Problem 1A. Suppose that $f$ is a continuous real function on $[0, 1]$. Prove that

$$
\lim_{\alpha \to 0^+} \alpha \int_0^1 x^{\alpha-1} f(x) \, dx = f(0).
$$

Solution: This is obvious for $f$ a constant, so by subtracting $f(0)$ from both sides we can assume $f(0) = 0$. Choose any $\epsilon > 0$ and choose $\delta$ so that $|f(x)| < \epsilon$ whenever $x \leq \delta$ and choose $M$ so that $f(x) < M$ for all $x$. Then

$$
\alpha \int_0^\delta x^{\alpha-1} f(x) \, dx \leq \alpha \int_0^1 x^{\alpha-1} \epsilon \, dx = \epsilon,
$$

while

$$
\alpha \int_\delta^1 x^{\alpha-1} f(x) \, dx \leq \alpha \int_\delta^1 x^{\alpha-1} M \, dx = M(1 - \delta^\alpha)
$$

which tends to 0 as $\alpha$ tends to 0. So for any $\epsilon > 0$ the limit is less than $\epsilon$ in absolute value, so the limit is 0.

(Assuming that $f$ is differentiable allows an easier solution by integrating by parts.)

Problem 2A. Prove that if an $n \times n$ matrix $X$ over $\mathbb{R}$ satisfies $X^2 = -I$, then $n$ is even.

Solution: Method 1: Since $X^2 + 1 = (X + i)(X - i) = 0$, the Jordan form $A$ of $X$ over $\mathbb{C}$ is diagonal with eigenvalues $\pm i$. Since $X$ is real, the eigenvalues come in conjugate pairs, hence $n$ is even.

Method 2: The identity $X^2 = -I$ implies that $a + bi \mapsto aI + bX$ is a ring homomorphism from $\mathbb{C}$ to $M_n(\mathbb{R}) = \text{End}(\mathbb{R}^n)$. Using this to define complex scalar multiplication, $\mathbb{R}^n$ becomes a vector space over $\mathbb{C}$ such that restriction of scalars to $\mathbb{R}$ recovers its original real vector space structure. Since a complex vector space has even real dimension, $n$ is even.

Method 3: $\det X^2 = (\det X)^2 = \det(-I) = (-1)^n$. Therefore $n$ is even. Problem 3A. Show that if $f : \mathbb{C} \to \mathbb{C}$ is a meromorphic function in the plane, such that there exists $R, C > 0$ so that for $|z| > R$, $|f(z)| \leq C|z|^m$, then $f$ is a rational function.

Solution: Since $f$ is meromorphic, and $|f(z)| < \infty$ for $|z| > R$, $f$ must have only finitely many poles $a_1, \ldots, a_m$ (with multiplicity) in the disk $|z| \leq R$. Let $g(z) = (z - a_1) \cdots (z - a_m) f(z)$, then $g(z)$ is entire, and $|g(z)| \leq C'|z|^{m+n}$ for $|z|$ large enough, and therefore $g(z)$ is a polynomial of degree a most $m + n$, using Cauchy’s estimate $|f^{N}(0)| \leq C' r^{m+n} N! r^{-N} \to 0$ as $r \to \infty$ if $N > m + n$. Thus, $f(z) = g(z)/((z - a_1) \cdots (z - a_n))$ must be a rational function.

Problem 4A. Let $G$ be a finite group, and for each positive integer $n$, let

$$
X_n = \{(g_1, \ldots, g_n) : g_i g_j = g_j g_i \ \forall i, j\}.
$$

Show that the formula

$$
h \cdot (g_1, \ldots, g_n) = (hg_1 h^{-1}, \ldots, hg_n h^{-1}),
$$

is well defined.
defines an action of $G$ on $X_n$, and that $|X_{n+1}| = |G| \cdot |X_n/G|$ for all $n$, where $X_n/G$ denotes the set of $G$-orbits in $X_n$.

**Solution:**

The given formula defines the action of $G$ on $G^n$ by coordinatewise conjugation, so one only has to verify that if $(g_1, \ldots, g_n) \in X_n$, then $h \cdot (g_1, \ldots, g_n) \in X_n$. Since $g_i$ commutes with $g_j$ implies $hg_ih^{-1}$ commutes with $hg_hh^{-1}$, this is clear.

For the counting assertion, we have $|X_n/G| = \frac{1}{|G|} \sum_{h \in G} f_n(h)$, by Burnside’s Lemma, where $f_n(h)$ is the number of elements of $X_n$ fixed by $h$. Now $(g_1, \ldots, g_n)$ is fixed by $h$ if and only if $h$ commutes with each $g_i$, that is, if and only if $(g_1, \ldots, g_n, h)$ belongs to $X_{n+1}$. Thus $|X_{n+1}| = \sum_{h \in G} f_n(h) = |G| \cdot |X_n/G|$.

**Problem 5A.** There is a “folk theorem” that a four-footed table can always be rotated into a stable position on an uneven floor. Prove the following mathematical formulation of this theorem.

Define four points in $\mathbb{R}^2$, depending on an angle $\theta$, by $P_1(\theta) = (\cos \theta, \sin \theta)$, $P_2(\theta) = (\sin \theta, \cos \theta)$, $P_3(\theta) = (\sin \theta, -\sin \theta)$, $P_4(\theta) = (\sin \theta, -\cos \theta)$. Show that given any continuous function $h : \mathbb{R}^2 \to \mathbb{R}$, there exists a value of $\theta$ such that the four points $Q_i(\theta) = (P_i(\theta), h(P_i(\theta)))$ on the graph of $h$ are co-planar in $\mathbb{R}^3$.

**Solution:** Let $\tilde{h}(\theta) = h(\cos \theta, \sin \theta)$ and $g(\theta) = \tilde{h}(\theta) - \tilde{h}(\theta + \pi/2) + \tilde{h}(\theta + \pi) - \tilde{h}(\theta + 3\pi/2)$. Then $g$ is continuous and satisfies $g(\theta + \pi/2) = -g(\theta)$. In particular, for any real number $x$ that is a value of $g$, we see that $-x$ is also a value of $g$, hence by the Intermediate Value Theorem, there exists $\theta$ such that $g(\theta) = 0$. For this $\theta$ we have $(\tilde{h}(\theta) + \tilde{h}(\theta + \pi))/2 = (\tilde{h}(\theta + \pi/2) + \tilde{h}(\theta + 3\pi/2))/2$. Call the quantity on both sides of this equality $z$. Then the point $(0, 0, z) \in \mathbb{R}^3$ lies on both the lines $Q_1(\theta)Q_3(\theta)$ and $Q_2(\theta)Q_4(\theta)$, showing that the four points $Q_i(\theta)$ are co-planar.

**Problem 6A.** Let $M_2(\mathbb{C})$ be the set of $2 \times 2$ matrices over the complex numbers. Given $A \in M_2(\mathbb{C})$, define $C(A) = \{B \in M_2(\mathbb{C}) : AB = BA\}$.

(a) Prove that $C(A)$ is a linear subspace of $M_2(\mathbb{C})$, for every $A$.

(b) Determine, with proof, all possible values of the dimension $\dim C(A)$.

(c) Formulate a simple and explicit rule to find $\dim C(A)$, given $A$. “Simple” means the rule should yield the answer with hardly any computational effort.

**Solution:**

(a) Either check directly that $C(A)$ is closed under matrix addition and scalar multiplication, or just note that for fixed $A$, the matrix equation $AB = BA$ is a system of linear equations in the entries of $B$.

(b) If $A' = SAS^{-1}$ is similar to $A$, then it is easy to verify that $B \mapsto SBS^{-1}$ is a linear isomorphism of $C(A)$ on $C(A')$. Hence we may assume w.o.l.o.g. that $A$ is in Jordan canonical form. This leads to three cases:
Case I.

\[ A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad \text{where } a \neq b. \]

Then \( C(A) \) consists of the diagonal matrices, \( \dim C(A) = 2. \)

Case II.

\[ A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}. \]

Then \( C(A) \) is the set of matrices \( B \) of the form

\[ B = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}, \]

so again \( \dim C(A) = 2. \)

Case III. \( A \) is a scalar multiple of \( I \). Then \( C(A) = M_2(\mathbb{C}), \dim C(A) = 4. \)

(c) The result of part (b) can be reformulated as follows: if \( A \) is a scalar multiple of \( I \), then \( \dim C(A) = 4, \) otherwise \( \dim C(A) = 2. \)

**Problem 7A.** Compute

\[ \int_0^\pi \frac{d\theta}{a + \cos \theta} \]

for \( a > 1 \) using the method of residues.

**Solution:** Using \( z = e^{i\theta}, \ dz = ie^{i\theta}d\theta, \ \cos \theta = (z + 1/z)/2, \) we may rewrite the integral as a line integral

\[ -2i \int_{|z|=1} \frac{dz}{z^2 + 2az + 1}. \]

Factor the denominator as \((z - \alpha)(z - \beta), \) where \( \alpha = -a + \sqrt{a^2 - 1}, \ \beta = -a - \sqrt{a^2 - 1}, \) and \( |\alpha| < 1, |\beta| > 1. \) Then

\[ Res_{z=\alpha} = \frac{1}{z^2 + 2az + 1} = \frac{1}{\alpha - \beta}; \]

and we see that the integral is equal to \( -2i \cdot 2\pi i \cdot \frac{1}{\alpha - \beta} = 2\pi / \sqrt{a^2 - 1}. \)

**Problem 8A.** Let \( \mathbb{Q}(x) \) be the field of rational functions of one variable over \( \mathbb{Q}. \) Let \( i: \mathbb{Q}(x) \to \mathbb{Q}(x) \) be the unique field automorphism such that \( i(x) = x^{-1}. \) Prove that the subfield of elements fixed by \( i \) is equal to \( \mathbb{Q}(x + x^{-1}). \)

**Solution:** Let \( F \) denote the fixed subfield, and set \( y = x + x^{-1} \in F. \) Clearly \( \mathbb{Q}(y) \subseteq F \neq \mathbb{Q}(x). \) The equation \( x^2 - yx + 1 = 0 \) shows that \( \mathbb{Q}(x) \) is an algebraic extension of degree 2 over \( \mathbb{Q}(y). \) Hence the only extension of \( \mathbb{Q}(y) \) properly contained in \( \mathbb{Q}(x) \) is \( \mathbb{Q}(y) \) itself, so \( F = \mathbb{Q}(y). \)

**Problem 9A.** Let \( d_k := \text{LCM}\{1, 2, \ldots, k\} \) (the least common multiple) and \( I_m = \int_0^1 x^m(1-x)^m dx. \) Show \( d_{2m+1}I_m \) is an integer, and use this to show that \( d_{2m+1} \geq 2^{2m}. \)
Solution:

\[ I_m = \sum_{n=0}^{2m} \frac{a_n}{n+1} \]

for some integers \( a_n \) so \( d_{2m+1}I_m \in \mathbb{Z} \). Also if \( f(x) = x(1 - x) \), \( f(0) = f'(1/2) = f(1) = 0 \), \( f(1/2) = 1/4 \) and \( 1/2 \) is the only critical point of \( f \) on \((0, 1)\) so \( 0 < I_m \leq (1/4)^m \) and so \( d_mI_m \geq 1 \) and the statement follows.

**Problem 1B.** Let \( I \subseteq \mathbb{R} \) be an open interval, and let \( f : I \to \mathbb{R} \) have continuous \( k \)-th derivatives \( f^{[k]} \) on \( I \) for \( k \leq n - 1 \). Let \( a \in I \) be a point such that \( f^{[k]}(a) = 0 \) for all \( 1 \leq k \leq n - 1 \), \( f^{[n]}(a) \) exists and \( f^{[n]}(a) > 0 \). Prove that \( f \) has a local minimum at \( a \) if \( n \) is even, and has no local extremum at \( a \) if \( n \) is odd.

**Solution:** Since \( f^{[n-1]}(a) = 0 \), the definition of derivative gives

\[
\lim_{x \to a} \frac{f^{[n-1]}(x)}{x - a} = f^{[n]}(a) > 0,
\]

and hence there exists \( \epsilon > 0 \) such that \( f^{[n-1]}(x)/(x - a) > 0 \) for all \( x \in (a - \epsilon, a + \epsilon) \setminus \{a\} \). Taylor’s Theorem with remainder yields

\[
f(x) = f(a) + f^{[n-1]}(c)(x - a)^{n-1}/(n-1)!
\]

for some \( c \in [a, x] \) if \( x \geq a \), or \( c \in [x, a] \) if \( x \leq a \). For \( x \in (a - \epsilon, a) \) we have \( f^{[n-1]}(c) \leq 0 \), whence \( f(x) \geq f(a) \) if \( n \) is even, \( f(x) \leq f(a) \) if \( n \) is odd. Similarly, we find for \( x \in (a, a + \epsilon) \) that \( f(x) \geq f(a) \) for any \( n \). For \( n \) even, this implies that \( f \) has a local minimum at \( a \). For \( n \) odd, it implies that either \( f \) has no local extremum at \( a \), or else \( f \) is constant on \((a - \epsilon, a + \epsilon)\). But the hypothesis \( f^{[n]}(a) > 0 \) rules out the latter possibility.

**Problem 2B.** Let \( A \) and \( B \) be \( n \times n \) matrices over a field of characteristic zero. Prove that the condition \( BA - AB = A \) implies that \( A \) is nilpotent. (Hint: what does \( A \) do to eigenvectors of \( B \)?)

**Solution:** Without loss of generality we can enlarge the field, say \( k \), to be algebraically closed, since this does not change the hypothesis or the conclusion. Let \( v \) be an eigenvector of \( B \), say \( Bv = \lambda v \). Then \( BAv = A(B + I)v = (\lambda + 1)Av \), in other words, if \( Av \neq 0 \) then \( Av \) is also an eigenvector of \( B \) with eigenvalue \( \lambda + 1 \). Since \( B \) has finitely many eigenvalues, we must have \( A^kv = 0 \) for some \( k \), so \( A \) is singular. Since \( AB = BA - A \) we also see that \( B \) preserves the nullspace \( W \) of \( A \). Then \( A \) and \( B \) induce linear transformations \( A', B' \) of \( k^n/W \) which again satisfy \( B'A' - A'B' = A' \). It follows by induction on \( n \) that \( A' \) is nilpotent, hence so is \( A \).

**Problem 3B.** How many roots of the equation \( z^4 - 5z^3 + z - 2 = 0 \) lie in the disk \( |z| < 1 \)?

**Solution:** By Rouche’s theorem, since \( |z^4 - 5z^3 + z - 2 - (-5z^3)| = |z^4 + z - 2| \leq 4 < 5 = | -5z^3| \) for \( |z| = 1 \), then \( z^4 - 5z^3 + z - 2 \) has the same number of zeroes for \( |z| < 1 \) as \(-5z^3\), which has three zeroes (counted with multiplicity).
Problem 4B. Consider a polynomial expression $H(\alpha) = A + B\alpha + C\alpha^2 + \cdots + D\alpha^N$ with rational coefficients $A, B, C, \ldots, D$, where $\alpha$ is an algebraic number, in other words a root of some polynomial with rational coefficients. Prove that if $H(\alpha) \neq 0$, then the reciprocal $1/H(\alpha)$ can be expressed as a polynomial in $\alpha$ with rational coefficients.

Solution: Let $P(x)$ be a polynomial of minimal degree such that $P(\alpha) = 0$. Then $P$ must be irreducible in $\mathbb{Q}[x]$ (since otherwise $\alpha$ would be a root of a polynomial of a smaller degree). By the Euclidean algorithm in $\mathbb{Q}[x]$, there exist polynomials $F(x)$ and $G(x)$ with rational coefficients such that the greatest monic common divisor $D$ of $P$ and $H$ is written as $D(x) = F(x)P(x) + G(x)H(x)$. Then $D$ cannot be a scalar multiple of $P$ (since otherwise we would have $H(\alpha) = 0$), and cannot be a proper divisor of $P$ (since $P$ is irreducible), and so $D = 1$. Thus $1/H(\alpha) = G(\alpha)$.

Problem 5B. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function of compact support. Show that $u(x) = \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy$ is well defined and that $\lim_{|x| \to \infty} u(x)|x| = \int_{\mathbb{R}^3} f(y)dy$.

Solution: Choose $R$ so that $f(y) = 0$ if $|y| \geq R$. Then for $a = R + |x|$, and using polar coordinates,
\[ \int \left| \frac{f(y)}{|x-y|} \right| dy \leq (\max |f|)4\pi \int_0^a \frac{1}{r^2} r^2 dr < \infty, \]
shows that the integral exists. On the other hand, for $|x| \geq L \geq R$ and $y$ satisfying $f(y) \neq 0$, we have
\[ \left| \frac{|x|}{|x-y|} - 1 \right| \leq \frac{R}{L-R}, \]
which implies
\[ \left| u(x)|x| - \int f(y)dy \right| \leq \frac{R}{L-R} \int |f(y)|dy. \]
The result follows by sending $L \to \infty$.

Problem 6B. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be eigenvalues of a symmetric real $n \times n$-matrix $A$. Prove that
\[ \lambda_k = \max_{V^k} \min_{x \in (V^k-0)} \frac{(Ax,x)}{(x,x)}, \]
where the maximum is taken over all $k$-dimensional linear subspaces $V^k$, the minimum over all non-zero vectors in the subspace, and $(x,y)$ denotes the Euclidean dot-product. (Hint: any $k$-dimensional subspace intersects the space spanned by the eigenvectors of the $n+1-k$ smallest eigenvalues in a space of dimension at least 1.)

Solution: In the orthonormal basis of eigenvectors of $A$ (provided by the orthogonal diagonalization theorem) we have:
\[ (Ax,x) = \lambda_1 x_1^2 + \cdots + \lambda_n x_n^2. \]
Let $W$ denotes the subspace of codimension $k-1$ given by the equations $x_1 = \cdots = x_{k-1} = 0$. On $W$, we have:

$$(Ax, x) = \lambda_k x_k^2 + \cdots + \lambda_n x_n^2 \leq \lambda_k (x_k^2 + \cdots + x_n^2) = \lambda_k (x, x),$$

i.e. the ratio $(Ax, x)/(x, x) \leq \lambda_k$. Since every $k$-dimensional subspace $V^k$ has a non-trivial intersection with $W$, we conclude that $\min(Ax, x)/(x, x)$ on every $V^k$ does exceed $\lambda_k$. On the other hand, in the $k$-dimensional subspace $V_0^k$ given by the equations $x_{k+1} = \cdots = x_n = 0$, we have:

$$(Ax, x) = \lambda_1 x_1^2 + \cdots + \lambda_k x_k^2 \geq \lambda_k (x_1^2 + \cdots + x_k^2) = \lambda_k (x, x),$$

the ratio $(Ax, x)/(x, x) \geq \lambda_k$.

**Problem 7B.** If is a univalent (1-1 analytic) function with domain the unit disc such that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then prove that

$$g(z) = \sqrt{f(z^2)}$$

is an odd analytic univalent function on the unit disc.

**Solution:** The function $f(z^2) = z^2 \phi(z)$, where $\phi(0) = 1$. Since $f$ is univalent, $\phi(z) \neq 0$ for $z \neq 0$. Then we may define a well-defined branch $h(z) = \sqrt{\phi(z)}$, since $B_1(0)$ is simply-connected. Since $f(z^2)$ is even, and $z^2$ is even, we must have $\phi(z)$ is even. Since $\phi(z) = \phi(-z)$ for $z$ near zero, we must have $\sqrt{\phi(z)} = \sqrt{\phi(-z)}$ for $z$ near zero, and therefore everywhere. Thus $\sqrt{\phi(z)}$ is an even function too. Thus, $g(z) = 1/\sqrt{f(z^2)} = 1/z \sqrt{\phi(z)}$ is an odd function defined for $z \in B_1(0) - \{0\}$.

Suppose that $g(z_1) = g(z_2)$. Then $f(z_1^2) = f(z_2^2)$, which means that $z_1^2 = z_2^2$, since $f$ is univalent. If $z_1 = -z_2$, then $g(z_1) = -g(z_2)$, which may hold only if $z_1 = z_2 = 0$. Otherwise, $z_1 = z_2$, so in either case $g$ is univalent.

**Problem 8B.**

1. Let $G$ be a non-abelian finite group. Show that $G/Z(G)$ is not cyclic, where $Z(G)$ is the center of $G$.

2. If $|G| = p^n$, with $p$ prime and $n > 0$, show that $Z(G)$ is not trivial.

3. If $|G| = p^2$, show that $G$ is abelian.

**Solution:**

1. If $g$ is an element of $G$ whose image generates the cyclic group $G/Z$, then $G$ is generated by the commuting set $g$ and $Z$, so is abelian.

2. All conjugacy classes have order a power of $p$ (as their order is the index of a centralizer of one of their elements) so the number of conjugacy classes with just one element is a multiple of $p$. These conjugacy classes form the center, so the center has order a multiple of $p$ so is non-trivial.
3. By part 2 the center has order at least \( p \), so \( G/Z \) has order at most \( p \) and is therefore cyclic. By part 1 the group must be abelian.

**Problem 9B.** Prove that the sequence of functions \( f_n(x) = \sin nx \) has no pointwise convergent subsequence. (Hint: show that given any subsequence and any interval of positive length there is a subinterval such that some element of the subsequence is at least \( 1/2 \) on this subinterval, and another element is at most \(-1/2\).)

**Remark.** This is an example from Ch. 7 of W. Rudin’s *Principles of Mathematical Analysis*, which is treated by the author using a result from the more advanced chapter on Lebesgue measure, namely the bounded convergence theorem. According to it, if a sequence of bounded continuous functions \( g_k (= (\sin n_k x - \sin n_{k+1} x)^2 \) in this example) tends to 0 pointwise, then \( \int g_k(t)dt \) tend to 0 too. (In the example, the integral over the period \([0, 2\pi]\) is equal to \(2\pi\) regardless of \( k \).) Below, an elementary proof is given; it is due to Evan O’Dorney (a high-school student taking Givental’s H104 class).

**Solution:** Given a subsequence \( \sin n_k x \), we find a subsequence \( \sin n_{k_l} \) in it and a point \( x_0 \) where \( \lim_{l \to \infty} \sin n_{k_l}x_0 \) does not exist. Start with picking an interval \([a_1, b_1]\) where \( \sin n_1 x \geq 1/2 \). Passing to a term \( \sin n_k x \) which oscillates sufficiently many times on the interval \([a_1, b_1]\), find in it an interval \([a_2, b_2]\) where \( \sin n_k x \leq -1/2 \). Passing to a term \( \sin n_m x \) which oscillates sufficiently many times on \([a_2, b_2]\), find in it an interval \([a_3, b_3]\) where \( \sin n_m x \geq 1/2 \), and so on. Call the selected functions \( \sin n_{k_1} x \), \( \sin n_{k_2} x \), \( \sin n_{k_3} x \), etc., and let \( x_0 \) be a common point of the nested sequence of intervals \([a_1, b_1] \supset [a_2, b_2] \supset \ldots \). Since \( \sin n_{k_l}x_0 \geq 1/2 \) for odd \( l \) and \( \leq -1/2 \) for even \( l \), the limit at \( x_0 \) does not exist.