Augmentations are sheaves (with Lenhard Ng, Dan Rutherford, Vivek Shende, and Eric Zaslow)

Steven Sivek

Princeton University

April 19, 2015

Singular support

A sheaf $\mathcal{F}$ on M is constructible if M is a finite union of locally closed sets U_i such that $\mathcal{F}|_{U_i}$ is locally constant.

Definition (Kashiwara-Schapira)

To define the microlocal stalk of $\mathcal{F}^\bullet$ at $(x, \xi) \in T^*_x M$:

- Let $f : U_x \to \mathbb{R}$ be Morse near x with $f(x) = 0$, $df_X = \xi$.
- Then

$$\mu_{x,\xi}\mathcal{F}^\bullet = H^*_{\mathrm{Morse}}(U_x^{(-\infty,\epsilon]}, U_x^{(-\infty,-\epsilon]}; \mathcal{F}^\bullet).$$

- The singular support $ss(\mathcal{F}^\bullet) \subset T^*M$ is defined as

$$ss(\mathcal{F}^\bullet) = \{(x, \xi) \in T^*M \mid \mu_{x,\xi}\mathcal{F}^\bullet \neq 0\}.$$

- This is a closed, conic Lagrangian.

Microlocalization

Theorem (Nadler-Zaslow, Nadler '06)

There is an equivalence of categories

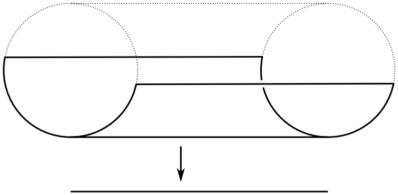
$$\mu : D_c(M) \xrightarrow{\sim} DFuk(T^*M)$$

where $D_c(M)$ is the derived category of constructible sheaves on M.

- What about a relative version?
- $T^\infty M$ is naturally a contact manifold. Given a Legendrian Λ, we have:
  - sheaves with microlocal support along Λ at infinity;
  - Lagrangians in T^*M which are asymptotic to Λ at infinity.

Sheaves and Legendrian knots [STZ]

Fix $M = \mathbb{R}^2$ and embed $(\mathbb{R}^3, \xi_{\text{std}}) \hookrightarrow T^\infty \mathbb{R}^2$ as the bottom half of the cotangent fibers.



Definition (Shende-Treumann-Zaslow)

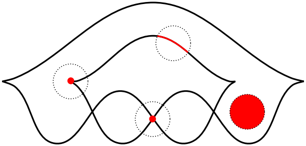
Given Legendrian $\Lambda \subset \mathbb{R}^3$, let $Sh^\bullet_{\mathbb{R}^2}(\Lambda; r)$ be the dg-category of complexes of constructible sheaves of r-modules on $\mathbb{R}^2$ with singular support at infinity contained in $\Lambda \subset T^{\infty,-}\mathbb{R}^2$.

Theorem (STZ): $Sh^\bullet_{\mathbb{R}^2}(\Lambda; r)$ is an invariant of Λ up to Legendrian isotopy.

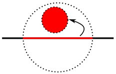
Sheaves on $\mathbb{R}^2$ [STZ]

What does the definition of $Sh^\bullet_{\mathbb{R}^2}(\Lambda; r)$ mean?

- The front projection of Λ stratifies $\mathbb{R}^2_{xz}$ into {cusps and crossings}, {strands}, {everything else}.
- A sheaf $\mathcal{F}$ should be locally constant on each stratum.
- Sections $\mathcal{F}(U)$ are determined by the smallest stratum intersecting U:



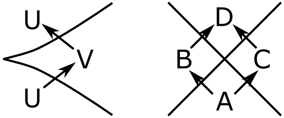
- Restriction maps point from a component of a stratum into the higher-dimensional strata:



The singular support condition [STZ]

The fact that $ss(\mathcal{F}) \subset T^{\infty,-}\mathbb{R}^2$ means that all *downward*-pointing restriction maps are isomorphisms.

- Sections of $\mathcal{F}$ on each stratum are determined by sections on open regions.
- Restriction maps are determined by the *upward* restriction maps.
- At cusps and crossings:



1. $U \to V \to U$ is the identity;
2. $A \to B \oplus C \to D$ is acyclic.

Note: $\mathrm{Cone}(U \to V) \cong \mathrm{Cone}(V \to U)$ up to a degree shift, and $\mathrm{Cone}(A \to B) \cong \mathrm{Cone}(C \to D)$.

Microlocal rank [STZ]

The cones of the upward restriction maps are quasi-isomorphic all along the front projection.

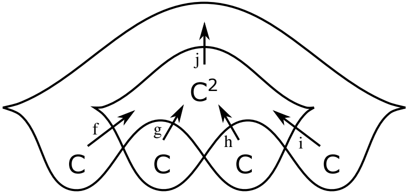
- They determine a local system of complexes on Λ, the microlocal monodromy $\mu\mathrm{mon}(\mathcal{F})$.
- If $\mu\mathrm{mon}(\mathcal{F})$ is supported in degree zero, its rank is the microlocal rank of $\mathcal{F}$.

Definition

$\mathcal{C}_1(\Lambda; r) \subset Sh^{\bullet}_{\mathbb{R}^2}(\Lambda; r)$ is the full subcategory of sheaves with microlocal rank 1 whose stalks vanish for $z \ll 0$.

- $\mathcal{C}_1(\Lambda; r)$ is a dg-category and a Legendrian isotopy invariant of Λ up to equivalence.

Example: the trefoil



- Images of f, g, h, i and $\ker(j) \leftrightarrow$ points $f, g, h, i, j \in \mathbb{CP}^1$.
- Cusp conditions: $f \neq j$, $i \neq j$.
- Crossing conditions: $f \neq g$, $g \neq h$, $h \neq i$.

So $\mathrm{Ob}(\mathcal{C}_1(\Lambda;\mathbb{C})) \cong \{(a_0, \ldots, a_4) \in (\mathbb{CP}^1)^5 \mid a_i \neq a_{i+1 \pmod 5}\}$.
For $\mathcal{C}_1(\Lambda;\mathbb{F}_2)$: exactly five objects, up to equivalence.

Conjecture (STZ)

$\mathcal{C}_1(\Lambda;k) \cong Aug(\Lambda;k)$.

The augmentation category

Motivation (or hindsight?) for conjecturing $\mathcal{C}_1(\Lambda; k) \cong Aug(\Lambda; k)$: $Aug(\Lambda; k)$ is an algebraic version of relative Fukaya category.

- $\mathrm{Ob}(Fuk(T^*\mathbb{R}^2, \Lambda))$: Lagrangians L asymptotic to Λ at infinity
- $\mathrm{Ob}(Aug(\Lambda; k))$: augmentations $\epsilon : \mathcal{A}(\Lambda) \to k$.
  - Thm (Ekholm-Honda-Kálmán): L induces an augmentation ϵ_L.

- $\mathrm{Hom}_{Fuk}(L_1, L_2)$: Lagrangian Floer chain complex
- $\mathrm{Hom}_{Aug}(\epsilon_1, \epsilon_2)$: bilinearized contact chain complex of Λ.
  - Thm: $H^*\mathrm{Hom}_{Aug}(\epsilon_L, \epsilon_L) \cong H^*(L) \cong HF^*(L, L)$.

Theorem (Ng-Rutherford-Shende-S-Zaslow)

$\mathcal{C}_1(\Lambda; k) \cong Aug(\Lambda; k)$.

Aug=Sh: basic strategy

- $\mathcal{C}_1(\Lambda; k)$ consists of sheaves, which satisfy a gluing axiom.
- We can define $\mathcal{C}_1$ over arbitrary open sets in $\mathbb{R}^2$, and they glue together tautologically to produce the full $\mathcal{C}_1$.
- In other words: $\mathcal{C}_1(\Lambda; k)$ is itself a sheaf! (of dg-categories on $\mathbb{R}^2$)
- We'll restrict $\mathcal{C}_1$ to vertical strips $U \times \mathbb{R} \subset \mathbb{R}_{xz}$, so $\mathcal{C}_1(\Lambda; k)$ is a sheaf of dg-categories over $\mathbb{R}_x$.

Proposition

$Aug(\Lambda; k)$ is also a sheaf of A_∞-categories over $\mathbb{R}_x$.

Then we can break $\mathbb{R}$ into pieces where Λ is very simple, compare $\mathcal{C}_1$ to *Aug* for each of them, and glue back together.

Aug is a sheaf

Since $Aug(\Lambda)$ is determined by the (fully non-commutative) DGA $\mathcal{A}(\Lambda)$, it will suffice to prove that $\mathcal{A}(\Lambda)$ is a cosheaf of DGAs over $\mathbb{R}_x$. This means:

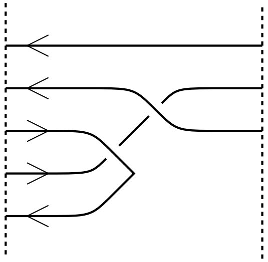
- To each open $U \subset \mathbb{R}_x$ we assign a DGA $\mathcal{A}(\Lambda|_U)$.
- For $U \subset V$ we have inclusions $\iota_{UV} : \mathcal{A}(\Lambda|_U) \to \mathcal{A}(\Lambda|_V)$.
- If $W = U \cup_X V$ then we have a pushout square

$$\begin{array}{ccc} \mathcal{A}(\Lambda|_X) & \longrightarrow & \mathcal{A}(\Lambda|_V) \\ \downarrow & & \downarrow \\ \mathcal{A}(\Lambda|_U) & \longrightarrow & \mathcal{A}(\Lambda|_W). \end{array}$$

Dualizing this will produce the (pre-)sheaf of categories $Aug(\Lambda)$.

$\mathcal{A}(\Lambda)$ is a cosheaf

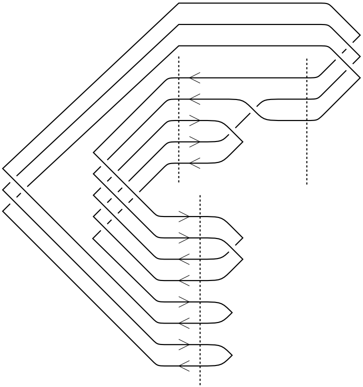
We associate algebras to "bordered" tangles as follows:



- Generators: crossings, right cusps, plus a_{ij} for $1 \leq i < j \leq n_L$.
    - The a_{ij} represent chords from the ith endpoint on the left to the jth.
- ∂c and the inclusion map from the right count properly embedded disks whose x-coordinate has a unique local maximum at c.
    - $\partial a_{ij} = \sum_{i<k<j} (-1)^{\sigma} a_{ik} a_{kj}$.

$\mathcal{A}(\Lambda)$ is a cosheaf, continued

We prove that $\partial^2 = 0$ and that inclusion maps are DGA morphisms by staring at:

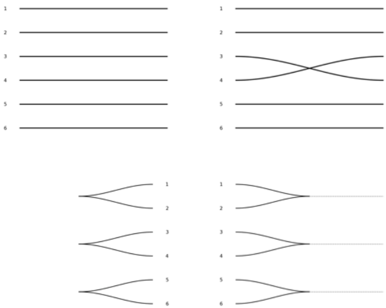


Pushouts: maps send crossings/cusps to themselves, uniquely specifying

$$\mathcal{A}(\Lambda|_U) \to \mathcal{A}(\Lambda|_{U \cup_X V}) \leftarrow \mathcal{A}(\Lambda|_V).$$

Simple pieces of a front

After putting a knot in plat position, we can break it into:



So we'll only have to study $\mathcal{C}_1$ and Aug for "strands", a crossing, "left cusps", and "right cusps".

Simplest piece: n strands

The DGA $\mathcal{A}(\equiv_n;\mu)$ has generators a_{ij}, $1 \le i < j \le n$, and (over $\mathbb{F}_2$)

$$\partial a_{ij} = \sum_{i<k<j} a_{ik}a_{kj}.$$

- $\mathrm{Ob}(Aug(\equiv))$: augmentations $\epsilon : \mathcal{A} \to \mathbb{F}_2$. Packaging generators into an upper triangular $A = (a_{ij})$, we have $\partial \circ \epsilon = 0$ iff $\epsilon(A)^2 = 0$.
- $\mathrm{Hom}_{Aug}(\epsilon, \epsilon')$: generated over $\mathbb{F}_2$ by a_{ij}^+, $1 \le i \le j \le n$.
- Multiplication: $m_2(a_{kj}^+, a_{ik}^+) = a_{ij}^+$ up to sign.
- Define the Morse complex category $MC(\equiv;\mu)$:
  - Let V be the free graded r-module with $\deg |i\rangle = -\mu(i)$, filtration

$$^kV = \mathrm{Span}(|n\rangle, \ldots, |k+1\rangle).$$

  - $\mathrm{Ob}(MC)$: filtered, degree-1 differentials $d : V \to V$.
  - $\mathrm{Hom}_{MC}(d, d') = \mathrm{Hom}_{filt}(V, V)$ with differential $D\phi = d' \circ \phi + \phi \circ d$.

Simplest piece: n strands

Define a dg functor $Aug(\equiv) \to MC(\equiv)$ on objects by

$$\left(\epsilon : \mathcal{A}(\equiv) \to \mathbb{F}_2\right) \mapsto \epsilon(A)^T$$

We view $\epsilon(A)^T$ as a graded, filtered differential $V \to V$. On morphisms:

$$a_{ij}^+ \mapsto |j\rangle\langle i|.$$

This (with appropriate signs) identifies $Aug(\equiv) = MC(\equiv)$:

Proposition

$Aug(\equiv, \mu)$ is the dg-category of filtered, degree-1 differentials on the vector space V with grading $\deg |i\rangle = -\mu(i)$.

Sheaves on n strands

An object of $\mathcal{C}_1(\equiv)$ associates a complex of sheaves to each region, and upward morphisms between them: it's a representation in chain complexes of the A_{n+1} quiver

$$\bullet \to \bullet \to \cdots \to \bullet.$$

In fact: $Sh(\equiv)$ is equivalent to $Rep_{ch}(A_{n+1})$.

Define a functor $MC(\equiv, \mu) \to Rep_{ch}(A_{n+1})$ by

$$(V, d) \mapsto ({}^n V \hookrightarrow {}^{n-1} V \hookrightarrow \ldots \hookrightarrow {}^0 V).$$

(Recall: each ${}^k V = \mathrm{span}(|n\rangle, \ldots, |k+1\rangle)$ is a dg-vector space.)

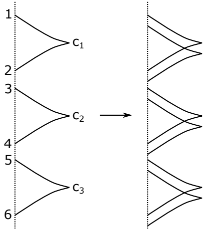
Proposition

If $r = k$ is a field, this is an equivalence onto the part of $Rep_{ch}(A_{n+1})$ where $\mathrm{Cone}(R_i \to R_{i-1}) = k[-\mu(i)]$, which is equivalent to $\mathcal{C}_1(\equiv, \mu)$.

The proof uses Gabriel's theorem classifying indecomposable A_{n+1} quiver reps, so k must be a field.

Right cusps

- We place a base point (hence a $t_i^{\pm 1}$) at each right cusp.
- Augmentations: $\partial c_i = t_i^{\pm 1} + a_{2i-1,2i}$, so $\epsilon(a_{2i-1,2i})$ must be invertible.
- Generators of $\mathrm{Hom}_{Aug}(\epsilon_1, \epsilon_2)$ are $2 \to 1$ Reeb chords: $a_{ij}^{+} = (a_{ij}^{12})^{\vee}$, $i \leq j$.



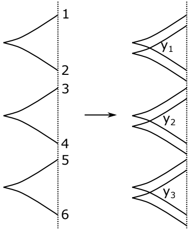
So $\mathrm{Hom}_{Aug(\succ)}(\epsilon_1, \epsilon_2) = \mathrm{Hom}_{Aug(\equiv)}(\epsilon_1|_{\equiv}, \epsilon_2|_{\equiv})$.

Conclusion 1: $Aug(\succ)$ is a full subcategory of $Aug(\equiv)$.
Conclusion 2: All objects in $Aug(\succ)$ are isomorphic!

From this it is not hard to show that $Aug(\succ) \cong \mathcal{C}_1(\succ)$.

Left cusps

- $\mathcal{A}(\prec) = k$ – there are no Reeb chord generators!
- So $\mathrm{Ob}(Aug(\prec)) = \{\epsilon\}$.
- $\mathrm{Hom}_{Aug}(\epsilon, \epsilon)$ is generated by $y_1, \ldots, y_n$.

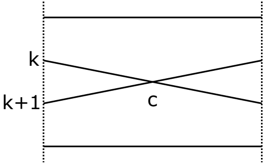


So $\mathrm{Hom}(\epsilon, \epsilon) = \bigoplus_{i=1}^{n} \mathbb{F} y_i^+$, with $m_2(y_i^+, y_j^+) = \delta_{ij}$.

Restriction map $Aug(\prec) \to Aug(\equiv)$: $y_i^+ \mapsto a_{2i-1,2i-1}^+ + a_{2i,2i}^+$.

Again, we have $Aug(\prec) \cong \mathcal{C}_1(\prec)$.

Crossings



- Augmentations: determined by $\epsilon(c)$ and $\epsilon|_{\equiv_{\text{left}}}$ with $\epsilon(a_{k,k+1}) = 0$.
- Differential on $Aug(\times)$: complicated but explicit.
- Multiplication: $m_2(a^+_{kj}, a^+_{ik}) = \pm a^+_{ij}$,
  $m_2(c^+, a^+_{kk}) = m_2(a^+_{k+1,k+1}, c^+) = -c^+$.

Note: $Aug(\times)$ is a dg-category, but the restriction

$$\rho_R : Aug(\times) \to Aug(\equiv_{\text{right}})$$

is an A_∞ functor – it has higher maps!

In any case: $Aug(\times)$ is equivalent to $\mathcal{C}_1(\times)$.

Conclusion

Since $Aug \cong \mathcal{C}^1$ for each of the basic pieces

$$\equiv, \prec, \times, \succ$$

compatibly with restriction maps, we glue these all together to get:

Theorem (Ng-Rutherford-Shende-S.-Zaslow)

$Aug(\Lambda; k) \cong \mathcal{C}_1(\Lambda; k)$.

Thank you!