Please do all five problems on this midterm. You have 80 minutes to work on the exam and 15 minutes to upload your work to Gradescope. You may consult the textbook, all the material on bCourses, the class piazza and your own notes. In case of questions, post a private note to instructors on piazza. Any clarifications or corrections that need to be promulgated will be added to a pinned post on piazza.

Explain all your answers fully; write in complete English sentences.

Not permitted: online searches, other uses of the internet, collaboration with other people (electronic or otherwise). Please act with honesty, integrity and respect for others.
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1. Using only pencil and paper (and possibly a primitive hand calculator), calculate \(7^{888}\) modulo 4, 25 and 100. Be sure to explain your work and your reasoning. (This applies to all problems.)

2. The number 6 is the product of its proper divisors: \(6 = 1 \times 2 \times 3\). So is 8: \(8 = 1 \times 2 \times 4\). Which integers \(> 1\) have this property?

3. What is the largest power of 2 dividing the binomial coefficient \(\binom{45}{23}\)?

4. Since \(15^2 - 15 - 1 = 225 - 16 = 209\), 15 is a solution to the congruence

\[ x^2 - x - 1 \equiv 0 \pmod{209}. \]

Without doing a major calculation, find another solution to the congruence. Taking note of the factorization \(209 = 11 \cdot 19\), determine the number of solutions to the congruence mod 209.

5. Let \(p\) be a prime number \(\geq 5\). This problem is inspired by the first Challenge on the slides for our class last Thursday:

   a. If there is an element \(a \in \mathbb{Z}/p\mathbb{Z}\) satisfying \(a^2 + a + 1 = 0\), show that \(p \equiv 1 \pmod{3}\).

   b. If \(-3\) is a square mod \(p\), show that there is an element \(a \in \mathbb{Z}/p\mathbb{Z}\) as in part (a).

To finish: Please copy and sign the statement below.

“As a member of the UC Berkeley community, I acted with honesty, integrity, and respect for others during this exam. The work that I am uploading is my own work. I did not collaborate with or contact anyone during the exam. I did not seek or obtain solutions from chegg.com or other sites. I adhered to all instructions for this examination.”