An $(\infty,4)$-Category

Towards Knot Homology for 3-Manifolds

joint work with:

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"A Braided $(\infty,2)$-Category of Soergel Bimodules"

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§ 1 : Knot polynomials
§ 2 : Knot homologies
§ 3 : Main theorem

* knot := link := "knot or link"

* ignore orientations & gradings

* q always generic

§ 1.1 : Knot polynomials for $L^1 \subset \mathbb{R}^3$

['84] Jones polynomial of $L^1 \subset \mathbb{R}^3$, $J_L(q) \in \mathbb{Z}[q^{\pm}]$

characterized by:

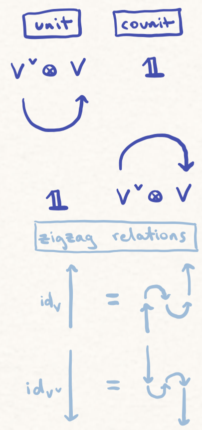
↳ normalization: $J\ (q) = 1$

unknot

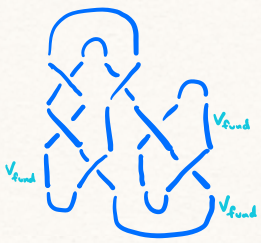
↳ skein relation: $(q-q^{-1})\cdot J_{\uparrow\uparrow}(q) = q^{-2}\cdot J_{\times}(q) - q^{2}\cdot J_{\times}(q)$

['89] Witten: Jones poly. from Chern-Simons theory (QFT)

['90] Reshetikhin–Turaev made this rigorous using quantum groups, namely the braided(-monoidal) category $\mathcal{C} := \mathrm{Rep}^{f.d.}(\underbrace{U_q(sl_2)}_{\text{quantum } sl_2}) \ni \underbrace{V_{\text{fund.}}}_{\text{dualizable!!}}$:



present $L^1 \subset \mathbb{R}^3$ as a "string diagram", label by $V_{\text{fund.}}$.



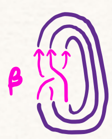
$\rightsquigarrow$ $J_L(q) \in \mathrm{end}_{\mathcal{C}}(\mathbb{1}) \cong \overbrace{\mathbb{C}(q)}^{\text{ground field}}$

Alternatively,

using braiding of $\mathcal{C}$ — using dualizability of $V_{\text{fund.}}$

$$\begin{array}{ccccc} Br_n & \xrightarrow{f} & \mathrm{end}_{\mathcal{C}}(V_{\text{fund.}}^{\otimes n}) & \xrightarrow{\text{trace}} & \mathrm{end}_{\mathcal{C}}(\mathbb{1}) \cong \overbrace{\mathbb{C}(q)}^{\text{ground field}} \\ \Psi & & & & \Psi \qquad \Psi \\ \beta & \longmapsto & & & \mathrm{tr}(f(\beta)) = J_{\bar\beta}(q) \end{array}$$

Thm. (Alexander '23, Markov '36): $\{\text{braids}\}/\sim \xrightarrow[\text{closure}]{\cong} \{\text{links in } \mathbb{R}^3\}$.



$\bar\beta$ the braid-closure

✯ gen$^{\text{zns}}$: sl_N-poly. $\xleftarrow[\text{of variables}]{\text{specialization}}$ $\underbrace{\text{HOMFLY-PT}}_{\text{"}sl_\infty\text{"}}$ poly. ← two variables

e.g. Jones poly $(N=2)$

§1.2: Knot "polynomials" for $L^1 \subset M^3$ via skein modules

Given $\mathcal{C}$ a braided $\mathbb{k}$-linear cat. w/ duals ... ← e.g. $\mathrm{Rep}^{f.d.}(U_q(sl_2))$

classical $\mathrm{sk}_{\mathcal{C}}(M^3) := \Bbbk\left\{\begin{matrix}\text{links in } M \ \overset{\text{use duality}}{\text{labeled}} \\ \text{by obj's of } \mathcal{C}\end{matrix}\right\} \Big/ \underbrace{\text{local relations}}$

isotopy, $\underbrace{\text{skein relations}}_{\text{use braiding}}$, merge strands and tensor their labels

$\uparrow H_0$

derived $\mathrm{sk}_{\mathcal{C}}(M^3) := \underbrace{\int_{M^3} \mathcal{C}} \in D_{\geq 0}(\mathrm{Mod}_{\Bbbk})$

considered as a pointed $\Bbbk$-linear 3-cat. (2-fold delooping)

factorization homology: $\int_{n\text{-mfld}} \mathcal{V}\text{-enriched } n\text{-cat} \in \mathcal{V}$

e.g. $\mathrm{sk}_{\mathcal{C}}(\mathbb{R}^3) = \mathrm{end}_{\mathcal{C}}(\mathbb{1})$

[$n=1$: AMR '17]
[$n \geq 2$: AFMR, w.i.p.]

A = Ayala
F = Francis
R = Rozenblyum

$\qquad \Psi \qquad\qquad\qquad \Psi$

generalized Jones poly.

$J^{(\vec{V}\in\mathcal{C})}_{(L^1\subset M^3)} := \int_{(L^1\subset M^3)} (\vec{V}\in\mathcal{C}) \quad \forall\ \pi_0(L^1) \xrightarrow[\text{labeling of path components}]{\vec{V}} \mathrm{obj}(\mathcal{C})$

<u>Toy version</u>: invariants for $L^0 \in M^1$, given $A \in \mathrm{Alg}_{\Bbbk} \ldots$

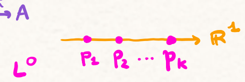
considered as a pointed $\Bbbk$-linear 1-cat.

$\int_{\mathbb{R}^1} A = A$

$\Psi \qquad \Psi$

$\int_{(L^0\subset\mathbb{R}^1)} (\vec{a}\in A) = a_1 \cdot a_2 \cdot \ldots \cdot a_k$

$\forall\ L^0 \xrightarrow{\vec{a}} A$



$A/[A,A] \in \mathrm{Mod}_{\Bbbk}$

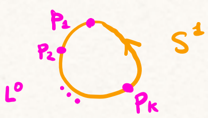
$H_0 \uparrow$

$\int_{S^1} A = HH(A) \in D_{\geq 0}(\mathrm{Mod}_{\Bbbk})$

$\Psi \qquad \Psi$

$\int_{(L^0\subset S^1)} (\vec{a}\in A) = [a_1 \cdot \ldots \cdot a_k]$

$\forall\ L^0 \xrightarrow{\vec{a}} A$



e.g. for $A = \mathrm{Mat}_{d\times d}(\Bbbk)$,

$HH(A) \cong \Bbbk$

$\Psi \qquad \Psi$

$[a_1 \cdot \ldots \cdot a_k] \leftrightsquigarrow \underbrace{\mathrm{tr}(a_1 \cdot \ldots \cdot a_k)}$

invariant under cyclic permutations

isotopies of $L^0 \subset S^1$

§2: Knot homologies

['99] Khovanov homology $Kh^{i,j}(L)$ $(i,j\in\mathbb{Z})$, a categorification

of Jones poly.: $J_L(q) = \chi_{gr}(Kh(L)) := \sum_{i,j} (-1)^i \cdot \mathrm{rk}(Kh^{i,j}) \cdot q^j$

✱ gen$^{\text{zns}}$: $\mathfrak{sl}_N$-homology (bigraded) $\xleftarrow[\text{['06 Rasmussen]}]{\text{spectral sequence}}$ HOMFLY-PT homology (triply-graded)

['04 Kh.-Rozansky] "$\mathfrak{sl}_\infty$" ['05 Kh.-Roz., Kh., Dunfield–Gukov–Rasmussen]

Major Q.: How to extend these to $L^1 \subset M^3$?

✱ approaches via physics (gauge thy, BPS states, ...): Witten ['11], Gaiotto–Witten ['11], Gukov–Pei–Putrov–Vafa ['17], Gukov–Manolescu ['19], Aganagic ['20], ...

Today's A.: seek braided $\Bbbk$-linear $(\infty,2)$-cat $\mathcal{C}$ with duals

cat$_{\Bbbk}$-enr. 3-cat.

$\rightsquigarrow \underbrace{Sk_{\mathcal{C}}(M^3)}_{\text{skein } \Bbbk\text{-linear } \infty\text{-category}} := \left(\int_{M^3} \mathcal{C}^{\mathcal{C}}\right) \ni \left(\int_{(L^1 \subset M^3)} (\vec{V} \in \mathcal{C})\right)$ ← gen'zd knot hlgy

$\forall\ \pi_0(L^1) \xrightarrow{\vec{V}} \mathrm{obj}(\mathcal{C})$

✱ for $\mathfrak{sl}_N$-hlgy (e.g. Kh (N=2)), want $\mathcal{C} = \mathrm{Rep}^{f.d.}(\mathcal{U}_q(\mathfrak{sl}_N))$

categorified quantum group: $\mathcal{U}_q(\mathfrak{g})$ a 2-cat [Chuang–Rouquier '04] [Lauda '08] [Khovanov–Lauda '08] ...

"2-"rep"ns" $:=$ fctrs $\mathcal{U}_q(\mathfrak{g}) \longrightarrow 2\mathrm{Vect} := \mathrm{Morita} := \begin{cases} \mathrm{obj} = \Bbbk\text{-alg's} \\ \mathrm{hom}(A,B) = \mathrm{Bim}_{(A,B)} \end{cases}$

Problem: No monoidal str. known – let alone braiding!

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§3: Main theorem: braiding on "$\mathrm{Rep}^{f.d.}(\mathcal{U}_q(\mathfrak{sl}_\infty))$"

§3.1: Background on HHH := HOMFLY-PT homology

$\Bbbk$ a field of char. 0, $n \geq 0$

$\mathrm{SBim}_n \subseteq {}_{g}\mathrm{Bim}_{\Bbbk[x_1,\dots,x_n]}$ ($|x_i| = 2$) the Soergel bimodules for S_n } additive $\Bbbk$-linear monoidal subcat.

$\otimes := \otimes_{\Bbbk[x_1,\dots,x_n]}$

$\mathcal{H}_n := K^b(\mathrm{SBim}_n)$ the Hecke (∞-)cat. for S_n } monoidal $\Bbbk$-linear stable ∞-cat.

$K^b(\mathrm{SBim}_n) \simeq D^b_{cbl}(B\backslash G/B)$, monoidal by convolution

Thm (Rouquier '04): a monoidal fctr $\mathrm{Br}_n \xrightarrow{F} \mathcal{H}_n$.

[Main Thm $\Rightarrow \mathcal{H}_n \simeq \mathrm{end}_{\mathcal{U}}(V_{fund.}^{\otimes n})$ and F categorifies $\mathrm{Br}_n \xrightarrow{f} \mathrm{end}_{\mathcal{C}}(V_{fund.}^{\otimes n})$]

Rmk.: A "deformation" of boring action: $\mathrm{Br}_n \xrightarrow{F} \mathcal{H}_n$

$\Big\downarrow \pi_\infty$ ← "underlying weak stable htpy type" (i.e. invert q.i.'s)

$$S_n \xrightarrow[\{x_1,\dots,x_n\} \curvearrowleft S_n]{} D^b(\mathrm{Bim}_{\Bbbk[x_1,\dots,x_n]}) .$$

Ex.: For $\sigma = \times \in Br_2$, $F(\sigma) := (\underbrace{R \otimes_{R^s} R}_{\text{degree } 0} \xrightarrow{\text{mult.}} R) \in K^b(\mathrm{SBim}_2)$,

$R := \Bbbk[x_1,x_2] \curvearrowleft S_2 = \{e,s\}$
$\cup$
$R^s = \Bbbk[x_1+x_2,\, x_1 \cdot x_2]$

boring! $\rightsquigarrow$ $H_i(F(\sigma)) \cong \begin{cases} \Bbbk[x_1,x_2] \overset{e}{\curvearrowright} \Bbbk[x_1,x_2] \overset{s}{\curvearrowleft} \Bbbk[x_1,x_2], & i=0 \\ \Bbbk[x_1,x_2] \curvearrowright 0 \curvearrowleft \Bbbk[x_1,x_2], & i\neq 0 \end{cases}$.

triple grading: q bims, ch cx., $\mathbb{R}^i$hom

$F(e) = \mathbb{1}$ β

Thm. (Kh. '05): $HHH(\bar{\beta}) := \mathbb{R}\mathrm{hom}_{\mathcal{H}_n}(\mathbb{1}, F(\beta))$ a well-defined link invariant, categorifying HOMFLY-PT poly.

§3.2: Main theorem

Recall: $\mathcal{H}_n := K^b(\mathrm{SBim}_n) \hookrightarrow K(\mathrm{Bim}_{\Bbbk[x_1,\dots,x_n]}) \xrightarrow{\Pi_\infty} D(\mathrm{Bim}_{\Bbbk[x_1,\dots,x_n]})$ "underlying weak stable htpy type"

Def.: $\mathcal{H} := \left\{ \begin{array}{l} \mathrm{ob} = \mathbb{N} \\ \mathrm{hom}(m,n) = \begin{cases} \mathcal{H}_n, & m=n \\ 0, & m\neq n \end{cases} \end{array} \right\}$ a "stable $\Bbbk$-linear $(\infty,2)$-cat" (hom's are st. $\Bbbk$-lin. $(\infty,1)$-cats)

"the Hecke 2-cat" (in type A)

Easy: $\mathcal{H}_i \times \mathcal{H}_j \xrightarrow{\boxtimes} \mathcal{H}_{i+j}$ $\rightsquigarrow$ monoidal str. on $\mathcal{H}$ ("parabolic induction")

s.t. $\Pi_\infty : \mathcal{H} \hookrightarrow K(\text{Morita}) \xrightarrow{\Pi_\infty} D(\text{Morita})$ ($n \longmapsto \Bbbk[x_1,\dots,x_n]$)

$\Bbbk[x_1,\dots,x_i] \otimes_{\Bbbk} \Bbbk[x_1,\dots,x_j] \cong \Bbbk[x_1,\dots,x_{i+j}]$

is monoidal.

$K(\text{Morita})$: $\left\{ \begin{array}{l} \mathrm{ob} = \Bbbk\text{-alg's} \\ \mathrm{hom}(A,B) = K(\mathrm{Bim}_{(A,B)}) \end{array} \right.$ $\otimes := \otimes_{\Bbbk}$

$D(\text{Morita})$: $\left\{ \begin{array}{l} \mathrm{ob} = \Bbbk\text{-alg's} \\ \mathrm{hom}(A,B) = D(\mathrm{Bim}_{(A,B)}) \end{array} \right.$ $\otimes := \otimes_{\Bbbk}$

∞-categorical uniqueness: contractible ∞-groupoid (but massive as a simplicial set!!)

Main Thm.: $\exists!$ braiding on $\mathcal{H}$ s.t.:

Cor: naturality of Rouquier's Br_n-actions, e.g.

in $2\mathrm{hom}_{\mathcal{H}}(\ast,\ast)$ i.e. $\mathrm{hom}_{\mathcal{H}_3}$

1. braiding $(1 \boxtimes 1 \to 1 \boxtimes 1) \in \mathrm{hom}_{\mathcal{H}}(2,2) := \mathcal{H}_2$ is the Rouquier complex $F(\times) := (R \otimes_{R^s} R \xrightarrow{\text{mult.}} R)$;

② $\mathcal{H} \xrightarrow{\Pi_\infty} D(\text{Morita})$ is braided. " $\Pi_\infty\left(\begin{smallmatrix}\text{braiding}\\ \text{on } \mathcal{H}\end{smallmatrix}\right) = $ boring (symmetric) braiding on $D(\text{Morita})$ "

Rmk.: Expect $\mathcal{H} \subset \mathrm{Rep}(``\mathcal{U}_q(\mathfrak{sl}_\infty)")$ full braided sub-$(\infty,2)$-cat. on $V_{\text{fund.}} := \mathbb{1} \in \mathcal{H}$. But $\mathcal{H}$ (and its braiding) <u>much</u> easier to construct!

↳ Q.: Comparison via Tannakian reconstruction?

Check: $2\mathrm{end}_{\mathcal{H}}(V_{\text{fund.}}) := \hom_{\mathcal{H}_1}(\mathbb{1},\mathbb{1}) = \mathbb{k}[x]$ ← inf.-dim^{l}

But still only for $L^1 \subset \mathbb{R}^3$: $V_{\text{fund.}} \in \mathcal{H}$ not 2-dualizable!

★ for <u>any</u> $X \xrightarrow{F} X$ in 2-cat $\mathcal{C}$, have

$\underbrace{2\hom(\mathrm{id}_X, F)}_{\text{e.g.}} \simeq \underbrace{2\hom(\mathrm{id}_{\mathbb{1}}, \mathrm{tr}(F))}_{\text{if } \mathcal{C} \text{ is monoidal and if } X \text{ is 2-dzbl!}}$

$2\hom_{\mathcal{H}}(\mathrm{id}_n, F(\beta)) \cong \hom_{\mathcal{H}_n}(\mathbb{1}, F(\beta))$

2-dzbility ⇒ functoriality w.r.t. embedded cobordisms

↳ Q.: $\mathfrak{sl}_\infty \rightsquigarrow \mathfrak{sl}_N$?

$\underset{\text{inf.-dim}^l}{\mathbb{k}[x]} \rightsquigarrow \underset{\text{f. dim}^l}{\mathbb{k}[x]/x^{N+1}}$

$\rightsquigarrow$ $V_{\text{fund.}} \in \mathcal{H}_{\mathfrak{sl}_N}$ should be 2-dualizable

$\rightsquigarrow$ hlgy of knots in 3-mflds!!!

$\mathrm{Cat}_{(\infty,1)} \ni \mathrm{Cob}^1(M^3) \xrightarrow{\mathfrak{sl}_N\text{-homology!}} \mathrm{Sk}_{\mathcal{H}_{\mathfrak{sl}_N}}(M^3) := \int_{M^3} \mathcal{H}_{\mathfrak{sl}_N}$

$\begin{cases} \text{ob's} = \text{links in } M^3 \\ \text{1-mor's} = \text{link. cob's in } M^3 \times [0,1] \\ \text{2-mor's} = \text{isotopies} \\ \vdots \qquad\qquad \vdots \end{cases}$

the $\mathfrak{sl}_N$ skein category of M^3

stable $\mathbb{k}$-linear $(\infty,1)$-

$(L^1 \subset M^3) \longmapsto \int_{(L^1 \subset M^3)} (V_{\text{fund.}} \in \mathcal{H}_{\mathfrak{sl}_N})$

E.g. for $N=2$, expect:

$$\begin{array}{ccccc}
\left\{\begin{smallmatrix}\text{gr}^{d}\\ \text{cxes}\end{smallmatrix}\right\} & \xleftarrow{\text{forget filtr}^n} & \left\{\begin{smallmatrix}\text{filt}^{d}\ \text{gr}^{d}\\ \text{cxes}\end{smallmatrix}\right\} & \xrightarrow{\text{gr}} & \left\{\begin{smallmatrix}\text{bigraded}\\ \text{cxes}\end{smallmatrix}\right\} \\
\wr\wr & & \wr\wr & & \cup \\
\mathrm{Sk}(\mathbb{R}^3) & \xleftarrow{\mathrm{Sk}(\mathbb{R}^3 \hookleftarrow \text{solid torus})} & \mathrm{Sk}\left(\begin{smallmatrix}\text{solid}\\ \text{torus}\end{smallmatrix}\right) & & \mathrm{Kh}^{\text{ann}}(L \subset \text{solid torus}) \\
\cup & & \cup & & \\
\mathrm{Kh}(L \subset \mathbb{R}^3) & \longleftarrow & \mathrm{Kh}(L \subset \text{solid torus}) & &
\end{array}$$

$\mathrm{Kh}(L) \in \left\{\begin{smallmatrix}\text{gr}^{d}\\ \text{cxes}\end{smallmatrix}\right\}$

annular Kh. hlgy

§3.3: Pf. idea

Main Thm.: $\exists!$ braiding on $\mathcal{H}$ s.t.: ① braiding $(1\boxtimes 1 \to 1\boxtimes 1) \in \hom_{\mathcal{H}}(2,2) := \mathcal{H}_2$ is the Rouquier complex $F(\times) := (R\otimes_{R^s} R \xrightarrow{\text{mult}} R)$; ② $\mathcal{H} \xrightarrow{\Pi_\infty} D(\text{Morita})$ is braided.

Rouquier's Pf. idea:

$$\begin{tikzcd}
Br_n \arrow[rr, "2:\text{ check this fact}^{2n}"] \arrow[dr, "3:\ \otimes \Rightarrow \text{ get this lifted mon. fctr!}", "\text{Thm.}"'] \arrow[ddr, bend right, "1:\text{ construct this monoid homomorphism}"'] & & S_n \arrow[dr, "\text{boring}"] & \\
& \mathcal{H}_n \arrow[rr, "\Pi_\infty", "\text{a ``partial covering space''!}"'] \arrow[d] & & D(Bim_{\Bbbk[x_1,\dots,x_n]}) \arrow[d] \\
& h_0(\mathcal{H}_n) \arrow[rr, "\Pi_\infty"] & & h_0(D(Bim_{\Bbbk[x_1,\dots,x_n]}))
\end{tikzcd}$$

$h_0 :=$ set of isoclasses of objects
(ignore all noninvertible 1-morphisms)

I.e.: If $L, L' \in \mathcal{H}_n$ $\otimes$-invertible s.t. $\exists$ equiv^{ce} $L \simeq L'$,

then $\hom_{\mathcal{H}_n}(L, L') \xrightarrow{\cong} \hom_D(\Pi_\infty(L), \Pi_\infty(L'))$.

$$\begin{array}{ccc}
\hom_{\mathcal{H}_n}(L,L') & \longrightarrow & \hom_D(\Pi_\infty(L),\Pi_\infty(L')) \\
\wr\| & & \wr\| \\
\hom_{\mathcal{H}_n}(\mathbb{1}, L^{-1}\otimes L') & \longrightarrow & \hom_D(\mathbb{1}, \Pi_\infty(L)^{-1}\otimes \Pi_\infty(L')) \\
\wr\| & & \wr\| \\
\hom_{\mathcal{H}_n}(\mathbb{1},\mathbb{1}) & \longrightarrow & \hom_D(\mathbb{1},\mathbb{1}) \\
\wr\| & & \wr\| \\
\Bbbk & \longrightarrow & \Bbbk
\end{array}$$

Pf. idea:

$$\begin{tikzcd}
\mathbb{E}_2 \arrow[rr, "2:\text{ check this fact}^{2n}"] \arrow[dr, squiggly, "3:\ \otimes \Rightarrow \text{ get this lifted action!}", "\text{Thm.}"'] \arrow[ddr, squiggly, "1:\text{ construct this action}"'] & & \mathbb{E}_\infty = \text{Comm} \arrow[dr, squiggly, "\text{boring}"] & \\
& \mathcal{H} \arrow[rr, "\Pi_\infty", "\text{a ``partial covering space''!}"'] \arrow[d] & & D(\text{Mor}) \arrow[d] \\
& h_1(\mathcal{H}) \arrow[rr, "\Pi_\infty"] & & h_1(D(\text{Mor}))
\end{tikzcd}$$

$h_1 :=$ homotopy $(1,1)$-cat
(ignore all noninvertible 2-morphisms)

$\rightsquigarrow \quad \hom_{h_1(\mathcal{H})}(n,n) := h_0(\mathcal{H}_n)$

Thanks for listening!

THESE SLIDES: etale.site/writing/utrecht-4-cat.pdf

PREPRINT: "A Braided $(\infty,2)$-Category of Soergel Bimodules" arXiv:2401.02956

EXIT: etale.site/exit.html

APPLICATION INSPIRATION: Barwick, "The Future of Homotopy Theory"
maths.ed.ac.uk/~cbarwick/papers/future.pdf

OPEN DIRECTIONS:

1. categorical context for <u>triply</u> graded HHH
2. Tannakian reconstruction of $\mathcal{U}_q(\mathfrak{sl}_\infty)$
3. $\mathcal{U}_q(\mathfrak{sl}_\infty) \rightsquigarrow \mathcal{U}_q(\mathfrak{sl}_N)$ for $\mathfrak{sl}_N$-homology (e.g. Khovanov for $N=2$)
4. lift from $\mathbb{Q}$ to $\mathbb{S}$ (or MU?) using flag varieties & Schubert stratifications
5. general deformation theory for "$\underset{\mathbb{E}_2}{K(\mathrm{SBim})} \to \underset{\mathbb{E}_\infty}{D(\mathrm{SBim})} \simeq \Bbbk[\mathrm{Fin}^{\simeq}]$", analogous to $\underset{\mathrm{Assoc}}{H_n} \xrightarrow{q \mapsto 1} \underset{\mathrm{Comm}}{\Bbbk[S_n]}$