

UNIFORM BOUNDEDNESS OF RATIONAL POINTS

ESSAY DESCRIPTION. It was conjectured by Mordell and proved by Faltings that every smooth curve C of genus $g \geq 2$ defined over a number field K has only finitely many rational points. A different proof was later found by Vojta, and simplified by Faltings and Bombieri [2]. The proofs are ineffective in that no bounds are given on the height of rational points on C . Nonetheless the proofs do give an effective upper bound on the number of rational points on C . This raises the interesting question of what bounds can be given for C varying in a family of curves. This essay should concentrate on proving the results of Caporaso, Harris and Mazur [3] and Pacelli [8] on uniform boundedness. These results are conditional on conjectures of Lang. Further developments (see [9] for references) might also be discussed.

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1. INTRODUCTION

The question of how many rational points lie on a given algebraic curve is easily motivated by a basic attempt to solve polynomial equations in two variables. Almost as soon as one asks this simple question however, one sees that it will not have a simple answer. It is easy to create examples of curves exhibiting the full range of behaviors, from having no rational points, to finitely many, to infinitely many. Indeed, it is far beyond the scope of current mathematics to hope for an algorithm that will tell us the exact number of rational points on any curve.

A better question is to ask what can be said in a more general sense: What kinds of curves will have finitely many rational points, or what kinds will have infinitely many rational points? Can we bound the number of rational points on certain curves? One of the first observations to be made is that the genus of a curve relates to the possible behavior. It has been known since very early on that if one can find one rational point on a curve

with genus zero, then there will be infinitely many, and they can be easily parameterized. The question of finding the first rational point can be answered exactly by searching in every completion of \mathbb{Q} , and similarly over any number field K .

Unfortunately already by genus one there is no such nice procedure for finding rational points. Even the question of whether there are finitely many or infinitely many can only be answered in general assuming the truth of the Birch and Swinnerton-Dyer Conjecture. One may be worried that things only get worse as the genus increases further, but a major theorem of Faltings' proves that this is not the case.

Theorem 1.1 (Faltings). *Let K be a number field and let C be an algebraic curve of genus $g(C) \geq 2$ defined over K . Then the number of rational points $|C(K)|$ on C is finite.*

This was first conjectured by Mordell in 1922 (and still often goes by the name *the Mordell conjecture*), and first proven by Faltings in 1983. A more elementary proof, with methods developed by both Faltings and Bombieri can be found in [2], with a description of the method in Section 4.

As soon as it is known that $|C(K)|$ is finite for curves of genus two or more, the next immediate question is whether we can bound the number of rational points. Even more optimistically, does there exist a uniform bound, i.e. one that doesn't depend on the individual curve? The answer to this is currently unknown.

Bombieri's proof of Falting's theorem, like all known proofs, does not bound the height of the rational points on a curve, and so is ineffective in producing a uniform bound. Given a particular curve C , the paper does express a bound on $|C(K)|$ in terms of the rank and torsion of the Mordell-Weil group of the Jacobian variety of C , but again there is no known way to control this across many curves.

This is essentially the current state of what has been proven in this area, but rather than stop here, we look at what further results can be had by assuming a well known conjecture.

Conjecture 1.2 (Bombieri–Lang Conjecture). *Let K be a number field, and X a variety of general type defined over K . Then the set $X(K)$ of K -rational points on X is not Zariski dense in X .*

The meaning of a variety of general type will be discussed shortly. This conjecture, made by Lang in [6] in 1986, and independently (though less generally) by Bombieri at the same time, was quite controversial for a time. Many of the results proven assuming it, including the main result here, were not thought of as certainly being true, but as being as much possible ways towards a counterexample to the conjecture as a statement of an independently true result. The conjecture can probably said to be more widely believed to be true today, though there is still little direct evidence.

1.1. Main Results. We are now ready to introduce the main result to be discussed here, which shows that the Bombieri–Lang Conjecture allows us to uniformly bound the finitely many rational points of Faltings’ theorem.

Theorem 1.3 (Uniform Bound Theorem). *If the Bombieri–Lang conjecture holds, then there exists an integer $B(d, g)$ for every integer $d \geq 1$ and integer $g \geq 2$ such that no smooth curve of genus g defined over a number field K of degree d has more than $B(d, g)$ K -rational points.*

The main work here will be taken up by proving a slightly weaker result of Caporaso, Harris, and Mazur in [3]. They prove the same statement as above, except getting a bound $B(K, g)$ depending on the individual number field K , not just $[K : \mathbb{Q}]$. After proving this we will sketch the method of Pacelli used in [8], which generalizes the result of [3] to the above form.

In order to derive a bound that applies to all curves of a fixed genus over a fixed number field, we want to work with many curves at once, and the natural thing to use is a family of curves. Let $f : X \rightarrow B$ be a proper morphism of integral varieties defined over a number field K . Denote the fiber of f over $b \in B$ by $X_b = f^{-1}(b)$. Since B is integral, it has a unique generic point, and the fiber over this point is called the generic fiber. A family of smooth curves of genus g is given by such $f : X \rightarrow B$, whose generic fiber is a smooth curve of genus g . We next define an object that encodes information about a family of curves, and will be one of our main objects of study.

Definition 1.4. Denote by X_B^n the irreducible component of the n -th fiber product

$$\underbrace{X \times_B X \times_B \cdots \times_B X}_n$$

which dominates B . We further specify that $X_B^0 := B$.

The main tool in proving the Uniform Bound Theorem will be the Correlation Theorem, and the bulk of our work will go towards proving this.

Theorem 1.5 (Correlation). *Let $f : X \rightarrow B$ be a family of smooth curves of genus $g \geq 2$. Then there is some N such that for $n \geq N$, X_B^n admits a dominant rational map h to a variety W of general type, where if X is defined over a number field K , then so are W and h .*

The idea behind the Correlation Theorem is that Faltings’ Theorem only tells us about each fiber X_b in a family of curves individually. In order to say something about all curves at once, we want to be able to say something about how rational points on different curves relate to each other, and how rational points are distributed throughout X , not just on individual fibers. The fiber product X_B^n encodes some of this information, essentially ‘correlating’ n points at a time. Proving the Correlation Theorem and then applying the Bombieri–Lang Conjecture will tell us about the locations of

rational points on X_B^n , which in turn will tell us about the correlated locations of rational points on many fibers X_b at once. Working out the details of this will give us the Uniform Bound Theorem.

1.2. Definitions. We now discuss what it means for a variety to be of general type, with terminology coming mostly from [4].

Definition 1.6. Let \mathcal{L} be an invertible sheaf on a scheme X , and let \mathcal{F} be any coherent sheaf on X . If $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections for all n greater than some n_0 , then \mathcal{L} is called *ample*.

When \mathcal{L} is a globally generated invertible sheaf over X , we can use global sections of \mathcal{L} to define a map from X into projective space, and \mathcal{L} being ample means that we can make this map a closed immersion.

Recall that when X is an integral scheme, we have an isomorphism

$$\begin{aligned} \text{Div}(X) &\xrightarrow{\sim} \text{Pic}(X), \\ D &\mapsto \mathcal{L}(D) \end{aligned}$$

between invertible sheaves and equivalence classes of Cartier divisors. Since we can then write a divisor as a sum

$$D = \sum n_i Y_i,$$

we can extend the notion of the degree $\deg D = \sum n_i$ to $\mathcal{L}(D)$, and we have the basic fact that when X is a curve, $\mathcal{L}(D)$ is ample exactly when it has positive degree. (See [4] page 308 for a basic proof of this fact.)

We further recall that a divisor $D = \sum n_i Y_i$ (and when appropriate the invertible sheaf $\mathcal{L}(D)$) is called *effective* if $n_i \geq 0$ for all i , allowing us to state the following definition, a generalization of the concept of ampleness:

Definition 1.7. An invertible sheaf \mathcal{L} is called *big* provided that it is the tensor product of an ample invertible sheaf and an effective invertible sheaf.

We can now define a variety of general type.

Definition 1.8 (Variety of General Type). A smooth projective variety is of general type if its Serre dualizing sheaf is big. An arbitrary projective variety is of general type if a desingularization of it is.

We also introduce a second definition of bigness, and will state the equivalence of the two without proof.

Definition 1.9 (Iitaka and Kodaira dimension). Let X be an algebraic variety and \mathcal{L} an invertible sheaf on X . The *Iitaka dimension* of \mathcal{L} is one less than the dimension (the transcendence degree over the base field) of the ring

$$R(X, \mathcal{L}) := \bigoplus_{d=0}^{\infty} H^0(X, \mathcal{L}^d).$$

In the special case where \mathcal{L} is the dualizing sheaf of X , we say that

$$R(X) := R(X, \omega_X)$$

is called the *canonical ring* of X , and the value $\dim(R(X)) - 1$ is called the *Kodaira dimension* of X .

Proposition 1.10. *Let X be an integral variety and \mathcal{L} an invertible sheaf on X . The following are equivalent:*

- (1) \mathcal{L} is big.
- (2) The Iitaka dimension of \mathcal{L} is maximal, i.e. equal to the dimension of X .

In particular, X is of general type exactly when it has maximal Kodaira dimension.

See [4] for a proof of this fact. The general method is to observe that the Iitaka dimension is the largest dimension of the image of X in projective space under the map determined by global sections of \mathcal{L}^n for some $n \geq 1$, and then this can be related to our original definitions of ampleness and bigness.

1.3. Outline of Proof. The proof of the uniform bound theorem will consist of five sections. The first three will be dedicated to proving the Correlation Theorem. We first prove in section 2 that the theorem holds in the special case of families of stable curves with maximal variation of moduli. This section is the real heart of the proof of the Correlation Theorem, where we will see how to show that X_B^n is of general type, and the next two sections will expand the class of families to which this can be applied.

Section 3 will not prove the correlation theorem for any class of families, but will construct a series of base changes that can be applied to a general family to relate it to a family of smooth curves with maximal variation of moduli as considered in the preceding section. These base changes will introduce an action by a finite group G , and realize an arbitrary family as a quotient by this group.

To complete the proof of the Correlation Theorem, Section 4 will show that the results of section 2 can be carried through the quotient by the action of G , and as such will demonstrate that the Correlation Theorem holds for an arbitrary family of curves.

Next, Section 5 will take a universal family $f : X \rightarrow B$ of curves of a fixed genus $g \geq 2$, apply first the Correlation Theorem and then Lang's Conjecture, and show that this gives a uniform bound on the number of rational points on any fiber

$$|X_b(K)| \leq B(K, g).$$

Finally, we will sketch Pacelli's methods in Section 6 to show that we can get a uniform bound which only depends on the degree $[K : \mathbb{Q}]$ of the number field K , not the individual field.

2. FAMILIES OF STABLE CURVES WITH MAXIMAL VARIATION OF MODULI

We now begin the proof of the Correlation Theorem, starting by proving it in the case of families of stable curves with maximal variation of moduli. This case is where the heart of the proof of the Correlation Theorem for an arbitrary family lies, and the next two sections will not prove correlation directly but instead extend the space of curves which can be related to those in this section, so that the results of this section can be applied. We first introduce moduli spaces and what it means for a family to have maximal variation of moduli.

2.1. Moduli Spaces. Moduli spaces can be thought of as solutions to classification problems of varieties. For example, in our case we would like a geometric object which classifies or parameterizes isomorphism classes of smooth algebraic curves of genus g . If we have such an object, we can look at the image of our particular family of curves in it, and gain information about how the curves in our family vary up to isomorphism.

If we consider smooth curves with $g = 0$, all such curves are isomorphic to \mathbb{P}^1 , and so their moduli space will be trivial.

In the case $g = 1$, recall that isomorphism classes of elliptic curves can be exactly described by their j -invariant, and so here we see how the affine line can be taken to be the moduli space via the map given by the j -invariant.

If $g \geq 2$, which is precisely the case which concerns us here, it is less clear how a similar construction can be made. Indeed, studying the existence and properties of moduli spaces for smooth curves of genus $g \geq 2$ is an entire subject unto itself, and to avoid going too far afield from our main objective we will only state here without proof a few results. While moduli spaces will be important to what follows, it will be possible to take the stated facts about them as a black box and continue without further knowledge of the subject. We begin by defining (and hence also purporting the existence of) the moduli spaces of smooth curves of genus g .

Definition 2.1. Let $g \geq 2$. Define the moduli space M_g to be the set of isomorphism classes of smooth curves of genus g . We further endow M_g with a topological and algebraic structure so that M_g is an integral quasiprojective variety.

Since M_g is not projective or complete, it will be convenient to work with the *Deligne-Mumford compactification*, \overline{M}_g . This is an integral, projective variety in which M_g is a dense subset. \overline{M}_g now no longer describes all smooth curves of genus g , but is the moduli space of *Deligne-Mumford stable* (often just called *stable*) curves of genus g . The necessary properties of stable curves will be taken in what follows from [10]. Briefly, stable curves have only nodes as singularities and have finitely many automorphisms.

Now if we let $f : X \rightarrow B$ be a family of stable curves (a family where the fibers of f are each stable curves), we have a *canonical morphism to moduli*, the map which sends a point in B to the isomorphism class of the

corresponding fiber, so that

$$\begin{aligned}\phi : B &\longrightarrow \overline{M}_g \\ b &\longmapsto [X_b]\end{aligned}$$

We are now ready to define a family of stable curves with maximal variation of moduli

Definition 2.2. Let $f : X \rightarrow B$ be a family of stable curves. Such a family is said to have *maximal variation of moduli* provided that the preimage under the canonical morphism to moduli of the generic point in \overline{M}_g has finite cardinality (i.e. the map is generically finite). This is equivalent to saying that there are no isotrivial subfamilies which include the generic fiber.

2.2. Correlation for Smooth Families with Maximal Variation of Moduli. We now turn to proving the Correlation Theorem for such families.

The first step is to construct an invertible sheaf on X_B^n using one on X , and then use this to prove a lemma constructing ample and big invertible sheaves on X_B^n . Recall that we have a natural morphism

$$f_n : X_B^n \rightarrow B,$$

and let

$$p_i : X_B^n \rightarrow X$$

be the natural projection maps onto the i -th factor. Then given an invertible sheaf \mathcal{L} on X ,

$$\mathcal{L}_n := p_1^* \mathcal{L} \otimes \cdots \otimes p_n^* \mathcal{L}$$

will be an invertible sheaf on X_B^n . Now we look at this construction when \mathcal{L} is ample or big.

Lemma 2.3. *Given a flat, projective family $X \rightarrow B$ of integral varieties, an ample (resp. big) invertible scheme \mathcal{L} on B and an arbitrary invertible scheme \mathcal{M} on B ,*

$$\mathcal{L}_n \otimes f_n^* \mathcal{M}$$

is ample (resp. big) on X_B^n for sufficiently large n .

Proof. First assume that \mathcal{L} is ample. \mathcal{L} will be ample over every curve C in the family X . If we use the equivalent definition of ample in terms of the positivity of the degree $\deg_C(\mathcal{L})$, then it is easy to see that by taking enough copies of \mathcal{L} , any negativity of $\deg_C(\mathcal{M})$ (i.e. \mathcal{M} not being ample) will eventually be overcome, and

$$\deg_C(\mathcal{L}^{\otimes e}) = e \deg_C(\mathcal{L}) \geq \deg_C(f^* \mathcal{M})$$

for sufficiently large e . Now Let C be a curve in X_B^n and take the projections $C_i := p_i(C)$, with d_i the degree of C over C_i . Since \otimes is the group operation on $\text{Pic}(X_B^n)$ and this group is isomorphic to the abelian group $\text{Div}(X_B^n)$,

$$\deg_C(\mathcal{L}_n \otimes f_n^* \mathcal{M}) = \deg_C \mathcal{L}_n + \deg_C f_n^* \mathcal{M}.$$

We can now project from C to the C_i s, and write

$$1 = \frac{e}{n} + \frac{n-e}{n}$$

for some $n > e$ to get

$$\begin{aligned} \deg_C(\mathcal{L}_n \otimes f_n^* \mathcal{M}) &= \frac{e}{n} \sum_{i=1}^n d_i \deg_{C_i} \mathcal{L} + \frac{n-e}{n} \deg_C \mathcal{L}_n + \deg_C f_n^* \mathcal{M}, \\ &\geq \frac{-1}{n} \sum_{i=1}^n d_i \deg_{C_i} f^* \mathcal{M} + \deg_C f_n^* \mathcal{M} + \frac{n-e}{n} \deg_C \mathcal{L}_n \\ &= \frac{n-e}{n} \deg_C \mathcal{L}_n > 0 \end{aligned}$$

since $e \deg_{C_i}(\mathcal{L}) \otimes \geq \deg_{C_i}(f^* \mathcal{M})$ and \mathcal{L}_n is itself ample.

Now suppose \mathcal{L} is instead a big invertible sheaf. Recalling the definition of big, this means we can write \mathcal{L} as the tensor product of an ample invertible sheaf \mathcal{A} and an effective one $\mathcal{L}(D)$, where D is an effective Cartier divisor. Writing

$$\mathcal{L}_n = p_1^* \mathcal{A} \otimes \cdots \otimes p_n^* \mathcal{A} \otimes p_1^* \mathcal{L}(D) \otimes \cdots \otimes p_n^* \mathcal{L}(D)$$

and noting that each pullback $p_i^*(\mathcal{L}(D))$ will still be effective, it is immediately clear that the proof above will give us $\mathcal{L}_n \otimes f_n^* \mathcal{M}$ as the tensor product of an ample and an effective invertible sheaf, making $\mathcal{L}_n \otimes f_n^* \mathcal{M}$ big as desired. \square

This construction of ample and big invertible sheaves on X_B^n will be our main tool in proving correlation for complete families of stable curves with maximal variation of moduli.

Let us first assume for simplicity that both X and B are smooth (we've already assumed them to be projective) so that the Serre dualizing sheaves of both X and B are just the invertible sheaves corresponding to the canonical divisors K_X and K_B . We can write

$$K_X = \omega_f \otimes f^* K_B$$

where ω_f is the relative dualizing sheaf of X over B , and then it is easy to see that the previous lemma will be exactly what we want if we can show

Lemma 2.4. *The relative dualizing sheaf ω_f of $f : X \rightarrow B$ is big.*

This result in fact will not require that X is smooth.

Proof. We begin with a result shown by Viehweg in [10] that gives us the relation

$$\frac{g(g+1)}{2} \omega_f = f^* \mathcal{L} + W,$$

where g is the genus of the curves in the family $X \rightarrow B$, \mathcal{L} is big, and W is effective. W and \mathcal{L} can be identified specifically, but this won't be important to us. Write

$$\mathcal{L} = A + E,$$

where A is ample and E is effective. Since A is in particular ample on every fiber of f , we also have that f^*A is ample, and so

$$m \cdot \deg_X(f^*A) \geq -\deg_X(\omega_f)$$

for sufficiently large m , making $mf^*A + \omega_f$ ample. Multiply Viehweg's relation by m and add ω_f to both sides:

$$\left(m \cdot \frac{g(g+1)}{g} + 1\right) \omega_f = (mf^*A + \omega_f) + (f^*E + mW),$$

Which is a sum of an ample divisor and an effective divisor, making

$$\left(m \cdot \frac{g(g+1)}{g} + 1\right) \omega_f,$$

and hence ω_f itself, big. \square

We now have that the relative dualizing sheaf of X over B , ω_f is big. In order to prove that X_B^n is of general type, we must prove that, if $\widetilde{X_B^n}$ is a desingularization of X_B^n , $\omega_{\widetilde{X_B^n}}$ is big. We will actually prove a slightly more general lemma for use later, of which correlation for families of stable curves with maximal variation of moduli will be a direct consequence.

Lemma 2.5. *Let $X \rightarrow B$ be a family of stable curves with maximal variation of moduli. Let σ be the birational morphism from the desingularization $\widetilde{X_B^n} \rightarrow X_B^n$, and*

$$g_n := f_n \circ \sigma : \widetilde{X_B^n} \rightarrow B.$$

Then if \mathcal{M} is any invertible sheaf on B ,

$$\omega_{g_n} \otimes g_n^* \mathcal{M}$$

will be a big invertible sheaf on $\widetilde{X_B^n}$.

Proof. We can write

$$\omega_{f_n} \otimes f_n^* \mathcal{M} = p_1^* \omega_f \otimes \cdots \otimes p_n^* \omega_f \otimes f_n^* \mathcal{M}.$$

By the previous two lemmas, we immediately see that this will be big on X_B^n for sufficiently large n . To move to a desingularization, we can explicitly calculate blow-ups to show that the singularities of X_B^n are all canonical, which means that every regular section of $\omega_{f_n} \otimes f_n^* \mathcal{M}$ pulls back through σ to a regular section of $\omega_{g_n} \otimes g_n^* \mathcal{M}$ on $\widetilde{X_B^n}$, making this invertible sheaf big. \square

Corollary 2.6. *Correlation holds for families stable curves with maximal variation of moduli.*

Proof. Letting $\mathcal{M} = K_B$, the invertible sheaf corresponding to the canonical Cartier divisor of B , we see that

$$\omega_{\widetilde{X_B^n}} = \omega_{g_n} \otimes g_n^* K_B$$

is big. \square

3. TREATING GENERAL FAMILIES THROUGH BASE CHANGE

In the previous section we've shown that correlation holds for families of stable curves with maximal variation of moduli. Our goal is to show that correlation holds for an arbitrary family $f : X \rightarrow B$ of curves of genus $g \geq 2$. Here we take an arbitrary family to mean that f is any morphism of integral varieties such that the fiber of the (unique) generic point of B is a smooth curve of genus $g \geq 2$. In order to deduce the full result we will show that any such arbitrary family can be related to a quotient of a stable family with maximal variation of moduli by the action of a finite group.

Remark 3.1. Note that in the previous section we've actually proven something more direct than the statement of the correlation theorem. Namely, we've proven that X_B^n is itself of general type. The correlation theorem, and importantly the uniform bound theorem that follows from it, only requires that there exists a dominant rational map from X_B^n to a variety of general type. This map was simply the identity in the previous section, but here the allowance that it be nontrivial will provide the extra freedom needed to relate any family to one of the type we've considered.

The method used to relate an arbitrary family to one of the type we've considered will be making base changes to the family $f : X \rightarrow B$. A base change is given by a map

$$B' \longrightarrow B$$

with B' irreducible and such that the preimage of the generic point of B has finite cardinality (the map is said to be generically finite). We make one further definition to fully specify what is meant by a base change.

Definition 3.2. Let $f : X \rightarrow B$ be a family of curves and $B' \rightarrow B$ a base change. Define the *essential pullback* to be the unique irreducible component X' of the fiber product

$$X \times_B B' \longrightarrow B$$

which dominates B along with the map $X' \rightarrow B'$, the restriction of the usual projection map.

Using the essential pullback is necessary as if the fiber dimension of f jumps, the standard pullback given by the direct product $X \times_B B'$ itself may not be irreducible, and we don't want to work with non-dominating components. With the above notation, a base change $B \rightarrow B'$ will give us an essential pullback $X' \rightarrow B'$.

We now describe the three base changes that will be used to relate to a stable family with maximal variation of moduli. Some facts about moduli spaces will be assumed along the way.

We begin with an arbitrary family $f : X \rightarrow B$.

3.1. First Base Change. Since one of the few assumptions we do have about an arbitrary family is that the fiber of the generic point of B is a smooth connected curve, there will exist an open subset $U \subset B$ over which

f is smooth, with all fibers being smooth connected curves of genus g . This means that the map to the moduli space M_g of smooth curves of genus g gives us a regular map $U \rightarrow \overline{M}_g$ to the moduli space of stable curves, and any such regular map can be extended to a rational map $B \rightarrow \overline{M}_g$ on all of B . Define

$$B_1 \subset B \times \overline{M}_g$$

to be the graph of this map, such that we get the essential pullback $X_1 \rightarrow B_1$.

Locally this looks exactly like what we want, that is every point $b \in B_1$ gives us (through its image Σ_1 in \overline{M}_g) an isomorphism class of stable curves which includes the fiber $X_{1,b}$, but we don't have enough algebraic information globally about where other such isomorphic curves may be found as fibers, and cannot conclude that the map $B_1 \rightarrow \Sigma_1 \subset \overline{M}_g$ is actually the canonical morphism to moduli for this family. We will use this local information to construct a second base change using a cover of Σ_1 in \overline{M}_g . First we state (without proof) a lemma about moduli spaces to provide such a cover.

Lemma 3.3 (Existence of tautological family of stable curves). *There exists a family of stable curves $\mathcal{T} \rightarrow \Omega$ such that the canonical map to moduli,*

$$\phi : \Omega \rightarrow \overline{M}_g$$

is finite and surjective. Such a family is called a tautological family of stable curves.

3.2. Second Base Change. Using the above, we have that $\Omega \rightarrow \overline{M}_g$ is the cover we desire, and we can take $\Sigma_2 := \phi^{-1}(\Sigma_1)$, Which gives by restriction the family of stable curves

$$\mathcal{T} \supset \mathcal{T}_2 \rightarrow \Sigma_2.$$

Now we make the base change $B_2 \rightarrow B_1$ to the family $X_1 \rightarrow B_1$ with

$$B_2 := B_1 \times_{\Sigma_1} \Sigma_2$$

and take the essential pullback $X_2 \rightarrow B_2$, so that on any open set $U \subset B_2$, if $b \in U$, then $X_{2,b}$ is isomorphic to the fiber of \mathcal{T}_2 given by the image $\mu(b)$ of b in Σ_2 , where $\mu : B_2 \rightarrow \Sigma_2$. This is even closer to the relation we want, but knowing that the individual fibers are isomorphic is not a strong enough condition to make the family $X_2 \rightarrow B_2$ birationally isomorphic to the family $\mathcal{T}_2 \rightarrow \Sigma_2$ pulled back to B_2 . It will take one final base change to fix this discrepancy, and this is where we will see the introduction of a group action.

3.3. Third Base Change. To speak in heuristic terms, the reason why the previous family $X_2 \rightarrow B_2$ is not sufficient is that while $X_{2,b}$ will be isomorphic to $\mathcal{T}_{2,\mu(b)}$ for each b , there may be multiple such isomorphisms. In order to globally relate the families we will need to give a consistent way to pick which isomorphism we want and the constructions given so far do not provide a way to do so.

The way to get around this is to consider the automorphism group of each fiber. Change to a new base

$$B' := \{(b, \psi) : b \in B_2, \psi \in \text{Aut}(X_{2,b})\}.$$

This now has the multiple possible isomorphisms $X_{2,b} \cong \mathcal{T}_{2,\mu(b)}$ built into it. The previous problem will then be solved by taking the quotient by the Galois group of the extension of function fields corresponding to $B' \rightarrow B$.

In order to guarantee that we actually get a finite group with the proper quotient we must slightly alter the above construction and let B_3 be the Galois normalization of a connected component of B' which dominates B_2 over B . Then we have indeed that with

$$G := \text{Gal}(K(B_3)/K(B)),$$

B_3/G is birationally isomorphic to B , and taking the essential pullback $X_3 \rightarrow B_3$, we have X_3/G is birationally isomorphic to our original X .

Finally, we want to change base of our family $\mathcal{T}_2 \rightarrow \Sigma_2$ of stable curves with maximal variation of moduli to match. Without going into explicit details, we create the base Σ_3 so that $B_2 \rightarrow \Sigma_2$ factors through it, with G acting on both B_3 and Σ_3 . Let

$$\mathcal{T}_3 \rightarrow \Sigma_3$$

be the essential pullback of this base change.

The thing to check is that the action of G on $X_3 \rightarrow B_3$ gives a birational isomorphism with $\mathcal{T}_3 \rightarrow \Sigma_3$. Geometrically, we can see heuristically that the previous problem of choosing which isomorphism of fibers to choose is solved as now automorphisms of fibers over Σ_1 are specified exactly by an element $g \in G$.

To summarize the previous three base changes, we have changed base to $X_3 \rightarrow B_3$ such that G acts on both X_3 and B_3 with quotient birationally isomorphic to our original family, and that $X_3 \rightarrow B_3$ is birationally isomorphic to the pullback of the family $\mathcal{T}_3 \rightarrow \Sigma_3$ of stable curves with maximal variation of moduli over B_3 . Putting this together and lifting to fiber powers, we get

$$X_B^n \xrightarrow{\sim} (X_3)_{B_3}^n / G \longrightarrow (\mathcal{T}_3)_{\Sigma_3}^n / G,$$

where the right map is a dominant rational map. Since $\mathcal{T}_3 \rightarrow \Sigma_3$ is a family of stable curves with maximal variation of moduli, if we can show that the Correlation result of the previous section descends through a quotient by G , we will conclude that $(\mathcal{T}_3)_{\Sigma_3}^n / G$ is a variety of general type when n is sufficiently large, which will imply the Correlation Theorem for an arbitrary family $f : X \rightarrow B$.

4. GROUP ACTIONS

4.1. Groups Acting on Families. The aim of this section is to show that our correlation result on stable curves descends to the quotient by the action

of a finite group. This is all that remains to prove the Correlation Theorem for an arbitrary family. Specifically, we will prove:

Proposition 4.1. *Suppose $f : X \rightarrow B$ is a family of stable curves of genus $g \geq 2$ with projective base B and smooth general fiber, and that the family has maximal variation of moduli. Let G be a finite group which acts birationally on both X and B such that*

$$\begin{array}{ccc} X & \xrightarrow{f} & B \\ \vdots & & \vdots \\ g & & g \\ \vdots & & \vdots \\ X & \xrightarrow{f} & B \end{array}$$

commutes for every $g \in G$. Then the quotient X_B^n/G of X_B^n by the action of G (where G acts diagonally on X_B^n) is of general type for sufficiently large n .

Proof. We would first like to strengthen the conditions on the action of G to further assume that the action of G on both X and B is regular instead of just rational. For each $g \in G$ we let $U_g \subset B$ be the open set on which the birational map $B \rightarrow B$ given by the action of g is regular. Since G is finite, write $G = \{g_1, \dots, g_\ell\}$, and

$$U := \bigcap_{i=1}^{\ell} U_{g_i}$$

will be nonempty and open, and this allows us to define a new base B' as the closure of the set

$$\{(p^{g_1}, \dots, p^{g_\ell}) : p \in U\} \subset B^\ell.$$

The action of G on B' is then simply a permutation of factors, and hence regular.

We can further change base to B'' taking a resolution of singularities of B' , and let

$$X'' = X \times_B B'' \rightarrow B''$$

be the pullback of this base change. Hironaka shows in [5] that this can be done so that the discriminant locus

$$\Delta(X''/B'') \subset B'',$$

i.e. the subset of B'' over which all fibers are singular, is a divisor with normal crossings, and so that the action of G lifts from B' to B'' , and remains regular. A divisor with normal crossings is one for which components intersect transversally. We refrain from discussing further properties of divisors with normal crossings, as the only property we will require is that in our situation of stable curves, having normal crossings implies that X has canonical singularities.

The last thing to verify for the above base changes is that the regular action of G on X lifts to a regular action on X'' . As stated earlier, one of the fundamental properties of stable curves is that their automorphism group is finite. Hence for $g \in G$ we can define a finite cover

$$\Gamma := \{(b, \phi) : b \in B'', \phi : X_b \rightarrow X_{bg} \text{ is an isomorphism}\}$$

of B'' . Since we already have that G acts regularly on B'' and on X , if this action lifts to $\Gamma \rightarrow B''$ it will verify that we get a well defined action on

$$X'' = X \times_B B''$$

given by the action on each factor. It lifts to a regular action on an open subset of B'' , and since B'' is smooth and the cover is finite, the lift can be extended over all of B'' .

So far we've reduced from having a rational group action on a stable family $f : X \rightarrow B$ with maximal variation of moduli to further assuming that X has canonical singularities and that G acts regularly on both X and B . In order to complete the proof that X_B^n/G is of general type, we introduce a lemma, the proof of which will only be discussed briefly.

4.2. Correlation for Quotients.

Lemma 4.2. *Suppose X be a variety with canonical singularities and G is a finite automorphism group acting regularly on X . Let θ be a G -invariant, regular section of the m -th power of the dualizing sheaf of X , and suppose that θ vanishes with*

$$\text{vanishing order}_p(\theta) \geq m \cdot (|G_p| - 1)$$

at $p \in X$, where G_p is the stabilizer of p under the action of G . Then θ induces a regular section of the dualizing sheaf on the quotient X/G .

To prove this, first replace X locally by its tangent space $T_p(X)$ at each point p , and then take a linear representation of G . With this setup, the idea is then to take sections of the m -th power of the canonical invertible sheaf on $T_p(X)/G$ and look at their pullback to $T_p(X)$. This pullback will be the product of the canonical invertible sheaf of $T_p(X)$ with the invertible sheaf Γ of the divisor derived from the fixed points of the action of G . One can then bound the order of vanishing coming from Γ based on G being finite, and guarantee that a section with order of vanishing exceeding this must come from the canonical invertible sheaf and descend as desired. Full details of this proof can be found in [3, Corollary 4.2].

We now continue to prove the main result of this section. Define a subset of B ,

$$\Phi := \{b \in B : \text{there exists } g \in G \setminus \{1\} \text{ fixing a component of } X_b \text{ pointwise}\}.$$

Note that this is potentially a smaller set than the set of $b \in B$ fixed by some non-identity $g \in G$, as such a g may still act nontrivially on all

components of X_b . This is constructed exactly so that if $p \in X$ is a point such that $|G_p| > 1$, since each g is an automorphism of X , we will have that

$$f(p) \in \Phi \subset B.$$

We now let D_0 be an effective divisor such that the support of D_0 contains this set Φ . Multiply by $|G|$ to get

$$D := |G| \cdot D_0,$$

and now use this divisor to define an invertible sheaf

$$\mathcal{M} := |G| \cdot K_B \otimes \mathcal{L}(-D_0).$$

We recall again Lemma 2.5 (really its proof up to the point of lifting to a desingularization of X_B^n) and conclude that the invertible sheaf

$$\omega_{f_n} \otimes f_n^* \mathcal{M} = \omega_{X_B^n}(-|G| \cdot f_n^* D_0)$$

is big on X_B^n . If we use the Iitaka dimension version of the definition of bigness, this means that the ring

$$R = R(X_B^n, \omega_{f_n} \otimes f_n^* \mathcal{M}) := \bigoplus_{m=0}^{\infty} (X_B^n, (\omega_{f_n} \otimes f_n^* \mathcal{M})^m)$$

has maximal (Krull) dimension $\dim(X_B^n) + 1$. Take R^G to be the subring of R consisting of elements fixed under G . Since G is finite, R is an integral ring extension of R^G , and it is a general fact that then R and R^G will have the same dimension.

Using the equivalence $\omega_{f_n} \otimes f_n^* \mathcal{M} = \omega_{X_B^n}(-|G| \cdot f_n^* D_0)$ above, we see that \mathcal{M} has been designed specifically so that the sections of $(\omega_{f_n} \otimes f_n^* \mathcal{M})^m$ will vanish to order at least $|G| \cdot m$ at all points $p \in X$ for which $|G_p| > 1$. Since then

$$\text{vanishing order}_p(\text{section}) \geq m \cdot (|G_p| - 1)$$

for all $p \in X$, we may apply Lemma 4.2 to conclude that each section descends to a regular section on the quotient X/G . Since X has canonical singularities, these lift to sections on Z , any resolution of singularities of X_B^n/G (also lifting through f_n), and we can say explicitly that this gives an inclusion

$$R^G \rightarrow \bigoplus_{m=0}^{\infty} H^0(Z, (K_Z)^m) =: R(Z),$$

so that Z has maximal Kodaira dimension. This means that Z is of general type, and this makes X_B^n/G a variety of general type as well, concluding the proof of the Correlation Theorem. \square

5. PROOF OF THE UNIFORM BOUND THEOREM

We now prove the Uniform Bound Theorem, assuming both the Correlation Theorem and the Bombieri–Lang Conjecture. This will be relatively straightforward, as the main depth of the proof of the Uniform Bound Theorem has already come in the proof of the Correlation Theorem. We begin with a lemma.

5.1. Main Lemma.

Lemma 5.1. *Let $f : X \rightarrow B$ be a proper morphism of integral varieties, whose general fiber is a smooth curve of genus $g \geq 2$. Then there exist a non-empty Zariski-open $U_0 \subset B$, and integer N such that for every $b \in U_0(K)$,*

$$|X_b(K)| \leq N.$$

Proof. The Correlation theorem gives us a dominant rational map h from X_B^n to a variety W of general type, all defined over K , when n is sufficiently large, and the Bombieri–Lang Conjecture purports the existence of a non-empty open set

$$W' \subset W \setminus W(K).$$

We can find a non-empty subset of X_B^n on which the map h is regular so that rational points of X_B^n map to rational points of W , and within this we may further restrict to the open set U_n which maps into W' . This must also be non-empty by construction since W' is non-empty and open, and h is dominant. Call the complement of this set Z_n , which is a proper closed subvariety of X_B^n in which all of $X_B^n(K)$ is contained.

If we define maps by dropping the last coordinate,

$$\pi_i : X_B^i \rightarrow X_B^{i-1},$$

We may recursively define Z_{i-1} to be the largest closed subvariety whose preimage under π_i is contained in Z_i . Here Z_i may be empty if $i < n$. Let

$$U_i := X_B^i \setminus Z_i.$$

As the notation may suggest $U_0 \subset X_B^0 = B$ will be the non-empty open set we want. Since U_{i-1} is then the smallest open set containing $\pi_i(U_i)$, we can conclude that

$$\pi_i : \pi_i^{-1}(U_{i-1} \cap Z_i) \rightarrow U_{i-1}$$

is finite over its image. Splitting

$$Z_i = \bigcup_j Z_{i,j}$$

into irreducible components, we can bound the size of the fibers of Z_i over U_{i-1} by

$$\beta_i := \sum_j \deg(\pi_i : \pi_i^{-1}(U_{i-1} \cap Z_{i,j}) \rightarrow U_{i-1}).$$

Consider what we have so far. All rational points of X_B^n are contained in Z_n , and as we follow the path of a rational point as we successively map

$X_B^i \rightarrow X_B^{i-1}$, once the path crosses from some Z_i to some U_{i-1} , it will remain in sets U_k down to U_0 . With this in mind, Pick a

$$b \in U_0(K)$$

and let $\alpha(b)$ be the largest i such that there is a rational point $u \in U_{i-1}$ that maps down to b . We have that $1 \leq \alpha(b) \leq n$ since U_n contains no rational points.

consider the fiber X_u of $\pi_{\alpha(b)}$ over the point u . We have that $X_u(K) \subset Z_i$, which means that

$$|X_u(K)| \leq \beta_i.$$

But X_u is isomorphic to the fiber of $X \rightarrow B$ over b , and so letting N be the maximum over the β_i s, we've proven the lemma. □

5.2. Global Families and Induction from Main Lemma. The above lemma states exactly the uniform bound result we want, except only for the curves given by fibers over the open subset $U_0 \subset B$. To complete the proof we will show that this can be made to apply to all curves. To do so, our main tool will be the existence of global families of curves (which will be taken without proof), and then an induction argument to carry us beyond the subset U_0 .

Proposition 5.2 (Existence of Global Families). *Fix a genus $g \geq 0$ and a number field K . Then there exists a family $f : X \rightarrow B$ of integral projective varieties over K such that every curve of genus g is realized (up to isomorphism) as the fiber of f over some $b \in B$. Such a family is called a global family.*

The proof of this fact is based on the existence of what is called the Hilbert Scheme, and can be found in [7]. Note that this is similar to the concept of a tautological family defined earlier, but here we get a surjective map to M_g not to \overline{M}_g . A tautological family can be thought of as a global family of stable curves with maximal variation of moduli.

Now take $f : X \rightarrow B$ to be such a global family for $g \geq 2$ over K . The previous lemma gives us the uniform bound N_0 on the number of K -rational points on any curve which can be found as a fiber of f over a point $b \in U_0 \subset B$. Recall that then

$$X_0 = B \setminus U_0$$

is a proper closed subvariety of B . Since B is integral (so in particular irreducible), X_0 has strictly smaller dimension than B .

We would like to now apply the lemma repeatedly, inducting on dimension. First though, X_0 may not be irreducible. Throw out any components of X_0 for which the fiber of the generic point is not a smooth curve, as these don't contribute fibers which matter to us. Let $B_{1,i}$ be one of the remaining components, and $X_{1,i}$ the restriction of f to the preimage of $B_{1,i}$, and

now the lemma can be applied to get a uniform bound $N_{1,i}$. Let N_1 be the maximum of the $N_{1,i}$ s over all remaining components i .

Repeating the above on each component now, we get a sequence $\{N_j\}$ which terminates since at each step the dimension of the components is made strictly smaller. Since $f : X \rightarrow B$ was a general family, we have now addressed every curve of genus g , and conclude that there is a uniform bound

$$|C(K)| \leq B(K, g) \leq \max_j N_j$$

such that every curve C of genus $g \geq 2$ has at most $B(K, g)$ rational points.

6. PACELLI'S GENERALIZATION

The uniform bound $B(K, g)$ proven conditionally upon the Bombieri–Lang Conjecture by Caporaso, Harris, and Mazur in [3] depends on the genus as well as the individual number field over which a curve is defined. One can hope to generalize this result even further by weakening the dependencies of the bound $B(K, g)$. We can see immediately that it is too optimistic to hope that the dependence on K can be removed entirely, for given any curve C , we know that $C(\overline{\mathbb{Q}})$ is infinite, so we may extend the field of definition of C by adjoining as many $\overline{\mathbb{Q}}$ points as we want to make a number field K such that the number of K -rational points is arbitrarily large. It does however turn out to be possible to weaken the dependence on K .

Abramovich proved in 1995 in [1] that there exists a bound $B'(K, g)$ that still holds if K is replaced with any quadratic extension of K .

Using similar methods to those of Abramovich, Pacelli proved the following further generalization in [8] in 1996, which makes the uniform bound result in [3] only dependent on the degree of the number field not the field itself.

Theorem 6.1. *Let $d \geq 1$ $g \geq 2$. There is a uniform bound $B(d, g)$ such that if C is any smooth curve of genus g defined over a number field K of degree d , then $B(d, g)$ bounds the number of K rational points on C , with*

$$|C(K)| \leq B(d, g).$$

Here we only sketch the ideas behind proving this. Let $f : X \rightarrow B$ be a family of curves. Instead of X_B^n , one of the main objects we will work with in this generalization is

$$\text{Sym}^d(X_B^n) := (X_B^n)^d / \sim,$$

where two points in (X_B^n) are equivalent under the relation \sim if one is a permutation of the factors of the other. Suppose K is a degree d number field, $b \in B(K)$, and

$$P_1, \dots, P_n \in X_b(K),$$

so that

$$(P_1, \dots, P_n) \in X_B^n.$$

Let $\sigma_1, \dots, \sigma_d$ be the d embeddings of K in \mathbb{C} and consider the point

$$y_{(P_1, \dots, P_n)} := (\sigma_1(P_1, \dots, P_n), \dots, \sigma_d(P_1, \dots, P_n)) \in \text{Sym}^d(X_B^n).$$

Since the Galois action of K/\mathbb{Q} will only permute the coordinates of $y_{(P_1, \dots, P_n)}$, it is Galois stable, and we can conclude that

$$y_{(P_1, \dots, P_n)} \in \text{Sym}^d(X_B^n)(\mathbb{Q}).$$

We would like to show that this construction cannot actually be realized for arbitrarily large n when we insist that the points $P_1, \dots, P_n \in X_b(K)$ are all distinct. To this end, we make the following definition.

Definition 6.2. Fix an integer $d \geq 1$. Let $y \in \text{Sym}^d(X_B^n)(\mathbb{Q})$, and let $m > n$. We say that y is *m-prolongable* provided that there exists a number field K of degree d and distinct points $P_i \in X$ for $1 \leq i \leq m$ each defined over K and not over any subfield which produce a point

$$y_{(P_1, \dots, P_m)} \in \text{Sym}^d(X_B^m)(\mathbb{Q}),$$

and such that

$$y_{(P_1, \dots, P_n)} = y \in \text{Sym}^d(X_B^n)(\mathbb{Q}).$$

Define

$$F_n^{(m)} \subset \text{Sym}^d(X_B^n)(\mathbb{Q})$$

to be the closure of the set of m -prolongable points.

Since an m -prolongation gives rise to an ℓ -prolongation for any $n \leq \ell \leq m$, we have a decreasing sequence of closed sets

$$\dots \subset F_n^{(m+1)} \subset F_n^{(m)} \subset \dots,$$

And since $\text{Sym}^d(X_B^n)(\mathbb{Q})$ is Noetherian, this sequence must eventually stabilize at some $F_n^{(m(n)})$, call this set simply F_n , the set of points which can be prolonged arbitrarily.

If we have m distinct rational points on a curve $X_b(K)$, these clearly give rise to an m -prolongable point, so proving that there are no arbitrarily prolongable points, i.e. that

$$F_n = \emptyset$$

will prove Pacelli's generalization.

The rest of Pacelli's work is dedicated to proving that F_n is of general type when n is large enough. This follows from many of the same methods as used in [3], but we leave out the details here. If F_n is of general type, then the Bombieri–Lang Conjecture says that the rational points of F_n can be contained in a proper closed subset. But F_n was constructed as the closure of a subset of $\text{Sym}^d(X_B^n)(\mathbb{Q})$ so this is a direct contradiction. By construction F_n surjects onto F_{n-1} , and so $F_n = \emptyset$ for all n , proving our result.

7. FURTHER RESULTS

We've completed the proof of the Uniform Bound Theorem, including Pacelli's generalization to only depend on the degree of the number field, and now turn to further directions this field can be taken. Perhaps the most obvious is to look for a proof of the Bombieri–Lang Conjecture, so as to prove these results unconditionally, but as of yet there seems to have been little progress in this direction.

7.1. Curves with Many Rational Points. Assuming the Bombieri–Lang Conjecture to be true so that our results here hold, one can ask about finding the actual sizes of the bounds $B(d, g)$ or $B(K, g)$. In relatively small genus, there are often updated records for curves with the most currently known rational points, but there is no general way of constructing such curves.

Unlike in the case of Elliptic curves, if a curve of genus $g > 1$ is a candidate for having many rational points, one can reasonably expect to find many of them using a basic search algorithm. This is due to Vojta's Conjecture.

Conjecture 7.1 (Vojta). *Let $f : X \rightarrow B$ be a family of curves of genus $g \geq 2$ over a number field K , and let H_X and H_B be height functions on X and B respectively. Then there exist constants a and ℓ such that*

$$H_X(P) \leq aH_B(b)^\ell$$

for all $P \in X_b(K)$.

Besides being only conjectural, the constants a and ℓ are not known, so this does not actually imply a finite search for rational points. It does however heuristically suggest that by searching for rational points in a space indexed by increasing height one is likely to find rational points (if they exist) in relatively short time.

7.2. Correlation in Higher Dimensions. We can also ask about generalizing our results here beyond curves. [3] asks about extending their Correlation Theorem to families of higher-dimensional varieties, and conjecture that the analogous statement will still hold.

Conjecture 7.2. *Suppose $f : X \rightarrow B$ is a family of integral varieties over a number field K , and that the fiber X_ν of the generic point $\nu \in B$ is an integral variety of general type. Then there exists a dominant rational map*

$$h : X_B^n \rightarrow W$$

from the n -th fiber product X_B^n to a variety W of general type, where h and W are defined over K . Further, $h|_{X_\nu}$ is generically finite.

Remark 7.3. The general fiber X_ν being of general type is indeed the analogous statement, as in dimension one, a smooth curve is of general type exactly when it has genus two or more. Hence we could have stated our hypothesis this way in the Correlation Theorem as well.

We can attempt to walk through our proof above in this more general setting to see what holds and what doesn't. Suppose that we're given such a family $f : X \rightarrow B$, and following our earlier procedure, further suppose that the family has maximal variation of moduli. The following should only be taken a suggestion and not a statement of results, as we leave out the specific details and hypotheses that may be required.

We still find that the relative dualizing sheaf ω_f is big, and we can follow the exact same procedure of Lemma 2.3 to build $\omega_{X_B^n}$ out of ω_f and conclude that the positivity of ω_f will eventually dominate, making $\omega_{X_B^n}$ big, and demonstrating this generalized correlation conjecture for families of maximal variation of moduli.

Before looking at Section 3, we look at Section 4 to see what reductions might need to be made by base changes or other means. Provided we can reduce to the case of families where the discriminant locus $\Delta(X/B)$ is a divisor with normal crossings, then [11] again tells us that the singularities of X are canonical, and so Lemma 4.2 will again conclude that if a finite group G acts on both X and B , the quotient X_B^n/G will be of general type for sufficiently large n .

In Section 4 we were able to make the required reduction to when $\Delta(X/B)$ is a divisor with normal crossings using the results of [5]. In the more general setting, however, this doesn't apply and our argument breaks down. A version of Section 3 would have to include a base change or some other method making this reduction in order to complete a proof.

If we go ahead and assume the generalized correlation conjecture, what results might this imply? We first introduce a second conjecture of Lang, related to the first.

Corollary 7.4 (Geometric Lang Conjecture). *Let X be any variety of general type, and let Ξ_X be the union of all irreducible subvarieties of X which have positive dimension and are not of general type. Then $\Xi_X \subset X$ is a proper closed subvariety.*

[3] proves the following.

Theorem 7.5. *Suppose both the Generalized Correlation Conjecture and the Geometric Lang Conjecture hold. Then there is a bound $B(d, k)$ such that*

$$\deg(\Xi_X) \leq B(d, k)$$

whenever X is a projective variety with degree $\leq d$ and dimension $\leq k$.

In particular, cases of this would have consequences such as bounding the degree of a (geometric) genus zero or one curve lying on any surface in \mathbb{P}^3 of a fixed degree d , and similar consequences in higher dimensions.

7.3. Uniform Bounds in Higher Dimensions. Instead of working through generalizing the Correlation Theorem and the results of this, Poonen looks directly at the possibility of results like the Uniform Bound Theorem for

higher dimensional varieties in [9]. There is no result like Falting's theorem for general varieties which guarantees a finite number of rational points, so a boundedness result may seem hopeless at first, but Poonen avoids this by asking a slightly different question, essentially ignoring varieties with infinitely many points.

Question 7.6. *Let $f : X \rightarrow B$ be a family of finite-type schemes over a number field K . Is*

$$\{|X_b(K)| : b \in B(K)\}$$

a finite set?

Note that if $X_b(K)$ is infinite this only contributes the single value \aleph_0 to this set so this is really a question of uniform boundedness on varieties which have finitely many rational points. Poonen then proves that this is equivalent to what is perhaps an even more basic or natural question.

Question 7.7. *Let*

$$\mathcal{P}_n \subset \mathbb{Q}[x_1, \dots, x_n]$$

be the subset consisting of polynomials of total degree 4, and let $\mathcal{P}'_n \subset \mathcal{P}_n$ consist of those polynomials with finitely many rational zeros. Is there a bound B_n on the number of rational zeros for all $f \in \mathcal{P}'_n$?

One can see how these are both related to uniform boundedness, and these questions currently remain open.

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