 \\
0 & -5 & -10 & 0 \\
0 & -3 & -22 & 0
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & 2 & 7 & 0 \\
0 & 1 & 2 & 0 \\
0 & -3 & -22 & 0
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & 2 & 7 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & -16 & 0
\end{bmatrix}.
\]

The equivalent system is

\[
\begin{aligned}
x + 2y + 7z &= 0 \\
y + 2z &= 0 \\
-16z &= 0.
\end{aligned}
\]

The unique solution is (0, 0, 0) thus the vectors are linearly independent.

False. The linear span of these vectors contains the point (0, 0, 0) and to find its equation it suffices to find a normal vector \( \vec{n} \). The cross product gives it:

\[
\vec{n} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ -5 \end{pmatrix}.
\]

The plane \( 7x - 6y - 5z = 0 \) has normal vector \( \begin{pmatrix} 7 \\ -6 \\ -5 \end{pmatrix} \), which is not colinear to \( \vec{n} \). Therefore the planes are different and the answer is false. Alternative solution: the linear span of the vectors \( \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \) contain the point (1, 2, 0). Therefore if the equation of this linear span was \( 7x - 6y - 5z = 0 \) then this point should satisfy the equation. But

\[
7 \cdot 1 - 6 \cdot 2 - 5 \cdot 0 = -5 \neq 0.
\]

Therefore the equation is not satisfied and the answer is false.

False. For instance, take

\[
x = 1, \ y = 1, \ z = 2, \ \vec{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \vec{w} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.
\]

Problem 3 (20 points) Let \( T \) be the 90 degree rotation clockwise. Find

\[
T \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \ T \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \ T \begin{pmatrix} x \\ y \end{pmatrix}, \ x, y \text{ are any numbers}.
\]
(10 points). Then find a matrix $A$ such that

$$ T \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} $$

(5 points). Find $A^4$ and interpret geometrically (5 points).

We can draw a picture to find

$$ T \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \ T \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}, \ x, y \text{ are any numbers.} $$

The matrix $A$ is given by

$$ A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. $$

We check that

$$ A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \ A^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. $$

$A^4$ is the identity matrix, which is expected: indeed it should represent the clockwise rotation of 360 degrees – which does not move vectors at all.

**Problem 4.** (20 points) Let $A$ be the matrix

$$ A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. $$

Find $A^n$ where $n$ is any number. Hint: first find $A^2, A^3, ...$

We have

$$ A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \ A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \ A^4 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}. $$

Thus we conjecture that

$$ A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}. \quad (0.1) $$

We prove this by recursion. It is clearly true for $n = 1$. No assume that (0.1) holds for a certain integer $n$. We need to check that it also holds for $n + 1$. For that we note that

$$ A^n+1 = A^n \cdot A = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n + 1 \\ 0 & 1 \end{bmatrix}. $$

This shows (0.1) for $n + 1$ and therefore it closes the recursion.

**Problem 5.** Find the inverse of the matrix

$$ \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 3 & 5 \end{bmatrix} \quad (15 \text{ points}). \text{ Double check your answer (5 points).} $$
We use the row reduction algorithm. We have:

\[
\begin{bmatrix}
1 & 0 & 2 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
2 & 3 & 5 & 0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 2 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
2 & 0 & 5 & 0 & -3 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & -2 & -3 & 1 \\
0 & 0 & 1 & -2 & -3 & 1
\end{bmatrix}.
\]

Therefore the inverse is given by

\[
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 0 \\
2 & 3 & 5
\end{bmatrix}^{-1} = \begin{bmatrix}
5 & 6 & -2 \\
0 & 1 & 0 \\
-2 & -3 & 1
\end{bmatrix}.
\]

To double check this answer we compute the matrix times its inverse:

\[
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 0 \\
2 & 3 & 5
\end{bmatrix} \cdot \begin{bmatrix}
5 & 6 & -2 \\
0 & 1 & 0 \\
-2 & -3 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

As the right side is the identity matrix we are fine.