Prelim Workshop
Summer 2018

Analysis Worksheet 4: Real Analysis II

1. Prove that a real valued C^3 function f on $\mathbb{R}^2$ whose Laplacian,
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2},$$
is everywhere positive, cannot have a local maximum.

   SdS 2.2.12, Fa88. Taylor expansion.

2. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, 0) = 0$ and
$$f(x, y) = \left(1 - \cos \frac{x^2}{y}\right) \sqrt{x^2 + y^2}$$
for $y \neq 0$.
1) Show that f is continuous at $(0, 0)$.
2) Calculate all the directional derivatives of f at $(0, 0)$.
3) Show that f is not differentiable at $(0, 0)$.

   SdS 2.2.3, Sp03. Definition of continuity, differentiability.

3. Let $M_{2\times 2}$ be the space of 2×2 matrices over $\mathbb{R}$, identified in the usual way with $\mathbb{R}^4$. Let the function F from $M_{2\times 2}$ into $M_{2\times 2}$ be defined by
$$F(X) = X + X^2.$$
Prove that the range of F contains a neighborhood of the origin.

   SdS 2.2.43, Sp96. Inverse Function Theorem.

4. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable. Assume the Jacobian matrix $(\partial f_i / \partial x_j)$ has rank n everywhere. Suppose f is proper; that is, $f^{-1}(K)$ is compact whenever K is compact. Prove $f(\mathbb{R}^n) = \mathbb{R}^n$.

   SdS 2.2.8, Sp80, Fa92. Closed and open. Also see Hadamard Global Inverse Thm.

5. Let f be a continuous real valued function on $\mathbb{R}$ such that
$$f(x) = f(x + 1) = f(x + \sqrt{2})$$
for all x. Prove that f is constant.

   SdS 1.7.4, Sp86. Fourier series.

6. Prove or supply a counterexample: If f and g are C^1 real valued functions on $(0, 1)$, $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$, $g, g' \neq 0$, and $\lim_{x\to 0} f(x)/g(x) = c$, then
$$\lim_{x\to 0} f'(x)/g'(x) = c.$$

   SdS 1.4.14, Fa83, Fa84. L'Hopital's Rule.