SPRING 2007 PRELIMINARY EXAMINATION

1A. Let $x_1, x_2, \ldots$ be an infinite sequence of real numbers such that every subsequence contains a subsequence converging to 0. Must the original sequence converge?

2A. Find a matrix U such that $U^{-1}AU = J$ is in Jordan canonical form, where

$$A = \begin{pmatrix} 0 & -3 & 5 \\ -1 & -6 & 11 \\ 0 & -4 & 7 \end{pmatrix}.$$

3A. Suppose $f : \mathbb{R} \to \mathbb{R}$ is real analytic and periodic with period 2π. Prove that f has an analytic continuation F defined on a strip

$$S = \{x + iy \in \mathbb{C} \; : \; |y| < \rho\}$$

with $\rho > 0$, and that $F(z + 2\pi) = F(z)$ for $z \in S$.

4A. Define six fields as follows:

- Let $A = \mathbb{Q}(\alpha)$ where $\mathbb{Q}$ is the field of rational numbers and α is the real cube root of 2.
- Let B be a splitting field of $x^3 - 2$ over $\mathbb{Q}$.
- Let C be an algebraic closure of the field $\mathbb{F}_2$ of 2 elements.
- Let D be the subfield of C generated over $\mathbb{F}_2$ by the set of $a \in C$ such that there exists $n \geq 1$ with $a^n = 1$.
- Let E be the field $\mathbb{R}$ of real numbers.
- Let F be the field $\mathbb{Q}[[T]](T^{-1})$ of formal Laurent series with rational coefficients.

For each pair of these, determine with proof whether or not they are isomorphic.

5A. Let $a_0(x)$, $a_1(x)$, $\ldots$, $a_{r-1}(x)$ and $b(x)$ be C^m functions on $\mathbb{R}$. Prove that if $y(x)$ is a solution of the differential equation

$$y^{(r)} + a_{r-1}(x)y^{(r-1)} + \cdots + a_1(x)y' + a_0(x)y = b(x)$$

(in particular, assuming that the derivatives y', y'', $\ldots$, $y^{(r)}$ exist), then $y(x)$ is C^{m+r}.

6A. Let $A = \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3$ where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ and $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Let $\beta \in \mathbb{C}$ be any square root of $\alpha_1^2 + \alpha_2^2 + \alpha_3^2$.

(a) Prove that $\exp(A) = \cosh(\beta) + \frac{\sinh \beta}{\beta} A$, where $\frac{\sinh(\beta)}{\beta}$ is interpreted as 1 if $\beta = 0$. (Hint: First show that A^2 is a scalar multiple of the identity.)

(b) Evaluate $\exp(A)$ explicitly in the case $\alpha_1 = i\pi$, $\alpha_2 = i\pi$, and $\alpha_3 = \pi$.

7A. Let a and b be complex numbers, and let $f\colon \mathbb{C} \to \mathbb{C}$ be a non-constant entire function such that $f(az+b) = f(z)$ for all $z \in \mathbb{C}$. Prove that there is a positive integer n such that $a^n = 1$.

8A. Let $n \geq 3$, and let A_n be the alternating subgroup of the symmetric group on n letters. Prove that A_n is generated by (123) and $(12\cdots n)$ if n is odd, or by (123) and $(2\cdots n)$ if n is even.

9A. Suppose b and L are positive constants and $f : [0,b] \to \mathbb{R}$ is continuous and satisfies

$$f(x) \geq L \int_0^x f(t)\,dt, \qquad (0 \leq x \leq b).$$

Show that $f(x) \geq 0$ for $0 \leq x \leq b$.

1B. If $c \in \mathbb{R}$, say that a real-valued function $f\colon \mathbb{R} \to \mathbb{R}$ is periodic with period c if it satisfies $f(x+c) = f(x)$ for all $x \in \mathbb{R}$.

(i) Let V be the set of continuous real-valued functions f having a positive integer as a period. Prove that V is a vector space.

(ii) Let $p_1 < p_2 < \ldots < p_n < \ldots$ be the sequence of prime numbers, and for each i, let f_i be a function whose minimal positive period is p_i. Prove that the functions $f_1, f_2, \ldots$ are linearly independent in V.

2B. Given any real number a_0, define $a_1, a_2, \ldots$ by the rule $a_{n+1} = \cos a_n$ for all $n \geq 0$. Prove that the sequence (a_n) converges, and that the limit is the unique solution of the equation $\cos x = x$.

3B. Let k and l be positive integers. Let $\mathbb{Q}(x)(\sqrt[k]{1-x^l})$ be any extension field of $\mathbb{Q}(x)$ generated by a k-th root of $1-x^l$. Define $\mathbb{Q}(x)(\sqrt[l]{1-x^k})$ similarly. Prove that $\mathbb{Q}(x)(\sqrt[k]{1-x^l})$ and $\mathbb{Q}(x)(\sqrt[l]{1-x^k})$ are isomorphic.

4B. Let E be the $\mathbb{C}$-vector space of entire functions. Let V be a nonzero finite-dimensional $\mathbb{C}$-subspace of E with the property that $f \in V$ implies $f' \in V$. Prove that V contains a function that is everywhere nonzero.

5B. Let $\mathbb{F}_q$ denote the finite field with q elements, where q is a power of a prime. Let $\mathrm{SL}_n(\mathbb{F}_q)$ be the group of $n \times n$ matrices with entries in $\mathbb{F}_q$ and determinant 1, under matrix multiplication. Determine (with proof) a simple necessary and sufficient condition on n and q for the center of $\mathrm{SL}_n(\mathbb{F}_q)$ to be trivial.

6B. Let U be a non-empty open subset of $\mathbb{R}^d$ and let $f : U \to \mathbb{R}^d$ be a continuous vector field defined on U. Let K be a compact subset of U and let $b > 0$. Suppose $\varphi : [0,b) \to K$ is a continuous function satisfying

$$\varphi(t) = \varphi(0) + \int_0^t f(\varphi(s))\,ds, \qquad (0 \leq t < b).$$

Prove that $\lim_{t \to b^-} \varphi(t)$ exists, where $t \to b^-$ means t approaches b from the left.

7B. Given any group G, define a binary operation $*$ on the set $H = G \times G$ by $(g_1, h_1) * (g_2, h_2) = (g_1 g_2, g_2^{-1} h_1 g_2 h_2)$.

(a) Show that $(H, *)$ is group.

(b) In the case that G is the alternating group A_n on n letters with $n \geq 5$, prove that H has no subgroup of index 2.

8B. Let A be the set of $z \in \mathbb{C}$ such that $|z| \leq 1$, $\operatorname{Im}(z) \geq 0$, and $z \notin \{1, -1\}$. Find an explicit continuous function $u \colon A \to \mathbb{R}$ such that

- u is harmonic on the interior of A,
- $u(z) = 3$ for $z \in A \cap \mathbb{R}$
- $u(z) = 7$ for z in the intersection of A with the unit circle.

9B. Let k and n be integers with $n \geq k \geq 0$. Let A and B be $n \times k$ matrices with real coefficients. Let A^t be the transpose of A. For each size-k subset $I \subseteq \{1, \ldots, n\}$, let A_I be the $k \times k$ matrix obtained by discarding all rows of A except those whose index belongs to I. Define B_I similarly. Prove that

$$\det(A^t B) = \sum_I \det(A_I) \det(B_I),$$

where the sum is over all size-k subsets $I \subseteq \{1, \ldots, n\}$. (Suggestion: use linearity to reduce to the case where the columns of A and B are particularly simple.)