Prelim Workshop
Summer 2018

Analysis Worksheet 3: Real Analysis I

1. 1) Prove that there is no continuous map from the closed interval $[0,1]$ onto the open interval $(0,1)$.
   2) Find a continuous surjective map from the open interval $(0,1)$ onto the closed interval $[0,1]$.
   3) Prove that no map in Part 2 can be bijective.

   SdS 1.1.10, Fa82. Compactness properties, Intermediate Value Theorem.

2. Let f be a C^2 function on the real line. Assume f is bounded with bounded second derivative. Let
$$A = \sup_{x\in\mathbb{R}} |f(x)|, \quad B = \sup_{x\in|R} |f''(x)|.$$
   Prove that
$$\sup_{x\in\mathbb{R}} |f'(x)| \leq 2\sqrt{AB}.$$

   SdS 1.1.17, Fa84, Fa97. Taylor expansion.

3. Let x_n be a sequence of real numbers so that $\lim_{n\to\infty} 2x_{n+1} - x_n = x$. Show that $\lim_{n\to\infty} x_n = x$.

   SdS 1.3.8, Sp03. Limsup, liminf.

4. Let a and x_0 be positive numbers, and define the sequence $(x_n)_{n=1}^{\infty}$ recursively by
$$x_n = \frac{1}{2}\left(x_{n-1} + \frac{a}{x_{n-1}}\right).$$
   Prove that this sequence converges, and find its limit.

   SdS 1.3.9, Sp00. Recurrence relations.

5. Suppose f is a continuous real valued function. Show that
$$\int_0^1 f(x)x^2\,dx = \frac{1}{3}f(\xi)$$
   for some $\xi \in [0,1]$.

   SdS 1.5.3, Fa90. Intermediate Value Theorem.

6. Let $0 \leq a \leq 1$ be given. Determine all nonnegative continuous functions f on $[0,\ 1]$ which satisfy the following three conditions:
$$\int_0^1 f(x)\,dx = 1, \qquad \int_0^1 xf(x)\,dx = a, \qquad \int_0^1 x^2 f(x)\,dx = a^2.$$

   SdS 1.5.9, Fa85. Cauchy-Schwarz or Jensen's Inequalities.