

Introduction to Hyperbolic Knot Theory

Herman Malik

May 4, 2023

Table of contents

What is Knot Theory?

Intro to Hyperbolic Geometry

Hyperbolic Knot Theory

What is Knot Theory?

What is Knot Theory?

- Knot theory began with Gauss and Peter Tait in the 1870s

What is Knot Theory?

- Knot theory began with Gauss and Peter Tait in the 1870s
- Knot theory started with classifying and tabulating knots



What is Knot Theory?

- Knot theory began with Gauss and Peter Tait in the 1870s
- Knot theory started with classifying and tabulating knots
- Surprisingly difficult question: How can we tell two knots apart?



What is Knot Theory?

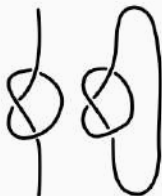
What actually is a mathematical knot?

What is Knot Theory?

What actually is a mathematical knot?

Definition

A *knot* is an (smooth) embedding of \mathbb{S}^1 , the circle, into \mathbb{S}^3 , the 3-sphere.

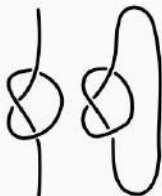


What is Knot Theory?

What actually is a mathematical knot?

Definition

A *knot* is an (smooth) embedding of \mathbb{S}^1 , the circle, into \mathbb{S}^3 , the 3-sphere.

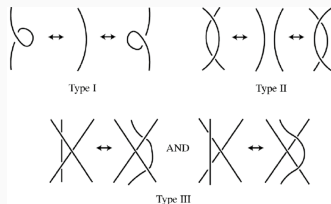


Two knots are *equivalent* if there is a continuous deformation (ambient isotopy) turning one knot into the other.

What is Knot Theory?

How can we tell whether two knots are equivalent?

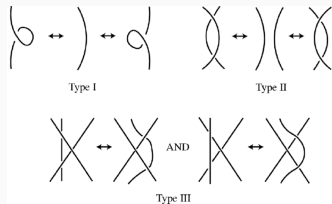
Reidemeister moves:



What is Knot Theory?

How can we tell whether two knots are equivalent?

Reidemeister moves:

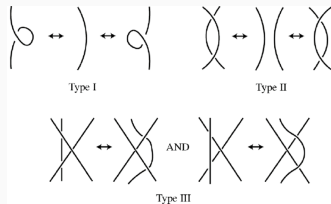


The difficulty is in showing that this set of moves is complete. Two knots are equivalent (isotopic) if and only if their diagrams are the same by Reidemeister moves. (1927)

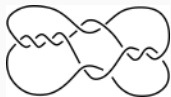
What is Knot Theory?

How can we tell whether two knots are equivalent?

Reidemeister moves:



One of these things is knot like the other:



The Goeritz Knot

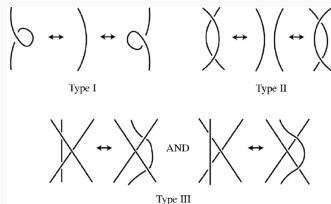


The Conway Knot

What is Knot Theory?

How can we tell whether two knots are equivalent?

Reidemeister moves:

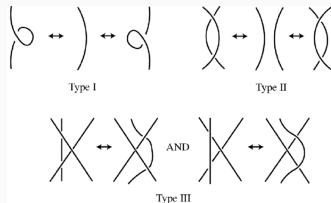


Even telling if a knot is an unknot is very hard!

What is Knot Theory?

How can we tell whether two knots are equivalent?

Reidemeister moves:



Even telling if a knot is an unknot is very hard!

Instead of trying sequences of Reidemeister moves, we look for *invariants*.

What is Knot Theory?

- One way of telling knots apart is by looking at the *knot complement*

What is Knot Theory?

- One way of telling knots apart is by looking at the *knot complement*
- Knot complements are 3 dimensional manifolds \longrightarrow often easier to study than knots themselves

What is Knot Theory?

- One way of telling knots apart is by looking at the *knot complement*
- Knot complements are 3 dimensional manifolds \rightarrow often easier to study than knots themselves
- Knot complements are unique - no two knots have the same complement (Gordon-Luecke, 1989)

What is Knot Theory?

- One way of telling knots apart is by looking at the *knot complement*
- Knot complements are 3 dimensional manifolds \rightarrow often easier to study than knots themselves
- Knot complements are unique - no two knots have the same complement (Gordon-Luecke, 1989)
- Every 3-manifold can be given one of eight possible geometric structures (Thurston)

What is Knot Theory?

- One way of telling knots apart is by looking at the *knot complement*
- Knot complements are 3 dimensional manifolds \rightarrow often easier to study than knots themselves
- Knot complements are unique - no two knots have the same complement (Gordon-Luecke, 1989)
- Every 3-manifold can be given one of eight possible geometric structures (Thurston)
- *We get topological invariants by looking at knot complements*

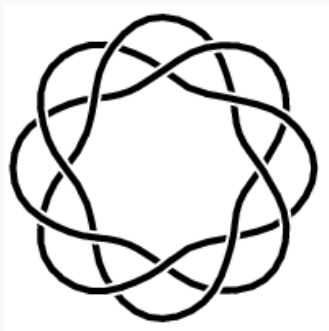
What is Knot Theory?

Every knot can be classified into one of 3 types (based on the geometric structure of the knot complement) (Thurston)

What is Knot Theory?

Every knot can be classified into one of 3 types (based on the geometric structure of the knot complement) (Thurston)

Torus knots

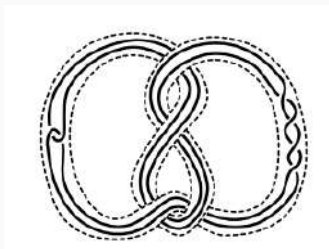


3-8 Torus Knot

What is Knot Theory?

Every knot can be classified into one of 3 types (based on the geometric structure of the knot complement) (Thurston)

Satellite knots

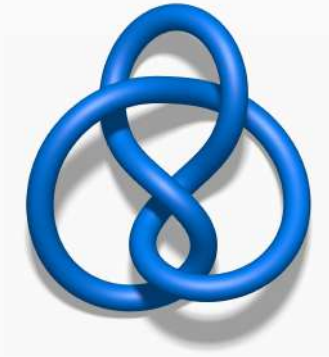


A Satellite Knot

What is Knot Theory?

Every knot can be classified into one of 3 types (based on the geometric structure of the knot complement) (Thurston)

Hyperbolic knots



The Figure-8 Knot is the smallest hyperbolic knot

What is Knot Theory?

Knot type by crossing number

type \ $n =$	3	4	5	6	7	8	9	10	11	12	13	14	15	16
all prime	1	1	2	3	7	21	49	165	552	2176	9988	46972	253293	1388705
hyperbolic	0	1	1	3	6	20	48	164	551	2176	9985	46969	253285	1388694
prime satellite	0	0	0	0	0	0	0	0	0	0	2	2	6	10
torus	1	0	1	0	1	1	1	1	1	0	1	1	2	1

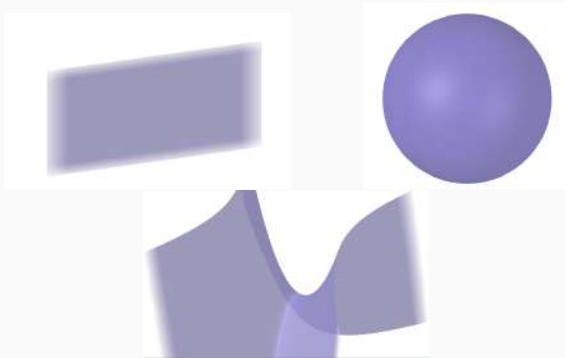
TABLE 1. Number of prime knots

This suggests that we should study hyperbolic manifolds in order to understand knot theory!

Intro to Hyperbolic Geometry

Intro to Hyperbolic Geometry

E^2 vs S^2 vs H^2



Intro to Hyperbolic Geometry



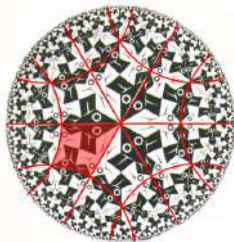
Crochet by Daina Taimina



A Dutch collar



The game HyperRogue



Artwork by Escher

Intro to Hyperbolic Geometry

- Hyperbolic structure given by *hyperbolic metric*



Intro to Hyperbolic Geometry

- Hyperbolic structure given by *hyperbolic metric*
- *Geodesics* - shortest path between two points



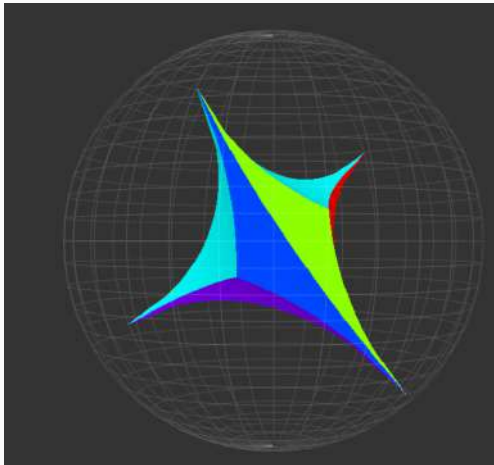
Intro to Hyperbolic Geometry

- Hyperbolic structure given by *hyperbolic metric*
- *Geodesics* - shortest path between two points
- Example: *Area of ideal triangle*



Intro to Hyperbolic Geometry

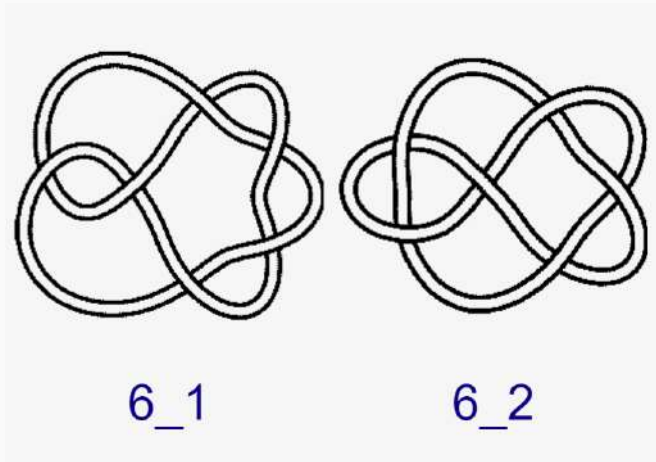
The Poincare disk as a model of \mathbb{H}^2 extends to the *Poincare ball* as a model for \mathbb{H}^3 .



Hyperbolic Knot Theory

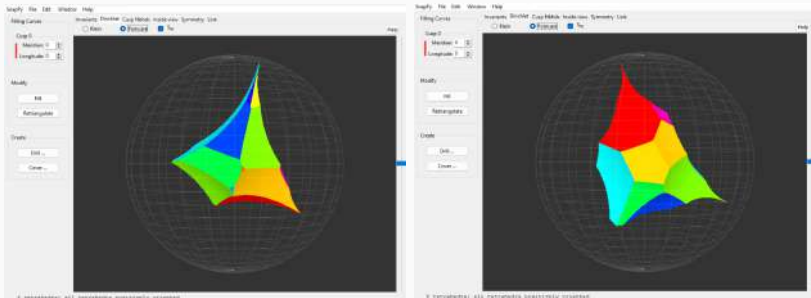
Hyperbolic Knot Theory

There are three knots with six crossings. Let's use software to tell two of them apart!



Hyperbolic Knot Theory

There are three knots with six crossings. Let's use software to tell two of them apart!



Hyperbolic Knot Theory

There are three knots with six crossings. Let's use software to tell two of them apart!

The image shows two side-by-side screenshots of a knot theory software interface. Each screenshot displays the same set of controls and panels for a different knot. The left screenshot shows a knot with Volume 3.16396122888 and Chern-Simons 0.15987702. The right screenshot shows a knot with Volume 4.40661251812 and Chern-Simons -0.20248250. Both knots are oriented and have a length spectrum of 4 tetrahedra.

Left Screenshot:

- Filling Curves:** Meridian 0, Longitude 0
- Basic Invariants:** Volume: 3.16396122888, Chern-Simons: 0.15987702, H: Z, Orientable: Yes
- Also Known As:**

Manifold	Same Link
K3a3	Yes
m022	Yes
6A_1	Yes
6_1	Yes
- Fundamental Group:** Generators: a, b; Relators: a5Ab2a3A2b2a5Aabbb
- Length Spectrum:** 4 tetrahedra; all tetrahedra positively oriented

Right Screenshot:

- Filling Curves:** Meridian 0, Longitude 0
- Basic Invariants:** Volume: 4.40661251812, Chern-Simons: -0.20248250, H: Z, Orientable: Yes
- Also Known As:**

Manifold	Same Link
m259	Yes
6Aa2	Yes
6A_19	Yes
6_2	Yes
- Fundamental Group:** Generators: a, b; Relators: a4B2Ba5AabbbAabbbA
- Length Spectrum:** 5 tetrahedra; all tetrahedra positively oriented

What's the Point?

- Original problem: classifying knots up to equivalence

What's the Point?

- Original problem: classifying knots up to equivalence
- In addition to combinatorial invariants (e.g. crossing number), we now have topological invariants (e.g. hyperbolic volume)

What's the Point?

- Original problem: classifying knots up to equivalence
- In addition to combinatorial invariants (e.g. crossing number), we now have topological invariants (e.g. hyperbolic volume)
- Why are these actually invariants?

What's the Point?

- Original problem: classifying knots up to equivalence
- In addition to combinatorial invariants (e.g. crossing number), we now have topological invariants (e.g. hyperbolic volume)
- Why are these actually invariants?
- Theorem (Gordon-Luecke, 1989): A knot is completely determined by its complement in \mathbb{S}^3

What's the Point?

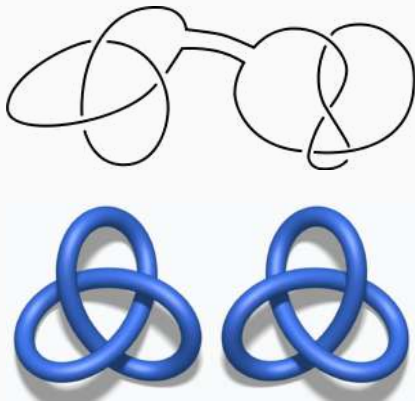
- Original problem: classifying knots up to equivalence
- In addition to combinatorial invariants (e.g. crossing number), we now have topological invariants (e.g. hyperbolic volume)
- Why are these actually invariants?
- Theorem (Gordon-Luecke, 1989): A knot is completely determined by its complement in \mathbb{S}^3
- Theorem (Mostow-Prasad Rigidity, 1968, 1973): A (complete, finite volume) hyperbolic manifold of dimension > 2 has a *unique* hyperbolic structure (up to isometry)

What's the Point?

- Original problem: classifying knots up to equivalence
- In addition to combinatorial invariants (e.g. crossing number), we now have topological invariants (e.g. hyperbolic volume)
- Why are these actually invariants?
- Theorem (Gordon-Luecke, 1989): A knot is completely determined by its complement in \mathbb{S}^3
- Theorem (Mostow-Prasad Rigidity, 1968, 1973): A (complete, finite volume) hyperbolic manifold of dimension > 2 has a *unique* hyperbolic structure (up to isometry)
- Modern knot theory research focuses on topological, combinatorial, and algebraic invariants (polynomials, homologies) and the connections between them

What is Knot Theory (Cut for time)

A brief note here: we only care about classifying *prime* knots, and we usually treat *chiral* knots as the same knot.



Hyperbolic Knot Theory (Cut for time)

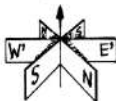
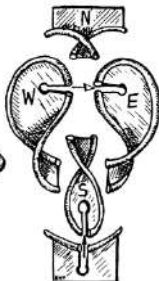
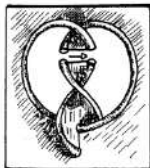
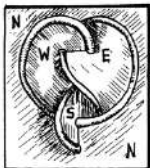
We can visualize knot complements by decomposing them into ideal polyhedra.

Hyperbolic Knot Theory (Cut for time)

We can visualize knot complements by decomposing them into ideal polyhedra.

152

A TOPOLOGICAL PICTUREBOOK



Hyperbolic Knot Theory (Cut for time)

We can visualize knot complements by decomposing them into ideal polyhedra.

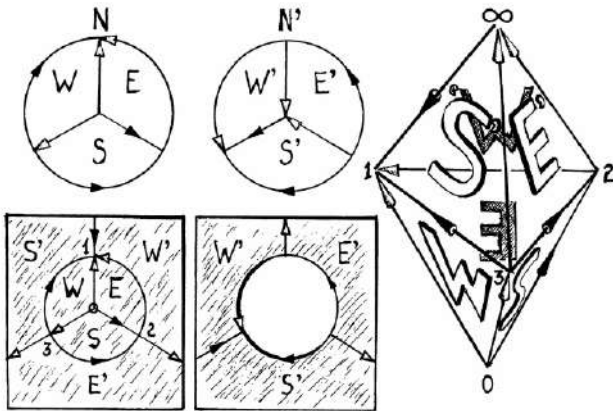
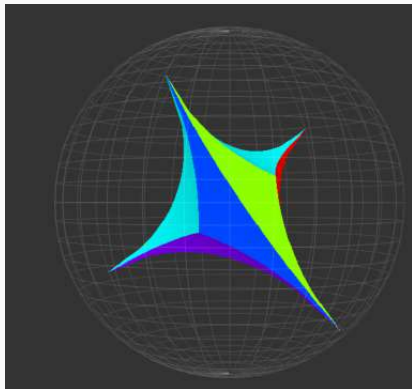


Figure 2

HEXAHEDRAL COMPLEMENT

Hyperbolic Knot Theory (Cut for time)

We can do this with software!



Decomposition of the figure 8 knot complement into two ideal tetrahedra