Chain Conditions of Horn and Tarski

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Global Chain Conditions

Definition

A subset A of a Boolean algebra $\mathbb{B}$ is **centered** if $\bigwedge F > 0$ for all finite $F \subseteq A$.
For a positive integer n, a subset A of a Boolean algebra $\mathbb{B}$ is n-**centered** if $\bigwedge F > 0$ for all $F \subseteq A$ of cardinality at most n.

When $n = 2$, the n-centered subsets are also known as **linked subsets**.

Definition

A Boolean algebra $\mathbb{B}$ is σ-**centered** if $\mathbb{B}^+$ can be decomposed into countably many centred subsets.
For a positive integer n, a Boolean algebra $\mathbb{B}$ is σ-n-**centered** if $\mathbb{B}^+$ can be decomposed into countably many n-centred subsets.

Proposition (Horn-Tarski, 1948)

Every σ-centered Boolean algebra supports a strictly positive finitely additive measure.

Proposition

A Boolean algebra $\mathbb{B}$ is σ-centered if and only if its Stone space is **separable**

Proposition

A σ-2-centered Boolean algebras can have cardinality at most continuum.

Proposition (Dow-Steprans, 1994)

The following are equivalent for a measure algebra $\mathbb{B}$:

1. $\mathbb{B}$ *is σ-2-centered.*
2. $\mathbb{B}$ *is σ-n-centered for all n.*
3. $\mathbb{B}$ *has density at most continuum.*

Definition (Horn and Tarski, 1948)

A Boolean algebra $\mathbb{B}$ satisfies σ-**finite chain condition** if it can be decomposed as $\mathbb{B} = \bigcup_{n=1}^{\infty} \mathbb{B}_n$ such that no $\mathbb{B}_n$ contains an infinite sequence (a_k) such that $a_k \wedge a_l = 0$ for $k \neq l$.

Definition (Horn and Tarski, 1948)

A Boolean algebra $\mathbb{B}$ satisfies σ-**bounded chain condition** if it can be decomposed as $\mathbb{B} = \bigcup_{n=1}^{\infty} \mathbb{B}_n$ such that for every n there is $k(n)$ such that $\mathbb{B}_n$ contains no sequence $(a_k)_{k=1}^{k(n)}$ such that $a_k \wedge a_l = 0$ for $k \neq l$.

Local Chain Conditions

Definition

A Boolean algebra $\mathbb{B}$ satisfies the **countable chain condition** or **Souslin condition** if every uncountable subset of $\mathbb{B}^+$ contains two elements with positive intersection.

Problem (Souslin, 1920)

Are the following two conditions equivalent for every Boolean algebra with an **ordered set of generators***?*

1. $\mathbb{B}$ *satisfies the* **countable chain condition***.*
2. $\mathbb{B}$ *is* σ-**centered.**

Souslin Hypothesis is the statement that the answer to this question is **positive**.

Definition

A Boolean algebra $\mathbb{B}$ satisfies the **Knaster condition** if every **uncountable subset** of $\mathbb{B}^+$ contains an **uncountable** 2-**centered subset**.

Proposition

Every measure algebra satisfies Knaster condition.

Proposition (Horn-Tarski, 1948)

Every Boolean algebra satisfying the σ-**finite chain condition** *satisfies also* **Knaster condition**.

Proposition (Knaster, 1941)

The following two conditions equivalent for every Boolean algebra with an **ordered set of generators**:

1. $\mathbb{B}$ *satisfies the* **Knaster condition**.
2. $\mathbb{B}$ *is* σ-**centered.**

Problem (Knaster and Szpilrajn, 1941)

Are the following two conditions equivalent for **every** *Boolean algebra?*

1. $\mathbb{B}$ *satisfies the* **countable chain condition**.
2. $\mathbb{B}$ *satisfies the* **Knaster condition.**

Let the **Knaster Hypothesis** be the statement that **every** Boolean algebra satisfying the **countable chain condition** satisfies also **Knaster condition**

Proposition (Knaster, 1941)

Knaster Hypothesis *implies* **Souslin Hypothesis**.

Definition

A Boolean algebra $\mathbb{B}$ satisfies the **Shanin condition** if every **uncountable subset** of $\mathbb{B}^+$ contains an **uncountable centered subset**.

Example

Compact groups satisfy **Shanin condition**.

Theorem (Shanin, 1948)

Shanin condition *is preserved by arbitrary Tychonoff* **products**.

Let the Shanin Hypothesis be the statement that every Boolean algebra satisfying the **countable chain condition** satisfies **Shanin condition**.

Clearly, **Shanin Hypothesis** implies **Knaster Hypothesis** .

Problem

Are the two hypotheses equivalent?

Products

Theorem (Szpilrajn, 1941)

Knaster condition *is preserved by arbitrary Tychonoff* **products***.*

Problem (Szpilrajn, 1941)

Is the **countable chain condition** *invariant under* **products***?*

Let the **Szpilrajn Hypothesis** be the hypothesis that the **countable chain condition** is invariant under **products**.

Theorem (Kurepa, 1950)

Szpilrajn Hypothesis *implies* **Souslin Hypothesis***.*

Borel theory

Theorem (Shelah, 1995)

The **countable chain condition** *is* **productive** *in the class of* **Borel-generated** *Boolean algebras.*

In particular, **Szpilrajn Hypothesis is true** for Borel generated Boolean algebras.

Theorem (T., 1991)

If $\mathfrak{b} = \omega_1$, *the* **Borel-generated** *Boolean algebra* $\mathbb{T}([0,1])$ **fails** *to satisfy* **Knaster condition**

In particular, the **Knaster Hypothesis is not provable** on the basis of standards axioms of set theory even if we restrict it to the **Borel context**. Recall from the first lecture:

Theorem (T., 1991)

For every separable metric space X, *the Boolean algebra* $\mathbb{T}(X)$ *satisfies the* **countable chain condition**.

Theorem (Erdös, 1963)

The measure algebra of $[0, 1]$ *is a Borel-generated Boolean algebra satisfying* **Knaster condition** *but* **failing Shanin condition** *if the Lebesgue measure is not* $\aleph_1$*-additive.*

Theorem (Solovay, 1970)

The **amoeba algebra** *also has these properties but it does not support a measure.*

Theorem (T., 1986)

For every positive integer $n \geq 2$ *there is a Borel-generated Boolean algebra that is* σ-n-**centered** *but is* **not** σ-$(n+1)$-**centered**, *and in fact fails to have* $(n+1)$-**Knaster property** *if* $\mathfrak{b} = \omega_1$

Theorem (T., 1991)

There is a Borel-generated Boolean algebra satisfying **Shanin condition** *but* **fails** σ**-finite chain condition**.

Martin's axiom

Definition

A subset D of a **partially ordered set** $\mathbb{P}$ is **dense** if

$$(\forall p \in \mathbb{P})(\exists q \in D) \;\; q \leq p.$$

A subset F of $\mathbb{P}$ is a **filter** if

1. $(\forall p \in F)(\forall q \in F)(\exists r \in F) \;\; r \leq p \wedge r \leq q$,
2. $(\forall p \in F)(\forall q \in \mathbb{P}) \;\; p \leq q \to q \in F$.

Martin's axiom asserts that for every poset $\mathbb{P}$ satisfying the **countable chain condition** and every family $\mathcal{D}$ of **cardinality less than continuum** many dense subsets of $\mathbb{P}$ there is a filter of F of $\mathbb{P}$ intersecting all the sets from $\mathcal{D}$.

$\mathrm{MA}_{\aleph_1}$ is Martin's axiom for families $\mathcal{D}$ of cardinality at most $\aleph_1$.

Theorem (Solovay-Tennenbaum, 1971)

Souslin Hypothesis *is consistent with the* **negation of the Continuum Hypothesis**.

Theorem (Jensen, 1974)

Souslin Hypothesis *is consistent with the* **Continuum Hypothesis**.

Theorem (Solovay-Tennenbaum, 1971)

Martin's axiom *is consistent with the* **negation of the Continuum Hypothesis**.

Theorem (Martin-Solovay, 1971)

$\mathrm{MA}_{\aleph_1}$ *implies* **Souslin Hypothesis** .

Theorem (Martin-Solovay, 1971)

Martin's axiom *implies that the* **Lebesgue measure is κ-additive** *for every cardinal κ less than the continuum.*

Theorem (Kunen, 1971)

$\mathrm{MA}_{\aleph_1}$ *implies* **Shanin Hypothesis**.

Theorem (Hajnal-Juhasz, 1971)

Martin's axiom *implies that every Boolean algebra of* **cardinality less that the continuum** *satisfying the* **countable chain condition** *is* σ-**cetered**.

Theorem (Todorcevic-Velickovic, 1987)

The following are equivalent

1. *Martin's axiom.*
2. *Every Boolean algebra of* **cardinality less that the continuum** *satisfying the* **countable chain condition** *is* σ-**cetered**.

Theorem (Todorcevic-Velickovic, 1987)

The following are equivalent

1. $\mathrm{MA}_{\aleph_1}$.
2. **Shanin Hypothesis**.

Compact Spaces

Definition

A topological space X has **countable tightness** or is **countably tight** if for every set $A \subseteq X$ and $x \in \overline{A}$ there is countable $B \subseteq A$ such that $x \in \overline{B}$.

Example

If X is a **sequential space** (i.e., if sequentially closed subsets of X are closed), then X is **countably tight**.

Definition

A π-**basis** for a topologicl space X is a collection $\mathcal{P}$ of nonempty open subsets of X such that for every nonempty open subset U of X there is P in $\mathcal{P}$ such that $P \subseteq U$.

Theorem (Shapirovski, 1987)

Every compact countably tight space has a **point-countable** *π-basis.*

Corollary (Shapirovski, 1987)

$\mathrm{MA}_{\aleph_1}$ *implies that every* **countably tight** *compact space satisfying the* **countable chain condition** *is* **separable**.

Corollary (Juhasz, 1970)

$\mathrm{MA}_{\aleph_1}$ *implies that every* **first countable** *compact space satisfying the* **countable chain condition** *is* **separable**.

Theorem (T., 2000)

The following are equivalent

1. $\mathrm{MA}_{\aleph_1}$.
2. *Every compact* **first countable** *space satisfying the* **countable chain condition** *is* **separable**.
3. *Every compact* **first countable** *space satisfying the* **countable chain condition** *satisfies the* **Shanin condition.**

Two subsets A and B of a topological space X are **separated** whenever

$$\overline{A} \cap B = \emptyset = A \cap \overline{B}.$$

We say that X satisfies the separation axiom T_5 whenever separated subsets of X are **separated by open sets**.

Example

Metric spaces and ordered space are T_5.

Theorem (Shapirovski, 1975 ; T., 2000)

$\mathrm{MA}_{\aleph_1}$ *implies that every* T_5 *compact space satisfying the* **countable chain condition** *is* **separable** *and, in fact, has a* **countable** π**-base**.

Biorthogonal systems

A **biorthogonal system** in a Banach space X is a sequence

$$\{(x_i, f_i) : i \in I\} \subseteq X \times X^*$$

such that $f_i(x_j) = \delta_{ij}$.

Example

The function space $C(S)$ of the **split interval** $S = [0,1] \times \{0,1\}$ has biorthogonal system

$$\{(f_x, \mu_x) : x \in [0,1]\}$$

where f_x is the characteristic function of the set $\{s \in S : s < (x,1)\}$ and $\mu_x = \delta_{(x,1)} - \delta_{(x,0)}$.

Theorem (T., 1998)

$\mathrm{MA}_{\aleph_1}$ *implies that any function space* $C(K)$ *in which all* **biorthogonal systems are countable** *must be* **separable**.

Radon measures

A **Radon measure** on a topological space X is a σ-additive probability measure μ defined of all open subsets of X such that

$$\mu(A) = \sup\{\mu(K) : K \subseteq A \text{ is compact } \}$$

for every measurable set $A \subseteq X$.

Problem

Which conditions on a compact space X guarantee that $L^1(X,\mu)$ is **separable** *for every Radon probability measure μ on X?*

Proposition

if a compact space X maps onto $[0,1]^{\omega_1}$ then it supports a **non-separable** *Radon probability measure .*

Theorem (Fremlin, 1997)

$\mathrm{MA}_{\aleph_1}$ *implies that every compact space that supports* **non-separable Radon measure** *maps onto $[0,1]^{\omega_1}$.*

Higher Forcing Axiom

Definition (Shelah, 1980)

A poset $\mathbb{P}$ is **proper** if
for all $p \in \mathbb{P}$ and every **countable elementary submodel** M of some large enough structure of the form $(H(\theta), \in)$ such that $p, \mathbb{P} \in M$
there is $q \leq p$ that is $(M, \mathbb{P})$-**generic** i.e.,
for every dense-open subset D of $\mathbb{P}$ such that $D \in M$ and every $r \leq q$ there is $\bar{r} \in D \cap M$ that is **compatible** with r.

Example

Every poset satisfying the **countable chain condition** is **proper.**

Theorem (Shelah, 1980)

Any **countable-support iteration** *of proper poset is proper.*

Definition (Baumgartner, 1980)

Let the **Proper Forcing Axiom** be the statement that for every **proper poset** $\mathbb{P}$ and every family $\mathcal{D}$ of **no more than** $\aleph_1$ many dense subsets of $\mathbb{P}$ there is a filter of F of $\mathbb{P}$ intersecting all the sets from $\mathcal{D}$.

Thus, PFA is a strengthening of $\mathrm{MA}_{\aleph_1}$.

Theorem (Baumgartner, 1980)

The Proper Forcing Axiom is consistent relative to the consistency of a supercompact cardinal.

Theorem (T., 1984)

PFA implies that $\square_\kappa$ *fails for all cardinals* $\kappa \geq \omega_1$.

Theorem (Baumgartner, 1973, 1984)

PFA implies that all $\aleph_1$*-dense sets of reals are order-isomorphic.*

Theorem (Woodin, 1982, 1987)

PFA implies that every norm on a Banach algebra of the form $C(K)$ *is equivalent to the supremum norm.*

Theorem (T., 1989)

PFA implies that every **open graph** *on a separable metric space in which all* **complete subgraphs are countable** *must be* **countably chromatic**.

Theorem (Farah, 2008)

The **Open Graph Axiom** *implies that all* **automorphisms** *of the* **Calkin algebra** *are* **inner**.

Five Cofinal Types

Theorem (T., 1985)

PFA implies that for every **directed set** *D of cardinality at most $\aleph_1$ there is a directed set E on the list*

$$1, \omega, \omega_1, \omega \times \omega_1 \text{ and } [\omega_1]^{<\omega}$$

such that D and E are of the same **cofinal type** *i.e., can be isomorphically embedded as cofinal subsets of a single directed set.*

Example

Given a directed set D, let

$$\mathcal{I}_D = \{A \subseteq D : A \text{ is strongly unbounded}\}.$$

Then D is of cofinal type $[\omega_1]^{<\omega}$ if and only if there is uncountable $X \subseteq D$ such that all countable subsets of X are in $\mathcal{I}_D$.

Theorem (T., 1985, 2000)

PFA implies the **P-Ideal Dichotomy**.

Five Linear Orderings

Theorem (Baumgartner, 1973, 1984; Moore, 2005)

PFA implies that every uncountable linear ordering contains an isomorphic copy of the one on the list

$$\omega_1, \omega_1^*, B, C \text{ and } C^*,$$

where B is any set of reals of cardinality $\aleph_1$, and where C is any uncountable linear ordering whose cartesian square can be covered by countably many chains.

Theorem (Martinez-Ranero, 2012)

PFA implies that the class of all uncountable linear orderings that are **orthogonal** *to ω_1, ω_1^* and $\mathbb{R}$ is* **well-quasi-ordered***.*

Biorthogonal systems again

Theorem (T., 2006)

PFA implies that **every Banach space** *in which all* **biorthogonal systems are countable** *must be* **separable**.

Theorem (T., 2006)

PFA implies that every Banach space X if density at most $\aleph_1$ has a **quotient** *with a* **Schauder basis** *which can be assumed to be or length ω_1 if X is not separable.*

Problem (Banach, 1930; Pelczynski, 1964)

Does every infinite-dimensional Banach space has an infinite-dimensional quotient with a Schauder basis?

Chain conditions again

Theorem (T., 1983)

PFA implies that every regular space in which all **discrete subspaces are countable** *must be* **Lindelöf***.*

Theorem (Balog, 1989)

PFA implies that every **countably compact** *space in which all* **free sequences are countable** *must be* **Lindelöf** *and, therefore,* **compact.**

Definition

A **free sequence** in a topological space X is a subset Y with a well-ordering $<$ such that

$$\overline{\{y \in Y : y < x\}} \cap \overline{\{y \in Y : x \leq y\}} = \emptyset.$$

It is known that a compact space X is **countably tight** if and only **all free frequencies in X are countable**.

Corollary (Balog, 1989)

PFA implies that all compact **countably tight** *spaces are* **sequential**.

Corollary (Dow, 1990)

PFA implies that every compact **countably tight** *space has a* G_δ-**point**.

Theorem (Nyikos, Soukup, Velickovic, 1995)

OGA implies every compact **separable** T_5*-space is* **countabley tight**.

Definition

Sequential space of **sequential index 1** are called **Fréchet spaces**
Thus, X is a **Fréchet space** if for every set $A \subseteq X$ and $x \in \overline{A}$ there is a sequence $(a_n) \subseteq A$ such that $a_n \to x$.

Example

A complete Boolean algebra satisfying the **countable chain condition** and the **weak countable distributive law**
is a **Fréchet space** in its topology of sequential convergence.

Theorem (T., 2000)

PFA implies that every compact T_5-space satisfying the **countable chain condition** *is* **Fréchet**.

A relativization of MA and PFA

A **coherent Souslin tree** is a downwards closed subtree $\mathbb{S}$ of $\omega^{<\omega_1}$ such that for all $s, t \in \mathbb{S}$,

$$\{\alpha \in \mathrm{dom}(s) \cap \mathrm{dom}(t) : s(\alpha) \neq t(\alpha)\} \textbf{ is finite } .$$

Fix a coherent Souslin tree $\mathbb{S}$ from now on.

Let $\mathrm{MA}_{\aleph_1}(\mathbb{S})$ be the restriction of $\mathrm{MA}_{\aleph_1}$ to posets $\mathbb{P}$ such that $\mathbb{P} \times \mathbb{S}$ satisfies the **countable chain condition.**

Similarly, let $\mathrm{PFA}(\mathbb{S})$ be the restriction of PFA to all proper posets $\mathbb{P}$ that **preserve** $\mathbb{S}$, a condition essentially equivalent to the **product** $\mathbb{P} \times \mathbb{S}$ **is proper.**

Theorem (Larson-Todorcevic, 2002)

If $\mathrm{MA}_{\aleph_1}(\mathbb{S})$ *holds then* $\mathbb{S}$ *forces that a compact space* X *is* **metrizable** *if and only if* **its square** X^2 **is** T_5.

Theorem (T., 2011)

$\mathrm{PFA}(\mathbb{S})$ *implies that* $\mathbb{S}$ *forces the* **Open Graph Axiom**.

Theorem (T., 2011)

$\mathrm{PFA}(\mathbb{S})$ *implies that* $\mathbb{S}$ *forces the* **P-Ideal Dichotomy**.

Theorem (T., 2011)

$\mathrm{PFA}(\mathbb{S})$ *implies that* $\mathbb{S}$ *forces that if* X *is a compact* T_5*-space satisfying* **countable chain condition** *then* **all closed subsets of** X **are** G_δ.

How critical is the split interval $[0,1] \times \{0,1\}$?

Problem

Is the split interval the only absolute obstruction towards the metrizability of a compact T_5 space satisfying the countable chain condition?
More precisely, can one find a non-metrizable compact space X with all closed subsets G_δ but X is **orthogonal to the split interval**, *i.e., shares no uncountable subspace with* $[0,1] \times \{0,1\}$.

We shall come back to this problem in the third lecture.