Department of Mathematics, University of California, Berkeley

GRADUATE PRELIMINARY EXAMINATION, Part A — Spring Semester 2023

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1. Answer six of the nine problems each day. You will get no extra credit for attempting more than 6 problems.
2. The exam lasts 3 hours each day, including time to enter questions in gradescope.
3. Do not answer more than one question on any given piece of paper, as this will confuse the examiners.
4. Submit your answers by uploading pictures or a PDF file to gradescope.
5. The exam is open book: you may use notes or books or calculators or the internet, but may not consult anyone else.
6. In case of questions or unexpected problems during the prelim send email to the chair of the prelim committee at `borcherds@berkeley.edu`. If a correction or announcement is needed during the exam it will be sent as an email to the address you use on gradescope for the prelim, so please keep an eye on this during the prelim.

YOUR EXAM NUMBER ___

Please cross out this problem if you do not wish it graded

Problem 1A. *Score:*

Evaluate the infinite product

$$\prod_{n=3}^{\infty} \frac{n^2 - 4}{n^2 - 1}$$

Solution:

Problem 2A. *Score:*

Let $f\colon [0,1]\times[0,1]\to\mathbb{R}$ be a continuous function, and let $g\colon [0,1]\to\mathbb{R}$ be the function

$$g(x) = \min_{0\le y\le 1} f(x,y)\ .$$

Show in detail that g is continuous on $(0,1)$. (It is also continuous at the endpoints, but don't worry about them.)

Solution:

YOUR EXAM NUMBER ____
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Problem 3A. *Score:*

Prove Taylor's theorem with the remainder on the form of Peano: If a real-valued function on the number line has a well-defined nth derivative at $x = 0$, then the error of approximating the function near $x = 0$ by its degree-n Taylor polynomial is $o(x^n)$. (A function f is $o(x^n)$ at a point if f/x^n tends to 0 at this point.) [Do not assume continuity or even existence of the nth derivative in any neighborhood of $x = 0$.]

Solution:

YOUR EXAM NUMBER ____

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Problem 4A. *Score:*

Suppose that f is a complex polynomial all of whose roots have real part at most 0. Show that if $r > 0$ then $|f(r)| \geq |f(-r)|$. Give an example to show that can be false if the condition that f is a polynomial is replaced by the condition that f is entire.

Solution:

YOUR EXAM NUMBER ____
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Problem 5A. *Score:*

Evaluate the contour integral

$$\lim_{R \mapsto +\infty} \int_{c-iR}^{c+iR} \frac{1}{z} dz$$

where c is a nonzero real number. (Warning: the answer depends on c.)

Solution:

YOUR EXAM NUMBER ___

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Problem 6A. *Score:*

A plane passing through the origin in $\mathbb{R}^3$ intersects the ellipsoid $x^2/4 + y^2/9 + z^2/16 = 1$ by an ellipse. Determine how many such sections are circles and find their radiuses.

Solution:

YOUR EXAM NUMBER ____
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Problem 7A. *Score:*

Suppose that the square complex matrix A is similar to A^n for some integer $n > 1$. Prove all eigenvalues of A are either zero or roots of unity.

Solution:

YOUR EXAM NUMBER ____
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Problem 8A. *Score:*

Let $\alpha\colon G \to G_1$ and $\beta\colon G \to G_2$ be group homomorphisms.

(a). Show that if $\ker\alpha \subseteq \ker\beta$ and α is surjective (onto) then there is a well-defined group homomorphism $\phi\colon G_1 \to G_2$ such that $\beta = \phi \circ \alpha$.

(b). Show that if $\ker\alpha \not\subseteq \ker\beta$ then there is no such homomorphism ϕ.

Solution:

YOUR EXAM NUMBER ____
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Problem 9A. *Score:*

Recall that S_5 and A_5 are the symmetric group and alternating group on 5 letters, respectively.

Prove or give a counterexample: For every $\sigma \in A_5$ there is a $\tau \in S_5$ such that $\tau^2 = \sigma$.

Solution:

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GRADUATE PRELIMINARY EXAMINATION, Part B Spring Semester 2023

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YOUR EXAM NUMBER ____

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Problem 1B. *Score:*

Let $f\colon (a,b] \to \mathbb{R}$ be a function. Assume that f is strictly increasing on (a,b). (This means that $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in (a,b).) Assume also that f is continuous from the left at b. Then show that f is strictly increasing on $(a,b]$.

Solution:

YOUR EXAM NUMBER ____
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Problem 2B. *Score:*

By definition, the *unit cube* in the space $C[0,1]$ of continuous real-valued functions on the closed interval $[0,1]$ consists of those functions whose norm $\|f\| := \max_{0\leq t\leq 1}|f(t)|$ doesn't exceed 1. Find a linear map $\mathbb{R}^3 \to C[0,1]$ such that the inverse image of the unit cube is the unit ball in $\mathbb{R}^3$.

Solution:

YOUR EXAM NUMBER ___
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Problem 3B. *Score:*

Show that the recursive sequence $x_{n+1} = x_n/2+1/x_n$ with the initial value $x_0 = 1.5$ converges to $\sqrt{2}$, and that x_{10} has at least 1000 correct decimal digits.

Solution:

YOUR EXAM NUMBER ____

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Problem 4B. *Score:*

Show that there is a function holomorphic on the open unit disc that is a bijection from the open unit disc to the vertical strip $0 < \Re z < 1$, where $\Re z$ is the real part of z. You may not use the Riemann mapping theorem.

Solution:

YOUR EXAM NUMBER ____

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Problem 5B. *Score:*

Find the radius of convergence of the Taylor series of $1/(e^x + e^{-x})$ at the point $x = 1$.

Solution:

YOUR EXAM NUMBER ___
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Problem 6B. *Score:*

Prove that if two real square matrices are similar by conjugation by a complex matrix, then they are similar by conjugation by a real matrix.

Solution:

Problem 7B. *Score:*

Given an example of two square complex matrices that have the same minimal polynomial and the same characteristic polynomial but are not similar.

Solution:

Problem 8B. *Score:*

Let n be a positive integer, and p any prime. Prove that over a finite field F of $p^{\phi(n)}$ elements the polynomial $x^n - 1$ factors into linear factors. (Here ϕ is Euler's totient function.)

Solution:

YOUR EXAM NUMBER ____
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Problem 9B. *Score:*

Give an example of a commutative ring which has an infinite descending chain of distinct prime ideals $I_1 \supset \cdots \supset I_n \supset \cdots$ (Recall that an ideal I of a commutative ring R is called prime if R/I is an integral domain.)

Solution: