Prelim Workshop
Summer 2018

Algebra Worksheet 4: Linear Algebra I

Bases and dimension

- Any independent set can be completed to a basis, any spanning set contains a basis
- Recognizing linear constraints. Recognizing dimension questions. (7.1.1, $*$)

Rank-nullity theorem

Tricks for computing rank

- Find nonsingular submatrix (7.2.7)

Tricks for computing determinants

- Induction (7.2.11, $**$)
- Vandermonde matrix (7.2.12)
- Add a multiple of a row/column to another row/column
- Find eigenvalues instead ($***$)

Characteristic and minimal polynomial

- Cayley-Hamilton theorem
- Minimal polynomial divides characteristic polynomial. Characteristic polynomial divides power of minimal polynomial. (What is the smallest power that must work?) Implications for eigenvalues. (7.5.3)
- Use minimal polynomial of $A \in M_n(F)$ to reduce polynomials in A with coefficients in F. What is the dimension as an F vector space of this ring? (7.6.5)
- Read off trace and determinant from characterstic polynomial

7.1.1 • Let p, q, r and s be polynomials of degree at most 3. Which, if any, of the following two conditions is sufficient for the conclusion that the polynomials are linearly dependent?

1. At 1 each of the polynomials has value 0.

2. At 0 each of the polynomials has the value 1.

7.2.7 • Let T be a real, symmetric, $n \times n$ *tridiagonal* matrix:

$$T = \begin{pmatrix} a_1 & b_1 & 0 & \cdots & 0 & 0 \\ b_1 & a_2 & b_2 & \cdots & 0 & 0 \\ 0 & b_2 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & 0 & b_{n-1} & a_n \end{pmatrix}$$

Assume $b_j \neq 0$ for all j. Prove

1. that rank $T \geq n-1$ and

2. that T has n distinct eigenvalues.

7.2.11 • Let $\mathbb{R}[x_1, ..., x_n]$ be the polynomial ring over the real field $\mathbb{R}$ in the n variables $x_1, ..., x_n$. Let the matrix A be the $n \times n$ matrix whose ith row is $(1, x_i, x_i^2, ..., x_i^{n-1})$, $i = 1, ..., n$. Show that

$$\text{Det A} = \prod_{i>j} (x_i - x_j).$$

7.2.12 • A matrix of the form

$$T = \begin{pmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^n \\ 1 & a_1 & a_1^2 & \cdots & a_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^n \end{pmatrix}$$

where the a_i are complex numbers, is called a Vandermonde matrix.

1. Prove that the Vandermonde matrix is invertible if the a_i are all different. (Do not use 7.2.11.)

2. In this case, prove that for any n complex numbers $b_0, ..., b_n$, there exists a unique complex polynomial f of degree n such that $f(a_i) = b_i$ for $i = 0, ..., n$.

For applications of the Vandermonde matrix, see 7.2.5 and 7.2.14.

7.5.3 • Let F be a field, n and m positive integers, and A an $n \times n$ matrix with entries in F such that $A^m = 0$. Prove that $A^n = 0$.

7.6.5 • Compute A^{10} for the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}.$$

(∗) • (*[L] 15.4.6*) Prove that every polynomial $f \in \mathbb{R}[x]$ has a nonzero multiple such that every exponent is prime. That is, $0 \neq g = f \cdot h$,

$$g = \sum_{\text{p prime}} a_p x^p$$

for some coefficients a_p.

(∗∗) • Assume that $a^2 - 4bc \neq 0$. Compute the determinant of the tridiagonal matrix that has a on the main diagonal, b on the first above-diagonal, and c on the first below-diagonal:

$$T = \begin{pmatrix} a & b & 0 & \cdots & 0 & 0 \\ c & a & b & \cdots & 0 & 0 \\ 0 & c & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & 0 & c & a \end{pmatrix}$$

(∗ ∗ ∗) • (*[L] 6.3.4(a)*) Compute the determinant of the following matrix.

$$\begin{pmatrix} \beta & \alpha & \alpha & \cdots & \alpha & \alpha \\ \alpha & \beta & \alpha & \cdots & \alpha & \alpha \\ \alpha & \alpha & \beta & \cdots & \alpha & \alpha \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha & \alpha & \alpha & \alpha & \beta & \alpha \\ \alpha & \alpha & \alpha & \alpha & \alpha & \beta \end{pmatrix}$$

[L] *Discover Linear Algebra* (draft) by Laszlo Babai. http://people.cs.uchicago.edu/ laci/linalgbook.dir/book.pdf