Department of Mathematics, University of California, Berkeley

YOUR 1- OR 2-DIGIT EXAM NUMBER ____

GRADUATE PRELIMINARY EXAMINATION, Part A Spring Semester 2026

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1. Please write your 1- or 2-digit exam number on this cover sheet.
2. Answer five of the eight problems each day. You will get no extra credit for attempting more than 5 problems.
3. Do not answer more than one question on any given page, as this will confuse the examiners.
4. No notes, books, calculators, or electronic devices may be used during the exam.

Please cross out this problem if you do not wish it graded

Problem 1A. *Score:*

Let $f \in C([0,1])$ be continuous. Suppose

$$\int_0^1 f(x)\, x^n\, dx = 0 \quad \text{for every integer } n \geq 0.$$

Prove that $f \equiv 0$ on $[0,1]$.

You may use without proof that polynomials are uniformly dense in $C([0,1])$.

Solution: Since polynomials are dense in $C([0,1])$ (Weierstrass approximation theorem), the linear span of $\{x^n : n \geq 0\}$ is dense in $C([0,1])$.

Define a continuous linear functional $L : C([0,1]) \to \mathbb{R}$ by

$$L(g) := \int_0^1 f(x)\, g(x)\, dx.$$

The hypothesis says that $L(x^n) = 0$ for all $n \geq 0$, hence $L(p) = 0$ for all polynomials p. By continuity of L and density of polynomials, we conclude $L(g) = 0$ for all $g \in C([0,1])$.

In particular, taking $g = f$, we have

$$0 = L(f) = \int_0^1 f(x) \cdot f(x)\, dx = \int_0^1 |f(x)|^2\, dx.$$

Therefore $|f(x)|^2 = 0$ for almost every x, so $f(x) = 0$ almost everywhere. Since f is continuous, this implies $f \equiv 0$ on $[0,1]$.

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Problem 2A. *Score:*

Let (f_n) be a sequence of differentiable functions on $[0,1]$ with $f_n(0) = 0$ for all n. Suppose that (f_n') converges uniformly on $[0,1]$ to a continuous function g.

1. Prove that (f_n) converges uniformly on $[0,1]$ to a function f.
2. Show that f is C^1 on $[0,1]$, that $f(0) = 0$, and that $f' = g$.

Solution:

For each n and $x \in [0,1]$, by the Fundamental Theorem of Calculus,

$$f_n(x) - f_n(0) = \int_0^x f_n'(t)\,dt.$$

Since $f_n(0) = 0$, this gives

$$f_n(x) = \int_0^x f_n'(t)\,dt.$$

Define

$$f(x) := \int_0^x g(t)\,dt, \qquad x \in [0,1].$$

Because g is continuous, f is C^1 with $f'(x) = g(x)$ and $f(0) = 0$.

To prove $f_n \to f$ uniformly, note that for all $x \in [0,1]$,

$$\bigl|f_n(x) - f(x)\bigr| = \left|\int_0^x \bigl(f_n'(t) - g(t)\bigr)\,dt\right| \le \int_0^1 \sup_{u\in[0,1]} \bigl|f_n'(u) - g(u)\bigr|\,dt = \sup_{u\in[0,1]} \bigl|f_n'(u) - g(u)\bigr|.$$

Since $f_n' \to g$ uniformly, the right-hand side tends to 0 as $n \to \infty$, so $f_n \to f$ uniformly on $[0,1]$.

We have already observed that f is C^1, $f(0) = 0$, and $f' = g$, completing the proof.

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Problem 3A. *Score:*

Let f be an entire function such that

$$|f(z)| \leq A + B|z|^m$$

for some constants $A, B \geq 0$ and a nonnegative integer m.

1. Show that f is a polynomial of degree at most m.
2. Deduce that if $|f(z)| \leq C(1 + |z|^{1/2})$ for some $C > 0$, then f is constant.

Solution: **(a)** Since f is entire, we can use Cauchy's integral formula for derivatives:

$$f^{(k)}(0) = \frac{k!}{2\pi i} \int_{|z|=R} \frac{f(z)}{z^{k+1}}\, dz.$$

Taking absolute values and using the given bound:

$$|f^{(k)}(0)| \leq \frac{k!}{2\pi} \cdot \frac{2\pi R}{R^{k+1}} \cdot \max_{|z|=R} |f(z)| \leq \frac{k!}{R^k}\big(A + BR^m\big).$$

If $k > m$, let $R \to \infty$. Then the bound becomes

$$|f^{(k)}(0)| \leq \frac{k!}{R^{k-m}} \left(\frac{A}{R^m} + B \right) \to 0$$

because $k - m > 0$. Thus $f^{(k)}(0) = 0$ for all $k > m$, which means f is a polynomial of degree at most m.

(b) If $|f(z)| \leq C(1 + |z|^{1/2})$, then the growth condition holds with $m = \frac{1}{2}$. From part (a), m must be an integer bound for the degree; the only integer $\leq \frac{1}{2}$ is 0. Hence f is a polynomial of degree 0, i.e. a constant function.

Please cross out this problem if you do not wish it graded

Problem 4A. *Score:*

Let $0 < \alpha < 1$. Evaluate

$$I(\alpha) := \int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx$$

using contour integration (do not use Beta/Gamma function identities). **Solution:**

Define $f(z) = \dfrac{z^{\alpha-1}}{1+z}$ on $\mathbb{C} \setminus [0,\infty)$, using the branch

$$z^{\alpha-1} = \exp\big((\alpha-1)\log z\big), \qquad \log z = \ln|z| + i\arg z, \quad \arg z \in (0, 2\pi),$$

so the branch cut is the positive real axis. The only singularity of f in $\mathbb{C} \setminus [0,\infty)$ is a simple pole at $z = -1$, which lies inside the contours defined below.

Consider the standard keyhole contour $\Gamma = \Gamma_R \cup [\varepsilon, R] \cup \gamma_\varepsilon \cup [R, \varepsilon]$, consisting of:

- the upper bank of (ε, R) along the positive real axis,
- the large circle $\Gamma_R : \ |z| = R$ (counterclockwise),
- the lower bank of (R, ε) along the positive real axis,
- the small circle $\gamma_\varepsilon : \ |z| = \varepsilon$ (clockwise).

By the residue theorem,

$$\int_\Gamma f(z)\,dz = 2\pi i\,\mathrm{Res}_{z=-1} f.$$

Compute the residue at $z = -1$:

$$\mathrm{Res}_{z=-1} f = \lim_{z\to -1} \frac{z^{\alpha-1}}{1+z}\,(z+1) = (-1)^{\alpha-1} = -(-1)^\alpha.$$

On our branch, $-1 = e^{i\pi}$, so

$$(-1)^{\alpha-1} = e^{i\pi(\alpha-1)}.$$

Next, parametrize the integrals along the two banks. For $z \in (\varepsilon, R)$ on the upper bank, $\arg z = 0$ and hence $z^{\alpha-1} = x^{\alpha-1}$. On the lower bank we approach the positive axis with argument 2π, so

$$z^{\alpha-1}\big|_{\text{lower bank}} = x^{\alpha-1}e^{2\pi i(\alpha-1)} = x^{\alpha-1}e^{2\pi i\alpha}.$$

Thus

$$\int_{\text{upper bank}} f(z)\,dz \;=\; \int_\varepsilon^R \frac{x^{\alpha-1}}{1+x}\,dx, \qquad \int_{\text{lower bank}} f(z)\,dz \;=\; -\int_\varepsilon^R \frac{x^{\alpha-1}e^{2\pi i\alpha}}{1+x}\,dx,$$

(the minus sign comes from the right-to-left orientation along the lower bank).

We claim the circular arcs vanish in the limits $R \to \infty$ and $\varepsilon \to 0^+$. Indeed, on $|z| = R$,

$$|f(z)| = \frac{|z|^{\alpha-1}}{|1+z|} \le \frac{R^{\alpha-1}}{R-1} \le \frac{2}{R^{2-\alpha}} \quad (R \text{ large}),$$

so

$$\left|\int_{\Gamma_R} f(z)\,dz\right| \le \sup_{|z|=R} |f(z)| \cdot (2\pi R) = O\big(R^{\alpha-1}\big) \to 0$$

since $\alpha - 1 < 0$. Similarly, on $|z| = \varepsilon$,

$$|f(z)| \le \frac{\varepsilon^{\alpha-1}}{|1+z|} \le 2\,\varepsilon^{\alpha-1}, \quad \text{so} \quad \left|\int_{\gamma_\varepsilon} f(z)\,dz\right| \le 2\,\varepsilon^{\alpha-1} \cdot (2\pi\varepsilon) = O(\varepsilon^\alpha) \to 0.$$

Putting everything together and letting $R \to \infty$, $\varepsilon \to 0^+$,

$$\int_\Gamma f(z)\,dz = \left(1 - e^{2\pi i\alpha}\right) \int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx = -2\pi i\, e^{i\pi\alpha}.$$

Hence

$$I(\alpha) = \frac{2\pi i\, e^{i\pi\alpha}}{e^{2\pi i\alpha} - 1} = \pi \frac{2i}{e^{\pi i\alpha} - e^{-\pi i\alpha}} = \frac{\pi}{\sin(\pi\alpha)}.$$

Therefore, for $0 < \alpha < 1$,

$$\boxed{\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx \;=\; \frac{\pi}{\sin(\pi\alpha)}}.$$

Problem 5A. *Score:*

Let A be a real $n \times n$ matrix such that

$$A^3 = A.$$

Describe all possible Jordan canonical forms of A.

Solution: **(a)** If λ is an eigenvalue of A with eigenvector $v \neq 0$, then

$$Av = \lambda v, \quad A^3 v = \lambda^3 v.$$

Since $A^3 = A$, we have

$$\lambda^3 v = \lambda v \quad \Rightarrow \quad (\lambda^3 - \lambda)v = 0 \quad \Rightarrow \quad \lambda(\lambda^2 - 1) = 0.$$

Thus $\lambda \in \{-1, 0, 1\}$.

(b) The minimal polynomial $m_A(x)$ divides $x(x-1)(x+1)$, which splits over $\mathbb{R}$ into distinct linear factors. A real matrix whose minimal polynomial has distinct roots is diagonalizable over $\mathbb{R}$. Hence A is diagonalizable.

(c) Since A is diagonalizable with eigenvalues only in $\{-1, 0, 1\}$, its Jordan canonical form over $\mathbb{R}$ is a diagonal matrix with entries each equal to -1, 0, or 1. In other words, A is similar to

$$\operatorname{diag}(\underbrace{-1, \ldots, -1}_{p}, \underbrace{0, \ldots, 0}_{q}, \underbrace{1, \ldots, 1}_{r}),$$

where $p + q + r = n$ and $p, q, r \geq 0$ are integers.

Problem 6A. *Score:*

Orthogonally diagonalize the symmetric matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Specifically:

1. Find the eigenvalues and eigenspaces of A.
2. Within the eigenspace of the repeated eigenvalue, use the *Gram–Schmidt process* to produce an orthonormal basis.
3. Form an orthogonal matrix Q from your orthonormal eigenvectors and verify that $Q^\top AQ$ is diagonal. Give Q and $Q^\top AQ$ explicitly.

Solution: 1. Eigenvalues and eigenspaces. Observe $A = I + J$, where J is the 3×3 all-ones matrix. The spectrum of J is $\{3, 0, 0\}$ with eigenvector $(1, 1, 1)^\top$ for eigenvalue 3 and orthogonal complement $(1, 1, 1)^\perp$ for eigenvalue 0. Hence the eigenvalues of A are

$$\lambda_1 = 4 \quad \text{(once)}, \qquad \lambda_2 = \lambda_3 = 1 \quad \text{(twice)}.$$

An eigenvector for $\lambda = 4$ is $v_3 = (1, 1, 1)^\top$. The $\lambda = 1$ eigenspace is

$$E_1 = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\} = \operatorname{span}\{(1, -1, 0)^\top,\ (1, 0, -1)^\top\}.$$

2. Gram–Schmidt in the $\lambda = 1$ eigenspace. Set $u_1 = (1, -1, 0)^\top$ and $u_2 = (1, 0, -1)^\top$.

Normalize u_1:

$$e_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{2}}(1, -1, 0)^\top.$$

Orthogonalize u_2 against e_1:

$$w_2 = u_2 - \langle u_2, e_1 \rangle e_1 = (1,0,-1)^\top - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}(1,-1,0)^\top = (\tfrac{1}{2}, \tfrac{1}{2}, -1)^\top.$$

Normalize w_2 (note $\|w_2\|^2 = \frac{1}{4} + \frac{1}{4} + 1 = \frac{3}{2}$):

$$e_2 = \frac{w_2}{\|w_2\|} = \frac{\sqrt{2}}{\sqrt{3}}\Big(\tfrac{1}{2}, \tfrac{1}{2}, -1\Big)^\top = \Big(\tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, -\sqrt{\tfrac{2}{3}}\Big)^\top.$$

Finally, normalize the $\lambda = 4$ eigenvector:

$$e_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{3}}(1,1,1)^\top.$$

By construction, $\{e_1, e_2\}$ is an orthonormal basis of E_1, and $e_3 \perp E_1$, so $\{e_1, e_2, e_3\}$ is an orthonormal eigenbasis of $\mathbb{R}^3$.

3. Orthogonal diagonalization. Form the orthogonal matrix $Q = [\, e_1 \ e_2 \ e_3 \,]$:

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \qquad Q^\top Q = QQ^\top = I.$$

Because $Ae_1 = 1 \cdot e_1$, $Ae_2 = 1 \cdot e_2$, and $Ae_3 = 4 \cdot e_3$, we have

$$Q^\top A Q = \operatorname{diag}(1,1,4).$$

This gives the explicit orthogonal diagonalization $A = Q \operatorname{diag}(1,1,4)\, Q^\top$.

Please cross out this problem if you do not wish it graded

Problem 7A. *Score:*

Let G be a finite p-group, i.e. $|G| = p^n$ for some prime p and integer $n \geq 1$.

1. Show that the center $Z(G)$ of G is nontrivial.
2. Deduce that G has a normal subgroup of order p.

Solution: 1. Consider the action of G on itself by conjugation: $g \cdot x = gxg^{-1}$.

For any $x \in G$, the size of its orbit is the index of its centralizer:

$$|\mathrm{Orb}(x)| = [G : C_G(x)].$$

The size of each orbit divides $|G| = p^n$, hence is a power of p. Moreover, the orbits of elements of $Z(G)$ are singletons (size 1).

The *class equation* for G is

$$|G| = |Z(G)| + \sum_i [G : C_G(x_i)],$$

where each x_i is a representative of a noncentral conjugacy class. Each index $[G : C_G(x_i)]$ is a power of p greater than 1, hence divisible by p.

Thus the sum $\sum_i [G : C_G(x_i)]$ is divisible by p, and since $|G|$ is divisible by p, we must have $|Z(G)| \equiv |G| \pmod{p}$. In particular, $|Z(G)|$ is divisible by p. Because $|Z(G)| \geq 1$ and divisible by p, it follows that $|Z(G)| \geq p$. So $Z(G)$ is nontrivial.

2. For any element $z \neq e$ in $Z(G)$, its cyclic group (of order equal to a positive power of p) contains an element of order p, and the cyclic group of that element, being central, is a normal subgroup of order p in G.

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Problem 8A. *Score:*

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1. Let R be a finite integral domain (a finite commutative ring with 1 and no zero divisors). Prove that R is a field.
2. Show that if p is the characteristic of R then the Frobenius map $F : R \to R$, $F(x) = x^p$ is an automorphism (and in particular is bijective).

Solution: (1) Fix $0 \neq a \in R$. Consider the map

$$m_a : R \longrightarrow R, \qquad m_a(x) = ax.$$

Because R has no zero divisors, m_a is injective: if $m_a(x) = m_a(y)$, then $a(x-y) = 0$, hence $x - y = 0$ and $x = y$. Since R is finite, any injective map $R \to R$ is surjective; thus there exists $b \in R$ with $m_a(b) = ab = 1$. Hence every nonzero $a \in R$ has a multiplicative inverse b, so R is a field.

(2) Since fields have no non-trivial ideals, a non-zero homomorphism $\varphi : R \to R$ is injective. Since R is finite, it bijective, and hence an automorphism.

For the Frobenius map $F(x) = x^p$ in characteristic p, note that F is a ring homomorphism: $(xy)^p = x^p y^p$ and $(x+y)^p = x^p + y^p$ (by the binomial theorem and $\binom{p}{k} \equiv 0 \pmod{p}$ for $1 \leq k \leq p-1$).

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GRADUATE PRELIMINARY EXAMINATION, Part B   Spring Semester 2026

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Problem 1B. *Score:*

Let c_n be the number of ways of giving n cents change using coins of values $1, 5, 10, 25$ cents. Find $\lim_{n\to\infty} c_n/n^3$.

Solution: The generating function $\sum_n c_n z^n$ is $1/(1-z)(1-z^5)(1-z^{10})(1-z^{25})$. Write it as partial fractions $\frac{1}{1\times5\times10\times25}(1-z)^{-4}+$ terms with asymptotically smaller coefficients. Since the coefficients of $1/(1-z)^4$ are asymptotic to $n^3/3!$, the limit is $1/(1\times5\times10\times25\times3!) = 1/7500$.

Alternatively: c_n is the number of integer points in the 3D-pyramid

$$\{x, y, z \geq 0 \mid 5x + 10y + 25 \leq n\},$$

and in the limit $n \to \infty$, c_n is approximately proportional to its volume

$$\frac{(n/5)\cdot(n/10)\cdot(n/25)}{6}.$$

Problem 2B. *Score:*

The Catalan numbers $1, 1, 2, 5, 14, ...$ satisfy $C_n = 0$ if $n < 0$, $C_0 = 1$, $C_n = \sum_i C_i C_{n-1-i}$. Find the function $f(x) = \sum_n C_n x^n$, and use this to calculate C_n as the quotient of a binomial coefficient by a linear function. **Solution:** We are given

a sequence $(C_n)_{n\in\mathbb{Z}}$ with

$$C_n = 0 \text{ for } n < 0, \quad C_0 = 1, \quad C_n = \sum_i C_i\, C_{n-1-i} \text{ for } n \geq 1,$$

where the sum is over all integers i (but only $0 \leq i \leq n-1$ contribute, since $C_n = 0$ for $n < 0$).

Define the generating function

$$f(x) = \sum_{n=-\infty}^{\infty} C_n x^n = \sum_{n=0}^{\infty} C_n x^n,$$

since $C_n = 0$ for $n < 0$.

Step 1: Find a functional equation for $f(x)$.

Consider $f(x)^2$:

$$f(x)^2 = \left(\sum_{n\geq 0} C_n x^n\right)\left(\sum_{m\geq 0} C_m x^m\right) = \sum_{k\geq 0}\Big(\sum_{i=0}^{k} C_i C_{k-i}\Big)x^k.$$

Now

$$\sum_{i=0}^{k} C_i C_{k-i}$$

is almost the defining recurrence; compare with

$$C_n = \sum_i C_i C_{n-1-i}.$$

Set $n - 1 = k$, i.e. $n = k + 1$. Then

$$C_{k+1} = \sum_{i=0}^{k} C_i C_{k-i}.$$

Thus

$$f(x)^2 = \sum_{k\geq 0} C_{k+1}x^k = \frac{1}{x}\sum_{k\geq 0} C_{k+1}x^{k+1} = \frac{1}{x}\big(f(x) - C_0\big) = \frac{f(x)-1}{x}.$$

Therefore

$$f(x)^2 = \frac{f(x)-1}{x} \quad\Longrightarrow\quad xf(x)^2 - f(x) + 1 = 0.$$

Step 2: Solve the quadratic for $f(x)$.

We have a quadratic equation in f:

$$xf^2 - f + 1 = 0.$$

Solving for f:

$$f(x) = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

To determine the correct sign, note that $f(0) = C_0 = 1$, so $f(x)$ must be analytic at $x = 0$ and have $f(0) = 1$.

Expand for small x:

$$\sqrt{1-4x} = 1 - 2x + O(x^2).$$

Then

$$\frac{1+\sqrt{1-4x}}{2x} \sim \frac{1+(1-2x)}{2x} = \frac{2-2x}{2x} \sim \frac{1}{x}$$

which blows up at $x = 0$, so this cannot be the generating function.

On the other hand,

$$\frac{1-\sqrt{1-4x}}{2x} \sim \frac{1-(1-2x)}{2x} = \frac{2x}{2x} = 1,$$

which matches $f(0) = 1$ and is analytic at 0. Therefore

$$\boxed{f(x) = \sum_{n\geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}.}$$

Step 3: Extract a formula for C_n.

We now expand $\sqrt{1-4x}$ using the binomial series:

$$(1-4x)^{1/2} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} (-4x)^k.$$

Therefore

$$1-\sqrt{1-4x}=1-\sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}(-4x)^k=-\sum_{k=1}^{\infty}\binom{\frac{1}{2}}{k}(-4x)^k.$$

Hence

$$f(x)=\frac{1-\sqrt{1-4x}}{2x}=-\frac{1}{2x}\sum_{k=1}^{\infty}\binom{\frac{1}{2}}{k}(-4)^k x^k=-\frac{1}{2}\sum_{k=1}^{\infty}\binom{\frac{1}{2}}{k}(-4)^k x^{k-1}.$$

Reindex with $n=k-1$:

$$f(x)=-\frac{1}{2}\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n+1}(-4)^{n+1}x^n.$$

So

$$C_n=-\frac{1}{2}\binom{\frac{1}{2}}{n+1}(-4)^{n+1}.$$

We simplify this expression. Recall

$$\binom{\frac{1}{2}}{n+1}=\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-n\right)}{(n+1)!}.$$

A standard simplification (or comparison with known expansions) gives

$$C_n=\frac{1}{n+1}\binom{2n}{n}.$$

Thus the nth Catalan number is

$$\boxed{C_n=\frac{1}{n+1}\binom{2n}{n}.}$$

Problem 3B. *Score:*

Evaluate the integral

$$I = \int_0^{2\pi} \frac{d\theta}{5 - 4\cos\theta}.$$

(Hint: Use the substitution $z = e^{i\theta}$.) **Solution:** Use the substitution

$$z = e^{i\theta}, \qquad |z| = 1, \qquad d\theta = \frac{1}{iz}\, dz,$$

and

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

Then

$$5 - 4\cos\theta = 5 - 2\left(z + \frac{1}{z}\right) = \frac{5z - 2(z^2+1)}{z} = \frac{-2z^2 + 5z - 2}{z}.$$

Thus

$$\frac{d\theta}{5 - 4\cos\theta} = \frac{1}{5 - 4\cos\theta}\,\frac{1}{iz}\, dz = \frac{1}{i(-2z^2 + 5z - 2)}\, dz.$$

The integral becomes the contour integral over the unit circle:

$$I = \int_0^{2\pi} \frac{d\theta}{5 - 4\cos\theta} = \oint_{|z|=1} \frac{1}{i(-2z^2 + 5z - 2)}\, dz.$$

Factor the denominator:

$$-2z^2 + 5z - 2 = -(2z - 1)(z - 2),$$

whose roots are $z = \frac{1}{2}$ and $z = 2$. Only $z = \frac{1}{2}$ lies inside the unit circle.

The residue at a simple pole z_0 of a function of the form

$$\frac{1}{i\, p(z)}$$

is

$$\operatorname{Res}_{z=z_0} \frac{1}{i\,p(z)} = \frac{1}{i\,p'(z_0)}.$$

Here

$$p(z) = -2z^2 + 5z - 2, \qquad p'(z) = -4z + 5,$$

so at $z = \frac{1}{2}$,

$$p'\left(\tfrac{1}{2}\right) = -4\left(\tfrac{1}{2}\right) + 5 = -2 + 5 = 3.$$

Thus

$$\operatorname{Res}_{z=1/2} \frac{1}{i(-2z^2 + 5z - 2)} = \frac{1}{i \cdot 3} = -\frac{i}{3}.$$

By the residue theorem,

$$I = 2\pi i \cdot \left(-\frac{i}{3}\right) = \frac{2\pi}{3}.$$

$$\boxed{\int_0^{2\pi} \frac{d\theta}{5 - 4\cos\theta} = \frac{2\pi}{3}}.$$

Please cross out this problem if you do not wish it graded

Problem 4B. *Score:*

Let f be analytic on the disk $D = \{z : |z| < 1\}$ and suppose that

$$|f(z)| \leq 1 \quad \text{for all } z \in D,$$

and that

$$f(0) = 0.$$

1. Show that $|f'(0)| \leq 1$.
2. Find all functions f for which equality holds.

Solution: (1) Bound on the derivative.
Since f is analytic on D and $|f(z)| \leq 1$, and $f(0) = 0$, define the function

$$g(z) = \begin{cases} \frac{f(z)}{z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

This defines an analytic function on D. By the maximum modulus principle,

$$|g(z)| \leq 1 \quad \text{on } D.$$

In particular, at $z = 0$,

$$|f'(0)| = |g(0)| \leq 1.$$

(2) Equality case.
The functions for which equality holds are exactly the rotations:

$$f(z) = e^{i\theta} z.$$

$$\boxed{|f'(0)| \leq 1 \quad \text{and equality holds iff } f(z) = e^{i\theta} z.}$$

Problem 5B. *Score:*

Let $M = (m_{i,j})$, $0 \le i, j < n$ be an $n \times n$ complex matrix with $m_{i,j} = m_{i+j}$ for constants m_i satisfying $m_i = m_{n+i}$. Find the eigenvalues of M.

Solution:

Let $\omega = e^{2\pi i/n}$ and for $r = 0, 1, \ldots, n-1$ let $v_r = (1, \omega^r, \omega^{2r}, \ldots, \omega^{(n-1)r})^T$. We have

$$\begin{aligned}(Mv_r)_i &= \sum_{j=0}^{n-1} m_{i+j}\omega^{jr} \\ &= \sum_{k=0}^{n-1} m_k \omega^{(k-i)r} \\ &= \mu_r \omega^{-ir} \\ &= \mu_r (v_r)_i,\end{aligned}$$

where $\mu_r = \sum_{k=0}^{n-1} m_k \omega^{kr}$. Thus $Mv_r = \mu_r v_{n-r}$.

So up to scaling M swaps v_r and v_{n-r}. We thus get eigenvalues $\lambda_r = \pm\sqrt{\mu_r \mu_{n-r}}$ for $r = 1, \ldots, \lfloor \frac{n-1}{2} \rfloor$ with the edge cases $\lambda_0 = \sum m_j$ and, if n is even, $\lambda_{n/2} = \sum (-1)^j m_j$.

Problem 6B. *Score:*

For complex square matrices A_i, put $F_n(A_1, ..., A_n) = \sum_{\sigma \in S_n} sign(\sigma) A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(n)}$. The sum is over the symmetric group S_n, and $sign(\sigma)$ is 1 or -1 if σ is an even or odd permutation.

Show that F_n vanishes if all its arguments are $m \times m$ matrices and $n > m^2$. (Hint: first show F_n vanishes if two arguments are equal.)

Solution: F_n changes sign if two of its arguments are exchanged, so vanishes if two are equal. In particular it vanishes if its arguments are linearly dependent, which is true if $n > m^2$. (In fact it vanishes if $n \geq 2m$, but this is harder to prove.)

Please cross out this problem if you do not wish it graded

Problem 7B. *Score:*

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1. Let G be a finite group and $\varphi : G \to \mathrm{GL}_n(\mathbb{C})$ a group homomorphism. Show that $\ker(\varphi) = \{g \in G \mid \mathrm{tr}(\varphi(g)) = n\}$.
2. Show that this fails for infinite G.

Solution: (1) Show that $\ker(\varphi) = \{g \in G : \mathrm{tr}(\varphi(g)) = n\}$**.**

Let $\varphi : G \to \mathrm{GL}_n(\mathbb{C})$ be a group homomorphism. If $g \in \ker(\varphi)$, then $\varphi(g) = I_n$, the identity matrix. Hence

$$\mathrm{tr}(\varphi(g)) = \mathrm{tr}(I_n) = n.$$

Thus

$$\ker(\varphi) \subseteq \{g \in G : \mathrm{tr}(\varphi(g)) = n\}.$$

For the reverse inclusion, suppose $\mathrm{tr}(\varphi(g)) = n$. Since $\varphi(g)$ is a matrix whose eigenvalues $\lambda_1, \ldots, \lambda_n$ satisfy

$$|\lambda_i| = 1 \qquad \text{(because } \varphi(g) \text{ is invertible and } g \text{ has finite order)},$$

we have

$$\mathrm{tr}(\varphi(g)) = \lambda_1 + \cdots + \lambda_n = n.$$

But the only way a sum of n complex numbers of absolute value 1 can equal n is if

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = 1.$$

Thus all eigenvalues of $\varphi(g)$ are equal to 1, so $\varphi(g)$ is unipotent:

$$\varphi(g) = I_n + N,$$

where N is nilpotent.

Because G is finite, g has finite order, say $g^m = e$. Then

$$\varphi(g)^m = \varphi(g^m) = I_n,$$

so

$$(I_n + N)^m = I_n.$$

Expanding the left-hand side gives

$$I_n + mN + \binom{m}{2} N^2 + \cdots + N^m = I_n.$$

Since N is nilpotent, this implies $N = 0$, and hence $\varphi(g) = I_n$. Thus $g \in \ker(\varphi)$.

Therefore,

$$\{g \in G : \operatorname{tr}(\varphi(g)) = n\} \subseteq \ker(\varphi).$$

Combining both inclusions proves

$$\ker(\varphi) = \{g \in G : \operatorname{tr}(\varphi(g)) = n\}.$$

Another solution. Picking an Hermitian inner product $(\cdot, \cdot)$ and taking its average over the finite group,

$$\frac{1}{|G|} \sum_{g \in G} (\phi(g)x, \phi(g)y),$$

we may assume that each $\phi(g)$ is unitary, and hence diagonaliable with the eigenvalues $\lambda_1, \ldots, \lambda_n$ of absolute value 1. The trace $\lambda_1 + \cdots + \lambda_n = n$ only when all $\lambda_i = 1$.

(2) Show the statement fails for infinite groups.

Consider the infinite group $G = \mathbb{Z}$ and the representation

$$\varphi : \mathbb{Z} \to \mathrm{GL}_2(\mathbb{C}), \qquad \varphi(k) = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

Each matrix has trace

$$\operatorname{tr}(\varphi(k)) = 1 + 1 = 2,$$

but $\varphi(k) = I_2$ only when $k = 0$. Thus

$$\{g \in G : \operatorname{tr}(\varphi(g)) = 2\} = \mathbb{Z}, \qquad \ker(\varphi) = \{0\}.$$

Hence the equality in part (1) does *not* hold for infinite groups.

Please cross out this problem if you do not wish it graded

Problem 8B. *Score:*

Prove that if all elements of a group satisfy $g^2 = 1$ then the group is abelian. Given an example of a group such that $g^3 = 1$ for all elements but the group is not abelian.

Solution: Part 1. If $g^2 = 1$ for all $g \in G$, then G is abelian.

Assume G is a group in which every element satisfies $g^2 = 1$. Then for any $g \in G$ we have

$$g^2 = 1 \quad \Longrightarrow \quad g^{-1} = g.$$

In particular, every element is its own inverse.

Let $a, b \in G$. Consider the product ab. Then

$$(ab)^2 = 1 \quad \Longrightarrow \quad abab = 1.$$

On the other hand, since $(ab)^{-1} = b^{-1}a^{-1}$, but $a^{-1} = a$ and $b^{-1} = b$, we get

$$(ab)^{-1} = ba.$$

However, every element is its own inverse, so

$$(ab)^{-1} = ab.$$

Thus

$$ab = (ab)^{-1} = ba.$$

Since a and b were arbitrary, we conclude $ab = ba$ for all $a, b \in G$, so G is abelian.

Part 2. Example with $g^3 = 1$ for all g, but non-abelian.

We give a standard example of a non-abelian group in which every element has order dividing 3.

Let G be the group of all 3×3 upper triangular matrices over the field $\mathbb{F}_3$ with 1's on the diagonal, i.e.

$$G = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{F}_3 \right\}.$$

Matrix multiplication makes G into a group.

(i) Every element satisfies $x^3 = I$.

Take any $X \in G$. We can write $X = I + N$, where I is the identity matrix and

$$N = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that N is strictly upper triangular, so $N^3 = 0$ (in fact already N^3 is the zero matrix for 3×3 strictly upper triangular matrices).

We now compute $(I+N)^3$ in characteristic 3. Using the binomial formula in a ring:

$$(I+N)^3 = I + 3N + 3N^2 + N^3.$$

But we are working over $\mathbb{F}_3$, so $3 \equiv 0$ in this field, and thus

$$3N = 0, \quad 3N^2 = 0, \quad N^3 = 0.$$

Therefore,

$$(I+N)^3 = I.$$

Hence every $X \in G$ satisfies $X^3 = I$, i.e. X has order dividing 3.

(ii) G is not abelian.

Consider the matrices

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Compute AB and BA:

$$AB = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad BA = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

These are not equal, so $AB \neq BA$. Thus G is non-abelian.

We have exhibited a non-abelian group G in which every element satisfies $g^3 = 1$.