A Characterization of the Supercuspidal Local Langlands Correspondence

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Introduction and Motivation

Study of Known Cases

Scholze–Shin Equations

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Endoscopy and Reduction to the Singleton Packet Case
Notation

- Fix $F/\mathbb{Q}_p$ finite.
- Fix $G/F$ connected reductive (e.g. $G = GL_n, Sp_{2n}, U_n(E/F)$).
- Let $\hat{G}$ denote the dual group of $G$ over $\mathbb{C}$
  \[ \hat{GL}_n = GL_n(\mathbb{C}), \hat{Sp}_{2n} = SO_{2n+1}(\mathbb{C}) \].
- Let $W_F$ be the Weil group of $F$.
- Define the “$L$-group” of $G$ to be $^LG := \hat{G} \rtimes W_F$. 
The Local Langlands Correspondence (LLC)

- Idea: Relates “nice” irreducible representations of $G(F)$ and “nice” finite dimensional representations of $W_F$ valued in $\hat{G}$.
- Simplest Case: LCFT = LLC for $\mathbb{G}_m$! The local Artin map

$$\text{Art} : W_F^{ab} \cong F^\times$$

induces a bijection

$$\begin{align*}
\left\{ \text{Continuous characters of } \mathbb{G}_m(F) \right\} & \quad \leftrightarrow \quad \left\{ \text{Continuous homs } W_F \to \mathbb{G}_m(\mathbb{C}) = \hat{\mathbb{G}}_m \right\}
\end{align*}$$
General Case

- Exists finite to one map
  \[ R : \mathcal{A}_F(G) \rightarrow \mathcal{G}_F(G) \]
- \( \mathcal{A}_F(G) \) the set of equivalence classes of irreducible smooth \( G(F) \)-representations
- \( \mathcal{G}_F(G) \) equivalence classes of “\( L \)-parameters”:
  \[ \phi : W_F \times SL_2(\mathbb{C}) \rightarrow L G. \]
- Fibers \( \Pi(\phi) := R^{-1}(\phi) \) called “\( L \)-packets”.

Key Question of Talk: How to characterize \( R \)?
Our goal: Describe new characterization generalizing work of Scholze (2013).
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The $GL_n$ Case

- $GL_n$ case is special: The local Langlands map $R : \mathcal{A}_F(G) \rightarrow \mathcal{G}_F(G)$ is a bijection.
- $R$ constructed by Harris–Taylor, Henniart.
- Characterized by Henniart using $L$, $\epsilon$ factors.
- In 2013, Scholze gave a new characterization coming from geometry.
Beyond $GL_n$ Case

- $GSp_4$ Gan–Takeda.
- $Sp_{2n}, SO_{2n+1},$ and $SO_{2n}$ (almost) due to Arthur.
- Quasisplit $U_n(E/F)$ Mok.
- Inner forms of $U_n(E/F)$ Kaletha–Minguez–Shin–White.
- Inner forms of $SL_n$ Hiraga–Saito.
- Supercuspidal case for “almost all” groups Kaletha.
- Characterization typically by compatibility with $GL_n$. 
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Scholze’s Construction

- Given $\tau \in W_F^+$ and $h \in C^\infty(GL_n(O_F))$ constructs $f_{\tau,h} \in C^\infty_c(GL_n(F))$.
- Constructs (via Shimura varieties) $R$ for supercuspidals satisfying Key Equation
  \[ \text{tr}(\pi | f_{\tau,h}) = \text{tr}(R(\pi)) \cdot \left| \frac{1-n}{2} \right| \tau \text{tr}(\pi | h). \]
- Extends to all of $\mathcal{A}_F(G)$ by proving compatibility with parabolic induction.
Scholze’s Characterization in the Supercuspidal Case

- Suppose $R_1, R_2 : A_F(G) \to G_F(G)$ satisfy Key Equation:

$$\text{tr}(\pi | f_{\tau,h}) = \text{tr}(R_i(\pi) | \frac{1-n}{2} | \tau) \text{tr}(\pi | h).$$

- Pick $\pi \in A_F(GL_n)$ and $h \in C^\infty(GL_n(O_F))$ such that $\text{tr}(\pi | h) \neq 0$.

- We have

$$\text{tr}(R_1(\pi) | \frac{1-n}{2} | \tau) = \frac{\text{tr}(\pi | f_{\tau,h})}{\text{tr}(\pi | h)} = \text{tr}(R_2(\pi) | \frac{1-n}{2} | \tau).$$

- Implies $R_1(\pi) \sim R_2(\pi)$.  

Work of Scholze–Shin

- Scholze–Shin (2011) extend construction of \( f_{\tau,h} \) to unramified “PEL type” and get a function \( f_{\tau,h}^{\mu} \in C_c^\infty(G(F)) \) for each:
  - \( \tau \in W_F^+ \)
  - \( h \in C_c^\infty(G(\mathcal{O}_F)), \) (where \( G(\mathcal{O}_F) \) is hyperspecial)
  - \( \mu \in X^*(\hat{G}) \) minuscule

- Youcis (thesis) defines \( f_{\tau,h}^{\mu} \) in “Abelian type” cases.

- \( f_{\tau,h}^{\mu} \) described by cohomology of tubular neighborhoods inside of Rapoport–Zink spaces.
Scholze–Shin Conjecture (No Endoscopy Case)

- Let $\phi : W_F \to ^L G$ be a supercuspidal $L$-parameter, $G$ unramified.
- Let $S\Theta_{\phi} \approx \sum_{\pi \in \Pi(\phi)} \Theta_{\pi}$ be the “stable distribution of $\phi$”
  
  $$(\Theta_{\pi}(f) := \text{tr}(\pi | f))$$

- Conjecture (Scholze–Shin Equation)
  
  We have the following trace identity:
  
  $$S\Theta_{\phi}(f_{\tau,h}^\mu) = \text{tr}(r_\mu \circ \phi) \cdot |^{-\langle \mu, \rho \rangle} | \tau S\Theta_{\phi}(h).$$

- Known cases
  
  - EL, some PEL cases (Scholze, Scholze–Shin)
  - $G = D^\times$ appropriately interpreted (Shen)
  - Unramified $U_n(E/F)$ (BM, Youcis)
Hint of Proof

- Fix global group $G/F$ such that $G_p = G$ and exists nice Shimura datum $(G, X)$.
- Langlands–Kottwitz–Scholze method: for $K \subset G(F)$ compact

$$ \text{tr}(\tau \times f^P h \mid H^*(Sh_K)) = \sum SO(f^P f_\infty f^\mu_{\tau, h}) $$

- Study of cohomology of Shimura varieties (Kottwitz and others) gives:

$$ \sum \text{tr}(\pi \mid f^P h) \text{tr}(r_{-\mu} \circ \phi_\pi \mid \tau) \approx \text{tr}(\tau \times f^P h \mid H^*(Sh_K)) $$

- Stable trace formula gives:

$$ \sum SO(f^P f_\infty f^\mu_{\tau, h}) \approx \sum \text{tr}(\pi \mid f^P f_\infty f^\mu_{\tau, h}) $$

- “Localize at $p$” to get result.
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Endoscopy and Reduction to the Singleton Packet Case
Supercuspidal Parameters

- From now on, assume $G$ quasisplit (for simplicity)
- $L$-parameter $\phi : W_F \times SL_2(\mathbb{C}) \rightarrow {}^L G$ supercuspidal if trivial on $SL_2$ part and doesn’t factor through a Levi subgroup of ${}^L G$.
- Reasons for supercuspidal parameters:
  - Easy to work with (behaves well with elliptic endoscopy)
  - Can prove Scholze–Shin equations
  - Considered in literature (Kaletha, Scholze)
- Need “Backwards LLC”

\[\Pi : \left\{ \begin{array}{c}
\text{Supercuspidal} \\
\text{L-Parameters}
\end{array} \right\} \rightarrow \left\{ \begin{array}{c}
\text{Finite Subsets of} \\
\text{supercuspidal } G(F) \text{ reps}
\end{array} \right\} \]

\[\phi \rightarrow \Pi(\phi)\]
Desired Properties

- **Dis**: $\Pi(\phi) \cap \Pi(\phi') \neq \emptyset$ implies $\phi \sim \phi'$.
- **Bij**: Each Whittaker datum $\mathfrak{w}$ gives a bijection
  \[ i_{\mathfrak{w}} : \Pi(\phi) \cong \text{Irr}(\overline{C_\phi}), \]
  where $\overline{C_\phi} = Z_{\hat{G}}(\text{im}\phi)/Z(\hat{G})\Gamma_F$.
- **Stab**: $S\Theta_\phi := \sum_{\pi \in \Pi(\phi)} \text{dim}(i_{\mathfrak{w}}(\pi))\Theta_\pi$ is stable.
- **SS**: Each $\phi$ satisfies the Scholze–Shin equations.
  \[ S\Theta_\phi(f_{\tau,h}^\mu) = \text{tr}(r_{-\mu} \circ \phi) \cdot \langle -\langle \mu,\rho \rangle | \tau \rangle S\Theta_\phi(h). \]
- We will need to assume $G$ is “good”: If
  \[ \text{tr}(r_\mu \circ \phi | \tau) = \text{tr}(r_\mu \circ \phi' | \tau) \]
  for all $\mu, \tau$ then $\phi \sim \phi'$. 

Main Theorem (Imprecise Version)

Theorem (BM-Youcis)

For $G$ a “good” reductive group, a supercuspidal LLC is characterized by $\text{Dis}$, $\text{Bij}$, $\text{Stab}$, $\text{SS}$, + compatibility with endoscopy.

- **Dis**: Packets are disjoint.
- **Bij**: $i_w : \Pi(\phi) \cong \text{Irr}(\overline{C_\phi})$
- **Stab**: $S\Theta_\phi$ is stable.
- **SS**: $S\Theta_\phi(f^\mu_{\tau,h}) = \text{tr}(r_{-\mu} \circ \phi) \cdot \langle \mu, \rho \rangle \mid \tau \rangle S\Theta_\phi(h)$
Proof in the Singleton Packet Case

- Suppose $\Pi_1, \Pi_2$ are supercuspidal LLCs.
- Pick $\phi$ and suppose $\Pi_1(\phi) = \{\pi\}$ is a singleton.
- If we knew $\Pi_2(\phi') = \{\pi\}$ for some $\phi'$ then we could compare $\phi, \phi'$ using SS.
- Need Atomic Stability: If $\Theta = \sum_i a_i \Theta_{\pi_i}$ is stable then $\Theta$ is a linear combination of $S\Theta_{\phi}$s.
- Do NOT need AtomicStab axiom (Thanks to Prof. Hiraga!)
Proof Assuming Atomic Stability

- Suppose $\Pi_1(\phi) = \{\pi\}$.
- By $\textbf{Stab}$, we have $\Theta_\pi$ is stable.
- By $\textbf{AtomicStab}$ for $\Pi_2$ we have $\Pi_2(\phi') = \{\pi\}$ for some $\phi'$.
- By $\textbf{SS}$:

$$
\text{tr}(r_\mu \circ \phi \cdot |\langle \mu,\rho \rangle \rangle | \tau) = \frac{\text{tr}(\pi | f^\mu_{\tau,h})}{\text{tr}(\pi | h)} = \text{tr}(r_{-\mu} \circ \phi' \cdot |\langle \mu,\rho \rangle \rangle | \tau).
$$

- Implies $\phi \sim \phi'$ since $G$ is good.
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Introduction to Endoscopy

- Elliptic endoscopic groups of $G$ are auxiliary groups $H$ with a map $\eta : L^1 H \to L^1 G$ and $s \in Z(\hat{H})\Gamma_F$.
- $GL_n$ only elliptic endoscopic group of $GL_n$.
- Elliptic endoscopy of $U_n(E/F)$ of the form $U_{n_1}(E/F) \times U_{n_2}(E/F)$ with $n_1 + n_2 = n$. 
A Useful Lemma

Lemma
If $\phi : W_F \rightarrow L G$ is supercuspidal then there is a bijection:

$$\left\{(H, \phi^H) \text{ with } \eta \circ \phi^H = \phi \right\} \leftrightarrow \left\{ \text{conjugacy classes in } \overline{C_\phi} := Z(\hat{G}(\text{im}\phi)/Z(\hat{G})\Gamma_F} \right\}$$

- In particular, $\phi$ factors through an elliptic endoscopic $\eta : L H \rightarrow L G$ iff $\overline{C_\phi} \neq 1$.
- By Bij, we have $\Pi(\phi)$ a singleton iff $\phi$ does not factor through non trivial $L H$.
- Want to induct on dim $G$ using endoscopy.
An elliptic hyperendoscopic datum is a sequence 
$$(LH_1, s_1, \eta_1), \ldots, (LH_n, s_n, \eta_n)$$ so that
$$(LH_1, s_1, \eta_1)$$ is an elliptic endoscopic datum for $G$ and 
$$(LH_i, s_i, \eta_i)$$ an elliptic endoscopic datum for $H_{i-1}$.

**ECI**: Let 
$$(H, s, \eta)$$ an elliptic endoscopic datum for $G$ and 
$f \in C_c^\infty(G(F)), f^H \in C_c^\infty(H(F))$ a pair of match of matching functions. Then

$$S \Theta_{\phi^H}(f^H) = \sum_{\pi \in \Pi(\phi)} \text{tr}(i_{w}(\pi) \mid s) \Theta_{\pi}(f)$$
Supercuspidal LLC

- **Definition**
  A supercuspidal LLC for $G$ is a map for each elliptic hyperendoscopic $H$:
  \[ \Pi_H : \begin{cases} 
  \text{Supercuspidal} \\
  \text{L-parameters of } H 
\end{cases} \rightarrow \begin{cases} 
  \text{Finite subsets of} \\
  \text{supercuspidal } H(F) \text{ reps} 
\end{cases} \]

- **Theorem (BM – Youcis)**
  Let $G$ be such that each elliptic hyperendoscopic $H$ is good. Suppose $\Pi_1, \Pi_2$ are supercuspidal LLCs such that $\bigcup_{\phi} \Pi_{1,H}(\phi) \subset \bigcup_{\phi} \Pi_{2,H}(\phi)$ for all $H$ and $\Pi_{i,H}$ satisfy Dis, Bij, Stab, SS, and ECI. Then $\Pi_{1,H} = \Pi_{2,H}$ for all $H$. 
Groups with “good” elliptic hyperendoscopy: $PGL_n, GL_n, U_n, GU_n, SO_{2n+1}, G_2$.

Groups with “bad” elliptic hyperendoscopy: $Sp_{2n}, SO_{2n}, E_8$.

Corollary (BM – Youcis)

LLC for $U_n(E/F)$ as in Mok is characterized by the above.
Sketch of inductive step

- Suppose we have proven that $\Pi_{1,H} = \Pi_{2,H}$ for all elliptic endoscopic $H$ of $G$.
- Let $\phi$ be an $L$-parameter of $G$. If $C_\phi = 1$, done by singleton packet case.
- Otherwise pick $\pi \in \Pi_{1,G}(\phi)$ and $1 \neq s \in C_\phi$ such that $\text{tr}(i_{\phi}(\pi) \mid s) \neq 0$ and get $(H, \phi^H)$ from lemma.
- By $\text{ECI}$

$$\sum_{\pi' \in \Pi_{1,G}(\phi)} \text{tr}(i_{\phi}(\pi') \mid s) \Theta_{\pi'}(f) = S \Theta_{\phi^H}(f^H)$$

$$= \sum_{\pi' \in \Pi_{2,G}(\phi)} \text{tr}(i_{\phi}(\pi') \mid s) \Theta_{\pi'}(f)$$

- Hence $\pi \in \Pi_{2,G}(\phi)$ by independence of characters.
Some Questions

- Can one show in a direct way that Kaletha’s construction of LLC for supercuspidals satisfies $\text{SS}$?

- For $GL_n$ we know this indirectly since Kaletha is compatible with Harris–Taylor (by Oi–Tokimoto) and Harris–Taylor is known to agree with Scholze.

- Can one define a useful version of $\text{SS}$ that avoids the “good group” assumption? Perhaps this would look like Genestier-Lafforgue’s characterization in terms of Bernstein center elements: \( \{ \delta I, f, (\gamma_i)_{i \in I} \} \).