SPRING 2006 PRELIMINARY EXAMINATION SOLUTIONS

1A. Let G be the subgroup of the free abelian group $\mathbb{Z}^4$ consisting of all integer vectors (x, y, z, w) such that $2x + 3y + 5z + 7w = 0$.

(a) Determine a linearly independent subset of G which generates G as an abelian group.
(b) Show that $\mathbb{Z}^4/G$ is a free abelian group and determine its rank.

Solution:
(b) The linear map

$$\mathbb{Z}^4 \mapsto \mathbb{Z}, (x, y, z, w) \mapsto 2x + 3y + 5z + 7w$$

has kernel G, and is onto because 2 and 3 are relatively prime. Hence $\mathbb{Z}^4/G$ is isomorphic to the image $\mathbb{Z}$, which is a free abelian group of rank 1.

(a) There is a sequence of elementary column operations over $\mathbb{Z}$ (not involving divisions) that transforms the 1×4-matrix $\begin{pmatrix} 2 & 3 & 5 & 7 \end{pmatrix}$ into $\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$. For instance, subtract 3 times the first column from the fourth to get $\begin{pmatrix} 2 & 3 & 5 & 1 \end{pmatrix}$, and then subtract appropriate multiples of the fourth from each of the first three columns to make them zero. The same sequence of operations applied to the 4×4 identity matrix eventually yields a matrix

$$U = \begin{pmatrix} 7 & 9 & 15 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & -3 & -5 & 1 \end{pmatrix}$$

such that

$$\begin{pmatrix} 2 & 3 & 5 & 7 \end{pmatrix} U = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}.$$

Because of the way U was constructed, it has an inverse U^{-1} with integer entries.

The first three columns of U are in G, and we claim that they span G as an abelian group. Suppose $\mathbf{v} \in G$. Then

$$0 = \begin{pmatrix} 2 & 3 & 5 & 7 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} U^{-1}\mathbf{v},$$

so $U^{-1}\mathbf{v} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ 0 \end{pmatrix}$ for some $\alpha, \beta, \gamma \in \mathbb{Z}$. Thus

$$\mathbf{v} = U \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ 0 \end{pmatrix},$$

which is an integer combination of the first three columns of U.

Finally these first three columns of U are linearly independent, since U is invertible.

2A. Find (with proof) all real numbers c such that the differential equation with boundary conditions

$$f'' - cf' + 16f = 0, \qquad f(0) = f(1) = 1$$

has no solution.

Solution: First suppose that the characteristic equation $x^2 - cx + 16 = 0$ has a repeated root. This happens when $c = \pm 8$. If $c = 8$, the repeated root is 4, and the general solution to the differential equation without boundary conditions has the form

$$f(t) = (at + b)e^{4t}.$$

The boundary conditions impose

$$b = 1$$
$$(a + b)e^4 = 1,$$

and this system has a solution. Similarly, there is a solution in the case $c = -8$.

From now on, we suppose that the complex roots α, β of $x^2 - cx + 16 = 0$ are distinct. Then the general solution is

$$f(t) = ae^{\alpha t} + be^{\beta t},$$

where $a, b \in \mathbb{C}$, and the boundary conditions impose

$$a + b = 1 \tag{1}$$
$$ae^{\alpha} + be^{\beta} = 1.$$

This system is guaranteed to have a solution if $e^\alpha \neq e^\beta$. So assume $e^\alpha = e^\beta$. Then $\alpha - \beta = 2\pi i k$ for some $k \in \mathbb{Z}$. By interchanging α, β, we may assume $k > 0$. On the other hand, by the quadratic formula,

$$(\alpha - \beta)^2 = c^2 - 64.$$

Thus $4\pi^2 k^2 = 64 - c^2 \leq 64$. The only possibility is $k = 1$, which leads to $c = \pm\sqrt{64 - 4\pi^2}$. In this case $e^\alpha = e^\beta$, but the common value is not 1, since $e^\alpha e^\beta = e^{\alpha+\beta} = e^c \neq e^0 = 1$. So the system (1) has no solution.

Thus the set of values c for which the differential equation with boundary conditions has no solution is $\{\pm\sqrt{64 - 4\pi^2}\}$.

3A. Let $S = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\}$. Find (with justification) the $n \times n$ matrix P of the orthogonal projection from $\mathbb{R}^n$ onto S. That is, P has image S, and $P^2 = P = P^T$.

Solution: The orthogonal complement of S is one-dimensional, and spanned by the unit vector $w = \frac{1}{\sqrt{n}}(1, \ldots, 1)$, because $v \in S \Leftrightarrow \langle v, w \rangle = 0$. So the orthogonal projection is given by $Pv = v - \langle v, w \rangle w = v - \frac{v_1 + \cdots + v_n}{n}(1, \ldots, 1)$. Therefore

$$P = \mathrm{Id} - w^T w = \begin{pmatrix} \frac{n-1}{n} & \frac{-1}{n} & \cdots & \frac{-1}{n} \\ \frac{-1}{n} & \frac{n-1}{n} & \cdots & \frac{-1}{n} \\ \vdots & & \ddots & \vdots \\ \frac{-1}{n} & \frac{-1}{n} & \cdots & \frac{n-1}{n} \end{pmatrix}.$$

4A. Let $D = \{z \in \mathbb{C} : |z| < 1\}$. Find all holomorphic functions $f \colon D \to \mathbb{C}$ such that $f(\frac{1}{n} + ie^{-n})$ is real for all integers $n \geq 2$.

Solution: We show that the only such functions are the real constant functions. Let

$$f(z) = \sum a_n z^n$$

be the Taylor series for f around 0. We first prove by contradiction that a_k are real. Suppose that k is the smallest index so that $\mathrm{Im} a_k \neq 0$. Then we must have

$$\mathrm{Im} a_k = \lim_{x \to 0, x \in \mathbb{R}} x^{-k} \mathrm{Im} f(x)$$

On the other hand, because there is a bound on $f'(z)$ in a closed disk containing all the numbers $\frac{1}{n} + ie^{-n}$,

$$\mathrm{Im} f(\frac{1}{n}) = \mathrm{Im} f(\frac{1}{n} + ie^{-n}) + O(e^{-n}) = O(e^{-n})$$

as $n \to \infty$. Hence

$$\mathrm{Im}\, a_k = \lim_{n \to \infty} n^k \mathrm{Im} f(\frac{1}{n}) = 0,$$

which is a contradiction. As a consequence, $f(\frac{1}{n})$ must be real.

By bounding $f''(z)$ on a closed disk, we may write

$$f(\frac{1}{n} + ie^{-n}) = f(\frac{1}{n}) + ie^{-n} f'(\frac{1}{n}) + O(e^{-2n})$$

Taking imaginary parts we get

$$\mathrm{Re} f'(\frac{1}{n}) = O(e^{-n})$$

Arguing as above, the Taylor series at 0 for $f'(z)$ has purely imaginary coefficients. We conclude that all a_k's must vanish with the exception of a_0.

5A. Consider the following four commutative rings:

$$\mathbb{Z}, \;\; \mathbb{Z}[x], \;\; \mathbb{R}[x], \;\; \mathbb{R}[x, y].$$

Which of these rings contains a nonzero prime ideal that is not a maximal ideal?

Solution: In the ring of integers $\mathbb{Z}$ the nonzero prime ideals are $\langle p \rangle$, where p is a prime number. Each of these ideals is maximal since $\mathbb{F}_p = \mathbb{Z}/\langle p \rangle$ is a field. Hence every nonzero prime ideal in $\mathbb{Z}$ is maximal.

The polynomial ring $\mathbb{Z}[x]$ in one variable x over the ring of integers $\mathbb{Z}$ is not a principal ideal domain. For instance, $\langle 2, x \rangle$ is not a principal ideal; it strictly contains the ideal $\langle 2 \rangle$, which is therefore not a maximal ideal. The ideal $\langle 2 \rangle$ is a prime ideal, because $\mathbb{Z}[x]/\langle 2 \rangle = \mathbb{F}_2[x]$ is a polynomial ring over a field, and hence an integral domain. Hence $\langle 2 \rangle$ is a nonzero prime ideal in $\mathbb{Z}[x]$ which is not maximal.

The polynomial ring $\mathbb{R}[x]$ in one variable x over the field $\mathbb{R}$ is a principal ideal domain. Hence every nonzero ideal has the form $\langle f(x) \rangle$ where $f(x)$ is a nonzero polynomial with real coefficients. The ideal is prime if and only if $f(x)$ is an irreducible polynomial, i.e., if $f(x)$ is a linear polynomial or $f(x)$ is a quadratic polynomial with no real roots. In either case, the quotient $\mathbb{R}[x]/\langle f \rangle$ is a field, namely, either $\mathbb{R}$ or $\mathbb{C}$, which means that $\langle f \rangle$ is a maximal ideal. Hence every nonzero prime ideal in $\mathbb{R}[x]$ is a maximal ideal.

The polynomial ring $\mathbb{R}[x, y]$ in two variables x, y over $\mathbb{R}$ has many nonzero prime ideals which are not maximal ideals. For instance, $\langle x \rangle$ is a prime ideal, but it is not maximal since it is contained in the ideal $\langle x, y \rangle$.

6A. Let $u\colon \mathbb{R} \to \mathbb{R}$ be a function for which there exists $B > 0$ such that

$$\sum_{k=1}^{N-1} |u(x_{k+1}) - u(x_k)|^2 \leq B$$

for all finite increasing sequences $x_1 < x_2 < \cdots < x_N$. Show that u has at most countably many discontinuities.

Solution: Let A be the set of points of discontinuity for u. Then

$$A = \bigcup_{n \geq 1} A_n$$

where

$$A_n = \{x \in \mathbb{R} : |\limsup_{y \to x} u(y) - \liminf_{y \to x} u(y)| > \frac{1}{n}\}$$

To prove that A is countable, we will prove that

$$|A_n| \leq 4n^2 B.$$

If $y_1 < y_2 < \cdots < y_N$ are in A_n then we can choose a strictly increasing sequence $(x_k)_{k=1}^{2N}$ such that

$$x_{2k-1} < y_k < x_{2k}$$

and

$$|u(x_{2k}) - u(x_{2k-1})| > \frac{1}{2n}$$

for $k = 1, \ldots, N$. Summing over k gives the inequality on the right in

$$B \geq \sum_{k=1}^{2N-1} |u(x_{k+1}) - u(x_k)|^2 \geq \sum_{k=1}^{N} |u(x_{2k}) - u(x_{2k-1})|^2 \geq N \left(\frac{1}{2n}\right)^2.$$

Hence $N \leq 4n^2 B$, which concludes the proof.

7A. Recall that $\mathrm{SL}(2, \mathbb{R})$ denotes the group of real 2×2 matrices of determinant 1. Suppose that $A \in \mathrm{SL}(2, \mathbb{R})$ does not have a real eigenvalue. Show that there exists $B \in \mathrm{SL}(2, \mathbb{R})$ such that BAB^{-1} equals a rotation matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for some $\theta \in \mathbb{R}$.

Solution: Since the eigenvalues of A are solutions to a real quadratic equation, they are complex conjugates of each other, call them λ and $\overline{\lambda}$. Since $\det(A) = 1$, it follows that $\lambda\overline{\lambda} = 1$, i.e. λ and $\overline{\lambda}$ are on the unit circle. Write $\lambda = \cos\theta + i\sin\theta$. Pick a nonzero eigenvector $z \in \mathbb{C}^2$ with $Az = \lambda z$. Write $z = v + iw$ with $v, w \in \mathbb{R}^2$. Taking the real and imaginary parts of the equation $Az = \lambda z$ gives the equations $Av = (\cos\theta)v - (\sin\theta)w$, $Aw = (\sin\theta)v + (\cos\theta)w$. Note also that $A(v - iw) = \overline{\lambda}(v - iw)$ and $\lambda \neq \overline{\lambda}$, so $v + iw$ and $v - iw$ are linearly independent over $\mathbb{C}$, so v and w are linearly independent over $\mathbb{R}$. We can find $B \in \mathrm{SL}(2, \mathbb{R})$ taking the basis $\{v, w\}$ to a real multiple of the standard basis for $\mathbb{R}^2$. Then $BAB^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. This is of the desired form, with θ in place of $-\theta$.

8A. Let $D = \{z \in \mathbb{C} : |z| < 1\}$. Let $f\colon D \to \mathbb{C}$ be holomorphic, and suppose that the restriction of f to $D - \{0\}$ is injective. Prove that f is injective.

Solution: Suppose on the contrary that there is $a \in D - \{0\}$ such that $f(a) = f(0)$. Let α be the common value. Choose disjoint open disks D_0 and D_a contained in D, centered at 0 and a, respectively. By the Open Mapping Theorem $f(D_0)$ and $f(D_a)$ are open subsets of $\mathbb{C}$ containing α. Hence $G := f(D_0) \cap f(D_a)$ is a nonempty open subset of $\mathbb{C}$. Choose $\xi \in G$ with $\xi \neq \alpha$. Then there exist $z_0 \in D_0$ and $z_a \in D_a$ such that $f(z_0) = f(z_a) = \xi$. Since $\xi \neq \alpha$, neither z_0 nor z_a is 0. This contradicts the injectivity of f restricted to $D - \{0\}$.

9A. Let p be a prime. Let G be a finite non-cyclic group of order p^m for some m. Prove that G has at least $p + 3$ subgroups.

Solution: We will use the following two facts:

(i) A nontrivial p-group has a nontrivial center Z (nontrivial conjugacy classes have size divisible by p, as does the whole group, so $\{1\}$ cannot be the only trivial one).

(ii) If G is a group with center Z, and G/Z is cyclic, then G is abelian (since if $a \in G$ generates G/Z, every element of G is of the form $a^n z$ for some $n \in \mathbb{Z}$ and $z \in Z$).

We use induction on m.

Suppose $m \leq 2$. Since G has order 1, p, or p^2, it is abelian (for order p^2, combine (i) and (ii) above). Since it is not cyclic, we have $G \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. So G has one trivial subgroup, $(p^2 - 1)/(p - 1) = p + 1$ subgroups of order p, and G itself. Thus G has exactly $p + 3$ subgroups.

Now suppose $m > 2$. By (i), the center Z of G is nontrivial. Since G is a nontrivial p-group, it has a nontrivial center Z. If G/Z is non-cyclic, then by the inductive hypothesis it has $\geq p + 3$ subgroups, and their inverse images in G are distinct subgroups of G. If G/Z is cyclic, then G is abelian by (ii); but G is not cyclic, so by the structure theory of finite abelian groups, it must contain $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, which already contains $p + 3$ subgroups.

1B. Let $A_1 \supseteq A_2 \supseteq \cdots$ be compact connected subsets of $\mathbb{R}^n$. Show that the set $A = \bigcap A_m$ is connected.

Solution: The intersection A is nonempty, since otherwise $\{A_1 - A_m\}$ is a covering of A_1 (by sets open in A_1) with no finite subcover.

Suppose that A is not connected. Then there exist sets B_0, C_0 open in A such that $B_0 \cup C_0 = A$ and $B_0 \cap C_0 = \emptyset$. Then B_0, C_0 are also closed in A, which (as an intersection of closed sets) is closed in $\mathbb{R}^n$, so B_0, C_0 are closed in $\mathbb{R}^n$. Hence we can find disjoint sets B, C open in A_1 such that $B_0 \subseteq B$, $C_0 \subseteq C$: for instance, we could let B be the set of points in A_1 that are strictly closer to B_0 than to C_0, and vice versa for C.

Since $A = B_0 \cup C_0 \subseteq B \cup C$, the sets B, C, and $A_1 - A_m$ for $m \geq 1$ form a cover of A_1 by sets open in A_1; thus there is a finite subcover consisting of B, C, and $A_1 - A_m$ for $m = 1, \ldots, r$. So r is such that $A_r \subseteq B \cup C$. Since B, C are open, disjoint, and $B \cap A_r \supseteq B_0 \cap A \neq \emptyset$ and $C \cap A_r \supseteq C_0 \cap A \neq \emptyset$, we have that A_r is not connected, a contradiction.

2B. Let $\mathbb{F}_2$ be the field of 2 elements. Let n be a prime. Show that there are exactly $(2^n - 2)/n$ degree-n irreducible polynomials in $\mathbb{F}_2[x]$.

Solution: There is a unique field extension $\mathbb{F}_{2^n}$ of degree n over $\mathbb{F}_2$. It is Galois over $\mathbb{F}_2$ (this is because it is a splitting field for the separable polynomial $x^{2^n} - x$). If $a \in \mathbb{F}_{2^n} - \mathbb{F}_2$, then $\mathbb{F}_2(a)$ is a subfield of $\mathbb{F}_{2^n}$ of degree dividing n but not equal to 1, so $\mathbb{F}_2(a) = \mathbb{F}_{2^n}$. Hence the minimal polynomial f_a of a over $\mathbb{F}_2$ is an irreducible polynomial of degree n over $\mathbb{F}_2$. Thus we have a map

$$\begin{aligned}(\mathbb{F}_{2^n} - \mathbb{F}_2) &\to \{\text{degree-}n \text{ irreducible polynomials in } \mathbb{F}_2[x]\}\\ a &\mapsto f_a.\end{aligned}$$

On the other hand, if $f \in \mathbb{F}_2[x]$ is any degree-n irreducible polynomial, then f has a zero in $\mathbb{F}_{2^n}$ (since $\mathbb{F}_{2^n}$ is the unique degree-n extension of $\mathbb{F}_2$) and it follows that f has n distinct zeros in $\mathbb{F}_{2^n}$ (since $\mathbb{F}_{2^n}$ is Galois over $\mathbb{F}_2$). Moreover, f is automatically monic (the only nonzero element of $\mathbb{F}_2$ is 1) so it is the minimal polynomial of each of its zeros. Thus our map is n-to-1.

Its domain has size $2^n - 2$, so its range has size $(2^n - 2)/n$.

3B. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{e^x + e^{-x}} dx$$

for $t > 0$.

Solution: The integral converges absolutely, since the numerator has absolute value 1, while the denominator decays exponentially in both directions.

Use a rectangular contour C bounded by $x = R$, $x = -R$, $y = 0$ and $y = \pi$. As $R \to \infty$ the integrals along the vertical parts of the contour tend to 0, since

$$\left| \int_0^{\pi} \frac{e^{it(R+iy)}}{e^{R+iy} + e^{-R-iy}}\, dy \right| \le \int_0^{\pi} \frac{1}{e^R - e^{-R}}\, dy = \frac{\pi}{e^R - e^{-R}}.$$

The integral along the horizontal path $y = \pi$ equals

$$\int_R^{-R} \frac{e^{it(x+\pi i)}}{e^{(x+\pi i)} + e^{-(x+\pi i)}} dx = \int_R^{-R} \frac{e^{-\pi t} e^{itx}}{-e^x - e^{-x}} dx = e^{-\pi t} \int_{-R}^{R} \frac{e^{itx}}{e^x + e^{-x}} dx.$$

Let I denote the integral we have to find. Then

$$\lim_{R\to\infty} \oint_C \frac{e^{itz}}{e^z + e^{-z}}\, dz = \left(1 + e^{-\pi t}\right) I.$$

On the other hand,

$$\oint_C \frac{e^{itz}}{e^z + e^{-z}}\, dz = 2\pi i \operatorname{Res}_{\frac{\pi i}{2}},$$

since the only singular point inside the contour is $\frac{\pi i}{2}$. Now

$$\operatorname{Res}_{\frac{\pi i}{2}} = \frac{e^{-\frac{\pi t}{2}}}{2i},$$

so

$$\oint_C \frac{e^{itz}}{e^z + e^{-z}} dz = \pi e^{-\frac{\pi t}{2}},$$

$$I = \pi \frac{e^{-\frac{\pi t}{2}}}{1 + e^{-\pi t}} = \frac{\pi}{e^{\frac{\pi t}{2}} + e^{-\frac{\pi t}{2}}}.$$

4B. Let n be a positive integer, and let $\mathrm{GL}_n(\mathbb{R})$ be the group of invertible $n \times n$ matrices. Let S be the set of $A \in \mathrm{GL}_n(\mathbb{R})$ such that $A - I$ has rank ≤ 2. Prove that S generates $\mathrm{GL}_n(\mathbb{R})$ as a group.

Solution: By Gaussian elimination, $\mathrm{GL}_n(\mathbb{R})$ is generated by the elementary matrices obtained from the identity matrix by interchanging two rows, by multiplying one row by a nonzero scalar, or by adding a multiple of one row to a different row. For each such matrix A, the matrix $A - I$ has at most two nonzero rows and hence has rank ≤ 2.

5B. Prove that there exists no continuous bijection from $(0,1)$ to $[0,1]$. (Recall that a bijection is a map that is both one-to-one and onto.)

Solution: Suppose on the contrary that there exists a continuous bijection $f\colon (0,1) \to [0,1]$. Then there exists $x \in (0,1)$ such that $f(x) = 0$. Let $A = (0,x)$, $B = (x,1)$. We have $A \cap B = \emptyset$ and since f is injective we have

$$f(A) \cap f(B) = f(A \cap B) = \emptyset. \qquad (*)$$

Since f is continuous and $(0,x]$ is connected, $f((0,x])$ contains an interval $[0,a)$ for some $a > 0$. Hence $f(A)$ contains $(0,a)$. Similarly, $f(B)$ contains $(0,b)$ for some $b > 0$. This gives $f(A) \cap f(B) \neq \emptyset$. Contradiction to $(*)$.

6B. Let A be the subring of $\mathbb{R}[t]$ consisting of polynomials $f(t)$ such that $f'(0) = 0$. Is A a principal ideal domain?

Solution: No. Suppose A is a principal ideal domain. Then the A-ideal I generated by t^2 and t^3 would be principal. Let $p(t)$ be a generator of I. Then $t^2 = q(t)p(t)$ for some $q(t) \in A$, so $p(t)$ divides t^2 also in the unique factorization domain $\mathbb{R}[t]$. Hence $p(t) = ut^m$ for some unit u of $\mathbb{R}[t]$ and some $m \in \{0,1,2\}$. The case $m = 1$ is impossible, since $p(t) \in A$. If $m = 0$, then $p(t)$ is a unit also of A, and hence generates the unit ideal; this contradicts the fact that every element of I has constant term zero. If $m = 2$, then t^3 is not a multiple of $p(t)$, since the element $t^3/p(t) \in \mathbb{R}[t]$ is not in A.

7B. Let m be a fixed positive integer.

(a) Show that if an entire function $f\colon \mathbb{C} \to \mathbb{C}$ satisfies $|f(z)| \leq e^{|z|}$ for all $z \in \mathbb{C}$, then

$$|f^{(m)}(0)| \leq \frac{m!e^m}{m^m}.$$

(b) Prove that there exists an entire function f such that $|f(z)| \leq e^{|z|}$ for all z and

$$|f^{(m)}(0)| = \frac{m!e^m}{m^m}.$$

Solution:

(a) Write $f(z) = \sum_{n \geq 0} a_n z^n$ with $a_n \in \mathbb{C}$. Then a_m is the coefficient of z^{-1} in the Laurent series of $f(z)/z^{m+1}$, so

$$a_m = \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)}{z^m} \frac{dz}{z},$$

for any $R > 0$, and we get

$$|a_m| \le \frac{1}{2\pi}\left(\frac{e^R}{R^m}\right)\frac{2\pi R}{R} = \frac{e^R}{R^m}.$$

Taking $R = m$ (which calculus shows minimizes the right hand side) and multiplying by $m!$ gives

$$|f^{(m)}(0)| = |m!a_m| \le \frac{m!e^m}{m^m}.$$

(b) Examining the proof of part (a) shows also that in order to have equality, $\frac{f(z)}{z^m}$ must have constant modulus e^m/m^m and constant argument on the circle $|z| = m$. Thus we guess $f(z) = \frac{e^m}{m^m}z^m$, and it remains to prove that $|f(z)| \le e^{|z|}$ for all $z \in \mathbb{C}$. Equivalently, we must show that the minimum value of e^x/x^m on $(0,\infty)$ is e^m/m^m. This can be seen by observing that the only zero of the derivative of $\log(e^x/x^m) = x - m\log x$ is at $x = m$, while the second derivative is positive everywhere (it is m/x^2).

8B. Let $\langle\ ,\ \rangle$ be the standard Hermitian inner product on $\mathbb{C}^n$. Let A be an $n \times n$ matrix with complex entries. Suppose $\langle x, Ax\rangle$ is real for all $x \in \mathbb{C}^n$. Prove that A is Hermitian.

Solution: We have $\langle x, Ax\rangle = x^HAx = \overline{x^HAx}$ (since x^HAx is real) $= (x^HAx)^H = x^HA^Hx$. Thus $x^HAx = x^HA^Hx$. So $x^H(A - A^H)x = 0$ for *all* $x \in \mathbb{C}^n$. Let $B = A - A^H$. We have

$$x^HBx = 0 \tag{$*$}$$

for all $x \in \mathbb{C}^n$ and $B^H = A^H - A = -B$, so B is skew-Hermitian (hence normal). Let x be an eigenvector of B with the eigenvalue λ, so $Bx = \lambda x$. Then $0 = x^HBx$ (by $(*)$) $= \lambda x^Hx = \lambda\|x\|^2$. This gives $\lambda = 0$. Thus all eigenvalues of B are zero. Being normal, B is diagonalizable, so $B = 0$. By definition of B, we get $A = A^H$. Thus A is Hermitian.

9B. Find a bounded non-convergent sequence of real numbers $(a_n)_{n\ge1}$ such that

$$|2a_n - a_{n-1} - a_{n+1}| \le n^{-2}$$

for all $n \ge 2$.

Solution: We will let $a_n = f(n)$, where $f(x)$ is a function similar to the sine function but with oscillations that slow down as $x \to \infty$, so that $f''(x) \to 0$. To be precise, we take

$$f(x) := \frac{1}{2}\sin(\ln(x+1)).$$

This sequence is bounded. It also does not converge, since the *spacing* between values of $\ln n$ tends to zero, which means that the values of $(\ln n) \bmod (2\pi)$ are dense in $[0, 2\pi]$.

By Taylor's theorem with remainder (centered at n),

$$f(n+1) = f(n) + f'(n) + \frac{1}{2}f''(\xi_+) \quad \text{for some } \xi_+ \in (n, n+1), \text{ and}$$

$$f(n-1) = f(n) - f'(n) + \frac{1}{2}f''(\xi_-) \quad \text{for some } \xi_- \in (n-1, n), \text{ so,}$$

$$|2f(n) - f(n-1) - f(n+1)| = \frac{1}{2}|f''(\xi_+) + f''(\xi_-)| = |f''(\xi)| \quad \text{for some } \xi \in (\xi_-, \xi^+) \subseteq (n-1, n+1)$$

by the intermediate value theorem. We compute

$$\begin{aligned}
f'(x) &= \frac{1}{2(x+1)}\cos(\ln(x+1)) \\
f''(x) &= -\frac{1}{2(x+1)^2}\left(\cos(\ln(x+1)) + \sin(\ln(x+1))\right), \\
|f''(x)| &\le \frac{1}{(x+1)^2} \\
|f''(\xi)| &\le \frac{1}{(\xi+1)^2} \le \frac{1}{n^2}.
\end{aligned}$$

so

$$|2a_n - a_{n-1} - a_{n+1}| = |f''(\xi)| \le n^{-2}.$$