Department of Mathematics, University of California, Berkeley

YOUR 1- OR 2-DIGIT EXAM NUMBER ____

GRADUATE PRELIMINARY EXAMINATION, Part A Spring Semester 2026

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1. Please write your 1- or 2-digit exam number on this cover sheet.
2. Answer five of the eight problems each day. You will get no extra credit for attempting more than 5 problems.
3. Do not answer more than one question on any given page, as this will confuse the examiners.
4. No notes, books, calculators, or electronic devices may be used during the exam.

Please cross out this problem if you do not wish it graded

Problem 1A. *Score:*

Let $f \in C([0,1])$ be continuous. Suppose

$$\int_0^1 f(x)\, x^n \, dx = 0 \quad \text{for every integer } n \geq 0.$$

Prove that $f \equiv 0$ on $[0,1]$.

You may use without proof that polynomials are uniformly dense in $C([0,1])$.

Solution:

Please cross out this problem if you do not wish it graded

Problem 2A. *Score:*

Let (f_n) be a sequence of differentiable functions on $[0,1]$ with $f_n(0) = 0$ for all n. Suppose that (f'_n) converges uniformly on $[0,1]$ to a continuous function g.

1. Prove that (f_n) converges uniformly on $[0,1]$ to a function f.
2. Show that f is C^1 on $[0,1]$, that $f(0) = 0$, and that $f' = g$.

Solution:

Please cross out this problem if you do not wish it graded

Problem 3A. *Score:*

Let f be an entire function such that

$$|f(z)| \leq A + B|z|^m$$

for some constants $A, B \geq 0$ and a nonnegative integer m.

1. Show that f is a polynomial of degree at most m.
2. Deduce that if $|f(z)| \leq C(1 + |z|^{1/2})$ for some $C > 0$, then f is constant.

Solution:

Please cross out this problem if you do not wish it graded

Problem 4A. *Score:*

Let $0 < \alpha < 1$. Evaluate

$$I(\alpha) := \int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx$$

using contour integration (do not use Beta/Gamma function identities). **Solution:**

Please cross out this problem if you do not wish it graded

Problem 5A. *Score:*

Let A be a real $n \times n$ matrix such that

$$A^3 = A.$$

Describe all possible Jordan canonical forms of A.

Solution:

Please cross out this problem if you do not wish it graded

Problem 6A. *Score:*

Orthogonally diagonalize the symmetric matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Specifically:

1. Find the eigenvalues and eigenspaces of A.
2. Within the eigenspace of the repeated eigenvalue, use the *Gram–Schmidt process* to produce an orthonormal basis.
3. Form an orthogonal matrix Q from your orthonormal eigenvectors and verify that $Q^\top AQ$ is diagonal. Give Q and $Q^\top AQ$ explicitly.

Solution:

Please cross out this problem if you do not wish it graded

Problem 7A. *Score:*

Let G be a finite p-group, i.e. $|G| = p^n$ for some prime p and integer $n \geq 1$.

1. Show that the center $Z(G)$ of G is nontrivial.
2. Deduce that G has a normal subgroup of order p.

Solution:

Please cross out this problem if you do not wish it graded

Problem 8A. *Score:*

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1. Let R be a finite integral domain (a finite commutative ring with 1 and no zero divisors). Prove that R is a field.
2. Show that if p is the characteristic of R then the Frobenius map $F : R \to R$, $F(x) = x^p$ is an automorphism (and in particular is bijective).

Solution:

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YOUR 1- OR 2-DIGIT EXAM NUMBER ____

GRADUATE PRELIMINARY EXAMINATION, Part B Spring Semester 2026

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Problem 1B. *Score:*

Let c_n be the number of ways of giving n cents change using coins of values $1, 5, 10, 25$ cents. Find $\lim_{n\to\infty} c_n/n^3$.

Solution:

Please cross out this problem if you do not wish it graded

Problem 2B. *Score:*

The Catalan numbers $1, 1, 2, 5, 14, ...$ satisfy $C_n = 0$ if $n < 0$, $C_0 = 1$, $C_n = \sum_i C_i C_{n-1-i}$. Find the function $f(x) = \sum_n C_n x^n$, and use this to calculate C_n as the quotient of a binomial coefficient by a linear function. **Solution:**

Please cross out this problem if you do not wish it graded

Problem 3B. *Score:*

Evaluate the integral

$$I = \int_0^{2\pi} \frac{d\theta}{5 - 4\cos\theta}.$$

(Hint: Use the substitution $z = e^{i\theta}$.) **Solution:**

Please cross out this problem if you do not wish it graded

Problem 4B. *Score:*

Let f be analytic on the disk $D = \{z : |z| < 1\}$ and suppose that

$$|f(z)| \leq 1 \quad \text{for all } z \in D,$$

and that

$$f(0) = 0.$$

1. Show that $|f'(0)| \leq 1$.
2. Find all functions f for which equality holds.

Solution:

Please cross out this problem if you do not wish it graded

Problem 5B. *Score:*

Let $M = (m_{i,j})$, $0 \le i, j < n$ be an $n \times n$ complex matrix with $m_{i,j} = m_{i+j}$ for constants m_i satisfying $m_i = m_{n+i}$. Find the eigenvalues of M.

Solution:

Please cross out this problem if you do not wish it graded

Problem 6B. *Score:*

For complex square matrices A_i, put $F_n(A_1, ..., A_n) = \sum_{\sigma \in S_n} sign(\sigma) A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(n)}$. The sum is over the symmetric group S_n, and $sign(\sigma)$ is 1 or -1 if σ is an even or odd permutation.

Show that F_n vanishes if all its arguments are $m \times m$ matrices and $n > m^2$. (Hint: first show F_n vanishes if two arguments are equal.)

Solution:

Please cross out this problem if you do not wish it graded

Problem 7B. *Score:*

1. Let G be a finite group and $\varphi : G \to \mathrm{GL}_n(\mathbb{C})$ a group homomorphism. Show that $\ker(\varphi) = \{g \in G \mid \mathrm{tr}(\varphi(g)) = n\}$.

2. Show that this fails for infinite G.

Solution:

Please cross out this problem if you do not wish it graded

Problem 8B. *Score:*

Prove that if all elements of a group satisfy $g^2 = 1$ then the group is abelian. Given an example of a group such that $g^3 = 1$ for all elements but the group is not abelian.

Solution: