Problem 1A. Suppose that f is a continuous real function on $[0,1]$. Prove that

$$\lim_{\alpha\to 0^+} \alpha \int_0^1 x^{\alpha-1} f(x)dx = f(0).$$

Solution: This is obvious for f a constant, so by subtracting $f(0)$ from both sides we can assume $f(0) = 0$. Choose any $\epsilon > 0$ and choose δ so that $|f(x)| < \epsilon$ whenever $x \le \delta$ and choose M so that $f(x) < M$ for all x. Then

$$\alpha \int_0^\delta x^{\alpha-1} f(x)dx \le \alpha \int_0^1 x^{\alpha-1}\epsilon dx = \epsilon,$$

while

$$\alpha \int_\delta^1 x^{\alpha-1} f(x)dx \le \alpha \int_\delta^1 x^{\alpha-1} M dx = M(1-\delta^\alpha)$$

which tends to 0 as α tends to 0. So for any $\epsilon > 0$ the limit is less than ϵ in absolute value, so the limit is 0.

(Assuming that f is differentiable allows an easier solution by integrating by parts.)

Problem 2A. Prove that if an $n \times n$ matrix X over $\mathbb{R}$ satisfies $X^2 = -I$, then n is even.

Solution: Method 1: Since $X^2 + 1 = (X+i)(X-i) = 0$, the Jordan form A of X over $\mathbb{C}$ is diagonal with eigenvalues $\pm i$. Since X is real, the eigenvalues come in conjugate pairs, hence n is even.

Method 2: The identity $X^2 = -I$ implies that $a + bi \mapsto aI + bX$ is a ring homomorphism from $\mathbb{C}$ to $M_n(\mathbb{R}) = \mathrm{End}(\mathbb{R}^n)$. Using this to define complex scalar multiplication, $\mathbb{R}^n$ becomes a vector space over $\mathbb{C}$ such that restriction of scalars to $\mathbb{R}$ recovers its orginal real vector space structure. Since a complex vector space has even real dimension, n is even.

Method 3: $\det X^2 = (\det X)^2 = \det(-I) = (-1)^n$. Therefore n is even. **Problem 3A.** Show that if $f : \mathbb{C} \to \hat{\mathbb{C}}$ is a meromorphic function in the plane, such that there exists $R, C > 0$ so that for $|z| > R$, $|f(z)| \le C|z|^n$, then f is a rational function.

Solution: Since f is meromorphic, and $|f(z)| < \infty$ for $|z| > R$, f must have only finitely many poles $a_1, \dots, a_m$ (with multiplicity) in the disk $|z| \le R$. Let $g(z) = (z-a_1)\cdots(z-a_m)f(z)$, then $g(z)$ is entire, and $|g(z)| \le C'|z|^{m+n}$ for $|z|$ large enough, and therefore $g(z)$ is a polynomial of degree a most $m+n$, using Cauchy's estimate $|f^N(0)| \le C' r^{m+n} N! r^{-N} \to 0$ as $r \to \infty$ if $N > m+n$. Thus, $f(z) = g(z)/((z-a_1)\cdots(z-a_n))$ must be a rational function.

Problem 4A. Let G be a finite group, and for each positive integer n, let

$$X_n = \{(g_1, \dots, g_n) : g_i g_j = g_j g_i \ \forall i, j\}.$$

Show that the formula

$$h \cdot (g_1, \dots, g_n) = (h g_1 h^{-1}, \dots, h g_n h^{-1}),$$

defines an action of G on X_n, and that $|X_{n+1}| = |G| \cdot |X_n/G|$ for all n, where X_n/G denotes the set of G-orbits in X_n.

Solution:

The given formula defines the action of G on G^n by coordinatewise conjugation, so one only has to verify that if $(g_1, \ldots, g_n) \in X_n$, then $h \cdot (g_1, \ldots, g_n) \in X_n$. Since g_i commutes with g_j implies hg_ih^{-1} commutes with hg_hh^{-1}, this is clear.

For the counting asserstion, we have $|X_n/G| = \frac{1}{|G|}\sum_{h\in G} f_n(h)$, by Burnside's Lemma, where $f_n(h)$ is the number of elements of X_n fixed by h. Now $(g_1, \ldots, g_n)$ is fixed by h if and only if h commutes with each g_i, that is, if and only if $(g_1, \ldots, g_n, h)$ belongs to X_{n+1}. Thus $|X_{n+1}| = \sum_{h\in G} f_n(h) = |G| \cdot |X_n/G|$.

Problem 5A. There is a "folk theorem" that a four-footed table can always be rotated into a stable position on an uneven floor. Prove the following mathematical formulation of this theorem.

Define four points in $\mathbb{R}^2$, depending on an angle θ, by $P_1(\theta) = (\cos\theta, \sin\theta)$, $P_2(\theta) = (-\sin\theta, \cos\theta)$, $P_3(\theta) = (-\cos\theta, -\sin\theta)$, $P_4(\theta) = (\sin\theta, -\cos\theta)$. Show that given any continuous function $h\colon \mathbb{R}^2 \to \mathbb{R}$, there exists a value of θ such that the four points $Q_i(\theta) = (P_i(\theta), h(P_i(\theta)))$ on the graph of h are co-planar in $\mathbb{R}^3$.

Solution: Let $\tilde{h}(\theta) = h(\cos\theta, \sin\theta)$ and $g(\theta) = \tilde{h}(\theta) - \tilde{h}(\theta + \pi/2) + \tilde{h}(\theta + \pi) - \tilde{h}(\theta + 3\pi/2)$. Then g is continuous and satisfies $g(\theta + \pi/2) = -g(\theta)$. In particular, for any real number x that is a value of g, we see that $-x$ is also a value of g, hence by the Intermediate Value Theorem, there exists θ such that $g(\theta) = 0$. For this θ we have $(\tilde{h}(\theta) + \tilde{h}(\theta + \pi))/2 = (\tilde{h}(\theta + \pi/2) + \tilde{h}(\theta + 3\pi/2))/2$. Call the quantity on both sides of this equality z. Then the point $(0, 0, z) \in \mathbb{R}^3$ lies on both the lines $Q_1(\theta)Q_3(\theta)$ and $Q_2(\theta)Q_4(\theta)$, showing that the four points $Q_i(\theta)$ are co-planar.

Problem 6A. Let $M_2(\mathbb{C})$ be the set of 2×2 matrices over the complex numbers. Given $A \in M_2(\mathbb{C})$, define $C(A) = \{B \in M_2(\mathbb{C}) : AB = BA\}$.

(a) Prove that $C(A)$ is a linear subspace of $M_2(\mathbb{C})$, for every A.

(b) Determine, with proof, all possible values of the dimension $\dim C(A)$.

(c) Formulate a simple and explicit rule to find $\dim C(A)$, given A. "Simple" means the rule should yield the answer with hardly any computational effort.

Solution:

(a) Either check directly that $C(A)$ is closed under matrix addition and scalar multiplication, or just note that for fixed A, the matrix equation $AB = BA$ is a system of linear equations in the entries of B.

(b) If $A' = SAS^{-1}$ is similar to A, then it is easy to verify that $B \mapsto SBS^{-1}$ is a linear isomorphism of $C(A)$ on $C(A')$. Hence we may assume w.o.l.o.g. that A is in Jordan canonical form. This leads to three cases:

Case I.

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad \text{where } a \neq b.$$

Then $C(A)$ consists of the diagonal matrices, $\dim C(A) = 2$.

Case II.

$$A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}.$$

Then $C(A)$ is the set of matrices B of the form

$$B = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix},$$

so again $\dim C(A) = 2$.

Case III. A is a scalar multiple of I. Then $C(A) = M_2(\mathbb{C})$, $\dim C(A) = 4$.

(c) The result of part (b) can be reformulated as follows: if A is a scalar multiple of I, then $\dim C(A) = 4$, otherwise $\dim C(A) = 2$.

Problem 7A. Compute

$$\int_0^\pi \frac{d\theta}{a + \cos\theta}$$

for $a > 1$ using the method of residues.

Solution: Using $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$, $\cos\theta = (z + 1/z)/2$, we may rewrite the integral as a line integral

$$-2i \int_{|z|=1} \frac{dz}{z^2 + 2az + 1}.$$

Factor the denominator as $(z - \alpha)(z - \beta)$, where $\alpha = -a + \sqrt{a^2 - 1}, \beta = -a - \sqrt{a^2 - 1}$, and $|\alpha| < 1, |\beta| > 1$. Then

$$Res_{z=a} \frac{1}{z^2 + 2az + 1} = \frac{1}{\alpha - \beta},$$

and we see that the integral is equal to $-2i \cdot 2\pi i \cdot \frac{1}{\alpha-\beta} = 2\pi/\sqrt{a^2 - 1}$.

Problem 8A. Let $\mathbb{Q}(x)$ be the field of rational functions of one variable over $\mathbb{Q}$. Let $i\colon \mathbb{Q}(x) \to \mathbb{Q}(x)$ be the unique field automorphism such that $i(x) = x^{-1}$. Prove that the subfield of elements fixed by i is equal to $\mathbb{Q}(x + x^{-1})$.

Solution: Let F denote the fixed subfield, and set $y = x + x^{-1} \in F$. Clearly $\mathbb{Q}(y) \subseteq F \neq \mathbb{Q}(x)$. The equation $x^2 - yx + 1 = 0$ shows that $\mathbb{Q}(x)$ is an algebraic extension of degree 2 over $\mathbb{Q}(y)$. Hence the only extension of $\mathbb{Q}(y)$ properly contained in $\mathbb{Q}(x)$ is $\mathbb{Q}(y)$ itself, so $F = \mathbb{Q}(y)$.

Problem 9A. Let $d_k := \text{LCM}\{1, 2, \ldots, k\}$ (the least common multiple) and $I_m = \int_0^1 x^m(1-x)^m \mathrm{d}x$. Show $d_{2m+1}I_m$ is an integer, and use this to show that $d_{2m+1} \geq 2^{2m}$.

Solution:

$$I_m = \sum_{n=0}^{2m} \frac{a_n}{n+1}$$

for some integers a_n so $d_{2m+1} I_m \in Z$. Also if $f(x) = x(1-x)$, $f(0) = f'(1/2) = f(1) = 0$, $f(1/2) = 1/4$ and $1/2$ is the only critical point of f on $(0,1)$ so $0 < I_m \leq (1/4)^m$ and so $d_m I_m \geq 1$ and the statement follows.

Problem 1B. Let $I \subseteq \mathbb{R}$ be an open interval, and let $f : I \to \mathbb{R}$ have continuous k-th derivatives $f^{[k]}$ on I for $k \leq n-1$. Let $a \in I$ be a point such that $f^{[k]}(a) = 0$ for all $1 \leq k \leq n-1$, $f^{[n]}(a)$ exists and $f^{[n]}(a) > 0$. Prove that f has a local minimum at a if n is even, and has no local extremum at a if n is odd.

Solution: Since $f^{[n-1]}(a) = 0$, the definition of derivative gives

$$\lim_{x \to a} \frac{f^{[n-1]}(x)}{x-a} = f^{[n]}(a) > 0,$$

and hence there exists $\epsilon > 0$ such that $f^{[n-1]}(x)/(x-a) > 0$ for all $x \in (a-\epsilon, a+\epsilon) \setminus \{a\}$. Taylor's Theorem with remainder yields

$$f(x) = f(a) + f^{[n-1]}(c)(x-a)^{n-1}/(n-1)!$$

for some $c \in [a,x]$ if $x \geq a$, or $c \in [x,a]$ if $x \leq a$. For $x \in (a-\epsilon, a)$ we have $f^{[n-1]}(c) \leq 0$, whence $f(x) \geq f(a)$ if n is even, $f(x) \leq f(a)$ if n is odd. Similarly, we find for $x \in (a, a+\epsilon)$ that $f(x) \geq f(a)$ for any n. For n even, this implies that f has a local minimum at a. For n odd, it implies that either f has no local extremum at a, or else f is constant on $(a-\epsilon, a+\epsilon)$. But the hypothesis $f^{[n]}(a) > 0$ rules out the latter possibility.

Problem 2B. Let A and B be $n \times n$ matrices over a field of characteristic zero. Prove that the condition $BA - AB = A$ implies that A is nilpotent. (Hint: what does A do to eigenvectors of B?)

Solution: Without loss of generality we can enlarge the field, say k, to be algebraically closed, since this does not change the hypothesis or the conclusion. Let v be an eigenvector of B, say $Bv = \lambda v$. Then $BAv = A(B+I)v = (\lambda+1)Av$, in other words, if $Av \neq 0$ then Av is also an eigenvector of B with eigenvalue $\lambda + 1$. Since B has finitely many eigenvalues, we must have $A^k v = 0$ for some k, so A is singular. Since $AB = BA - A$ we also see that B preserves the nullspace W of A. Then A and B induce linear transformations A', B' of k^n/W which again satisfy $B'A' - A'B' = A'$. It follows by induction on n that A' is nilpotent, hence so is A.

Problem 3B. How many roots of the equation $z^4 - 5z^3 + z - 2 = 0$ lie in the disk $|z| < 1$?

Solution: By Rouche's theorem, since $|z^4 - 5z^3 + z - 2 - (-5z^3)| = |z^4 + z - 2| \leq 4 < 5 = |-5z^3|$ for $|z| = 1$, then $z^4 - 5z^3 + z - 2$ has the same number of zeroes for $|z| < 1$ as $-5z^3$, which has three zeroes (counted with multiplicity).

Problem 4B. Consider a polynomial expression $H(\alpha) = A + B\alpha + C\alpha^2 + \cdots + D\alpha^N$ with rational coefficients $A, B, C, \ldots, D$, where α is an algebraic number, in other words a root of some polynomial with rational coefficients. Prove that if $H(\alpha) \neq 0$, then the reciprocal $1/H(\alpha)$ can be expressed as a polynomial in α with rational coefficients.

Solution: Let $P(x)$ be a polynomial of minimal degree such that $P(\alpha) = 0$. Then P must be irreducible in $\mathbf{Q}[x]$ (since otherwise α would be a root of a polynomial of a smaller degree). By the Euclidean algorithm in $\mathbf{Q}[x]$, there exist polynomials $F(x)$ and $G(x)$ with rational coefficients such that the greatest monic common divisor D of P and H is written as $D(x) = F(x)P(x) + G(x)H(x)$. Then D cannot be a scalar multiple of P (since otherwirse we would have $H(\alpha) = 0$), and cannot be a proper divisor of P (since P is irreducible), and so $D = 1$. Thus $1/H(\alpha) = G(\alpha)$.

Problem 5B. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function of compact support. Show that $u(x) = \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy$ is well defined and that $\lim_{|x|\to\infty} u(x)|x| = \int_{\mathbb{R}^3} f(y)dy$.

Solution: Choose R so that $f(y) = 0$ if $|y| \geq R$. Then for $a = R + |x|$, and using polar coordinates,

$$\int \left| \frac{f(y)}{|x-y|} \right| dy \leq (\max |f|) 4\pi \int_0^a \frac{1}{r} r^2 dr < \infty,$$

shows that the integral exists. On the other hand, for $|x| \geq L \geq R$ and y satisfying $f(y) \neq 0$, we have

$$\left| \frac{|x|}{|x-y|} - 1 \right| \leq \frac{R}{L-R},$$

which implies

$$\left| u(x)|x| - \int f(y)dy \right| \leq \frac{R}{L-R} \int |f(y)| dy.$$

The result follows by sending $L \to \infty$.

Problem 6B. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be eigenvalues of a symmetric real $n \times n$-matrix A. Prove that

$$\lambda_k = \max_{V^k} \min_{x \in (V^k - 0)} \frac{(Ax, x)}{(x, x)},$$

where the maximum is taken over all k-dimensional linear subspaces V^k, the minimum over all non-zero vectors in the subspace, and (x, y) denotes the Euclidean dot-product. (Hint: any k-dimensional subspace intersects the space spanned by the eigenvectors of the $n+1-k$ smallest eigenvalues in a space of dimension at least 1.)

Solution: In the orthonormal basis of eigenvectors of A (provided by the orthogonal diagonalization theorem) we have:

$$(Ax, x) = \lambda_1 x_1^2 + \cdots + \lambda_n x_n^2.$$

Let W denotes the subspace of *co*dimension $k-1$ given by the equations $x_1 = \cdots = x_{k-1} = 0$. On W, we have:

$$(Ax, x) = \lambda_k x_k^2 + \cdots + \lambda_n x_n^2 \leq \lambda_k (x_k^2 + \cdots + x_n^2) = \lambda_k (x, x),$$

i.e. the ratio $(Ax, x)/(x, x) \leq \lambda_k$. Since every k-dimensional subspace V^k has a non-trivial intersection with W, we conclude that $\min(Ax, x)/(x, x)$ on every V^k does exceed λ_k. On the other hand, in the k-dimensional subspace V_0^k given by the equations $x_{k+1} = \cdots = x_n = 0$, we have:

$$(Ax, x) = \lambda_1 x_1^2 + \cdots + \lambda_k x_k^2 \geq \lambda_k (x_1^2 + \cdots + x_k^2) = \lambda_k (x, x),$$

the ratio $(Ax, x)/(x, x) \geq \lambda_k$.

Problem 7B. If is a univalent (1-1 analytic) function with domain the unit disc such that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then prove that

$$g(z) = \sqrt{f(z^2)}$$

is an odd analytic univalent function on the unit disc.

Solution: The function $f(z^2) = z^2 \phi(z)$, where $\phi(0) = 1$. Since f is univalent, $\phi(z) \neq 0$ for $z \neq 0$. Then we may define a well-defined branch $h(z) = \sqrt{\phi(z)}$, since $B_1(0)$ is simply-connected. Since $f(z^2)$ is even, and z^2 is even, we must have $\phi(z)$ is even. Since $\phi(z) = \phi(-z)$ for z near zero, we must have $\sqrt{\phi(z)} = \sqrt{\phi(-z)}$ for z near zero, and therefore everywhere. Thus $\sqrt{\phi(z)}$ is an even function too. Thus, $g(z) = 1/\sqrt{f(z^2)} = 1/z\sqrt{\phi(z)}$ is an odd function defined for $z \in B_1(0) - \{0\}$.

Suppose that $g(z_1) = g(z_2)$. Then $f(z_1^2) = f(z_2^2)$, which means that $z_1^2 = z_2^2$, since f is univalent. If $z_1 = -z_2$, then $g(z_1) = -g(z_2)$, which may hold only if $z_1 = z_2 = 0$. Otherwise, $z_1 = z_2$, so in either case g is univalent.

Problem 8B.

1. Let G be a non-abelian finite group. Show that $G/Z(G)$ is not cyclic, where $Z(G)$ is the center of G.
2. If $|G| = p^n$, with p prime and $n > 0$, show that $Z(G)$ is not trivial.
3. If $|G| = p^2$, show that G is abelian.

Solution:

1. If g is an element of G whose image generates the cyclic group G/Z, then G is generated by the commuting set g and Z, so is abelian.
2. All conjugacy classes have order a power of p (as their order is the index of a centralizer of one of their elements) so the number of conjugacy classes with just one element is a multiple of p. These conjugacy classes form the center, so the center has order a multiple of p so is non-trivial.

3. By part 2 the center has order at least p, so G/Z has order at most p and is therefore cyclic. By part 1 the group must be abelian.

Problem 9B. Prove that the sequence of functions $f_n(x) = \sin nx$ has no pointwise convergent subsequence. (Hint: show that given any subsequence and any interval of positive length there is a subinterval such that some element of the subsequence is at least $1/2$ on this subinterval, and another element is at most $-1/2$.)

`Remark.` This is an example from Ch. 7 of W. Rudin's *Principles of Mathematical Analysis*, which is treated by the author using a result from the more advanced chapter on Lebesgue measure, namely the bounded convergence theorem. According to it, if a sequence of bounded continuous functions g_k $(= (\sin n_k x - \sin n_{k+1} x)^2$ in this example) tends to 0 pointwise, then $\int g_k(t)dt$ tend to 0 too. (In the example, the integral over the period $[0, 2\pi]$ is equal to 2π regardless of k.) Below, an elementary proof is given; it is due to Evan O'Dorney (a high-school student taking Givental's H104 class).

Solution: Given a subsequence $\sin n_k x$, we find a subsequence $\sin n_{k_l}$ in it and a point x_0 where $\lim_{l\to\infty} \sin n_{k_l} x_0$ does not exist. Start with picking an interval $[a_1, b_1]$ where $\sin n_1 x \geq 1/2$. Passing to a term $\sin n_k x$ which oscillates sufficiently many times on the interval $[a_1, b_1]$, find in it an interval $[a_2, b_2]$ where $\sin n_k x \leq -1/2$. Passing to a term $\sin n_m x$ which oscillates sufficiently many times on $[a_2, b_2]$, find in it an interval $a_3, b_3]$ where $\sin n_m x \geq 1/2$, and so on. Call the selected functions $\sin n_{k_1} x$, $\sin n_{k_2} x$, $\sin n_{k_3} x$, etc., and let x_0 be a common point of the nested sequence of intervals $[a_1, b_1] \supset [a_2, b_2] \supset \ldots$. Since $\sin n_{k_l} x_0$ is $\geq 1/2$ for odd l and $\leq -1/2$ for even l, the limit at x_0 does not exist.