Math 110 midterm exam, February 24, 2020

1. Suppose that

$$v_1, \dots, v_n; x_1, \dots, x_n$$

is a linearly independent list of $2n$ vectors of the vector space V. Is it true that

$$v_1 + x_1, v_2 + x_2, \dots, v_n + x_n$$

is linearly independent? (Prove that your answer is correct.)

2. Suppose that U is a subspace of a finite-dimensional vector space V. If $S : U \to W$ is a linear map, prove that there is a linear map $T : V \to W$ whose restriction to U is S.

3. Let $T : V \to W$ be a linear map between finite-dimensional vector spaces. Let $w_1, \dots, w_n$ be a basis of W. Prove that there is a basis $v_1, \dots, v_m$ of V such that the matrix of T with respect to the two bases has its first row either entirely 0 or else of the form $(1, 0, 0, \dots, 0)$.

4. Suppose that V and W are 2-dimensional $\mathbf{F}$-vector spaces. Show that $\{\, T \in \mathcal{L}(V, W) \,|\, T \text{ is not surjective} \,\}$ is not a subspace of $\mathcal{L}(V, W)$.

5a. Let $T : V \to V$ be a linear operator such that $T \circ T$ is the identity map I on V. Show that the range of $T + I$ is contained in the null space of $T - I$ and that the range of $T - I$ is contained in the null space of $T + I$.

b. Show that V is the direct sum of the null space of $T - I$ and the null space of $T + I$.

6a. Let T be the operator on $\mathcal{P}(\mathbf{R})$ defined by the formula

$$Tp = \text{ the sum of the second and first derivatives of } p.$$

Find a basis for $\operatorname{null} T$.

b. Prove that T is surjective (onto).

As a member of the UC Berkeley community, I acted with honesty, integrity, and respect for others.