

An Introduction to Moduli Spaces of Riemann Surfaces

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Motivation

Let's write S_g for the genus- g closed oriented surface. If we cut out n open disks, we get a compact surface with boundary that we will denote $S_{g,n}$. These surfaces have interesting automorphisms. We will focus on $Diff^+ S_g$, the orientation-preserving diffeomorphisms of S_g , and $Diff^+ S_{g,n}$, the orientation-preserving diffeomorphisms of $S_{g,n}$ that are the identity near the boundary.

Certainly we can wiggle our surfaces to obtain diffeomorphisms that are isotopic to the identity. But there are other diffeomorphisms too. For example, there is the *Dehn twist*. (Look up a picture on The Internet.) This is obtained by cutting out a small cylinder, twisting it around a bunch of times (while fixing a neighborhood of its boundary), and then gluing it back in.

Our goal is to compute $H^*(B(Diff^+(S_{g,n})))$.

Remarkable fact. For $g \geq 2$, the connected component $Diff_{Id}^+ S_g$ of the identity in $Diff^+ S_g$ is contractible.

Definition 1. The mapping class group $MCG(S_g)$ (also denoted Γ_g) is $\pi_0 Diff^+(S_g)$.

Thanks to our remarkable fact, $BDiff^+ S_g \simeq B\Gamma_g$ for $g \geq 2$. On the other hand, $H^*(\Gamma_g)$ is very hard to compute (and not for lack of trying – people have been working on this for about a century).

Given $S_{g,1}$, we can adjoin $S_{1,2}$ (a torus with two disks removed) to obtain $S_{g+1,1}$. This induces a map $Diff^+ S_{g,1} \rightarrow Diff^+ S_{g+1,1}$ (by extending by the identity).

Remarkable fact (Harer stability, simple version). $H^k(\Gamma_{g,1}; \mathbb{Z}) \cong H^k(\Gamma_{g+1,1}; \mathbb{Z})$ if $g \geq 3k - 1$.

This is a really hard theorem.

Consider $\Gamma_\infty = \text{colim}_{g \rightarrow \infty} \Gamma_{g,1}$. Mumford conjectured (in algebro-geometric language) the following.

Conjecture (Mumford). $H^*(\Gamma_\infty; \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \dots]$.

Mumford had in mind an analogy these mapping class group problems with classifying spaces; the idea is that there should be certain canonical classes coming from certain canonical bundles. These are called *Miller-Morita-Mumford classes* (or *tautological classes*). Let's define κ_1 . We take the canonical S_g -bundle

$$\begin{array}{ccc} S_g & \longrightarrow & X \\ & & \downarrow \pi \\ & & BDiff^+ S_g \end{array}$$

If we pick a point in X , it lives on some copy of S_g , and so we obtain the relative tangent bundle, which gives us a fibration

$$\begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & T_{X/BDiff^+ S_g} \\ & & \downarrow \\ & & X \end{array}$$

This gives us e , the Euler class of $T_{X/BDiff}$. We define $\kappa_1 = \pi_!(e^{i+1})$, the pushforward.

Proposition (Tillmann, early 2000s). $\mathbb{Z} \times B\Gamma_\infty$ is an infinite loop space.

Let us write $Bord_n^{or}$ for the (∞, n) -category of bordisms of oriented n -manifolds. There is a notion of geometric realization of such a category.

Theorem (Galatius-Madsen-Tillmann-Weiss). $|Bord_n^{or}|$ is homotopy equivalent to the 0-space of the spectrum $\Sigma^n MTSO(n)$, where $MTSO(n)$ is the Thom spectrum for the virtual bundle $-\gamma$, where γ is the universal bundle over $BSO(n)$.

This is waaaaay better than Mumford’s conjecture.

Geometry

Definition 2. A Riemann surface Σ is a complex manifold with $\dim_{\mathbb{C}} \Sigma = 1$. Equivalently, Σ is an oriented real 2-manifold with coordinate chart $\varphi_{\alpha} : U_{\alpha} \rightarrow \mathbb{C}$ such that the transition maps $g_{\alpha\beta}$ are holomorphic.

Example. Examples include $\mathbb{C}P^1$, open sets in \mathbb{C} , \mathbb{C}/Λ where $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$ is a lattice (so $\tau \in \mathbb{C} - \mathbb{R}$).

Question. Let’s classify (compact) Riemann surfaces. More precisely, let’s describe the “moduli space” $\mathcal{M}_g = \{\text{genus-}g \text{ Riemann surfaces}\}/\text{iso.}$

This originally started in algebraic geometry. For example, at $g = 1$ we obtain things like elliptic cohomology in topology and modular forms in number theory. This led to the introduction of stacks. The notion of “derived” algebraic geometry was developed primarily to study \mathbf{tmf} . Also related is analytic geometry; Kodaira and Spencer introduced deformation theory into geometry to understand \mathcal{M}_g . Even further, geometric topology starts to look at 3-manifolds, Fuchsian groups, etc. String theory even comes out of this circle of idea.

From our point of view, \mathcal{M}_g shows up because we want to understand mapping class groups, and the best way to study groups is to find objects on which they act very naturally.

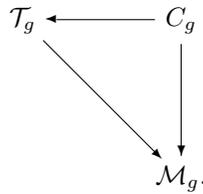
Definition 3. An almost-complex structure on a manifold M is a section $J \in \Gamma(M, \text{End}(T_m))$ such that $J^2 = -1$.

(In two dimensions, this is the same as a complex structure! This is false in higher dimensions; the Newlander-Nirenberg theorem gives that the Nijenhuis tensor is a complete local invariant: it vanishes iff our almost-complex structure comes from a complex structure. But this is a 4-tensor, which agrees with the fact that all almost-complex 2-manifolds are complex.)

Suppose we have a smooth oriented surface M and we pick a point $p \in M$. If we have a metric on M , it automatically gives us an almost-complex structure: we send $v \in T_p M$ to the orthogonal vector of the same length such that $\{v, Jv\}$ is an oriented basis of $T_p M$. Thus we can get from oriented Riemannian surfaces to 2-manifolds with almost-complex structure.

Now, write C_g for the space of complex structures on S_g . This is convex, so it is contractible. $\text{Diff}^+ S_g$ acts naturally on this space. Hence $\mathcal{M}_g = C_g / \text{Diff}^+ S_g$.

Definition 4. We define Teichmüller space to be $\mathcal{T}_g = C_g / \text{Diff}_1^+ S_g$. This fits into a diagram



Remarkable fact. $\mathcal{T}_g \cong \mathbb{R}^{6g-6}$ as real manifolds.

This fact comes from Fenchel-Nielsen coordinates. We will assume $g \geq 2$. Observe that we have a correspondence between conformal-structures-with-orientation and complex structures. (A conformal structure is an equivalence class of metrics that yield the same angles between tangent vectors.) In 2 dimensions, a conformal structure is equivalent to a complex structure!

There is a deep theorem, called the uniformization theorem, which tells us that every simply-connected Riemann surface is biholomorphic to either $\mathbb{C}P^1$, \mathbb{C} , or $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. So we can look at the universal cover of any Riemann surface and it will have to be biholomorphic to one of these. It turns out that if the universal cover is $\mathbb{C}P^1$ then the surface has genus 0, if the universal cover is \mathbb{C} then the surface has genus 1, and if the universal cover is \mathbb{D} then the surface has genus ≥ 2 . Under our bijective correspondence between conformal and complex structures,

every simply-connected Riemannian 2-manifold is conformally equivalent to S^2 (with the +1 constant curvature metric), \mathbb{R}^2 (with the flat metric), or \mathbb{D} (with the hyperbolic metric).

So for $g \geq 2$, instead of talking about complex structures for C_g we can talk about hyperbolic metrics. Now, given S_g , we can cut it up into a *pants decomposition*. (Google it.) This is determined by the simple closed curves that are the boundaries of the pants. Then, given a metric we can choose geodesics in each of these isotopy classes; it is not hard to see that these will be nonintersecting.

Cool fact. *Every pair of pants with geodesic boundary is determined by the lengths of the cuffs and the waist size. (You just cut along the side seams and the crotch line to get some chaps, and then by a direct hyperbolic geometry argument, this hyperbolic hexagon is totally determined.)*

So our Fenchel-Nielsen coordinates first contain the lengths of the cuffs/waistline, which are always positive. Then there's the question of how we glue the pants together. At first it might like seem like there are 2π worth of gluings, but recall that Dehn twists are nontrivial: this means that actually there's an \mathbb{R} 's worth of choices for each gluing, and so we obtain a fibration

$$\begin{array}{ccc} \mathbb{R}^{3g-3} & \longrightarrow & \mathcal{T}_g \\ & & \downarrow \\ & & \mathbb{R}_+^{3g-3} \end{array}$$

Of course this must be trivial, so $\mathcal{T}_g \cong \mathbb{R}^{6g-6}$.

References:

- Ulrike Tillmann, *Mumford Conjecture*.
- Allen Hatcher, *Short Exposition of Madsen-Weiss*.
- Lurie, *An Expository Article on Tautological Theories*.