Introduction to Hyperbolic Knot Theory

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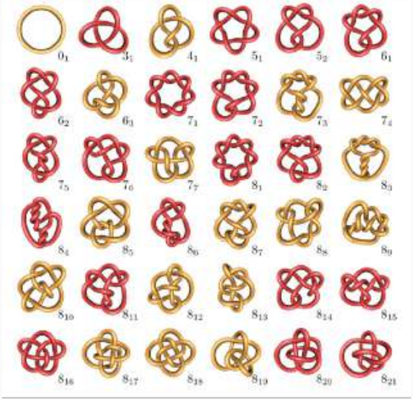
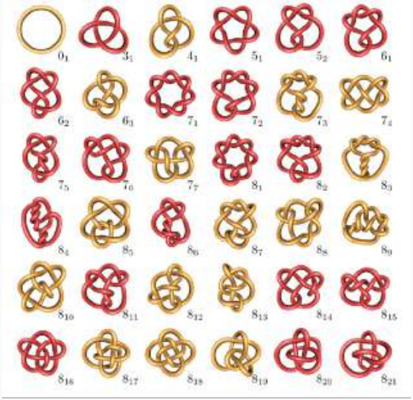


What is Knot Theory?

- Knot theory began with Gauss and Peter Tait in the 1870s
- Knot theory started with classifying and tabulating knots
- Surprisingly difficult question: How can we tell two knots apart?



What is Knot Theory?

What actually is a mathematical knot?

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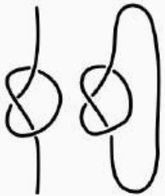
Definition

A *knot* is an (smooth) embedding of $\mathbb{S}^1$, the circle, into $\mathbb{S}^3$, the 3-sphere.



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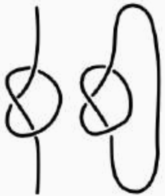
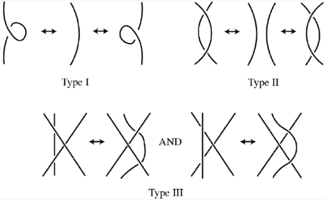


Two knots are *equivalent* if there is a continuous deformation (ambient isotopy) turning one knot into the other.

What is Knot Theory?

How can we tell whether two knots are equivalent?

Reidemeister moves:



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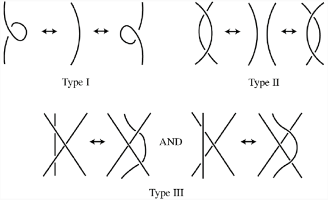
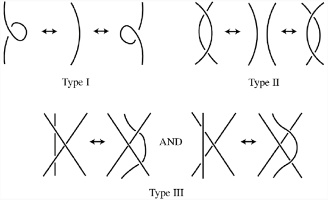


The difficulty is in showing that this set of moves is complete. Two knots are equivalent (isotopic) if and only if their diagrams are the same by Reidemeister moves. (1927)

What is Knot Theory?

How can we tell whether two knots are equivalent?

Reidemeister moves:



One of these things is knot like the other:

The Goeritz Knot The Conway Knot

What is Knot Theory?

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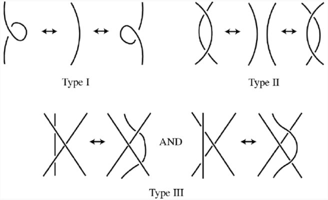

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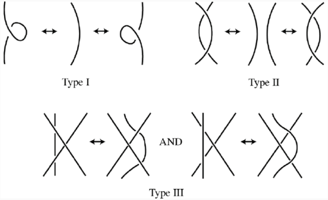

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Instead of trying sequences of Reidemeister moves, we look for *invariants*.

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- Every 3-manifold can be given one of eight possible geometric structures (Thurston)
- We get *topological invariants* by looking at knot complements

What is Knot Theory?

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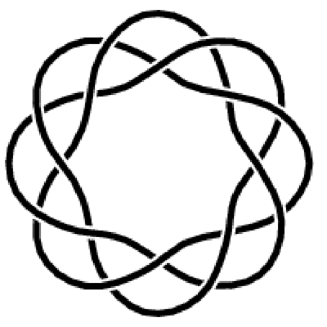
Torus knots



3-8 Torus Knot

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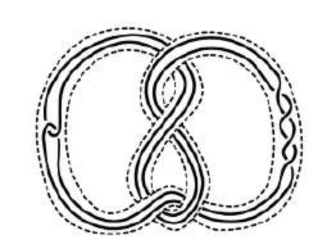
Satellite knots



A Satellite Knot

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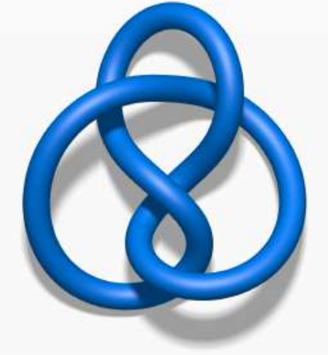
Hyperbolic knots



The Figure-8 Knot is the smallest hyperbolic knot

What is Knot Theory?

Knot type by crossing number

type \ $n =$	3	4	5	6	7	8	9	10	11	12	13	14	15	16
all prime	1	1	2	3	7	21	49	165	552	2 176	9 988	46 972	253 293	1 388 705
hyperbolic	0	1	1	3	6	20	48	164	551	2 176	9 985	46 969	253 285	1 388 694
prime satellite	0	0	0	0	0	0	0	0	0	0	2	2	6	10
torus	1	0	1	0	1	1	1	1	1	0	1	1	2	1

TABLE 1. Number of prime knots

This suggests that we should study hyperbolic manifolds in order to understand knot theory!

Intro to Hyperbolic Geometry

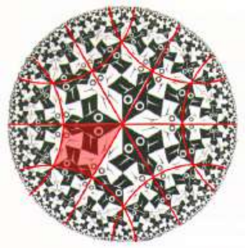
$\mathbb{E}^2$ vs $\mathbb{S}^2$ vs $\mathbb{H}^2$

Intro to Hyperbolic Geometry

Crochet by Daina Taimina

A Dutch collar

The game HyperRogue



Artwork by Escher

Intro to Hyperbolic Geometry

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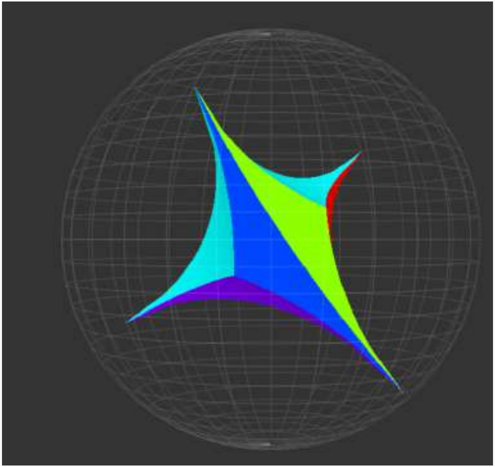
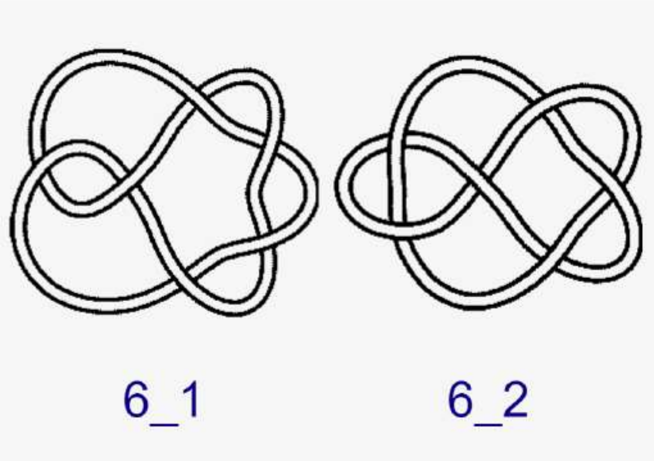
- Hyperbolic structure given by *hyperbolic metric*
- *Geodesics* - shortest path between two points
- Example: Area of *ideal triangle*

The Poincare disk as a model of $\mathbb{H}^2$ extends to the *Poincare ball* as a model for $\mathbb{H}^3$.



Hyperbolic Knot Theory

There are three knots with six crossings. Let's use software to tell two of them apart!



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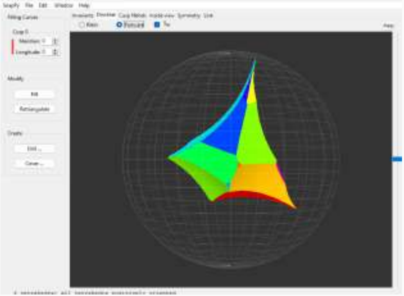

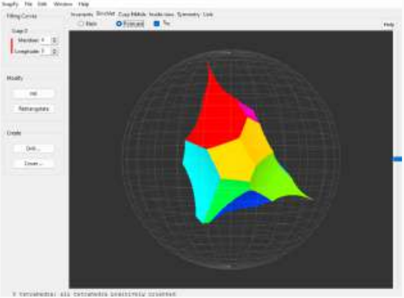

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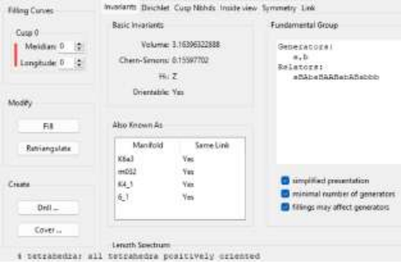

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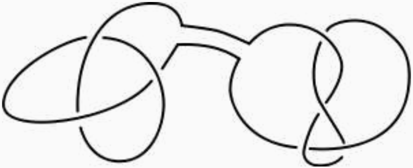
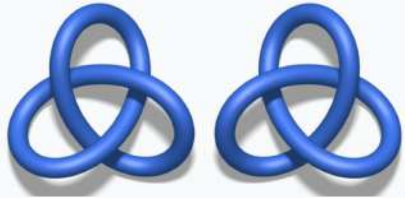
- Theorem (Mostow-Prasad Rigidity, 1968, 1973): A (complete, finite volume) hyperbolic manifold of dimension > 2 has a *unique* hyperbolic structure (up to isometry)
- Modern knot theory research focuses on topological, combinatorial, and algebraic invariants (polynomials, homologies) and the connections between them

What is Knot Theory (Cut for time)

A brief note here: we only care about classifying *prime* knots, and we usually treat *chiral* knots as the same knot.





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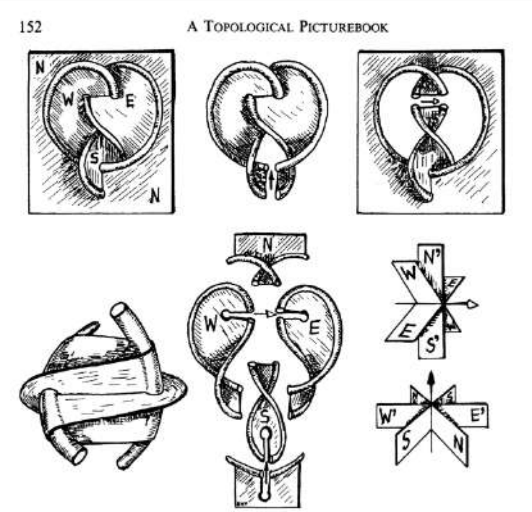

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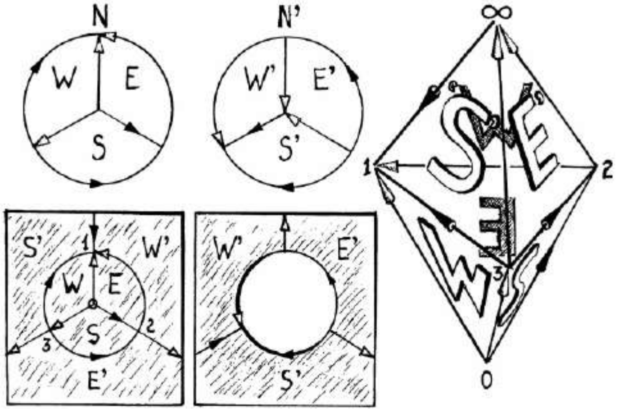
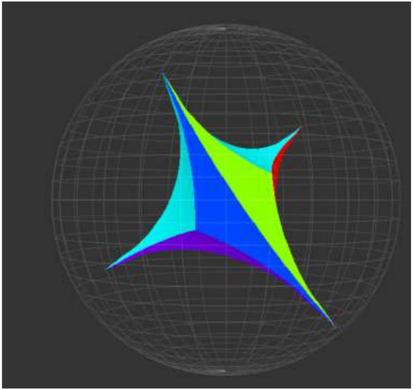


Figure 2 HEXAHEDRAL COMPLEMENT

Hyperbolic Knot Theory (Cut for time)

We can do this with software!



Decomposition of the figure 8 knot complement into two ideal tetrahedra