Let $d = 1$-manifold

and $\$ : d \to X$

$I = \frac{1}{2} \int dt \left( \frac{d\$}{dt} \right)^2$

Quantization leads to

functions on $X$ with

Hamiltonian = Laplacian

This case, and its supersymmetric extensions, are familiar mathematically.
LESS FAMILIAR IS THE CASE IN WHICH $\mathcal{D}$ IS REPLACED BY A TWO-MANIFOLD $\Sigma$

$$\Phi: \Sigma \rightarrow X$$

$$I = \int \langle d\Phi, +d\bar{\Phi} \rangle$$

$\Sigma$

= ACTION FOR HARMONIC MAP

QUANTUM THEORY:

FUNCTIONS ON FREE LOOP SPACE WITH APPROPRIATE ANALOG OF LAPLACIAN, ETC.
This second one is more typical of what physicists actually do and is the natural framework, for instance, for mirror symmetry.

Yesterday we started with a four-dimensional gauge theory

$$I = \frac{1}{4e^2} \int d^4 F_{\mu \nu} + ...$$

and "reduced" to one dimension solely to get something more familiar.
however, a different kind of reduction is essential for our problem!

Let $M = \Sigma \times C$

with a product metric multiply metric on $\Sigma$ by real constant $\tau$ and take $\tau$ large, to keep energy or action from diverging, the fields, restricted to $\Sigma \times C$ for $p \in \Sigma$
MUST OBEY CERTAIN MARVELOUS EQUATIONS DUE TO HITCHIN

AS $\phi$ VARIES, ONE GETS A SLOWLY VARYING MAP TO THE MODULI SPACE

OF HITCHIN'S EQUATIONS
EQUATIONS TO MINIMIZE THE ENERGY ARE HITCHIN'S EQUATIONS

\[ F - \phi \wedge \phi = 0 \]
\[ d_A \phi = d_A + \phi = 0 \]

THE MODULI SPACE OF SOLUTIONS OF THIS EQUATION IS A HYPER-KAHLER MANIFOLD \( \mathfrak{M}_H \) (OR \( \mathfrak{M}_H(G, C) \)) WITH MARVELOUS PROPERTIES
$\mathcal{N} = 4$ super Yang-Mills theory on $\mathcal{M} = \Sigma \times C$ can be usefully approximated, for our purposes, by a "sigma model" of maps

$$\Phi : \Sigma \to \mathcal{M}_H(G, C)$$

(actually $\Sigma$ should be extended to a supermanifold to give a supersymmetric model)

Bershadsky, Johansen, Sadov, Vafa (1995); Harvey, Moore (1995)
THE FOUR-DIMENSIONAL ELECTRIC-MAGNETIC DUALITY

\[ S : \varphi \rightarrow -\frac{1}{\eta^2} \]

BECOMES A "MIRROR SYMMETRY" OF THE TWO-DIMENSIONAL SIGMA MODEL, AND THIS INSTANCE OF "MIRROR SYMMETRY" (HAUSEL & THADDEUS 2001) GIVES ESSENTIALLY, AFTER SOME WORK, THE USUAL STATEMENTS OF GEOMETRIC LANDS....
B-MODEL OF $\mathcal{M}_H(LG, C)$

MIRROR SYMMETRY

A-MODEL OF $\mathcal{M}_H(G, C)$

Now start with

$\rho: \pi_1(C) \rightarrow LG_0$

According to Hitchin, $\rho$

defines a point in $\mathcal{M}_H(G)$. 
A "zero-brane" supported at this point is a brane of the B-model of $\mathcal{M}_H(\mathcal{G}, \mathcal{C})$. Its mirror will be an A-brane on $\mathcal{M}_H(\mathcal{G}, \mathcal{C})$. 
MH has some special properties, which I'll describe, such that A-branes on MH are naturally associated to "D-modules" on $\mathcal{M}(G, C) = \text{the moduli space of stable bundles}.$
Moreover, by returning to four dimensions and considering the "Wilson and 't Hooft operators" of the underlying four-dimensional gauge theory, one can argue that the $D$-module derived from $\rho : \pi_1(C) \to LG$ is a "Hecke eigensheaf."
MORE LINEAR COMBINATIONS
OF THE THREE REAL
HITCHIN'S EQUATIONS
EQUATIONS COMBINE TO A
SINGLE F HOMOMORPHIC
CONDITION \( \phi = \text{THE THIRD IS}
SOLUTION ENCLOSEMAP IS CONDITION.

HYPER-KAHLER.
LET'S SEE HOW THIS WORKS
IN ANY REAL PSEUDO-STRUCTURE
I, J, K (OR \( a I + b J + c K \)
USING HITCHIN'S NOTATION
\( a + b + c = 1 \))
COMPLEX STRUCTURE $J$

THE HOLOMORPHIC EQUATION IS

$$F - \phi \wedge \phi + i \partial \bar{\partial} \phi = 0$$

OR $$\mathcal{F} = 0$$

WHERE $$\mathcal{F} = \partial A + A \wedge A$$

$$A = A + i \phi$$

SO THE COMPLEX-VALUED CONNECTION $A$ IS FLAT.
BY CORLETTE & DONALDSON

THE SPACE OF COMPLEX FLAT CONNECTIONS WITH VANISHING "MOMENT MAP"

\[ D \phi = 0 \]

MOD \( G \)-VALUED GAUGE TRANSFORMATIONS

IS THE MODULI SPACE OF (STABLE) COMPLEX FLAT CONNECTIONS MODULO \( G \)-VALUED GAUGE TRANSFORMATIONS.
So \( mh \) in complex structure \( J \) is the moduli space of (stable) homomorphisms

\[ \rho : \pi_1(C) \to G_c \]

A homomorphism is stable if it is "irreducible" ... otherwise semistable (the moduli space of stable homomorphisms include points representing semi-stable equivalence classes.)
So this explains a statement I made before:

A homomorphism

\( \rho : \pi_1(C) \to G \mathbb{Z} \)

defines a point

in \( \mathcal{M}_H(G, C) \).
The holomorphic equation is

\[ dA \phi + i \times dA \phi = 0 \]

or more simply

\[ \partial_A \phi = 0 \]

where \( \phi \) is of type \( (1,0) \)

\[ \phi = \varphi + \overline{\varphi} \]

The moment map Eqn is

\[ F - \phi \Lambda \phi = 0 \]
THE HOLOMORPHIC EQN.

$$\bar{\Theta} A \phi = 0$$

DEPENDS ON A ONLY VIA ITS $$\bar{\Theta}_A$$ OPERATOR

$$\bar{\Theta}_A = \bar{\Theta} + A^{(0,1)}$$

i.e. ONLY VIA THE HOLOMORPHIC STRUCTURE WITH WHICH IT ENDOWS THE BUNDLE E.
Here we use the fact that in complex dimension one, there is no obstruction to integrability, i.e.

\((\bar{\partial}A)^2 = 0\) for any \(A\).

The equation

\[\bar{\partial}A \varphi = 0\]

tells us that \(\varphi\) represents an element of

\[H^1(C, K_C \otimes \omega (E))\]

\(K_C = \text{canonical bundle of } C\)
SO \((A, \varphi)\) with
\[
\bar{\partial}_A \varphi = 0
\]
Define a "Higgs Bundle"

i.e. a pair \((E, \varphi)\)
\[
\varphi \in \mathfrak{h}^*(K_C \oplus \text{ad}(E))
\]

Hitchin's theorem is that the moment map condition
\[
F - \phi \wedge \phi = 0
\]
plus the operation of dividing by \(G\)-valued gauge transformations
Gives us the "moduli space of stable Higgs bundles \((E, \phi)\)," up to holomorphic equivalence (i.e., \(G_\mathbb{C}\)-valued gauge transformations).
NOW IN COMPLEX STRUCTURE
I, M₄ HAS TWO MORE
MARVELOUS PROPERTIES:

(a) LET $E$ BE A STABLE BUNDLE

REPRESENTING A POINT $m = $ THE
MODULISPACE OF STABLE BUNDLES.

$H^2(C, K_C \otimes \text{ad}(E))$ IS
THE FIBER AT $E$ OF
$T^*m$ SO

$(E, \phi)$ DEFINE A POINT IN
$T^*m$
This construction gives an embedding of $T^*\mathcal{M}$ as a dense open set in $\mathcal{M}^4$.

For many purposes, one can approximate $\mathcal{M}^4$ by $T^*\mathcal{M}$. 
With respect to the holomorphic symplectic structure which extends that of the cotangent bundle $T^*M$, $\Omega^h$ is a completely integrable Hamiltonian system, commuting Hamiltonians from characteristic polynomial of $\Phi$. 
For example, for $G = SU(2)$

$$\dim_c M_H = 6g - 6$$

$g = \text{genus}(C)$

We need $3g - 3$ commuting Hamiltonians.

Let $V = H^0(C, K_C^2)$, of dimension $3g - 3$

We get a map

$$\pi : M_H \to V$$

By $(E, \varphi) \to T - \varphi^2 \in V$
And this map establishes complete integrability.

\[ X \times Y \times \cdots \]

Generic fiber = complex abelian variety.
Since $\mathfrak{M}$ is hyper-Kähler, in each complex structure $I, J, K$ there is a Kähler form $\omega_I, \omega_J, \omega_K$ and a holomorphic two-form $\Omega_I, \Omega_J, \Omega_K$.

We have $\Omega_I = \omega_J + i \omega_K$ and cyclic permutations.

$\Omega_I$, restricted to $T^*\mathfrak{M} \subset T^*\mathfrak{M}$, is the natural holomorphic two-form of $T^*\mathfrak{M}$. 
NOW OUR PICTURE

\[ \overline{1} \overline{1} \overline{1} \overline{1} \overline{1} \overline{1} \]

WITH GENERIC FIBER A TORUS
IS THE STROMINGER-YAU-ZASLOW
PICTURE OF MIRROR SYMMETRY

\[ \cdots \text{ BETWEEN THE B-MODEL IN} \]
\[ \text{COMPLEX STRUCTURE } J \text{ AND} \]
\[ \text{THE A-MODEL OF} \]
\[ w_K = \Im - \Omega I \]
However, it is drastically simpler than the generic case of mirror symmetry.

Because \( MH \) is hyperkähler and the fibers are holomorphic in complex structure \( J \).

This is actually a rare case of an \( SYZ \) fibration that can be described very explicitly (Hasegawa et al.).
The underlying four-dimensional S-duality $S: \mathbb{Z} \rightarrow -\mathbb{Z}$ reduces in two dimensions to the mirror symmetry.

B-model of $\mathcal{M}H(\mathbb{C}, \mathbb{C})$

in complex structure $J$

A-model of $\mathcal{M}H(\mathbb{C}, \mathbb{C})$

in symplectic structure $\omega$.
So \( \rho : \pi_1(C) \to \mathfrak{mH} \)

which determines a \( B \)-brane

\[ \text{ } \]

is dual to an \( A \)-brane

of SYZ type

\[ \text{ } \]

( fiber endowed with a flat line bundle )
Moreover, because

\[ m_H \cong T^* \mathcal{M} \]

there is a natural map

\[ \{ \text{A-branes on } m_H \} \]

to

\[ \{ \text{D-modules on } m \} \]

so this gives the association

\[ \{ \text{flat LG bundle } \langle E \rightarrow C \rangle \} \]

to

\[ \{ \text{D-module on } \mathcal{M}(G, C) \} \]

of the geometric Langlands program.
But I want to explain why the A-brane corresponding to a homomorphism \( p \) is actually a "Hecke eigensheaf" for this, we go back to four dimensions.