Introduction to Hyperbolic Knot Theory

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May 4, 2023
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What is Knot Theory?

Intro to Hyperbolic Geometry

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What is Knot Theory?
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- Knot theory began with Gauss and Peter Tait in the 1870s
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- Knot theory started with classifying and tabulating knots
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- Knot theory started with classifying and tabulating knots
- **Surprisingly difficult question:** How can we tell two knots apart?
What actually is a mathematical knot?
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**Definition**

A knot is an (smooth) embedding of $\mathbb{S}^1$, the circle, into $\mathbb{S}^3$, the 3-sphere.
What is Knot Theory?

What actually is a mathematical knot?

**Definition**

A *knot* is an (smooth) embedding of $S^1$, the circle, into $S^3$, the 3-sphere.

Two knots are *equivalent* if there is a continuous deformation (ambient isotopy) turning one knot into the other.
What is Knot Theory?

How can we tell whether two knots are equivalent?

Reidemeister moves:
What is Knot Theory?

How can we tell whether two knots are equivalent?

Reidemeister moves:

The difficulty is in showing that this set of moves is complete. Two knots are equivalent (isotopic) if and only if their diagrams are the same by Reidemeister moves. (1927)
What is Knot Theory?

How can we tell whether two knots are equivalent?

Reidemeister moves:

One of these things is knot like the other:

The Goeritz Knot    The Conway Knot
What is Knot Theory?

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Reidemeister moves:

Even telling if a knot is an unknot is very hard!
What is Knot Theory?

How can we tell whether two knots are equivalent?

Reidemeister moves:

Even telling if a knot is an unknot is very hard!

Instead of trying sequences of Reidemeister moves, we look for *invariants*. 
What is Knot Theory?

- One way of telling knots apart is by looking at the knot complement

Knot complements are 3 dimensional manifolds and are easier to study than knots themselves. Knot complements are unique - no two knots have the same complement (Gordon-Luecke, 1989).

Every 3-manifold can be given one of eight possible geometric structures (Thurston). We get topological invariants by looking at knot complements.
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(Gordon-Luecke, 1989)

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Every knot can be classified into one of 3 types (based on the geometric structure of the knot complement) (Thurston)
What is Knot Theory?

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**Torus knots**

3-8 Torus Knot
What is Knot Theory?

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**Satellite knots**

A Satellite Knot
What is Knot Theory?

Every knot can be classified into one of 3 types (based on the geometric structure of the knot complement) (Thurston)

Hyperbolic knots

The Figure-8 Knot is the smallest hyperbolic knot
What is Knot Theory?

Knot type by crossing number

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<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
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<th>13</th>
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<td>1</td>
<td>1</td>
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</tbody>
</table>

TABLE 1. Number of prime knots

This suggests that we should study hyperbolic manifolds in order to understand knot theory!
Intro to Hyperbolic Geometry
$E^2 \text{ vs } S^2 \text{ vs } H^2$
Intro to Hyperbolic Geometry

Crochet by Daina Taimina

A Dutch collar

The game HyperRogue

Artwork by Escher
• Hyperbolic structure given by \textit{hyperbolic metric}
• Hyperbolic structure given by *hyperbolic metric*

• *Geodesics* - shortest path between two points
Intro to Hyperbolic Geometry

- Hyperbolic structure given by *hyperbolic metric*
- *Geodesics* - shortest path between two points
- Example: Area of *ideal triangle*
The Poincare disk as a model of $\mathbb{H}^2$ extends to the Poincare ball as a model for $\mathbb{H}^3$. 

![Hyperbolic Geometry Diagram](image-url)
Hyperbolic Knot Theory
There are three knots with six crossings. Let’s use software to tell two of them apart!
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• Theorem (Mostow-Prasad Rigidity, 1968, 1973): A (complete, finite volume) hyperbolic manifold of dimension $> 2$ has a unique hyperbolic structure (up to isometry)
• Modern knot theory research focuses on topological, combinatorial, and algebraic invariants (polynomials, homologies) and the connections between them
A brief note here: we only care about classifying *prime* knots, and we usually treat *chiral* knots as the same knot.
We can visualize knot complements by decomposing them into ideal polyhedra.
Hyperbolic Knot Theory (Cut for time)

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We can do this with software!

Decomposition of the figure 8 knot complement into two ideal tetrahedra