Problem 1A.

A non-empty metric space X is said to be connected if it is not the union of two non-empty disjoint open subsets, and is said to be path-connected if for every two points a, b there is a continuous map f from the unit interval to X with $f(0) = a, f(1) = b$.

(a) Prove that every path-connected space is connected.

(b) If X is the subset of the plane consisting of the points (x, y) with $x = 0$ or $x > 0, y = \sin(1/x)$ show that X is connected but not path-connected.

Solution:

(a) If a space X is not connected, it is the union of 2 disjoint open subsets A and B.Choose a in A and b in B. Then for any continuous map f from the unit interval to X with $f(0) = a, f(1) = b$ the inverse images of A and B give a partition of the unit interval into 2 disjoint nonempty open subsets. This is not possible, as the supremum of one of the open subsets cannot be in either.

(b) This space is the union of the y axis A and the graph B of $y = \sin(1/x)$ both of which are connected. So the only possible partition into 2 disjoint nonempty open subsets is A union B, which is not possible as A and B are not open subsets. So the space is connected. To show it is not path connected, take any map from the unit interval to it with $f(0)$ in the y axis. Let x be the supremum of points whose image is in the y axis. For a small neighborhood of $f(x)$ the largest connected subset containing $f(x)$ is in the y axis, so some neighborhood of x must have image in the y axis. This forces x to be 1 otherwise there are points above it whose image is not in the y-axis. So there are no maps of the unit interval to X with $f(0)$ in the y axis and $f(1)$ not, so the space is not path connected.

Problem 2A.

Find an irreducible polynomial over the integers with $2\cos(2\pi/7)$ as a root, and use this to show that it is not contained in any extension of the rational numbers of degree a power of 2.

Solution:

Write $x = 2\cos(2\pi/7) = z + 1/z$ with $z^7 = 1$, $z \neq 1$. Then $x^3 + x^2 - 2x - 1 = z^{-3} + z^{-2} + z^{-1} + 1 + z + z^2 + z^3 = 0$. This polynomial is irreducible as it is irreducible mod 2. So x generates a field extension of degree 3, so any field containing x has degree divisible by 3, so the degree cannot be a power of 2.

Problem 3A.

Use residues to compute

$$\int_0^\infty \frac{dx}{x^4 + 1}.$$

Solution: This is half of $\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$, and therefore πi times the sum of residues in the upper half plane (using the usual semicircular contour and the residue theorem). The residues are at $(i \pm 1)/\sqrt{2}$ and have values $1/4(i \pm 1)$ so their sum is $-\sqrt{2}i/4$. The integral is therefore $\pi/2\sqrt{2}$.

Problem 4A.

Let $M_n(k)$ be the n by n matrices over a field k. Find (with proof) all linear maps f from $M_n(k)$ to k such that $f(AB) = f(BA)$ for all matrices A and B.

Solution:

Taking commutators $AB - BA$ of suitable matrices A and B each with just one nonzero entry shows that any matrix with just one nonzero entry off the diagonal, or with 2 nonzero entries on the diagonal with sum zero, is of this form. In other words all matrices of trace zero are linear combinations of matrices of the form $AB - BA$. Any matrix $AB - BA$ has image 0 under f. So the linear maps are just those that vanish on all matrices of trace 0, and so are multiples of the trace.

Problem 5A.

Show that the function equal to e^{-1/x^2} for $x \neq 0$ and equal to 0 at $x = 0$ is infinitely differentiable at all real numbers, and find its Taylor series at $x = 0$.

Solution: By induction any higher derivative is (polynomial in $1/x$)e^{-1/x^2} for $x \neq 0$. This has limit 0 at $x = 0$. So all higher derivatives exist and are all 0 at 0. The Taylor series at 0 is therefore $0 + 0x + 0x^2 + ...$

Problem 6A.

If N is the integer $2^4 \cdot 3^3 \cdot 5^2 \cdot 7$ find the smallest positive integer m such that $x^m \equiv 1 \bmod N$ for all integers x coprime to N.

Solution:

By the Chinese remainder theorem $Z/(mnZ)$ is $Z/(mZ) \times Z/(nZ)$ for m, n coprime, so it is enough to solve this question for prime powers. If N is 2^4 or 3^3 or 5^2 or 7 then the smallest m as above is 4, 2×3^2, 4×5, and 6 respectively. So the solution is the least common multiple of these, which is $m = 2^2 \times 3^2 \times 5 = 180$.

Problem 7A.

If $0 < r < 1$, find

$$\sum_{k=0}^{\infty} r^k \cos(k\theta).$$

Your final answer should not involve any complex numbers.

Solution:

Put $z = re^{i\theta}$. It's enough to find the real part of

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z} = \frac{1}{1-re^{i\theta}} \frac{1-re^{-i\theta}}{1-re^{-i\theta}} = \frac{1-r\cos(\theta)+ir\sin(\theta)}{1-2r\cos(\theta)+r^2},$$

so the answer is

$$\frac{1-r\cos(\theta)}{1-2r\cos(\theta)+r^2}.$$

Problem 8A.

For each of the following 4 statements, give either a counterexample or a reason why it is true.

(a) For every real matrix A there is a real matrix B with $B^{-1}AB$ diagonal.

(b) For every symmetric real matrix A there is a real matrix B with $B^{-1}AB$ diagonal.

(c) For every complex matrix A there is a complex matrix B with $B^{-1}AB$ diagonal.

(d) For every symmetric complex matrix A there is a complex matrix B with $B^{-1}AB$ diagonal.

Solution:

To generate conterexamples, observe that a nonzero 2 by 2 matrix with trace and determinant 0 cannot be diagonalizable as both eigenvalues vanish.

(a) False $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

(b) True as Hermitean matrices are diagonalizable

(c) False $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

(d) False $\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$

Problem 9A.

The Catalan numbers $C(n)$ satisfy $C(0) = 1$, $C(n) = C(0)C(n-1) + C(1)C(n-2) + \cdots + C(n-1)C(0)$ if $n > 0$. Find the function $\sum_{n=0}^{\infty} C(n)x^n$ and use this to evaluate $C(n)$.

Solution: If $f(x) = \sum_{n=0}^{\infty} C(n)x^n$ then $xf(x)^2 + 1 = f(x)$ so $f(x) = (1 - \sqrt{1-4x})/2x$. Expanding this by the binomial series shows that $C(n) = \frac{(2n)!}{n!(n+1)!}$.

Problem 1B.

Let D be an open subset of $\mathbb{R}^2$ (with the topology induced by the euclidean metric), and assume that it contains the closed unit square

$$[0,1] \times [0,1] = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1\}.$$

Show that D contains the partially-open rectangle

$$[0,1] \times [0,1+\epsilon) = \{(x,y) : 0 \le x \le 1,\ 0 \le y < 1+\epsilon\}$$

for some $\epsilon > 0$.

Solution: For each $x \in [0,1]$, we have $(x,1) \in D$, so there is an $\epsilon(x) > 0$ such that $B_{2\epsilon(x)}(x,1) \subseteq D$ (here $B_r(P)$ denotes the open ball of radius r centered at the point P). Therefore the open square with corners $(x \pm \epsilon(x), 1 \pm \epsilon(x))$ is also contained in D.

As x varies over $[0,1]$, the open sets $(x - \epsilon(x), x + \epsilon(x))$ cover the compact set $[0,1]$, so there is a finite subcollection $\{(x_i - \epsilon(x_i), x_i + \epsilon(x_i)) : i = 1, \ldots, n\}$ that covers $[0,1]$. Let ϵ be the smallest of $\epsilon(x_1), \ldots, \epsilon(x_n)$. Then D contains the set $[0,1] \times [0, 1+\epsilon)$.

Problem 2B.

Prove that every group is isomorphic to a group of permutations. Prove that every finite group is isomorphic to a group of even permutations of a finite set.

Solution: Let G be a finite group, and let S_G denote the group of permutations of G. For each $g \in G$ define σ_g $G \to G$ by $\sigma_g(x) = gx$. This function is one-to-one because $gx = gy$ implies $x = y$ by cancellation, and it is onto because it is one-to-one and G is a finite set. Therefore σ_g lies in S_G for all g. This is a group homomorphism $G \to S_G$ because $\sigma_g(\sigma_h(x)) = ghx = \sigma_{gh}(x)$ for all x, so $\sigma_g \circ \sigma_h = \sigma_{gh}$ for all $g, h \in G$. This group homomorphism is injective because if σ_g lies in the kernel then $\sigma_g(e) = e$ (where $e \in G$ denotes the identity element); however, $\sigma_g(e) = ge = g$, implying $g = e$ and therefore the kernel is trivial. Thus this homomorphism is an isomorphism of G with a subgroup of the permutation group S_G.

To get G isomorphic to a group of even permutations, do the same procedure to show that G is isomorphic to a subgroup of even permutations of S_{2G}, where $2G$ denotes the disjoint union of G with itself.

Problem 3B.

Prove that there are infinitely many complex numbers z with $e^z = z$. (Hint: consider the behavior of $e^z - z$ on the boundary of a large square.)

Solution: Consider the change of argument of $e^z = z$ on a large square of side $2R$ centered at 0. The change of argument on the left hand edge and the top and bottom is bounded as R tends to infinity by easy estimates. The change of argument on the right hand edge is about that of e^z which increases linearly with R. So up to a bounded term, the total change in argument increases linearly with R. As the number of zeros is (change in agument)$/2\pi$, the function has an infinite number of zeros.

Problem 4B.

For which real numbers x does the matrix-valued series $\Sigma_{n=0}^{\infty} x^n A^n$ converge, where A is the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$?

Solution: The matrix has eigenvalues $(1 \pm \sqrt{5})/2$. Diagonalizing the matrix shows that the series converges for $|x|$ less than the inverse of the absolute value of the largest eigenvalue, so for $|x| < (-1 + \sqrt{5})/2$.

Problem 5B.

(a) Evaluate $I(n) = \int_0^\pi \sin(x)^n dx$ for n a non-negative integer.
(b) Prove that $I(n) > I(n+1) > 0$
(c) Evaluate the infinite product $\frac{1}{2} \times \frac{3}{2} \times \frac{3}{4} \times \frac{5}{4} \times \frac{5}{6} \times \cdots$.

Solution: (a) $I(0) = \pi$, $I(1) = 2$. Integration by parts gives $I(n) = \int_0^\pi (n-1)\sin(x)^{n-2}\cos(x)\cos(x)dx = (n-1)(I(n-2) - I(n)$ so $I(n) = \frac{n-1}{n} I(n-2)$. So $I(2n) = \pi \frac{1}{2}\frac{3}{4} \cdots \frac{2n-1}{2n}$ and $I(2n+1) = 2\frac{2}{3}\frac{4}{5} \cdots \frac{2n}{2n+1}$.

(b) Follows because $\sin(x)^n > \sin(x)^{n+1} > 0$.

(c) The product of $2n-1$ terms of the product is $\frac{I(2n)/\pi}{I(2n-1)/2}$ which is less than $2/\pi$ and the product of $2n$ terms is $\frac{I(2n)/\pi}{I(2n+1)/2}$ which is greater then $2/\pi$. As the product converges by the "alternating product test" it is $2/\pi$.

Problem 6B.

Prove that the polynomial $x^4 + x + 2011$ is irreducible over $\mathbb{Q}$.

Solution:

It is sufficient to check irreducibility in $\mathbb{Z}[x]$ and for this it is enough to check irreducibility mod 2. For this just check it has no linear factors and is not divisible by the only irreducible degree 2 mod 2 polynomial $x^2 + x + 1$.

Problem 7B.

Prove that the real and imaginary parts of a holomorphic complex function are harmonic (solutions of Laplace's equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$). Find two linearly independent real solutions of Laplace's equation in two variables that are homogeneous polynomials of degree 6.

Solution:

The fact that the real and imaginary parts are harmonic follows easily from the Cauchy-Riemann equations. Two homogeneous harmonic polynomials of degree 6 are the real and imaginary parts of $(x+iy)^6$ which are $x^6 - 15x^4y^2 + 15x^2y^4 - y^6$ and $6xy^5 - 20x^3y^3 + 6x^5y$.

Problem 8B.

For p a prime show that the number of non-singular $n \times n$ matrices with entries in the field with p elements has the form $p^r s$ where $s \equiv (-1)^n \pmod{p}$, and find r.

Solution:

The number of nonsingular matrices is the number of bases which is $(p^n - 1)(p^n - p)...(p^n - p^{n-1})$ (product of number of ways to choose first, second, ...n'th basis vectors). So $r = 0 + 1 + \ldots (n-1) = (n-1)n/2$ and $s = (p^n - 1)(p^{n-1} - 1)...(p-1)$ is congruent to $(-1)^n \pmod{p}$.

Problem 9B.

For each of the following statements, either prove it or give a counterexample:

(a) If $f(x)$ and $f_n(x)$ are continuous real-valued functions on the unit interval, and $\lim_{n\to\infty} f_n(x) = f(x)$ for all x, then $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$.

(b) If $g(m,n)$ is real for all integers m,n, and $\sum_{m=0}^{\infty}(\sum_{n=0}^{\infty} g(m,n))$ and $\sum_{n=0}^{\infty}(\sum_{m=0}^{\infty} g(m,n))$ are both defined, then they are equal.

(c) If the functions $h_n(x)$ are continuous real-valued functions on the unit interval, and $\lim_{n\to\infty} h_n(x) = h(x)$ for all x, then $h(x)$ is a continuous function of x.

Solution:

(a) False. Take f_n to be 0 for $x \geq 1/n$ and $x = 0$ and to have integral 1. Then $f = 0$ does not have integral 1.

(b) False. Take $g(m,n)$ to be 1 if $m = n$, -1 if $m = n+1$, 0 otherwise.

(c) False. Take h_n to be 1 at 0, 0 at $1/n$, 0 at 1, and linear between these points. Then h is 1 at 0 and 0 elsewhere so is not continuous.