Problem 1A.

A non-empty metric space $X$ is said to be connected if it is not the union of two non-empty disjoint open subsets, and is said to be path-connected if for every two points $a$, $b$ there is a continuous map $f$ from the unit interval to $X$ with $f(0) = a$, $f(1) = b$.

(a) Prove that every path-connected space is connected.

(b) If $X$ is the subset of the plane consisting of the points $(x, y)$ with $x = 0$ or $x > 0$, $y = \sin(1/x)$ show that $X$ is connected but not path-connected.

Solution:

(a) If a space $X$ is not connected, it is the union of 2 disjoint open subsets $A$ and $B$. Choose $a$ in $A$ and $b$ in $B$. Then for any continuous map $f$ from the unit interval to $X$ with $f(0) = a$, $f(1) = b$ the inverse images of $A$ and $B$ give a partition of the unit interval into 2 disjoint nonempty open subsets. This is not possible, as the supremum of one of the open subsets cannot be in either.

(b) This space is the union of the $y$ axis $A$ and the graph $B$ of $y = \sin(1/x)$ both of which are connected. So the only possible partition into 2 disjoint nonempty open subsets is $A$ union $B$, which is not possible as $A$ and $B$ are not open subsets. So the space is connected.

To show it is not path connected, take any map from the unit interval to it with $f(0)$ in the $y$ axis. Let $x$ be the supremum of points whose image is in the $y$ axis. For a small neighborhood of $f(x)$ the largest connected subset containing $f(x)$ is in the $y$ axis, so some neighborhood of $x$ must have image in the $y$ axis. This forces $x$ to be 1 otherwise there are points above it whose image is not in the $y$-axis. So there are no maps of the unit interval to $X$ with $f(0)$ in the $y$ axis and $f(1)$ not, so the space is not path connected.

Problem 2A.

Find an irreducible polynomial over the integers with $2 \cos(2\pi/7)$ as a root, and use this to show that it is not contained in any extension of the rational numbers of degree a power of 2.

Solution:

Write $x = 2 \cos(2\pi/7) = z + 1/z$ with $z^7 = 1$, $z \neq 1$. Then $x^3 + x^2 - 2x - 1 = z^{-3} + z^{-2} + z^{-1} + 1 + z + z^2 + z^3 = 0$. This polynomial is irreducible as it is irreducible mod 2. So $x$ generates a field extension of degree 3, so any field containing $x$ has degree divisible by 3, so the degree cannot be a power of 2.

Problem 3A.

Use residues to compute

$$\int_0^\infty \frac{dx}{x^4+1}.$$
Solution: This is half of \( \int_\infty^{-\infty} \frac{dx}{x^4+1} \), and therefore \( \pi i \) times the sum of residues in the upper half plane (using the usual semicircular contour and the residue theorem). The residues are at \( (i \pm 1)/\sqrt{2} \) and have values \( 1/4(i \pm 1) \) so their sum is \( -\sqrt{2}i/4 \). The integral is therefore \( \pi/2\sqrt{2} \).

Problem 4A.

Let \( M_n(k) \) be the \( n \) by \( n \) matrices over a field \( k \). Find (with proof) all linear maps \( f \) from \( M_n(k) \) to \( k \) such that \( f(AB) = f(BA) \) for all matrices \( A \) and \( B \).

Solution:

Taking commutators \( AB - BA \) of suitable matrices \( A \) and \( B \) each with just one nonzero entry shows that any matrix with just one nonzero entry off the diagonal, or with 2 nonzero entries on the diagonal with sum zero, is of this form. In other words all matrices of trace zero are linear combinations of matrices of the form \( AB - BA \). Any matrix \( AB - BA \) has image 0 under \( f \). So the linear maps are just those that vanish on all matrices of trace 0, and so are multiples of the trace.

Problem 5A.

Show that the function equal to \( e^{-1/x^2} \) for \( x \neq 0 \) and equal to 0 at \( x = 0 \) is infinitely differentiable at all real numbers, and find its Taylor series at \( x = 0 \).

Solution: By induction any higher derivative is \( (\text{polynomial in } 1/x) e^{-1/x^2} \) for \( x \neq 0 \). This has limit 0 at \( x = 0 \). So all higher derivatives exist and are all 0 at 0. The Taylor series at 0 is therefore \( 0 + 0x + 0x^2 + .... \).

Problem 6A.

If \( N \) is the integer \( 2^4 \cdot 3^3 \cdot 5^2 \cdot 7 \) find the smallest positive integer \( m \) such that \( x^m \equiv 1 \mod N \) for all integers \( x \) coprime to \( N \).

Solution:

By the Chinese remainder theorem \( \mathbb{Z}/(mn\mathbb{Z}) \) is \( \mathbb{Z}/(m\mathbb{Z}) \times \mathbb{Z}/(n\mathbb{Z}) \) for \( m,n \) coprime, so it is enough to solve this question for prime powers. If \( N \) is \( 2^4 \) or \( 3^3 \) or \( 5^2 \) or \( 7 \) then the smallest \( m \) as above is \( 4, 2 \times 3^2, 4 \times 5, \) and \( 6 \) respectively. So the solution is the least common multiple of these, which is \( m = 2^2 \times 3^2 \times 5 = 180 \).

Problem 7A.

If \( 0 < r < 1 \), find

\[ \sum_{k=0}^{\infty} r^k \cos(k\theta). \]

Your final answer should not involve any complex numbers.

Solution:
Put \( z = re^{i\theta} \). It’s enough to find the real part of
\[
\sum_{k=0}^{\infty} z^k = \frac{1}{1 - z} = \frac{1 - re^{-i\theta}}{1 - re^{i\theta}} = \frac{1 - r \cos(\theta) + ir \sin(\theta)}{1 - 2r \cos(\theta) + r^2},
\]
so the answer is
\[
\frac{1 - r \cos(\theta)}{1 - 2r \cos(\theta) + r^2}.
\]

**Problem 8A.**

For each of the following 4 statements, give either a counterexample or a reason why it is true.

(a) For every real matrix \( A \) there is a real matrix \( B \) with \( B^{-1}AB \) diagonal.

(b) For every symmetric real matrix \( A \) there is a real matrix \( B \) with \( B^{-1}AB \) diagonal.

(c) For every complex matrix \( A \) there is a complex matrix \( B \) with \( B^{-1}AB \) diagonal.

(d) For every symmetric complex matrix \( A \) there is a complex matrix \( B \) with \( B^{-1}AB \) diagonal.

**Solution:**

To generate counterexamples, observe that a nonzero 2 by 2 matrix with trace and determinant 0 cannot be diagonalizable as both eigenvalues vanish.

(a) False \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \)

(b) True as Hermitian matrices are diagonalizable

(c) False \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \)

(d) False \( \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \)

**Problem 9A.**

The Catalan numbers \( C(n) \) satisfy \( C(0) = 1, C(n) = C(0)C(n-1) + C(1)C(n-2) + \cdots + C(n-1)C(0) \) if \( n > 0 \). Find the function \( \sum_{n=0}^{\infty} C(n)x^n \) and use this to evaluate \( C(n) \).

**Solution:**

If \( f(x) = \sum_{n=0}^{\infty} C(n)x^n \) then \( xf(x)^2 + 1 = f(x) \) so \( f(x) = \frac{(1 - \sqrt{1 - 4x})/2x}. \)

Expanding this by the binomial series shows that \( C(n) = \frac{(2n)!}{n!(n+1)!}. \)

**Problem 1B.**

Let \( D \) be an open subset of \( \mathbb{R}^2 \) (with the topology induced by the euclidean metric), and assume that it contains the closed unit square
\[
[0,1] \times [0,1] = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}.
\]

Show that \( D \) contains the partially-open rectangle
\[
[0,1] \times [0,1 + \epsilon] = \{(x,y) : 0 \leq x \leq 1, 0 \leq y < 1 + \epsilon\}
\]
for some \( \epsilon > 0. \)
Solution: For each $x \in [0, 1]$, we have $(x, 1) \in D$, so there is an $\epsilon(x) > 0$ such that $B_{\epsilon(x)}(x, 1) \subseteq D$ (here $B_r(P)$ denotes the open ball of radius $r$ centered at the point $P$). Therefore the open square with corners $(x \pm \epsilon(x), 1 \pm \epsilon(x))$ is also contained in $D$.

As $x$ varies over $[0, 1]$, the open sets $(x - \epsilon(x), x + \epsilon(x))$ cover the compact set $[0, 1]$, so there is a finite subcollection $\{(x_i - \epsilon(x_i), x_i + \epsilon(x_i)) : i = 1, \ldots, n\}$ that covers $[0, 1]$. Let $\epsilon$ be the smallest of $\epsilon(x_1), \ldots, \epsilon(x_n)$. Then $D$ contains the set $[0, 1] \times [0, 1 + \epsilon]$.

**Problem 2B.**

Prove that every group is isomorphic to a group of permutations. Prove that every finite group is isomorphic to a group of even permutations of a finite set.

Solution: Let $G$ be a finite group, and let $S_G$ denote the group of permutations of $G$. For each $g \in G$ define $\sigma_g : G \rightarrow G$ by $\sigma_g(x) = gx$. This function is one-to-one because $gx = gy$ implies $x = y$ by cancellation, and it is onto because it is one-to-one and $G$ is a finite set. Therefore $\sigma_g$ lies in $S_G$ for all $g$. This is a group homomorphism $G \rightarrow S_G$ because $\sigma_g(\sigma_h(x)) = ghx = \sigma_{gh}(x)$ for all $x$, so $\sigma_g \circ \sigma_h = \sigma_{gh}$ for all $g, h \in G$. This group homomorphism is injective because if $\sigma_g$ lies in the kernel then $\sigma_g(e) = e$ (where $e \in G$ denotes the identity element); however, $\sigma_g(e) = ge = g$, implying $g = e$ and therefore the kernel is trivial. Thus this homomorphism is an isomorphism of $G$ with a subgroup of the permutation group $S_G$.

To get $G$ isomorphic to a group of even permutations, do the same procedure to show that $G$ is isomorphic to a subgroup of even permutations of $S_{2G}$, where $2G$ denotes the disjoint union of $G$ with itself.

**Problem 3B.**

Prove that there are infinitely many complex numbers $z$ with $e^z = z$. (Hint: consider the behavior of $e^z - z$ on the boundary of a large square.)

Solution: Consider the change of argument of $e^z = z$ on a large square of side $2R$ centered at $0$. The change of argument on the left hand edge and the top and bottom is bounded as $R$ tends to infinity by easy estimates. The change of argument on the right hand edge is about that of $e^z$ which increases linearly with $R$. So up to a bounded term, the total change in argument increases linearly with $R$. As the number of zeros is (change in argument)/$2\pi$, the function has an infinite number of zeros.

**Problem 4B.**

For which real numbers $x$ does the matrix-valued series $\sum_{n=0}^{\infty} x^n A^n$ converge, where $A$ is the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$?

Solution: The matrix has eigenvalues $(1 \pm \sqrt{5})/2$. Diagonalizing the matrix shows that the series converges for $|x|$ less than the inverse of the absolute value of the largest eigenvalue, so for $|x| < (-1 + \sqrt{5})/2$. 


Problem 5B.

(a) Evaluate \( I(n) = \int_0^\pi \sin(x)^n \, dx \) for \( n \) a non-negative integer.
(b) Prove that \( I(n) > I(n + 1) > 0 \)
(c) Evaluate the infinite product \( \frac{1}{2} \times \frac{3}{2} \times \frac{3}{4} \times \frac{5}{4} \times \frac{5}{6} \times \cdots. \)

Solution: (a) \( I(0) = \pi, I(1) = 2 \). Integration by parts gives 
\[
I(n) = \int_0^\pi (n-1) \sin(x)^{n-2} \cos(x) \cos(x) \, dx = (n-1)(I(n-2) - I(n)) \quad \text{so} \quad I(n) = \frac{n-1}{n} I(n-2).
\]
So \( I(2n) = \frac{1}{2} \frac{3}{4} \cdots \frac{2n-1}{2n} \). \( I(2n+1) = \frac{2 \frac{1}{2} \frac{3}{4} \cdots \frac{2n-1}{2n+1}}{2} \).
(b) Follows because \( \sin(x)^n > \sin(x)^{n+1} > 0 \).
(c) The product of \( 2n-1 \) terms of the product is \( \frac{2\frac{1}{2} \frac{3}{4} \cdots \frac{2n-1}{2n+1}}{2} \) which is less than \( \frac{2}{\pi} \) and the product of \( 2n \) terms is \( \frac{I(2n)/\pi}{I(2n+1)/2} \) which is greater than \( \frac{2}{\pi} \). As the product converges by the “alternating product test” it is \( \frac{2}{\pi} \).

Problem 6B.

Prove that the polynomial \( x^4 + x + 2011 \) is irreducible over \( \mathbb{Q} \).

Solution:
It is sufficient to check irreducibility in \( \mathbb{Z}[x] \) and for this it is enough to check irreducibility mod 2. For this just check it has no linear factors and is not divisible by the only irreducible degree 2 mod 2 polynomial \( x^2 + x + 1 \).

Problem 7B.

Prove that the real and imaginary parts of a holomorphic complex function are harmonic (solutions of Laplace’s equation \( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \)). Find two linearly independent real solutions of Laplace’s equation in two variables that are homogeneous polynomials of degree 6.

Solution:
The fact that the real and imaginary parts are harmonic follows easily from the Cauchy-Riemann equations. Two homogeneous harmonic polynomials of degree 6 are the real and imaginary parts of \( (x + iy)^6 \) which are \( x^6 - 15x^4y^2 + 15x^2y^4 - y^6 \) and \( 6xy^5 - 20x^3y^3 + 6x^5y \).

Problem 8B.

For \( p \) a prime show that the number of non-singular \( n \times n \) matrices with entries in the field with \( p \) elements has the form \( p^r \) where \( s \equiv (-1)^n \pmod{p} \), and find \( r \).

Solution:
The number of nonsingular matrices is the number of bases which is \( (p^n - 1)(p^n - p)\cdots(p^n - p^{n-1}) \) (product of number of ways to choose first, second, ...n’th basis vectors). So \( r = 0 + 1 + \cdots (n - 1) = (n - 1)n/2 \) and \( s = (p^n - 1)(p^{n-1} - 1)\cdots(p - 1) \) is congruent to \( (-1)^n \) (mod \( p \)).

Problem 9B.
For each of the following statements, either prove it or give a counterexample:

(a) If \( f(x) \) and \( f_n(x) \) are continuous real-valued functions on the unit interval, and \( \lim_{n \to \infty} f_n(x) = f(x) \) for all \( x \), then \( \lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \int_0^1 f(x) \, dx \).

(b) If \( g(m, n) \) is real for all integers \( m, n \), and \( \sum_{m=0}^{\infty} (\sum_{n=0}^{\infty} g(m, n)) \) and \( \sum_{n=0}^{\infty} (\sum_{m=0}^{\infty} g(m, n)) \) are both defined, then they are equal.

(c) If the functions \( h_n(x) \) are continuous real-valued functions on the unit interval, and \( \lim_{n \to \infty} h_n(x) = h(x) \) for all \( x \), then \( h(x) \) is a continuous function of \( x \).

Solution:

(a) False. Take \( f_n \) to be 0 for \( x \geq 1/n \) and \( x = 0 \) and to have integral 1. Then \( f = 0 \) does not have integral 1.

(b) False. Take \( g(m, n) \) to be 1 if \( m = n \), \(-1\) if \( m = n + 1 \), 0 otherwise.

(c) False. Take \( h_n \) to be 1 at 0, 0 at \( 1/n \), 0 at 1, and linear between these points. Then \( h \) is 1 at 0 and 0 elsewhere so is not continuous.