1A. Let \( x_1, x_2, \ldots \) be an infinite sequence of real numbers such that every subsequence contains a subsequence converging to 0. Must the original sequence converge?

2A. Find a matrix \( U \) such that \( U^{-1}AU = J \) is in Jordan canonical form, where
\[
A = \begin{pmatrix}
0 & -3 & 5 \\
-1 & -6 & 11 \\
0 & -4 & 7 \\
\end{pmatrix}.
\]

3A. Suppose \( f : \mathbb{R} \to \mathbb{R} \) is real analytic and periodic with period \( 2\pi \). Prove that \( f \) has an analytic continuation \( F \) defined on a strip
\[
S = \{ x + iy \in \mathbb{C} : |y| < \rho \}
\]
with \( \rho > 0 \), and that \( F(z + 2\pi) = F(z) \) for \( z \in S \).

4A. Define six fields as follows:
- Let \( A = \mathbb{Q}(\alpha) \) where \( \mathbb{Q} \) is the field of rational numbers and \( \alpha \) is the real cube root of 2.
- Let \( B \) be a splitting field of \( x^3 - 2 \) over \( \mathbb{Q} \).
- Let \( C \) be an algebraic closure of the field \( \mathbb{F}_2 \) of 2 elements.
- Let \( D \) be the subfield of \( C \) generated over \( \mathbb{F}_2 \) by the set of \( a \in C \) such that there exists \( n \geq 1 \) with \( a^n = 1 \).
- Let \( E \) be the field \( \mathbb{R} \) of real numbers.
- Let \( F \) be the field \( \mathbb{Q}[[T]][(T^{-1})] \) of formal Laurent series with rational coefficients.

For each pair of these, determine with proof whether or not they are isomorphic.

5A. Let \( a_0(x), a_1(x), \ldots, a_{r-1}(x) \) and \( b(x) \) be \( C^m \) functions on \( \mathbb{R} \). Prove that if \( y(x) \) is a solution of the differential equation
\[
y^{(r)} + a_{r-1}(x)y^{(r-1)} + \cdots + a_1(x)y' + a_0(x)y = b(x)
\]
(in particular, assuming that the derivatives \( y', y'', \ldots, y^{(r)} \) exist), then \( y(x) \) is \( C^{m+r} \).

6A. Let \( A = \alpha_1 \sigma_1 + \alpha_2 \sigma_2 + \alpha_3 \sigma_3 \) where \( \alpha_1, \alpha_2, \alpha_3 \in \mathbb{C} \) and
\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},
\]
\[
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
Let \( \beta \in \mathbb{C} \) be any square root of \( \alpha_1^2 + \alpha_2^2 + \alpha_3^2 \).

(a) Prove that \( \exp(A) = \cosh(\beta) + \frac{\sinh(\beta)}{\beta} A \), where \( \frac{\sinh(\beta)}{\beta} \) is interpreted as 1 if \( \beta = 0 \). (Hint: First show that \( A^2 \) is a scalar multiple of the identity.)

(b) Evaluate \( \exp(A) \) explicitly in the case \( \alpha_1 = i\pi, \alpha_2 = i\pi, \) and \( \alpha_3 = \pi \).
7A. Let $a$ and $b$ be complex numbers, and let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function such that $f(az + b) = f(z)$ for all $z \in \mathbb{C}$. Prove that there is a positive integer $n$ such that $a^n = 1$.

8A. Let $n \geq 3$, and let $A_n$ be the alternating subgroup of the symmetric group on $n$ letters. Prove that $A_n$ is generated by $(123)$ and $(12\cdots n)$ if $n$ is odd, or by $(123)$ and $(2\cdots n)$ if $n$ is even.

9A. Suppose $b$ and $L$ are positive constants and $f : [0, b] \to \mathbb{R}$ is continuous and satisfies

$$f(x) \geq L \int_0^x f(t) \, dt, \quad (0 \leq x \leq b).$$

Show that $f(x) \geq 0$ for $0 \leq x \leq b$.

1B. If $c \in \mathbb{R}$, say that a real-valued function $f : \mathbb{R} \to \mathbb{R}$ is periodic with period $c$ if it satisfies $f(x + c) = f(x)$ for all $x \in \mathbb{R}$.

(i) Let $V$ be the set of continuous real-valued functions $f$ having a positive integer as a period. Prove that $V$ is a vector space.

(ii) Let $p_1 < p_2 < \ldots < p_n < \ldots$ be the sequence of prime numbers, and for each $i$, let $f_i$ be a function whose minimal positive period is $p_i$. Prove that the functions $f_1, f_2, \ldots$ are linearly independent in $V$.

2B. Given any real number $a_0$, define $a_1, a_2, \ldots$ by the rule

$$a_{n+1} = \cos a_n$$

for all $n \geq 0$. Prove that the sequence $(a_n)$ converges, and that the limit is the unique solution of the equation $\cos x = x$.

3B. Let $k$ and $l$ be positive integers. Let $\mathbb{Q}(x)(\sqrt[k]{1-x^l})$ be any extension field of $\mathbb{Q}(x)$ generated by a $k$-th root of $1-x^l$. Define $\mathbb{Q}(x)(\sqrt[l]{1-x^k})$ similarly. Prove that $\mathbb{Q}(x)(\sqrt[k]{1-x^l})$ and $\mathbb{Q}(x)(\sqrt[l]{1-x^k})$ are isomorphic.

4B. Let $E$ be the $\mathbb{C}$-vector space of entire functions. Let $V$ be a nonzero finite-dimensional $\mathbb{C}$-subspace of $E$ with the property that $f \in V$ implies $f' \in V$. Prove that $V$ contains a function that is everywhere nonzero.

5B. Let $\mathbb{F}_q$ denote the finite field with $q$ elements, where $q$ is a power of a prime. Let $\text{SL}_n(\mathbb{F}_q)$ be the group of $n \times n$ matrices with entries in $\mathbb{F}_q$ and determinant 1, under matrix multiplication. Determine (with proof) a simple necessary and sufficient condition on $n$ and $q$ for the center of $\text{SL}_n(\mathbb{F}_q)$ to be trivial.

6B. Let $U$ be a non-empty open subset of $\mathbb{R}^d$ and let $f : U \to \mathbb{R}^d$ be a continuous vector field defined on $U$. Let $K$ be a compact subset of $U$ and let $b > 0$. Suppose $\varphi : [0, b) \to K$ is a continuous function satisfying

$$\varphi(t) = \varphi(0) + \int_0^t f(\varphi(s)) \, ds, \quad (0 \leq t < b).$$

Prove that $\lim_{t \to b^-} \varphi(t)$ exists, where $t \to b^-$ means $t$ approaches $b$ from the left.
7B. Given any group $G$, define a binary operation $\ast$ on the set $H = G \times G$ by $(g_1, h_1) \ast (g_2, h_2) = (g_1g_2, g_2^{-1}h_1g_2h_2)$.

(a) Show that $(H, \ast)$ is group.

(b) In the case that $G$ is the alternating group $A_n$ on $n$ letters with $n \geq 5$, prove that $H$ has no subgroup of index 2.

8B. Let $A$ be the set of $z \in \mathbb{C}$ such that $|z| \leq 1$, $\text{Im}(z) \geq 0$, and $z \notin \{1, -1\}$. Find an explicit continuous function $u : A \to \mathbb{R}$ such that

- $u$ is harmonic on the interior of $A$,
- $u(z) = 3$ for $z \in A \cap \mathbb{R}$
- $u(z) = 7$ for $z$ in the intersection of $A$ with the unit circle.

9B. Let $k$ and $n$ be integers with $n \geq k \geq 0$. Let $A$ and $B$ be $n \times k$ matrices with real coefficients. Let $A^t$ be the transpose of $A$. For each size-$k$ subset $I \subseteq \{1, \ldots, n\}$, let $A_I$ be the $k \times k$ matrix obtained by discarding all rows of $A$ except those whose index belongs to $I$. Define $B_I$ similarly. Prove that

$$
\det(A^t B) = \sum_I \det(A_I) \det(B_I),
$$

where the sum is over all size-$k$ subsets $I \subseteq \{1, \ldots, n\}$. (Suggestion: use linearity to reduce to the case where the columns of $A$ and $B$ are particularly simple.)