POWERS OF THE ETA-FUNCTION AND HECKE OPERATORS

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Abstract. Half-integer weight Hecke operators and their distinct properties play a major role in the theory surrounding partition numbers and Dedekind's eta-function. Generalizing the work of Ono in [7], here we obtain closed formulas for the Hecke images of all negative powers of the eta-function. These formulas are generated through the use of Faber polynomials. In addition, congruences for a large class of powers of Ramanujan's Delta-function are obtained in a corollary. We further exhibit a fast calculation for many large values of vector partition functions.

1. Introduction and statement of results

Congruences of partition numbers have been the subject of many works since the famous congruences of Ramanujan from the early 1900s. In particular, Ramanujan [8] proved that

$$p(5n+4) \equiv 0 \pmod{5},$$

$$p(7n+5) \equiv 0 \pmod{7},$$

$$p(11n+6) \equiv 0 \pmod{11},$$

where $p(n)$ is the number of partitions of an integer n. Later, Atkin [2], using half-integral weight Hecke operators, showed that further congruences modulo some larger primes exist by making use of explicit modular equations and identities which are relevant for these primes. Ono [5] was able to build an overarching theory which explains all such congruences.

Much of the work of Ono and Atkin involves the action of the half-integer weight Hecke operators on the partition generating function. Atkin proved his results because he was able to derive explicit relations for small primes ℓ. To obtain the general theory of congruences, Ono had to work in the absence of a closed formula. Instead, he made use of the theory of ℓ-adic Galois representations, in the sense of Deligne and Serre, to conclude that certain modular forms are eigenfunctions modulo ℓ of special Hecke operators.

Following the proof laid out by Ono in [7], we extend his result which solves the problem of obtaining such closed formulas. In particular, we solve this problem for all positive powers of the generating function for the partition numbers.

We first fix notation. For a prime ℓ recall the integer and half-integer weight Hecke operators (for example, see [6]). Given a modular form of half-integer weight $-s/2 \in \frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}$ with Nebentypus character χ, the normalized Hecke action with respect to ℓ^2 is defined on its q-series (note $q := e^{2\pi i z}$ throughout) by

(1.1)

$$\sum_{n\gg-\infty} a(n)q^n \,\Big|\, T(\ell^2) := \sum_{n\gg-\infty}\left(\ell^{s+2}\chi(1/\ell^2)a(n\ell^2) + \ell^{\frac{s+1}{2}}\left(\frac{(-1)^{\frac{s+1}{2}}n}{\ell}\right)\chi(1/\ell)\,a(n) + a(n/\ell^2)\right)q^n.$$

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Throughout we define $a(x) := 0$ if $x \notin \mathbb{Z}$, $\left(\frac{a}{n}\right)$ denotes the usual Jacobi symbol, and we suppress the weight in the notation of the Hecke operator as it will be clear in context. For a modular form of integer weight $-s/2 \in \mathbb{Z}$ with Nebentypus character χ, the normalized Hecke action with respect to ℓ is defined on its q-series by

$$\sum_{n \gg -\infty} a(n)q^n \,\Big|\, T(\ell) := \sum_{n \gg -\infty} \left(\ell^{s/2+1}\chi(1/\ell)a(\ell n) + a(n/\ell)\right) q^n. \tag{1.2}$$

Remark. The normalization in both definitions is nonstandard, and is tailored so that the coefficient of the first power of q in the q-series we considered here remains the same under the Hecke action for modular forms with poles supported at ∞.

We begin by recalling Dedekind's eta-function

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \tag{1.3}$$

which generates the partition numbers as follows:

$$\frac{1}{\eta(24z)} = P(z) := \sum_{n=0}^{\infty} p(n) q^{24n-1}. \tag{1.4}$$

For a nonnegative integer s, we consider reciprocals of the modular form $\eta^s(\delta(s)z)$, where $\delta(s) := \frac{24}{\gcd(|s|,24)}$. Notice that $\delta(s)$ is chosen minimally so that the q-expansion has integer exponents. In the special cases when $s = 1$ and $s = 3$, Euler and Jacobi showed that $\eta^s(\delta(s)z)$ is a theta-function (see for example Theorem 1.60 in [6]). This allows us to say that

$$\eta(24z) \in S_{\frac{1}{2}}(\Gamma_0(576), \chi_{12}) \quad \text{and} \quad \eta^3(8z) \in S_{\frac{3}{2}}(\Gamma_0(64), \chi_0), \tag{1.5}$$

where $\chi_{12} = \left(\frac{12}{\cdot}\right)$ and χ_0 is the trivial character modulo 2. Here $S_k(\Gamma_0(N), \chi)$ denotes the space of weight k cusp forms on $\Gamma_0(N)$ with Nebentypus character χ.

We obtain closed formulas for the action of the Hecke operator on $1/\eta^s(\delta(s)z)$. In order to state these results, it is useful to first recall the Eisenstein series

$$E_4(z) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \quad \text{and} \quad E_6(z) := 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \tag{1.6}$$

where $\sigma_k(n) := \sum_{d|n} d^k$, and Ramanujan's weight 12 cusp form $\Delta(z)$ given by

$$\Delta(z) := \eta^{24}(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}. \tag{1.7}$$

We also make extensive use of Klein's modular function $j(z)$, defined by

$$j(z) := E_4^3(z)/\Delta(z) = q^{-1} + 744 + 196884q + \cdots \tag{1.8}$$

and Euler's Pentagonal Number generating function $(q;q)_\infty$

$$f(z) := (q;q)_\infty = \prod_{n=1}^{\infty} (1 - q^n) = \sum_{k \in \mathbb{Z}} (-1)^k q^{(3k^2+k)/2}. \tag{1.9}$$

Remark. We use the notation $f(z)$ in Section 2.3 rather than $(q;q)_\infty$ for convenience when working with Möbius transformations.

Central to our formulas are the Faber polynomials $J(m;x)$, which are the coefficients of the following series:

$$\sum_{m=0}^{\infty} J(m;x)q^m := \frac{E_4^2(z)E_6(z)}{\Delta(z)} \cdot \frac{1}{j(z)-x} \tag{1.10}$$
$$= 1 + (x-744)q + (x^2 - 1488x + 159768)q^2 + \cdots .$$

Using these polynomials, we now define polynomials $A(s,m;x) \in \mathbb{Z}[x]$ as the coefficients of the series

$$\mathcal{A}(s;q) = \sum_{m=0}^{\infty} A(s,m;x)q^m := (q;q)_\infty^s \cdot \sum_{m=0}^{\infty} J(m;x)q^m. \tag{1.11}$$

Remark. Notice that if $s=1$ or 3, then by classical identities of Euler and Jacobi, we have

$$A(s,m;j(z)) = \begin{cases} \displaystyle\sum_{k=-\infty}^{\infty} (-1)^k q^{(3k^2+k)/2} \cdot J(m;j(z)), & \text{if } s=1 \\ \displaystyle\sum_{k=0}^{\infty} (-1)^k (2k+1) q^{k(k+1)/2} \cdot J(m;j(z)), & \text{if } s=3. \end{cases}$$

In stating our main result it will be useful to define

$$B_{\ell^2}(s;x) := \sum_{m=0}^{\left\lfloor \frac{s}{24} \right\rfloor} p_s(m) A\left(s, s\nu_\ell - m\ell^2; x\right), \tag{1.12}$$

where $\nu_\ell := \frac{\ell^2-1}{24}$, and

$$B_{\ell}(s;x) := \sum_{m=0}^{\left\lfloor \frac{s}{24} \right\rfloor} p_s(m) A\left(s, s\mu_\ell - m\ell; x\right), \tag{1.13}$$

where $\mu_\ell := \frac{\ell-1}{24}$ and $p_s(m)$ is the number of vector partitions of length s. We also define the quadratic character $\chi^{(s)}$ by

$$\chi^{(s)} := \begin{cases} \left(\frac{(-1)^{s/2}}{\cdot}\right), & \text{if } s \in 2\mathbb{Z} \\ \chi_{12}, & \text{if } s \notin 2\mathbb{Z} \cup 3\mathbb{Z} \\ \chi_0, & \text{if } s \in 3\mathbb{Z} \setminus 2\mathbb{Z}. \end{cases} \tag{1.14}$$

Remark. For positive integers n and s, a length s vector partition of n is an s dimensional vector where each coordinate is a partition and the sum of these partitions is n. For $s=1$, $p_1(n) = p(n)$, the classical partition function. Furthermore, the generating function for $p_s(n)$ follows analogously to the $s=1$ case, so that we have

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^s} = \sum_{n=0}^{\infty} p_s(n)q^n.$$

The Hecke operators, for certain primes ℓ act as follows on the reciprocals of powers of the Dedekind eta-function.

Theorem 1.1. *Suppose s is a positive integer and ℓ is prime. Then the following are true.*

(1) *If s is odd and $\ell^2 \equiv 1 \pmod{\delta(s)}$, then*

$$\frac{1}{\eta^s(\delta(s)z)} \Big| T(\ell^2) = \frac{1}{\eta^s(\delta(s)z)} \left(\ell^{\frac{s+1}{2}} \left(\frac{(-1)^{\frac{s-1}{2}} \frac{s\delta(s)}{24}}{\ell} \right) \chi^{(s)}(\ell) + B_{\ell^2}(s; j(\delta(s)z)) \right).$$

(2) *If s is even and $\ell \equiv 1 \pmod{\delta(s)}$, then*

$$\frac{1}{\eta^s(\delta(s)z)} \Big| T(\ell) = \frac{1}{\eta^s(\delta(s)z)} \cdot B_\ell(s; j(\delta(s)z)).$$

Remark. In Theorem 1.1(1) note that $\ell^2 \equiv 1 \pmod{\delta(s)}$ for all $\ell \geq 5$. This condition never holds for $\ell = 2$. Also note that $p_s(0) = 1$ and that if $s < 24$ the sum on the right hand side in Theorem 1.1 is just a single term.

This gives us the following congruences for the Delta-function.

Corollary 1.2. *If ℓ is prime, then we have*

(1) *If s is odd and $\ell^2 \equiv 1 \pmod{\delta(s)}$, then*

$$\Delta^{\nu_\ell}(z) \equiv \frac{1}{B_{\ell^2}(s; j(z))} \pmod{\ell}.$$

(2) *If s is even and $\ell \equiv 1 \pmod{\delta(s)}$, then*

$$\Delta^{\mu_\ell}(z) \equiv \frac{1}{B_\ell(s; j(\delta(s)z))} \pmod{\ell}.$$

Theorem 1.1 also allows us to efficiently compute some $p_s(n)$ values. To this end, we define $c_\ell(s, n)$ as the coefficients of

$$\sum_{n \geq n_0} c_\ell(s, n) q^{\delta(s)n} := \begin{cases} \ell^{\frac{s+1}{2}} \left(\frac{(-1)^{\frac{s-1}{2}} \frac{s\delta(s)}{24}}{\ell} \right) \chi^{(s)}(\ell) + B_{\ell^2}(s; j(\delta(s)z)), & \text{if } s \text{ is odd} \\ B_\ell(s; j(\delta(s)z)), & \text{if } s \text{ is even.} \end{cases} \tag{1.15}$$

Then we have the following corollary which shows how certain vector partition function values can be quickly computed.

Corollary 1.3. *Let $s > 0$ and let ℓ be a prime. Then the following are true.*

(1) *If s is odd and $\ell^2 \equiv 1 \pmod{\delta(s)}$, then for $n \geq 0$ we have*

$$p_s(n\ell^2 - s\nu_\ell) = \frac{1}{\ell^{s+2}} \left[-\ell^{\frac{s+1}{2}} \left(\frac{(-1)^{\frac{s+1}{2}} \left(\delta(s)n - \frac{s\delta(s)}{24} \right)}{\ell} \right) \chi^{(s)}(\ell)\, p_s(n) - p_s\left(\frac{n + s\nu_\ell}{\ell^2} \right) + \sum_{k=-s\nu_\ell}^{n} c_\ell(s, k) p_s(n-k) \right].$$

(2) *If s is even and $\ell \equiv 1 \pmod{\delta(s)}$, then for $n \geq 0$ we have*

$$p_s(n\ell - \mu_\ell) = \chi(\ell)\ell^{s/2-1} \left[-p_s\left(\frac{n + \mu_\ell}{\ell} \right) + \sum_{k=-s\mu_\ell}^{n} c_\ell(s, k) p_s(n-k) \right].$$

Remark. Note that performing this calculation only requires knowing the first $n + s\nu_\ell$ values of $p_s(n)$ when s is odd and the first $n + s\mu_\ell$ values when s is even, and just the coefficients up to q^n in the q-expansion of j. In effect this allows us to calculate extremely large vector partition function values only using a few initial values (see Examples 4.3 and 4.4).

In Section 2 we recall results about the Faber Polynomials and describe properties of the polynomials $A(s, m; j)$. We then use the work of Atkin and Newman to create modular functions on $\mathrm{SL}_2(\mathbb{Z})$ from $1/\eta^s(\delta(s)z)$ and its Hecke action. These results are used to prove Theorem 1.1 and Corollaries 1.2 and 1.3 in Section 3. Finally, in Section 4 we give examples of Theorem 1.1 for the cases $s = 3$ and $s = 24$ and of Corollary 1.3 for $s = 1$ and $s = 3$.

2. Powers of the Dedekind eta-function and modular functions on $\mathrm{SL}_2(\mathbb{Z})$

In this section we first give results about special polynomials in j, and then derive the modularity relations for $1/\eta^s(\delta(s)z)$. Next, we use this to create polynomials in j, which give the Hecke action.

2.1. **Special polynomials in $j(z)$.** For $m \geq 0$, define $j(m; z)$ to be the unique modular function on $\mathrm{SL}_2(\mathbb{Z})$ such that

$$j(m; z) = q^{-m} + O(q).$$

We have the following relation to the Faber polynomials $J(m; x)$ defined in (1.10).

Theorem 2.1. *The following are true.*

(1) *If $m \geq 0$, then $j(m; z) = J(m; j(z))$.*

(2) *If $m \geq 2$, then*

$$j(m; z) = J(1; j(z)) \Big| T_0(m),$$

where $T_0(m)$ is the normalized mth weight 0 Hecke operator.

(3) *The polynomials $J(m, x)$ are the coefficients of*

$$\sum_{m=0}^{\infty} J(m; x) q^m := \frac{E_4^2(z) E_6(z)}{\Delta(z)} \cdot \frac{1}{j(z) - x}.$$

For a proof of this theorem, see [1].

Remark. These are just a few of the properties that we will require for the Faber polynomials. For example in [3], the authors make use of the polynomials $J(m; x)$ to automatically generate Fourier coefficients of modular forms.

Comparing the generating functions for $J(m; x)$ and $A(s, m; x)$ and plugging in j for x we have that

$$A(s, m; j) = J(m; j)(q; q)_\infty^s = j(m; z)(q; q)_\infty^s = q^{-m}(q; q)_\infty^s + O(q). \tag{2.1}$$

2.2. **Modularity properties of Dedekind's eta-function.** We begin by deriving the level and character for the modularity relations for $1/\eta^s(\delta(s)z)$. For the character, we refer back to $\chi^{(s)}$ defined in (1.14).

Lemma 2.2. *Suppose that s is a nonzero integer. Then the following are true.*

(1) *If $s > 0$, then*

$$\eta^s(\delta(s)z) \in S_{\frac{s}{2}}(\Gamma_0(\delta^2(s)), \chi^{(s)}).$$

(2) *If $s < 0$, then*

$$\eta^s(\delta(s)z) \in M^!_{\frac{s}{2}}(\Gamma_0(\delta^2(s)), \chi^{(s)}).$$

Remark. Here $M_k^!$ denotes the space of weight k weakly holomorphic modular forms, that is, forms whose poles are supported at the cusps.

Proof. We assume $s > 0$ because it is simple to read off the modularity property for the reciprocal of a modular form, and the condition coming from the poles at the cusps is clear. When s is even the result is immediate by Theorem 1.64 in [6], so let s be an odd positive integer. Notice that $\delta(s) \in \{8, 24\}$ and that the level can be computed from the levels of $\eta(24z)$ and $\eta^3(8z)$ in (1.5), so it remains to figure out the character. When $\delta(s) = 8$, the character only takes the values 0 and 1, depending on the parity of the input, so $(\eta^3(8z))^{s/3}$ will have the same character. When $\delta(s) = 24$, we are considering $\eta^s(24z)$, where s is odd, so its character is $(\chi_{12})^s = \chi_{12}$. □

2.3. **Producing modular forms on $\mathbf{SL}_2(\mathbb{Z})$.** Define $F_\ell(s, \delta(s)z)$ for even s and primes $\ell \equiv 1 \pmod{\delta(s)}$, and $G_{\ell^2}(s, \delta(s)z)$ for odd s and primes ℓ, with $\ell^2 \equiv 1 \pmod{\delta(s)}$, by

$$F_\ell(s, \delta(s)z) := \eta^s(\delta(s)z) \left(\frac{1}{\eta^s(\delta(s)z)} \,\Big|\, T(\ell) \right) \tag{2.2}$$

and

$$G_{\ell^2}(s, \delta(s)z) := \eta^s(\delta(s)z) \left(\frac{1}{\eta^s(\delta(s)z)} \,\Big|\, T(\ell^2) \right). \tag{2.3}$$

Theorem 2.3. *Letting $z \mapsto z/\delta(s)$, we have that $F_\ell(s, z)$ and $G_{\ell^2}(s, z)$ are modular functions on $SL_2(\mathbb{Z})$ with poles supported at $i\infty$. In particular, they are both polynomials in j with integer coefficients.*

Our main tool to prove this theorem will be a lemma of Atkin in [2]. For a prime ℓ, recall Atkin's U_ℓ-operator, which maps a q-series $\sum_{n \geq m} a(n)q^n$ to $\sum_{\ell n \geq m} a(\ell n)q^n$. In particular, $U_\ell f(\ell z) = f(z)$.

Lemma 2.4. *Suppose F is a modular form on $\Gamma_0(\ell)$, where ℓ is prime. Then*

$$\ell U_\ell F(z) + F\left(\frac{-1}{\ell z}\right) \text{ is on } SL_2(\mathbb{Z}).$$

Proof of Theorem 2.3. First suppose s is even and $\ell \equiv 1 \pmod{\delta(s)}$ is prime. Define

$$\phi(z) = \eta^s(\ell z)/\eta^s(z).$$

By Theorem 1.64 in [6] this is a modular form on $\Gamma_0(\ell)$. Moreover by the transformation law for the eta-function, $\eta(-1/z) = \sqrt{-iz} \cdot \eta(z)$, it follows that $\phi(-1/\ell z) = \ell^{-s/2}\phi^{-1}(z)$. Recall the definitions $f(z) := (q;q)_\infty$ and $\mu_\ell = \frac{\ell-1}{24}$. Then writing $f^s(z) = \sum_{n\geq 0} a_s(n)q^n$, we have that

$$\begin{aligned} F_\ell(s,z) &= f^s(z)\left(\ell^{\frac{s+2}{2}} \sum_{\ell n - s\mu_\ell \geq 0} a_{-s}\left(\ell n - s\mu_\ell\right) q^n + \sum_{n\geq 0} a_{-s}\left(n/\ell - s\frac{1/\ell - 1}{24}\right) q^n\right) \\ &= \ell^{s/2}\left(\ell U_\ell\left(f^s(\ell z)\right) U_\ell\left(\frac{q^{s\mu_\ell}}{f^s(z)}\right) + q^{s\frac{1/\ell-1}{24}} \frac{f^s(-1/z)}{f^s(-1/\ell z)}\right) \\ &= \ell^{s/2}\left(\ell U_\ell\left(q^{s\mu_\ell}\frac{f^s(\ell z)}{f^s(z)}\right) + q^{s\frac{1/\ell-1}{24}} \frac{f^s(-1/z)}{f^s(-1/\ell z)}\right) \\ &= \ell^{s/2}\left(\ell U_\ell \phi(z) + \phi(-1/\ell z)\right) \end{aligned}$$

is on $\mathrm{SL}_2(\mathbb{Z})$ by Lemma 2.4 (note that when $\ell \equiv 1 \pmod{\delta(s)}$, the character $\left(\frac{(-1)^{s/2}}{\ell}\right)$ is always 1). This proves the theorem for $F_\ell(s,z)$

Now suppose s is odd and $\ell^2 \equiv 1 \pmod{\delta(s)}$ is prime. Define $\phi(z) = \eta^s(z)/\eta^s(\ell^2 z)$, $\nu_\ell = \frac{\ell^2-1}{24}$, $R_k = \begin{pmatrix} 1 & 0 \\ -\ell k & 1 \end{pmatrix}$ and

$$S(z) = \sum_{n=0}^{p-1} \phi\left(R_n z\right).$$

Newman shows in [4] that $S(z)$ and $S(-1/\ell z)$ are on $\Gamma_0(\ell)$, and that

$$\phi\left(R_n z\right) = \ell^{s/2} e^{\pi i s(\ell-1)/4} e^{-\pi i s n'\ell/12} \left(\frac{n'}{\ell}\right) \frac{\eta^s(z)}{\eta^s(z-n'/\ell)},$$

where n' is the least positive inverse of n mod ℓ, using the modularity law for eta. He then uses the Gaussian summation formula on $S(z)$ to show that

$$S(z) = \phi(z) + \ell^{\frac{s+1}{2}} f^s(z) \sum_{n\geq 0} \left(\frac{(-1)^s n}{\ell}\right) \chi^{(s)}(\ell) a_{-s}(n) q^n,$$

and

$$U_\ell S(-1/\ell z) = \ell^{s+1} f^s(z) \sum_{\ell^2 n - s\nu_\ell \geq 0} a_{-s}(n\ell^2 - s\nu_\ell) q^n.$$

Since $\phi(z) = f^s(z)\left(q^{-s\nu_\ell} + \sum_{n\geq 1} a_{-s}((n+s\nu_\ell)/\ell^2) q^n\right)$, we can see that

$$\begin{aligned} G_{\ell^2}(s,z) &= f^s(z) \sum_{n\gg -\infty} \left(\ell^{s+2} a_{-s}(n\ell^2 - s\nu_\ell) + \ell^{\frac{s+1}{2}} \left(\frac{(-1)^s n}{\ell}\right) \chi^{(s)}(\ell) a_{-s}(n) + a_{-s}((n+s\nu_\ell)/\ell^2)\right) q^n \\ &= \ell\, U_\ell S(-1/\ell z) + S(z) \end{aligned}$$

is on $\mathrm{SL}_2(\mathbb{Z})$ by Lemma 2.4, completing the proof. □

3. Proofs of theorems and corollaries

Here we prove Theorem 1.1 and Corollaries 1.2 and 1.3.

3.1. **Proof of Theorem 1.1.** By the discussion in Section 2, it follows that both $F_\ell(s,z)$ and $G_{\ell^2}(s,z)$ are modular functions on $\mathrm{SL}_2(\mathbb{Z})$ whose poles are supported at $i\infty$, and hence are polynomials in $j(z)$.

To prove Theorem 1.1 for odd $s > 0$ and prime ℓ such that $\ell^2 \equiv 1 \pmod{\delta(s)}$, we find that

$$\frac{1}{\eta^s(\delta(s)z)} \Big| T(\ell^2) = \ell^{\frac{s+1}{2}} \left(\frac{(-1)^{\frac{s+1}{2}} \frac{-s\delta(s)}{24}}{\ell}\right) \chi^{(s)}(\ell) + \sum_{m=0}^{\left\lfloor \frac{s}{24} \right\rfloor} p_s(m) q^{\left(\frac{-s\delta(s)}{24}\ell^2 + m\delta(s)\ell^2\right)} + O\left(q^{\left(-\frac{s\delta(s)}{24}+1\right)}\right).$$

By (2.1), we then have that

$$\ell^{\frac{s+1}{2}} \left(\frac{(-1)^{\frac{s+1}{2}} \frac{-s\delta(s)}{24}}{\ell}\right) \chi^{(s)}(\ell) + B_{\ell^2}(s; j(\delta(s)z) - G_{\ell^2}(s,z) = O(q). \tag{3.1}$$

The right hand side of the above equation is a modular function on $\mathrm{SL}_2(\mathbb{Z})$, and since every nonconstant modular function on $\mathrm{SL}_2(\mathbb{Z})$ must have a pole, it must be 0.

Similarly, we have that when $s > 0$ is even and $\ell \equiv 1 \pmod{\delta(s)}$ is prime,

$$\frac{1}{\eta^s(\delta(s)z)}\,\Big|\,T(\ell) = \sum_{m=0}^{\lfloor \frac{s}{24} \rfloor} p_s(m) q^{\left(\frac{-s\delta(s)}{24}\ell + m\delta(s)\ell\right)} + O\left(q^{\left(-\frac{s\delta(s)}{24}+1\right)}\right). \tag{3.2}$$

By (2.1), it follows that

$$B_\ell(s; j(\delta(s)z)) - F_\ell(s,z) = O(q).$$

Again the right hand side of the above equation is a modular function on $\mathrm{SL}_2(\mathbb{Z})$ with no pole. Therefore, Theorem 1.1 follows.

3.2. **Proofs of Corollaries 1.2 and 1.3.** We begin with the proof of Corollary 1.2.

Proof of Corollary 1.2. Theorem 1.1 implies that for $s > 0$ odd and ℓ prime, where $\ell^2 \equiv 1 \pmod{\delta(s)}$, we have that

$$\frac{1}{\eta^s(\delta(s)z)}\,\Big|\,T(\ell^2) \equiv \frac{1}{\eta^s(\delta(s)z)} \cdot B_{\ell^2}(s; j(\delta(s)z)) \pmod{\ell}.$$

Additionally, from the definition of the Hecke operator, we have that

$$\frac{1}{\eta^s(\delta(s)z)}\,\Big|\,T(\ell^2) \equiv \frac{1}{\eta^s(\ell^2\delta(s)z)} \pmod{\ell}. \tag{3.3}$$

Combining these,

$$\frac{1}{B_{\ell^2}(s; j(\delta(s)z))} \equiv \frac{\eta^s(\ell^2\delta(s)z)}{\eta^s(\delta(s)z)} \equiv \eta^{\ell^2-1}(\delta(s)z) = \Delta^{s\nu_\ell}(\delta(s)z) \pmod{\ell},$$

and the corollary follows letting $z \mapsto z/\delta(s)$.

The proof of the second half of Corollary 1.2 follows in the same manner from Theorem 1.1 for when s is even and $\ell \equiv 1 \pmod{\delta(s)}$ is prime. □

Proof of Corollary 1.3. Fix a prime number ℓ, with $\ell^2 \equiv 1 \pmod{\delta(s)}$, and a nonnegative odd integer s. We begin by writing $1/\eta^s(\delta(s)z)$ as the generating function for $p_s(n)$, as follows:

$$\frac{1}{\eta^s(\delta(s)z)} = \sum_{n=0}^{\infty} p_s(n) q^{\delta(s)n - \frac{s\delta(s)}{24}}. \tag{3.4}$$

With Theorem 1.1 in mind we consider

$$\sum_{n=0}^{\infty} p_s(n) q^{\delta(s)n - \frac{s\delta(s)}{24}} \cdot \sum_{n \geq -s\nu_\ell} c_\ell(s,n) q^{\delta(s)n}. \tag{3.5}$$

Notice that the exponents of the above q-series are supported on the progression $-s\delta(s)/24 \pmod{\delta(s)}$, and so we can write the coefficient of $q^{\delta(s)n - s\delta(s)/24}$ as

$$\sum_{k=-s\nu_\ell}^{n} c_\ell(s,k) p_s(n-k).$$

By Theorem 1.1, this must equal the corresponding power of $q^{\delta(s)n - s\delta(s)/24}$ in the image of $1/\eta^s\delta(s)z$ under the Hecke operator. This gives the desired result.

For s even the proof of the corollary follows in the same manner, so we omit it. □

4. Examples

Here we provide a few examples of Theorem 1.1 and Corollary 1.3.

Example 4.1. We demonstrate Theorem 1.1 for the half-integer weight case by computing

$$\frac{1}{\eta^3(8z)} \Bigm| T(7^2).$$

We have that $s = 3$ and $\ell = 7$, and so

$$\begin{aligned} B_{7^2}(3;x) &= A(3,6;x) = J(6,x) - 3J(5,x) + 5J(3,x) - 7J(0,x) \\ &= x^6 - 4467x^5 + 7132896x^4 - 4863670351x^3 + 1310279865420x^2 \\ &\quad -97427798897598x + 458929136417417. \end{aligned}$$

This implies that

$$7^2\left(\frac{3}{7}\right)\chi_0(7)+B_{7^2}(3;j(8z)) = q^{-48} - 3q^{-40} + 5q^{-24} - 49 \\ + 20527064691126q^8 + 4907435201413149510q^{16} + \cdots,$$

giving

$$\frac{1}{\eta^3(8z)}\left(7^2\left(\frac{3}{7}\right)+B_{7^2}(3;j(8z))\right) = q^{-49} - 49q^{-1} \\ + 20527064690979q^7 + 4907496782607222447q^{15} + \cdots.$$

This illustrates Theorem 1.1 since we can compute directly that

$$\frac{1}{\eta^3(8z)} \Bigm| T(7^2) = q^{-49} - 49q^{-1} + 20527064690979q^7 + 4907496782607222447q^{15} + \cdots.$$

Example 4.2. We also demonstrate Theorem 1.1 for the integer weight case by computing

$$\frac{1}{\Delta^2(z)} \Bigm| T(3),$$

where $s = 48$ and $\ell = 3$. In this case we have that

$$\begin{aligned} B_3(48;x) &= A(48,4;x) + 48A(48,1;x) \\ &= x^4 - 3024x^3 + 2641896x^2 - 614424576x + 10460922660 + 48\,(x - 792) \\ &= x^4 - 3024x^3 + 2641896x^2 - 614424528x + 10460884644 \end{aligned}$$

and further that

$$\frac{1}{\Delta^2(z)}\,B_3(48;j(z)) = q^{-6} + 48q^{-3} + 1037081257959456 + 3098798798779197216q + \cdots.$$

Computing directly, we find that

$$\frac{1}{\Delta^2(z)} \Bigm| T(3) = q^{-6} + 48q^{-3} + 1037081257959456 + 3098798798779197216q + \cdots,$$

verifying the theorem.

Example 4.3. Here we use Corollary 1.3 to calculate $p(237) = p_1(237)$. This is obtained with $s = 1$, $n = 2$ and $\ell = 11$. We begin by computing

$$B_{11^2}(1,x) = x^5 - 3721x^4 + 4553915x^3 - 2031082648x^2 + 247243785602x - 1971682051559,$$

which yields

$B_{11^2}(1, j(24z)) = q^{-120} - q^{-96} - q^{-72} + 1 + 1582436878066q^{24} + 1112182468884561 92q^{48} + \cdots.$

These terms, along with the fact that $\left(\frac{1}{11}\right)\left(\frac{12}{11}\right) = 1$, is enough to give us $c_{11}(1, k)$ for $-5 \le k \le 2$. Hence

$$\begin{aligned} p(237) = -\frac{1}{11^2}\left(\frac{-(48-1)}{11}\right)\left(\frac{12}{11}\right)p(2) + \frac{1}{11^3}\,[p(7) - p(6) - p(5) + 12p(2) \\ + 1582436878066p(1) + 111218246888456192p(0)] \\ = 83561103925871. \end{aligned}$$

Computing directly, we get $p(237) = 83561103925871$, verifying the corollary.

Example 4.4. Finally we demonstrate Corollary 1.3 for a more general vector partition number. In particular, we calculate $p_3(92)$. Here we use $n = 2$ and $\ell = 7$. As we are in the case when $s = 3$ and $\ell = 7$, we obtain $c_7(3, -6)$ through $c_7(3, 2)$ from Example 4.1. Then we have that

$$\begin{aligned} p_3(92) = \frac{1}{7^3}\left(\frac{(8\cdot 4-1)}{7}\right)\chi_0(7)p_3(2) + \frac{1}{7^5}\,[p_3(8) - 3p_3(7) + 5p_3(5) \\ -49p_3(2) + 20527064691126p_3(1) + 4907435201413149510p_3(0)] \\ = 291991240709658. \end{aligned}$$

Calculating directly, $p_3(92)$ will be the coefficient of q^{735} in the q-expansion of $1/\eta^3(8z)$, which is 291991240709658, verifying the corollary.

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