Prelim Workshop
Summer 2018

Algebra Worksheet 6: Linear Algebra III

Symmetric/Hermitian matrices

- Positive (semi-)definiteness
  1. Characterization by eigenvalues (7.9.6)
  2. Characterization by factorization (7.5.34)
  3. Submatrix criterion ("Sylvester's Criterion") ($*$)
- Sylvester's law of interia, signature of a quadratic form
- Eigenvalues of a hermitian/symmetric/orthogonal/unitary matrix

Inner product spaces

- Working with orthonormal bases, Gram-Schmidt (7.8.5)
- Polarization identity (7.8.1)
- Rayleigh's theorem (7.5.33)

The matrix exponential (7.4.31)

- Proving convergence
- Trick: Jordan form/triangularization (7.9.24)

7.4.31 • Let V be the vector space of all polynomials of degree ≤ 10 and let D be the differentiation operator on V (i.e. $Dp(x) = p'(x)$). Show that tr $D = 0$ and find all eigenvectors of D and e^D.

7.5.33 • Let k be real, n an integer ≥ 2, and let $A = (a_{ij})$ be the $n \times n$ matrix such that all the diagonal entries $a_{ii} = k$, all entries $a_{i,i\pm1}$ immediately above or below the diagonal equal 1, and all other entires equal 0. For example, if $n = 5$,

$$A = \begin{pmatrix} k & 1 & 0 & 0 & 0 \\ 1 & k & 1 & 0 & 0 \\ 0 & 1 & k & 1 & 0 \\ 0 & 0 & 1 & k & 1 \\ 0 & 0 & 0 & 1 & k \end{pmatrix}.$$

Let $\lambda_{\min}$ and $\lambda_{\max}$ denote the smallest and largest eigenvalues of A, respectively. Show that $\lambda_{\min} \leq k - 1$ and $\lambda_{\max} \geq k + 1$.

7.5.34 • Let A and B be real $n \times n$ symmetric matrices with B positive definite. Conider the function defined for $x \neq 0$ by $G(x) = \frac{\langle Ax,x \rangle}{\langle Bx,x \rangle}$.

1. Show that G attains its maximum value.
2. Show that any maximum point U for G is an eigenvector for a certain matrix related to A and B and show which matrix.

7.8.1 • Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that

1. the function g defined by $g(x, y) = f(x + y) - f(x) - f(y)$ is bilinear and
2. for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $f(tx) = t^2 f(x)$.

Show that there is a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(x) = \langle x, Ax \rangle$, where $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$ (in other words, f is a quadratic form).

7.8.5 • Let w be a positive continuous function on $[0, 1]$, n a positive integer, and P_n the vector space of real polynomials whose degrees are at most n, equipped with the inner product

$$\langle p, q \rangle = \int_0^1 p(t)q(t)w(t)\, dt.$$

1. Prove that P_n has an orthnormal basis $p_0, p_1, ..., p_n$ such that deg $p_k = k$ for each k.
2. Prove that $\langle p_k, p_k' \rangle = 0$ for each k.

7.9.6 • A real symmetric $n \times n$ matrix is called *positive semi-definite* if $x^t Ax \geq 0$ for all $x \in \mathbb{R}^n$. Prove that A is positive semi-definite if and only if tr $AB \geq 0$ for every real symmetric positive semi-definite $n \times n$ matrix B.

7.9.24 • Show that

$$\det(\exp(M)) = e^{\mathrm{tr}(M)}$$

for any complex $n \times n$ matrix M.

($*$) • "Sylvester's Criterion": Given a symmetric matrix $A = (a_{ij})_{1 \leq i,j \leq n} \in M_n(\mathbb{R})$, let A_k denote the upper left submatrix $A_k = (a_{ij})_{1 \leq i,j \leq k}$.

1. Prove by induction on n that A is positive definite if and only if $\mathrm{Det}(A_k) > 0$ for $k = 1, ..., n$.

2. Prove that the analogous statement fails for positive semi-definite matrices. That is, find n and $A \in M_n(\mathbb{R})$ symmetric such that $\mathrm{Det}(A_k) \geq 0$ for all $1 \leq k \leq n$, but $v^t A v < 0$ for some $v \in \mathbb{R}^n$.