SPRING 2006 PRELIMINARY EXAMINATION

1A. Let G be the subgroup of the free abelian group $\mathbb{Z}^4$ consisting of all integer vectors (x, y, z, w) such that $2x + 3y + 5z + 7w = 0$.

(a) Determine a linearly independent subset of G which generates G as an abelian group.
(b) Show that $\mathbb{Z}^4/G$ is a free abelian group and determine its rank.

2A. Find (with proof) all real numbers c such that the differential equation with boundary conditions

$$f'' - cf' + 16f = 0, \qquad f(0) = f(1) = 1$$

has no solution.

3A. Let $S = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\}$. Find (with justification) the $n \times n$ matrix P of the orthogonal projection from $\mathbb{R}^n$ onto S. That is, P has image S, and $P^2 = P = P^T$.

4A. Let $D = \{z \in \mathbb{C} : |z| < 1\}$. Find all holomorphic functions $f \colon D \to \mathbb{C}$ such that $f(\frac{1}{n} + ie^{-n})$ is real for all integers $n \geq 2$.

5A. Consider the following four commutative rings:

$$\mathbb{Z}, \ \ \mathbb{Z}[x], \ \ \mathbb{R}[x], \ \ \mathbb{R}[x, y].$$

Which of these rings contains a nonzero prime ideal that is not a maximal ideal?

6A. Let $u \colon \mathbb{R} \to \mathbb{R}$ be a function for which there exists $B > 0$ such that

$$\sum_{k=1}^{N-1} |u(x_{k+1}) - u(x_k)|^2 \leq B$$

for all finite increasing sequences $x_1 < x_2 < \cdots < x_N$. Show that u has at most countably many discontinuities.

7A. Recall that $\mathrm{SL}(2, \mathbb{R})$ denotes the group of real 2×2 matrices of determinant 1. Suppose that $A \in \mathrm{SL}(2, \mathbb{R})$ does not have a real eigenvalue. Show that there exists $B \in \mathrm{SL}(2, \mathbb{R})$ such that BAB^{-1} equals a rotation matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for some $\theta \in \mathbb{R}$.

8A. Let $D = \{z \in \mathbb{C} : |z| < 1\}$. Let $f \colon D \to \mathbb{C}$ be holomorphic, and suppose that the restriction of f to $D - \{0\}$ is injective. Prove that f is injective.

9A. Let p be a prime. Let G be a finite non-cyclic group of order p^m for some m. Prove that G has at least $p + 3$ subgroups.

1B. Let $A_1 \supseteq A_2 \supseteq \cdots$ be compact connected subsets of $\mathbb{R}^n$. Show that the set $A = \bigcap A_m$ is connected.

2B. Let $\mathbb{F}_2$ be the field of 2 elements. Let n be a prime. Show that there are exactly $(2^n - 2)/n$ degree-n irreducible polynomials in $\mathbb{F}_2[x]$.

3B. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{e^x + e^{-x}} dx$$

for $t > 0$.

4B. Let n be a positive integer, and let $\mathrm{GL}_n(\mathbb{R})$ be the group of invertible $n \times n$ matrices. Let S be the set of $A \in \mathrm{GL}_n(\mathbb{R})$ such that $A - I$ has rank ≤ 2. Prove that S generates $\mathrm{GL}_n(\mathbb{R})$ as a group.

5B. Prove that there exists no continuous bijection from $(0, 1)$ to $[0, 1]$. (Recall that a bijection is a map that is both one-to-one and onto.)

6B. Let A be the subring of $\mathbb{R}[t]$ consisting of polynomials $f(t)$ such that $f'(0) = 0$. Is A a principal ideal domain?

7B. Let m be a fixed positive integer.

(a) Show that if an entire function $f : \mathbb{C} \to \mathbb{C}$ satisfies $|f(z)| \leq e^{|z|}$ for all $z \in \mathbb{C}$, then

$$|f^{(m)}(0)| \leq \frac{m! e^m}{m^m}.$$

(b) Prove that there exists an entire function f such that $|f(z)| \leq e^{|z|}$ for all z and

$$|f^{(m)}(0)| = \frac{m! e^m}{m^m}.$$

8B. Let $\langle\ ,\ \rangle$ be the standard Hermitian inner product on $\mathbb{C}^n$. Let A be an $n \times n$ matrix with complex entries. Suppose $\langle x, Ax \rangle$ is real for all $x \in \mathbb{C}^n$. Prove that A is Hermitian.

9B. Find a bounded non-convergent sequence of real numbers $(a_n)_{n \geq 1}$ such that

$$|2a_n - a_{n-1} - a_{n+1}| \leq n^{-2}$$

for all $n \geq 2$.