SPRING 2008 PRELIMINARY EXAMINATION

1A. Prove that it is not possible to find two linear operators A and B on a non-zero finite dimensional complex vector space with $AB - BA = I$, where I is the identity operator. Give an example of two such operators acting on an infinite dimensional complex vector space.

Solution: $Tr(AB - BA) = 0 \neq Tr(I)$. The operators $A = d/dx$ and $B = x$ acting on the ring of polynomials satisfy $AB - BA = I$.

2A. Evaluate

$$\int_{-\infty}^{+\infty} \frac{\cos(x)}{1+x^2} dx$$

Solution. The integral is unaffected if we replace $\cos(x)$ by e^{ix}. By the residue theorem the integral is equal to $2\pi i$ times the sum of the residues in the upper half plane (as e^{ix} is small there). The only residue is at $x = i$, where the residue is $1/2ie$. So the integral is $\pi/2e$.

3A. Find (without proof) the number of subgroups of each possible order of the symmetric group S_4 of all permutations of 4 points.

Solution: The order of the subgroup has to divide 24. Check each possible order. There is 1 (trivial) subgroup of order 1, 6 (type 2) +3 (type 2^2) of order 2, 4 of order 3 (cyclic), 3 (cyclic) +1 (normal 4-group) +3 (non-normal 4-group) of order 4, 4 of order 6 (fixing a point), 3 of order 8 (Sylow subgroups), 1 of order 12 (alternating group), and 1 of order 24 (whole group).

4A. Find the solution of the differential equation

$$y'' - 2y' + y = e^{-x}$$

satisfying $y(0) = y'(0) = 0$.

Solution: $y = e^{-x}/4 + ae^x + bxe^x$ is the general solution. $y(0) = 0$ forces $a = -1/4$, and $y'(0) = 0$ then forces $b = 1/2$.

5A. Suppose M is an $n \times n$ nilpotent matrix over $\mathbb{C}$. Show the set of matrices $C(M)$ which commute with M is the ring $\mathbb{C}[M]$ if and only if the null space of M has dimension one.

Solution.

Let $V = \mathbb{C}^n$. The ring $\mathbb{C}[M]$ is a vector space spanned by $1, M, \ldots, M^{n-1}$, because $M^{n-1} = 0$. Put M in Jordan form

$$\begin{matrix} M_1 & & \\ & \ddots & \\ & & M_r \end{matrix}$$

(M_j is an $s_j \times s_j$ matrix with 0's on the diagonal and 1's on the supradiagonal). The dimension of the nullspace of M is r.

Suppose $r = \dim(\mathrm{Null}\,(M)) = 1$. It follows that there is a vector $v \in V$ such that $v, Mv, \ldots, M^{n-1}v$ is a basis for V. Suppose A commutes with M and $Av = \sum_{i=0}^{n-1} a_i(M^i v)$. Claim:

$$A = \sum_{i=0}^{n-1} a_i M^i.$$

Indeed,

$$A(M^j v) = M^j(Av) = \sum_{i=0}^{n-1} a_i(M^{i+j}v) = (\sum_{i=0}^{n-1} a_i M^i)(M^j v),$$

and so $A \in \mathbb{C}[M]$.

On the other hand, the n matrices

$$\begin{pmatrix} \delta_{i\,1} M_1^{k_1} & & \\ & \ddots & \\ & & \delta_{i\,r} M_r^{k_r} \end{pmatrix}$$

where $1 \le i \le r$ and $0 \le k_i < s_i$ are linearly independent and commute with M. It follows that if $C(M) = \mathbb{C}[M]$, $M^{n-1} \neq 0$ so $1 = r = \dim(\mathrm{Null}\,(M))$.

6A. Suppose G is a finite group with only one automorphism. Show $|G| \le 2$.

Solution:

Since $h \in G \to ghg^{-1}$ is an automorphism for $g \in G$, G must be abelian. Then $h \in G \to h^k$ is an automorphism for $(k, |G|) = 1$. Thus $h^k = h$ for $(k, |G|) = 1$. In particular, $h = h^{-1}$ for $h \in G$. Thus $G = (\mathbb{Z}/2\mathbb{Z})^r$ for some $r \ge 0$. If $r > 1$, $(x_1, x_2, \ldots, x_r) \to (x_2, x_1, \ldots, x_r)$ is a non-trivial automorphism. Thus $r \le 1$.

7A. Find all irreducible polynomials of degree at most 4 over the field with 2 elements.

Solution: Using the sieve of Eratosthenes we find x, $x+1$ of degree 1. Therefore higher degree irreducible polynomials must have constant term 1 and sum of coefficients 1. This gives the irreducible polynomials x^2+x+1, x^3+x+1, x^3+x^2+1 in degrees 2 and 3. In degree 4 we also have to eliminate polynomials divisible by x^2+x+1; the only extra possibility eliminated by this is $(x^2+x+1)^2 = x^4+x^2+1$. So in degree 4 the irreducible polynomials are x^4+x+1, x^4+x^3+1, $x^4+x^3+x^2+x+1$.

8A. Let p, q be distinct prime numbers and let R be a commutative ring with 1 of characteristic pq. Show there are rings S, T of characteristic p, q respectively, such that R is isomorphic to $S \times T$.

Solution:

p and q are relatively prime, so there are integers m, n such that $1 = mp + nq$.

pR and qR are ideals of R. Let $S = R/pR$ and $T = R/qR$. So in S $p = 0$; since R does not have characteristic q, $1 \notin pR$ (since otherwise $q \in qpR = (0)$); thus S has characteristic exactly p. Similarly T has characteristic q.

There are onto homomorphisms $f : R \to S$ and $g : R \to T$ given by $f(a) = a + pR$ and $g(a) = a + qR$. So there is a homomorphism $h : R \to S \times T$ given by $h(a) =< f(a), g(a) >$.

If a is in the kernel of h then $a \in (pR \cap qR)$ so $a = 1a = mpa + nqa \in pqR = (0)$. Thus h is $1 - 1$.

Notice $f(mp) = 0_S$ and $g(mp) = g(1 - nq) = 1_T$ while $f(nq) = f(1 - mp) = 1_S$ and $g(nq) = 0_T$. So given $a, b \in R$ let $c = anq + bmp$, then $f(c) = f(a)1_S + f(b)0_S = f(a)$ and $g(c) = g(a)0_T + g(b)1_T = g(b)$. So $f(c) =< f(a), g(b) >$. Since f, g are onto S, T respectively, we get h is onto $S \times T$.

9A. For integers $n \geq 1$, let S_n be the symmetric group on n letters, and let $f(n) =$ the maximum order of elements of S_n. Show

$\liminf_{n\to\infty} \frac{n}{f(n)} = 0$.

Solution:

The product of a k-cycle and a $k+1$-cycle in S_{2k+1} or S_{2k+2} has order $k(k+1)$ as the cycles have coprime orders, so for $n = 2k+1$ or $n = 2k+2$, $n/f(n)$ is at most $(2k+2)/k(k+1) \leq 2/k \leq 4/(n-2)$. This tends to 0 as n tends to infinity, so $n/f(n)$ has limit 0 as n tends to infinity.

1B. For integers $n \geq 1$, let $P_n =$ the set of degree $\leq n$ polynomials with real coefficients. Show there is $q(x) \in P_n$ such that for all $p(x) \in P_n$

$$\int_0^1 p(x)q(x)dx = \int_0^1 \frac{p(x)}{x^2+1}dx$$

Solution:

P_n is a vector space over the reals of dimension $n+1$.

For $p(x) \in P_n$ let

$$T(p) = \int_0^1 \frac{p(x)}{x^2+1}dx$$

.

T is a linear map from P_n to the reals. So T is in the dual space P_n^*.

For $p, q \in P_n$ let

$$L_p(q) = \int_0^1 p(x)q(x)dx$$

. Each L_p is in P_n^*. The map L is linear from P_n to P_n^*.

If p is in the kernel of L then

$$0 = L_p(p) = \int_0^1 p(x)p(x)dx$$

and so $p(x)$ is 0 on the unit interval; since $p(x)$ is a polynomial, $p(x) = 0$. Thus L is $1 - 1$. Since P_n and P_n^* have the same dimension, L is onto.

Hence there is $q \in P_n$ such that $L_q = T$.

2B. (a) Let G be a finite commutative group, and let c be the product of all elements of G. Show that $c^2 = 1$.

(b) Let F be a finite field, and let c be the product of all nonzero elements in F. Show that $c = -1$.

Solution: (a) Let $Z \subset G$ be the subset of elements $g \in G$ for which $g \neq g^{-1}$ (equivalently $g^2 \neq 1$). Then

$$\prod_{g \in Z} g = 1$$

since for every $g \in Z$ we have $g^{-1} \in Z$ and $g^{-1} \neq g$. Therefore

$$c = \prod_{g \in G, g^2 = 1} g$$

so

$$c^2 = \prod_{g \in G, g^2 = 1} g^2 = 1.$$

(b) Consider the finite group F^*. Then the set of elements $g \in F^*$ such that $g^2 = 1$ is precisely the set $\{1, -1\}$ since $X^2 - 1 = (X-1)(X+1)$. Therefore by the proof of (a) we have $c = -1$.

3B. Let G be a group and $H \subset G$ a subgroup of finite index n. Show that G contains a normal subgroup N such that $N \subset H$ and the index of N is $\leq n!$.

4B. Let p and q be distinct primes. Show that any group G of order p^2q^2 is not simple.

Solution. Assume to the contrary that G is simple. Let s_q (resp. s_p) denote the number of q-Sylow (resp. p-Sylow) subgroups of G. Then s_q divides p^2 so either $s_q = p$ or $s_q = p^2$. Also $s_q \equiv 1 \pmod{q}$. Therefore either q divides $p-1$ or q divides $p^2 - 1 = (p-1)(p+1)$. We conclude that q divides one of $p-1$ and $p+1$. Similarly by symmetry we get that p divides either $q-1$ or $q+1$. This implies that either $q = p-1$ or $q = p+1$. This implies that (after possibly interchanging p and q) we have $q = 3$ and $p = 2$ (since both must be prime). Therefore $|G| = 36$. Let S be the set of 3-Sylow subgroups. Then the group $Aut(S)$ has order either $2! = 2$ or $4! = 24$. In either case the homomorphism $\rho : G \to Aut(S)$ given by the conjugation action on S must have nontrivial kernel as $|G| > Aut(S)$.

5B. Let $\zeta = e^{2\pi i/5}$ and let $\alpha = \sqrt[5]{2} \in \mathbb{R}$. Let E denote the subfield $\mathbb{Q}[\zeta, \alpha] \subset \mathbb{C}$ generated by ζ and α.

(a) Show that E is Galois over $\mathbb{Q}$.

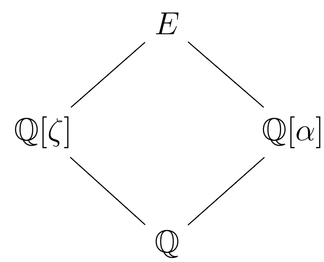
(b) What is $[E : \mathbb{Q}]$?

Solution. For (a) note that

$$X^5 - 2 = \prod_{i=0}^{4} (X - \zeta^i \alpha),$$

which implies that E is the splitting field of $X^5 - 2$ over $\mathbb{Q}$.

For (b) consider the diagram of fields



The irreducible polynomial of ζ is $X^4+X^3+X^2+X+1$ so the extension $\mathbb{Q}[\zeta]$ has degree 4 over $\mathbb{Q}$. On the other hand, the field extension $\mathbb{Q}[\alpha]$ has degree 5 over $\mathbb{Q}$. Since 4 and 5 are relatively prime it follows that $[E:\mathbb{Q}]=20$.

6B. The function $y(x)$ defined on $[0,\infty)$ is smooth and satisfies $y''-y=f(x)$ in $x>0$, $y(0)=0$, $y'(0)=0$ and $y(x)$ and $y'(x)$ tend to 0 as $x\to\infty$. Here $f(x)$ is a continuous function on $[0,\infty)$ which vanishes for $x>1$. Find a non-zero function $g(x)$ (not depending on y or f) such that $\int_0^1 f(x)g(x)dx=0$.

Solution. Multiply ODE by e^{-x} and integrate over $[0,L]$,

$$\int_0^L e^{-x}\ y''\ dx-\int_0^L e^{-x}\ y\ dx=\int_0^L e^{-x}\ f(x)\ dx. \tag{1}$$

Do two integrations by parts to the first integral on LHS, and use $y(0)=0$, $y'(0)=0$. When the dust settles, (1) becomes

$$e^{-L}(y'(L)+y(L))=\int_0^L e^{-x}\ f(x)\ dx. \tag{2}$$

Now take $L>1$. In $x>1$, $y''-y=0$ and the solutions which decay to zero as $x\to\infty$ are proportional to e^{-x}. Hence in LHS of (2), $y'(L)+y(L)=0$ for $L>1$. In RHS we can replace L by 1 since $f(x)=0$ for $x>1$, so we find the condition $\int_0^1 e^{-x}\ f(x)\ dx=0$.

7B. Evaluate $\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^5}$.

Solution. Lets avoid doing residue of 5th order pole at $z=i$. First, for $a>0$,

$$\int_{-\infty}^{\infty}\frac{dx}{a+x^2}=a^{-\frac{1}{2}}\int_{-\infty}^{\infty}\frac{dx}{1+x^2}=\pi a^{-\frac{1}{2}}.$$

Differentiate with respect to a:

$$\int_{-\infty}^{\infty}\frac{dx}{(a+x^2)^2}=\frac{1}{2}\pi\ a^{-\frac{3}{2}}.$$

Do it again three times:

$$\int_{-\infty}^{\infty}\frac{dx}{(a+x^2)^5}=\frac{1}{2}\frac{3}{2}\frac{5}{2}\frac{7}{2}\pi\ a^{-\frac{9}{2}}.$$

Now set $a=1$,

$$\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^5}=\frac{1.3.5.7}{2^4}\pi.$$

8B. Compute the sequence $\{x_n\}_0^\infty$ of real numbers so that $x_n = x_{n-1} - \frac{1}{2}x_{n-2}$ for $n \geq 2$, and $x_0 = 1$, $x_1 = 1$.

Solution. Seek elementary solutions of the difference equation in the form $x_n = r^n$. Get $r^2 - r + \frac{1}{2} = 0$, with solutions $r = \frac{1 \pm \sqrt{1-2}}{2} = \frac{1 \pm i}{2}$. General solution of difference equation is linear combination of $\left(\frac{1+i}{2}\right)^n$ and $\left(\frac{1-j}{2}\right)^n$, and the linear combination with $x_0 = 1$, $x_1 = 1$ is

$$x_n = \left(\frac{1+i}{2}\right)^n + \left(\frac{1-i}{2}\right)^n = \left(\frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}\right)^n + \left(\frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}}\right)^n = 2^{-\frac{n}{2}}\left(e^{i\frac{n\pi}{4}} + e^{-i\frac{n\pi}{4}}\right),$$

or

$$\boxed{x_n = 2^{1-\frac{n}{2}} \text{ as } \frac{n\pi}{4}}$$

Solution. Seek elementary solutions of the difference equation in the form $x_n = r^n$. Get $r^2 - r + \frac{1}{2} = 0$, with solutions $r = \frac{1 \pm \sqrt{1-2}}{2} = \frac{1 \pm i}{2}$. General solution of difference equation is linear combination of $\left(\frac{1+i}{2}\right)^n$ and $\left(\frac{1-j}{2}\right)^n$, and the linear combination with $x_0 = 1$, $x_1 = 1$ is

$$x_n = \left(\frac{1+i}{2}\right)^n + \left(\frac{1-i}{2}\right)^n = \left(\frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}\right)^n + \left(\frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}}\right)^n = 2^{-\frac{n}{2}}\left(e^{i\frac{n\pi}{4}} + e^{-i\frac{n\pi}{4}}\right),$$

or

$$\boxed{x_n = 2^{1-\frac{n}{2}} \text{ as } \frac{n\pi}{4}}$$

9B. Compute

$$\lim_{x \to 0} \frac{d^4}{dx^4} \frac{x}{\sin x}.$$

Solution: By Taylor's formula,

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5).$$

Therefore

$$\begin{aligned}
\frac{x}{\sin x} &= \frac{1}{1 - \frac{x^2}{6} + \frac{x^4}{120} + o(x^4)} \\
&= 1 + (\frac{x^2}{6} - \frac{x^4}{120} + o(x^4)) + (\frac{x^2}{6} + o(x^2))^2 + o(x^4) \\
&= 1 + \frac{x^2}{6} + \left[\frac{1}{36} - \frac{1}{120}\right] x^4 + o(x^4).
\end{aligned}$$