The Hitchin System, the Airy Equation, the Trefoil and a Cluster Variety

(After Gaiotto-Moore-Neitzke, Kontsevich-Soibelman, Gross-Siebert, Boalch, Auroux)

Eric Zaslow, Northwestern 10/9/14, in discussion with Neitzke-Shende-Treumann and Shen

Lord Kelvin (1867): knots are elements. Nope, but they seem elemental in mathematics. We'll find knots at the intersection of symplectic geometry, integrable systems, differential equations and sheaf theory. Isomorphic moduli spaces and wall-crossing phenomena are found in each setting. We'll focus on a simple example.

From the word $\beta = sss$ in the braid group on two strands we can form a Legendrian knot by rainbow closure of the front diagram on the plane $\mathbb{R}^2$. The topological type is the trefoil, or $(2,3)$ torus knot, which can be realized as $y^2 = x^3$ along the torus $\{|y| = |x| = \frac{1}{2}\}$ in $S^3 = \{|x|^2 + |y|^2 = 2\}$. Given a positive braid, we can also form the front diagram and take its closure on the cylinder $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$. To these Legendrian knots we associate categories of sheaves on the plane or cylinder with singular support lying in the knot. [Draw category.] It is Legendrian isotopy invariant, and equivalent to the Fukaya category of Lagrangian branes which at infinity lie in the Legendrian knot. In sheaves, can define a subcategory of "rank-one" objects, loosely corresponding to Lagrangians with rank-one local systems. (Not all such objects are necessarily geometric.) This subcategory has a concrete combinatorial description, and the space of objects is a variety: an open Bott-Samelson variety for the group SL_2: equivalently a necklace of five transverse two-step flags in $\mathbb{C}^2$, or put differently five points $p_i \in \mathbb{P}^1$, $i \in \mathbb{Z}/5$, $p_i \neq p_{i+1}$. This space is a cluster variety. It can be described as the hypersurface $X = \{1 + x + z = xyz\}$ in $\mathbb{C}^3$. [Draw necklace.]

Algebraically, torus knots appear as links of singularities of plane curves, and cables thereof appear as links-at-infinity of not-necessarily-singular plane curves. Our curves are spectral curves associated to a Hitchin system on the complex plane. Recall Hitchin set-up. For C a complex curve and G a compact reductive Lie group, recall the diffeomorphic versions of the Hitchin moduli space: Dolbeault) Holomorphic G-connections A with hol Higgs bundles ϕ mod G-gauge; de Rham) flat $G_{\mathbb{C}}$ connection $A + i\phi$ mod $G_{\mathbb{C}}$-gauge; Betti) Reps of $\pi_1(C)$ in $G_{\mathbb{C}}$. The Hitchin integrable system, seen in the Dolbeault description, has base a set of spectral curves in T^*C given by the characteristic polynomial of the Higgs field ϕ — but certain features are evident in the de Rham model, as we shall see. This all happens in physics (GMN).

C is often smooth but can have (tame case) punctures along which the more connection has simple poles, or (wild) higher poles. Boalch (after Deligne-Malgrange) found local moduli spaces for these singularities. Physics: GMN, AGT.

Consider the Airy family of ODE's $u'' = x^n u$, particularly when $n = 3$: $u'' = x^3 u$, arising from the SL_2 connection $d + \begin{pmatrix} 0 & 1 \\ -x^3 & 0 \end{pmatrix}$, with its naive spectral curve (characteristic polynomial) $y^2 = x^3$. (Naive curve is good enough since Higgs field captures higher order terms of meromorphic connection.) To look at infinity, set $z = 1/x$ and rewrite as $(x\partial_x)^2 - x\partial_x - x^{-5}$. The equation is singular, but some singular gauge transformations with fractional powers gives two power-series developments: $y = e^{\pm\frac{2}{5}z^{-5/2}} z^{3/4} \sum_{n\geq 0} a_n z^{n/2}$. Stokes says to look at asymptotics as $z \to 0$. Putting $z = re^{i\theta}$, the growth rate of solutions along the ray θ is $\pm\cos(5\theta/2)$. (Wasow: OK.) Plotting them leads to the (2,5) braid $sssss$ on the cylinder. The associated Legendrian category (restricting to trivial monodromy) *also* has moduli space X!

Why? Every spectral curve is a holo. filling of the knot. After hyperkahler rotation, it's Lagrangian. This, with a line bundle gives an object in the Fukaya category. Each smooth filling gives an open complex torus by complex flat line bundles. Summary: Spectral curve moduli space = Hitchin moduli space = Stokes moduli space (Boalch et al) = Constructible sheaf moduli space = Lagrangian filling moduli space = Augmentation moduli space (by SFT).

Since $\mathbb{P}^1 = G/B$ for SL_2, X is a necklace of transverse flags, a cluster variety. X has $\mathbb{Z}/5$ action: $x \mapsto y, y \mapsto z, z \mapsto xy - 1 = \frac{1+x}{z}$, preserving positivity. Cluster transformation (square) maps charts $(x, z) \in \mathbb{C}^{*2}$ to $(z, \tilde{x} = \frac{1+z}{x}) \in \mathbb{C}^{*2}$. Goncharov-Shen mirror conjecture: compactifying X to $\overline{\mathcal{M}}_{0,5} \overset{\text{mirror}}{\longleftrightarrow}$ superpotential $W = x + y + z + (xy - 1) + (zy - 1)$. Indeed, $\#Crit(W) = 7$, matching $dimH^*\overline{X}$, noting $\overline{X} = \overline{\mathcal{M}}_{0,5} \cong Bl_4(\mathbb{P}^2)$. Without W, X should be self-mirror (HK).

Mirror is dual torus fibration, X. Mirror coordinate: $z_\beta(L, \nabla) = e^{-\int_\beta \omega} \cdot Hol_\nabla(\partial\beta)$. Auroux: Hol. disks generate superpotential $W : X \to \mathbb{C}$ as sum over Maslov-2 disks. *Walls* appear separating regions where Maslov index 0 disks can bubble on. Wall crossing is a change of variables in the coordinates of the mirror so that W is single-valued. These coord. transformations are of cluster type! Cluster transform *is* blow-up!?

Ex: $\mathcal{M}_1 = \overline{\mathcal{M}}_{0,5} \backslash (K^{-1}) \cong Bl_2(\mathbb{C}^2) \backslash \overline{(xy)}$. Consider $Bl_{(-1,0)}(\mathbb{C}^{*2}) \backslash \overline{(xy)} = \{\frac{y}{x+1} = \frac{1}{\tilde{y}}\}$. [Draw picture.] Fibration: $T_{r,\mu}$ where $|y| = r$ and $\mu = 0$ when $|xy - 1| = |zy - 1|$, nodal at $T_{1,0}$. Mirror coordinate $z_\beta = \exp(-\int_\beta \omega) Hol_\nabla(\partial\beta)$. Try to verify Goncharov-Shen superpotential for $\overline{X}$ as $W = x + y + z + (xy - 1) + (zy - 1)$. Problems: Find torus fibration for all of X. Find Fukaya/sheaf/Legendrian interpretation of canonical basis (cf. Gross-Hacking-Keel, Goncharov-Shen).

More wall xing. Hitchin: nonabelianization means recovering flat connection from spectral curve. GMN: Walls are trajectories of π_* one-form on spectral curve. Microlocalization gives perverse sheaf F on C from Fukaya cat: restriction morphisms $F(U) \to F(V)$ give flat conn. Walls when critical disks contribute to these. Morse analogue: continuation maps give q-iso $MC^*(f_0) \sim MC^*(f_1)$ for family $\{f_t\}$. Walls when critical trajectories arise. [Pic near branch pt.]