SPRING 2007 PRELIMINARY EXAMINATION SOLUTIONS

1A. Let $x_1, x_2, \ldots$ be an infinite sequence of real numbers such that every subsequence contains a subsequence converging to 0. Must the original sequence converge?

Solution: Yes; in fact it must converge to 0. If not, there would exist $\epsilon > 0$ such that for infinitely many n, we have $|x_n| > \epsilon$. Choose a subsequence S consisting of such x_n. If T is a subsequence of S, then T also consists of numbers of absolute value greater than ϵ, so T cannot converge to 0. Thus S has no subsequence converging to 0. This contradicts the given hypothesis.

2A. Find a matrix U such that $U^{-1}AU = J$ is in Jordan canonical form, where

$$A = \begin{pmatrix} 0 & -3 & 5 \\ -1 & -6 & 11 \\ 0 & -4 & 7 \end{pmatrix}.$$

Solution: Expanding by minors along the first column shows that the characteristic determinant is given by

$$\det(A - \lambda I) = -\lambda^3 + \lambda^2 + \lambda - 1 = -(\lambda - 1)^2(\lambda + 1).$$

Thus $\lambda_1 = -1$ is an eigenvalue of algebraic (and hence geometric) multiplicity 1 while $\lambda_2 = 1$ is an eigenvalue of algebraic multiplicity 2. The eigenvectors of A belong to the kernels of the matrices

$$A - \lambda_1 I = \begin{pmatrix} 1 & -3 & 5 \\ -1 & -5 & 11 \\ 0 & -4 & 8 \end{pmatrix}, \qquad A - \lambda_2 I = \begin{pmatrix} -1 & -3 & 5 \\ -1 & -7 & 11 \\ 0 & -4 & 6 \end{pmatrix},$$

which can be row-reduced to

$$P_1(A - \lambda_1 I) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}, \qquad P_2(A - \lambda_2 I) = \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 0 \end{pmatrix},$$

where P_1 and P_2 are products of elementary row operations. We see that $u_1 = (1, 2, 1)^T$ and $u_{2,0} = (1, 3, 2)^T$ are the eigenvectors of A and the geometric multiplicity of λ_2 is 1. To put A in Jordan canonical form, we want $A(u_1, u_{2,0}, u_{2,1}) = AU = UJ = (\lambda_1 u_1, \lambda_2 u_{2,0}, \lambda_2 u_{2,1} + u_{2,0})$, so we need to find a vector $u_{2,1}$ satisfying $Au_{2,1} = \lambda_2 u_{2,1} + u_{2,0}$. This can be done by solving $P_2(A - \lambda_2 I)u_{2,1} = P_2 u_{2,0}$:

$$\left(\begin{array}{ccc|c} -1 & -3 & 5 & 1 \\ -1 & -7 & 11 & 3 \\ 0 & -4 & 6 & 2 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{ccc|c} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & -3/2 & -1/2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Thus $u_{2,1} = (1, 1, 1)^T$ works (as does $(1, 1, 1)^T + \alpha(1, 3, 2)^T$ for any $\alpha \in \mathbb{C}$) and we have

$$U = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{pmatrix}, \qquad U^{-1}AU = J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

3A. Suppose $f : \mathbb{R} \to \mathbb{R}$ is real analytic and periodic with period 2π. Prove that f has an analytic continuation F defined on a strip

$$S = \{x + iy \in \mathbb{C} \;:\; |y| < \rho\}$$

with $\rho > 0$, and that $F(z + 2\pi) = F(z)$ for $z \in S$.

Solution: Since f is real analytic, it possesses derivatives of all orders and agrees with its (convergent) Taylor series on a neighborhood $(x - r_x,\, x + r_x)$ of every point $x \in \mathbb{R}$. The same power series may be used to define F on the complex neighborhood $B(x, r_x)$ of radius r_x centered at x. Since f is periodic, the coefficients of the Taylor series at $x + 2\pi$ are the same as those at x, so we may assume that $r_{x+2\pi} = r_x$ for all $x \in \mathbb{R}$. Let us cover the compact interval $[-\pi, \pi] \subset \mathbb{C}$ with open squares $U_x = (x - \frac{1}{2}r_x,\, x + \frac{1}{2}r_x) \times (-\frac{1}{2}r_x,\, \frac{1}{2}r_x)$ and choose a finite sub-cover $U_{x_1}, \ldots, U_{x_n}$ of $[-\pi, \pi]$. We now define

$$\rho = \min\left\{\frac{1}{2}r_{x_i} \;:\; 1 \le i \le n\right\}$$

and note that since each square U_{x_i} has half-height $\ge \rho$ and satisfies $U_{x_i} \subset B(x_i, r_{x_i})$, the balls $\{B(x_{ik},\, r_{x_i}) \;:\; x_{ik} = x_i + 2\pi k,\; 1 \le i \le n,\; k \in \mathbb{Z}\}$ cover the strip S. For any $z \in S$, we define $F(z)$ using the Taylor series at any x_{ik} for which $z \in B(x_{ik}, r_{x_i})$. A different choice of x_{ik} will yield the same value $F(z)$ since the intersection of two balls containing z will contain a positive interval of the real axis on which the Taylor expansions agree with f, so they represent the same analytic function on this intersection. F satisfies $F(z + 2\pi) = F(z)$ for $z \in S$ since the Taylor expansion centered at x_{ik} defining F at $z \in B(x_{ik}, r_{x_i})$ has the same coefficients as the one centered at $x_{i,k+1}$ defining F at $z + 2\pi$.

4A. Define six fields as follows:

- Let $A = \mathbb{Q}(\alpha)$ where $\mathbb{Q}$ is the field of rational numbers and α is the real cube root of 2.
- Let B be a splitting field of $x^3 - 2$ over $\mathbb{Q}$.
- Let C be an algebraic closure of the field $\mathbb{F}_2$ of 2 elements.
- Let D be the subfield of C generated over $\mathbb{F}_2$ by the set of $a \in C$ such that there exists $n \ge 1$ with $a^n = 1$.
- Let E be the field $\mathbb{R}$ of real numbers.
- Let F be the field $\mathbb{Q}[[T]](T^{-1})$ of formal Laurent series with rational coefficients.

For each pair of these, determine with proof whether or not they are isomorphic.

Solution: We will show that the only isomorphic pair consists of C and D.

Let $S_1 = \{A, B\}$, $S_2 = \{C, D\}$, and $S_3 = \{E, F\}$. The fields in S_1 are of finite dimension over $\mathbb{Q}$, hence countable and of characteristic 0. The fields in S_2 are of characteristic 2. The fields in S_3 are uncountable and of characteristic 0. Hence no field in S_i is isomorphic to a field in S_j if $i \neq j$.

By Eisenstein's criterion, $x^3 - 2$ is irreducible, so $[A : \mathbb{Q}] = 3$. The zeros of this polynomial are $\omega^i\alpha$ where ω is a primitive cube root of unity. Thus $\omega \in B$. Since $[\mathbb{Q}(\omega) : \mathbb{Q}] = 2$, the degree $[B : \mathbb{Q}]$ is even. Hence $A \not\approx B$.

If $a \in C$, then $\mathbb{F}_2(a)$ is a finite extension of $\mathbb{F}_2$, hence finite, say of order q; if moreover $a \neq 0$, then $a^{q-1} = 1$. Hence $C \subseteq D$. But $D \subseteq C$, so $C = D$.

The square of a nonzero element of $\mathbb{Q}[[T]](T^{-1})$ has a leading coefficient that is a rational square. Thus 2 is not a square in $\mathbb{Q}[[T]](T^{-1})$. But 2 is a square in $\mathbb{R}$. So $E \not\approx F$.

5A. Let $a_0(x)$, $a_1(x)$, $\ldots$, $a_{r-1}(x)$ and $b(x)$ be C^m functions on $\mathbb{R}$. Prove that if $y(x)$ is a solution of the differential equation

$$y^{(r)} + a_{r-1}(x)y^{(r-1)} + \cdots + a_1(x)y' + a_0(x)y = b(x)$$

(in particular, assuming that the derivatives y', y'', $\ldots$, $y^{(r)}$ exist), then $y(x)$ is C^{m+r}.

Solution: Rewrite the differential equation as

$$y^{(r)} = b - \left(a_{r-1}y^{(r-1)} + \cdots + a_1y' + a_0y\right), \tag{1}$$

and proceed by induction on m. For $m = 0$, the derivatives of y on the right-hand side of (1) are differentiable and hence continuous. The functions a_i and b are continuous by assumption, so $y^{(r)}$ is continuous, *i.e.*, y is C^r.

For $m > 0$, assume by induction that y is C^{m+r-1}. Then the derivatives of y on the right-hand side of (1) are C^m. The functions a_i and b are C^m by assumption, so $y^{(r)}$ is C^m, hence y is C^{m+r}.

6A. Let $A = \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3$ where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ and $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Let $\beta \in \mathbb{C}$ be any square root of $\alpha_1^2 + \alpha_2^2 + \alpha_3^2$.

(a) Prove that $\exp(A) = \cosh(\beta) + \frac{\sinh\beta}{\beta}A$, where $\frac{\sinh(\beta)}{\beta}$ is interpreted as 1 if $\beta = 0$. (Hint: First show that A^2 is a scalar multiple of the identity.)

(b) Evaluate $\exp(A)$ explicitly in the case $\alpha_1 = i\pi$, $\alpha_2 = i\pi$, and $\alpha_3 = \pi$.

Solution: (a) An explicit calculation shows that $A^2 = (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)I = \beta^2 I$. Thus

$$\begin{aligned}
\exp(A) &= I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots \\
&= I + A + \frac{\beta^2}{2!}I + \frac{\beta^2}{3!}A + \frac{\beta^4}{4!}I + \frac{\beta^4}{5!}A + \cdots \\
&= \cosh(\beta) + \frac{\sinh\beta}{\beta}A,
\end{aligned}$$

where the last step is valid (with our convention) even if $\beta = 0$.

(b) The values $\alpha_1 = i\pi$, $\alpha_2 = i\pi$, $\alpha_3 = \pi$ give $\beta^2 = -\pi^2$, so we choose $\beta = i\pi$ and obtain $\cosh(\beta) = \cos(i\beta) = \cos(-\pi) = -1$, $\sinh(\beta) = -i\sin(i\beta) = -i\sin(-\pi) = 0$, and $\exp(A) = -I$.

7A. Let a and b be complex numbers, and let $f\colon \mathbb{C} \to \mathbb{C}$ be a non-constant entire function such that $f(az+b) = f(z)$ for all $z \in \mathbb{C}$. Prove that there is a positive integer n such that $a^n = 1$.

Solution: If $a = 1$, we are done, so assume that $a \neq 1$. Then $az + b = z$ has a unique solution, say c. Define $g(z) := f(z+c)$, so

$$g(az) = f(az+c) = f(az+ac+b) = f(a(z+c)+b) = f(z+c) = g(z).$$

If the Taylor series of $g(z)$ at $z = 0$ is $\sum_{i \geq 0} g_i z^i$, then equating coefficients of z^n in $g(az) = g(z)$ yields

$$a^n g_n = g_n.$$

Since f is not constant, g is not constant. Therefore for some $n \geq 1$ we have $g_n \neq 0$, and hence $a^n = 1$.

8A. Let $n \geq 3$, and let A_n be the alternating subgroup of the symmetric group on n letters. Prove that A_n is generated by (123) and $(12 \cdots n)$ if n is odd, or by (123) and $(2 \cdots n)$ if n is even.

Solution: We prove the statement by induction on n. The base case $n = 3$ is trivial.

Let G be the subgroup of A_n generated by these elements. Then G acts transitively on $\{1, \ldots, n\}$, so it suffices to show that the stabilizer of 1 in G is the full alternating group on $\{2, \ldots, n\}$. By induction we need only show for $n \geq 4$ that G contains (234) and either $(3 \cdots n)$ (if n is odd) or $(2 \cdots n)$ (if n is even).

Case 1: n is odd. Then conjugating (123) by $(12 \cdots n)$ yields $(234) \in G$. And

$$(3 \cdots n) = (123)^{-1}(12 \cdots n) \in G.$$

Case 2: n is even. Then conjugating (123) by $(2 \cdots n)$ yields $(134) \in G$, and conjugating (123) by (134) yields $(324) \in G$, and hence $(234) = (324)^{-1} \in G$. This time, $(2 \cdots n) \in G$ is part of the inductive hypothesis.

9A. Suppose b and L are positive constants and $f : [0, b] \to \mathbb{R}$ is continuous and satisfies

$$f(x) \geq L \int_0^x f(t)\, dt, \qquad (0 \leq x \leq b).$$

Show that $f(x) \geq 0$ for $0 \leq x \leq b$.

Solution: Let $F(x) = \int_0^x f(t)\, dt$. Since f is continuous, F is differentiable and we have

$$F'(x) = f(x) \geq LF(x), \qquad (0 \leq x \leq b).$$

Thus, for $0 \leq t \leq b$ we have

$$\begin{aligned} F'(t) - LF(t) &\geq 0, \\ \big(F'(t) - LF(t)\big)e^{-Lt} &\geq 0, \\ \frac{d}{dt}\Big(F(t)e^{-Lt}\Big) &\geq 0, \end{aligned}$$

and since definite integrals preserve inequalities:

$$F(x)e^{-Lx} - F(0)e^0 = \int_0^x \frac{d}{dt}\Big(F(t)e^{-Lt}\Big)\,dt \geq \int_0^x 0\,dt = 0, \quad (0 \leq x \leq b).$$

Since $F(0) = 0$ and $e^{-Lx} > 0$, we learn that $F(x) \geq 0$ for $0 \leq x \leq b$, hence the original inequality $f(x) \geq LF(x)$ gives the desired result.

An alternative proof might run as follows. Let $x_0 = \sup\{x < b \;:\; f(t) \geq 0 \text{ for } t \in [0, x]\}$. We know $x_0 \geq 0$ since $f(0) \geq 0$. We must show that $x_0 = b$. Suppose to the contrary that $x_0 < b$. Since $f(x)$ is continuous and non-negative to the left of x_0, $f(x_0) \geq 0$. On the other hand, there are points $x > x_0$ arbitrarily close to x_0 at which $f(x) < 0$. Thus $f(x_0) = 0$. The given inequality now implies that $f(x) = 0$ for $0 \leq x \leq x_0$. Now define $x_1 = \min(x_0 + L^{-1}, b)$. Then there is an x_2 in the interval $x_0 < x_2 < x_1$ which satisfies $f(x_2) < 0$. Let $\varepsilon = |f(x_2)|$. The given inequality implies that $f(x) \geq L \int_{x_0}^x f(t)\,dt$ for $x_0 \leq x \leq x_1$. Thus $u(x) = f(x) + \varepsilon$ satisfies $u(x_0) = \varepsilon$, $u(x_2) = 0$, and

$$u(x) \geq L \int_{x_0}^x u(t) - \varepsilon\,dt \,+\, \varepsilon = \int_{x_0}^x u(t)\,dt + \varepsilon[1 - L(x - x_0)] \geq \int_{x_0}^x u(t)\,dt$$

for $x_0 \leq x \leq x_1$. But since u is continuous and $u(x_0) = \varepsilon > 0$, it's impossible for u to reach 0 over the interval $x_0 < x < x_1$, for at the first crossing x_3 where $u(x_3) = 0$, the integral $\int_{x_0}^{x_3} u(x)\,dx > 0$. Thus the assumption that $u(x_2) = 0$ leads to a contradiction, and we conclude that $x_0 = b$.

1B. If $c \in \mathbb{R}$, say that a real-valued function $f\colon \mathbb{R} \to \mathbb{R}$ is periodic with period c if it satisfies $f(x + c) = f(x)$ for all $x \in \mathbb{R}$.

(i) Let V be the set of continuous real-valued functions f having a positive integer as a period. Prove that V is a vector space.

(ii) Let $p_1 < p_2 < \ldots < p_n < \ldots$ be the sequence of prime numbers, and for each i, let f_i be a function whose minimal positive period is p_i. Prove that the functions $f_1, f_2, \ldots$ are linearly independent in V.

Solution: (i) The set of all functions $\mathbb{R} \to \mathbb{R}$ is a vector space, so it suffices to check that V contains 0 and is closed under addition and scalar multiplication. The only nontrivial claim is closure under addition. Suppose $f, g \in V$, say with periods c and d. Any positive integer multiple of a period is a period of the same function, so f, g have cd as a common period. Thus $f + g$ has cd as a period (and it is continuous).

(ii) Suppose not. Then there exists a relation

$$a_1 f_1 + \cdots + a_n f_n = 0$$

where $a_i \in \mathbb{R}$ and $a_n \neq 0$. Solving for f_n shows that f_n has $p_1 p_2 \cdots p_{n-1}$ as a period. It also has p_n as a period. Now $p_1 p_2 \cdots p_{n-1}$ and p_n are relatively prime, so 1 is an integer combination of $p_1 p_2 \cdots p_{n-1}$ and p_n. Any integer combination of periods is a period, so in particular 1 is also a period of f_n. This contradicts the hypothesis that p_n is the minimal period of f_n.

2B. Given any real number a_0, define $a_1, a_2, \ldots$ by the rule $a_{n+1} = \cos a_n$ for all $n \geq 0$. Prove that the sequence (a_n) converges, and that the limit is the unique solution of the equation $\cos x = x$.

Solution: Let $g(x) = \cos x - x$. Then $g(1) < 0$, and since $\cos x$ is decreasing on $[0, \pi]$, we have $\cos(1/2) > \cos(\pi/3) = 1/2$, that is, $g(1/2) > 0$. By the Intermediate Value Theorem, there exists $1/2 < a < 1$ such that $\cos a = a$. To see that this a is the unique solution of $\cos x = x$, observe first that any solution must clearly lie in $[-1, 1]$. On $[-1, 0)$ we have $x < 0 < \cos x$, so all solutions lie in $[0, 1]$. But $g(x)$ is strictly decreasing on $[0, 1]$, so the solution is unique.

Consider any function f which is differentiable and satisfies $|f'(x)| < c$ for all x in an interval $(a - d, a + d)$, where $c < 1$, and $f(a) = a$. For any $a_0 \in (a - d, a + d)$, define a sequence (a_n) by $a_{n+1} = f(a_n)$. It follows easily by induction on n using the Mean Value Theorem that $|a_{n+1} - a| < c|a_n - a|$ for all n, hence (a_n) converges to a.

We'll apply this with $f(x) = \cos x$, a the solution of $\cos a = a$, and $d = 1/2$. Note that $[a - d, a + d] \subseteq (0, 3/2) \subseteq (0, \pi/2)$ since $a \in (1/2, 1)$, and therefore $|\cos'(x)| = |\sin x| < c$ for some $c < 1$.

The given sequence (a_n) satisfies $a_1 \in [-1, 1]$, hence $a_2 \in [\cos(1), 1] \subseteq [1/2, 1] \subseteq (a - 1/2, a + 1/2)$. We conclude that $(a_2, a_3, \ldots)$ converges to a.

3B. Let k and l be positive integers. Let $\mathbb{Q}(x)(\sqrt[k]{1 - x^l})$ be any extension field of $\mathbb{Q}(x)$ generated by a k-th root of $1 - x^l$. Define $\mathbb{Q}(x)(\sqrt[l]{1 - x^k})$ similarly. Prove that $\mathbb{Q}(x)(\sqrt[k]{1 - x^l})$ and $\mathbb{Q}(x)(\sqrt[l]{1 - x^k})$ are isomorphic.

Solution: The polynomial $y^k + x^l - 1$ is irreducible, for any positive integers k, l. One way to prove this is to regard $y^k + x^l - 1$ as a polynomial in y over $\mathbb{Q}[x]$ and apply Eisenstein's criterion, using the fact that $x - 1$ divides $x^l - 1$ but $(x - 1)^2$ does not. It follows that the fields in question are the fraction fields of $\mathbb{Q}[x, y]/(y^k + x^l - 1)$ and $\mathbb{Q}[x, y]/(y^l + x^k - 1)$, respectively, which are obviously isomorphic by the exchange of x and y.

4B. Let E be the $\mathbb{C}$-vector space of entire functions. Let V be a nonzero finite-dimensional $\mathbb{C}$-subspace of E with the property that $f \in V$ implies $f' \in V$. Prove that V contains a function that is everywhere nonzero.

Solution: The map $T \colon V \to V$ sending f to f' is a linear transformation. Since V is a $\mathbb{C}$-vector space, there exists an eigenvalue $\lambda \in \mathbb{C}$. Let $f \in V$ be a corresponding (nonzero) eigenvector. Then $f' = \lambda f$, so $f(z) = ce^{\lambda z}$ for some $c \in \mathbb{C}^\times$. This function is everywhere nonzero.

5B. Let $\mathbb{F}_q$ denote the finite field with q elements, where q is a power of a prime. Let $\mathrm{SL}_n(\mathbb{F}_q)$ be the group of $n \times n$ matrices with entries in $\mathbb{F}_q$ and determinant 1, under matrix multiplication. Determine (with proof) a simple necessary and sufficient condition on n and q for the center of $\mathrm{SL}_n(\mathbb{F}_q)$ to be trivial.

Solution: Let E_{ij} denote the $n \times n$ matrix with (i, j) entry equal to 1 and all other entries zero. For $i \neq j$, we have $I_n + E_{ij} \in \mathrm{SL}_n(\mathbb{F}_q)$. A matrix A commutes with $I_n + E_{ij}$ if and only if $E_{ij}A = AE_{ij}$. The latter condition implies that $A_{jk} = 0$ for $k \neq j$, that $A_{ki} = 0$ for

$k \neq i$, and that $A_{ii} = A_{jj}$. If this holds for all $i \neq j$, then $A = xI_n$ is a scalar multiple of the identity, and we have $A \in \mathrm{SL}_n(\mathbb{F}_q)$ if and only if $x^n = 1$ in $\mathbb{F}_q$.

The multiplicative group $\mathbb{F}_q^\times$ is cyclic, so the necessary and sufficient condition for $x = 1$ to be the unique solution of $x^n = 1$ in $\mathbb{F}_q$ is that $q - 1$ and n are relatively prime.

6B. Let U be a non-empty open subset of $\mathbb{R}^d$ and let $f : U \to \mathbb{R}^d$ be a continuous vector field defined on U. Let K be a compact subset of U and let $b > 0$. Suppose $\varphi : [0, b) \to K$ is a continuous function satisfying

$$\varphi(t) = \varphi(0) + \int_0^t f(\varphi(s))\, ds, \qquad (0 \leq t < b).$$

Prove that $\lim_{t \to b^-} \varphi(t)$ exists, where $t \to b^-$ means t approaches b from the left.

Solution: Let $M = \sup_{y \in K} \|f(y)\|$. For any two points $t_1, t_2 \in [0, b)$,

$$\|\varphi(t_2) - \varphi(t_1)\| = \left\| \int_{t_1}^{t_2} f(\varphi(t))\, dt \right\| \leq M|t_2 - t_1|$$

so φ is Lipschitz continuous on $[0, b)$ and hence preserves Cauchy sequences. Let $t_k \to b$ from the left. Then $\varphi(t_k)$ is Cauchy and hence converges to some $y_0 \in \mathbb{R}^d$. We claim that $\lim_{t \to b^-} \varphi(t) = y_0$. Let $\varepsilon > 0$ and choose k large enough that $|t_k - b| < \frac{\varepsilon}{M+1}$ and $\|\varphi(t_k) - y_0\| < \frac{\varepsilon}{M+1}$. Then for $0 < b - t < \delta = \frac{\varepsilon}{M+1}$ we have

$$\begin{aligned} \|\varphi(t) - y_0\| &\leq \|\varphi(t) - \varphi(t_k)\| + \|\varphi(t_k) - y_0\| \\ &\leq M|t - t_k| + \frac{\varepsilon}{M+1} \leq \varepsilon \end{aligned}$$

as required.

Alternative solution, based on a suggestion of Andre Kornell (using the dominated convergence theorem of Lebesgue integration, however): Because of the given integral equation, it suffices to apply the following claim to the function $g(s) = f(\varphi(s))$: for any continuous bounded function $g\colon [0, b) \to \mathbb{R}^d$, the limit $\lim_{t \to b^-} \int_0^t g(s)\, ds$ exists. To prove this, it suffices to prove that for every increasing sequence (t_k) in $[0, b)$ tending to b, the limit $\lim_{k \to \infty} \int_0^{t_k} g(s)\, ds$ exists. This follows from the dominated convergence theorem applied to the sequence of functions

$$g_k(s) := \begin{cases} g(s), & \text{if } s \in [0, t_k] \\ 0, & \text{if } s \in (t_k, b]. \end{cases}$$

7B. Given any group G, define a binary operation $*$ on the set $H = G \times G$ by $(g_1, h_1) * (g_2, h_2) = (g_1 g_2, g_2^{-1} h_1 g_2 h_2)$.

(a) Show that $(H, *)$ is group.

(b) In the case that G is the alternating group A_n on n letters with $n \geq 5$, prove that H has no subgroup of index 2.

Solution: (a) $(H, *)$ is the semidirect product $G \ltimes G$ where G acts on itself by conjugation.

(b) By the solution to part (a), H contains a normal subgroup N such that $N \cong H/N \cong A_n$. Since A_n is simple, the Jordan-Hölder theorem implies that $H/M \cong A_n$ for every nontrivial proper normal subgroup $M \subseteq H$. In particular, H cannot have a subgroup M of index 2, since such a subgroup is always normal.

(Alternatively, one could "avoid" the Jordan-Hölder theorem by essentially proving it in the special case needed, considering first the intersection of M with N, and then the image of M in H/N.)

8B. Let A be the set of $z \in \mathbb{C}$ such that $|z| \leq 1$, $\operatorname{Im}(z) \geq 0$, and $z \notin \{1, -1\}$. Find an explicit continuous function $u\colon A \to \mathbb{R}$ such that

- u is harmonic on the interior of A,
- $u(z) = 3$ for $z \in A \cap \mathbb{R}$
- $u(z) = 7$ for z in the intersection of A with the unit circle.

Solution: We use a conformal transformation to reduce to a problem on a different region. The transformation $w = f(z)$ where $f(z) := (1+z)/(1-z)$ maps the interval $(-1, 1)$ to $(0, \infty)$ and maps the upper half of the unit circle to the ray from $f(-1) = 0$ to $f(1) = \infty$ passing through $f(i) = i$. It therefore maps A to the first quadrant or its complement (ignoring boundaries); that it is the former can be determined by calculating $f(i/2)$, or by observing the orientation of the image of the path from -1 to 1.

Let $Q = f(A)$, so Q is the closed first quadrant minus the origin. The function $\operatorname{Im} \log w$ (where we use the standard branch of log) is a continuous function on Q, harmonic on the interior, whose values along the positive real and imaginary axes are 0 and $\pi/2$, respectively, so $3 + \frac{8}{\pi} \operatorname{Im} \log w$ is harmonic on the interior of Q and has the values 3 and 7 along those axes. Substituting $w = f(z)$, we find that

$$u = 3 + \frac{8}{\pi} \operatorname{Im} \log \left(\frac{1+z}{1-z} \right)$$

is a solution.

9B. Let k and n be integers with $n \geq k \geq 0$. Let A and B be $n \times k$ matrices with real coefficients. Let A^t be the transpose of A. For each size-k subset $I \subseteq \{1, \ldots, n\}$, let A_I be the $k \times k$ matrix obtained by discarding all rows of A except those whose index belongs to I. Define B_I similarly. Prove that

$$\det(A^t B) = \sum_I \det(A_I) \det(B_I),$$

where the sum is over all size-k subsets $I \subseteq \{1, \ldots, n\}$. (Suggestion: use linearity to reduce to the case where the columns of A and B are particularly simple.)

Solution: Let a_i be the i-th column vector of A. Let b_j be the j-th column vector of B. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. Both sides of the identity are linear in each a_i and b_j, so we may assume that each column is a standard basis vector, say $a_i = e_{f(i)}$ and $b_j = e_{g(j)}$. Then $A^t B$ is the matrix whose ij-entry is $a_i^t b_j$, which is 1 if $f(i) = g(j)$ and 0 otherwise.

If $f(1), \ldots, f(k)$ are not all different, then $A^t B$ has a repeated row, and every A_I has a repeated column, so both sides of the desired identity are 0. So assume that the $f(i)$ are all different.

Similarly, if $g(1), \ldots, g(k)$ are not all different, then $A^t B$ has a repeated column, and every B_I has a repeated column, so both sides of the desired identity are 0. So assume that the $g(j)$ are all different.

If $I \neq \{f(1), \ldots, f(k)\}$, then A_I has fewer than k nonzero entries, so $\det A_I = 0$. If $I \neq \{g(1), \ldots, g(k)\}$, then B_I has fewer than k nonzero entries, so $\det B_I = 0$.

Suppose that $\{f(1), \ldots, f(k)\} \neq \{g(1), \ldots, g(k)\}$. Then $A^t B$ has fewer than k nonzero entries. But also, by the previous paragraph, for every I, either $\det A_I$ or $\det B_I$ is 0. Thus the desired identity holds.

Finally, suppose that $\{f(1), \ldots, f(k)\} = \{g(1), \ldots, g(k)\}$. Let S be this common k-element subset of $\{1, \ldots, n\}$. Then $A^t B = (A_S)^t (B_S)$, so the left hand side of the identity equals $\det(A_S) \det(B_S)$. If $I \neq S$, then $\det(A_I) \det(B_I) = 0$, so the right hand side of the identity equals $\det(A_S) \det(B_S)$ too.