SPRING 2005 PRELIMINARY EXAMINATION SOLUTIONS

1A. (a) Let $(a_n)_1^\infty$ be a sequence in $\mathbb{R}$ such that

$$\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty.$$

Prove that $(a_n)_1^\infty$ is a Cauchy sequence.

(b) Is the converse true? Give a proof or a counterexample.

Solution: (a) Given $\varepsilon > 0$, there is an integer N such that

$$\sum_{k=N}^{\infty} |a_{k+1} - a_k| < \varepsilon.$$

Therefore, for any m, n with $N \le m < n$,

$$\left| \sum_{k=m}^{n-1} (a_{k+1} - a_k) \right| \le \sum_{k=m}^{n-1} |a_{k+1} - a_k| < \varepsilon.$$

The series on the left telescopes, giving

$$|a_n - a_m| < \varepsilon.$$

(b) Simple counterexample: $a_n = (-1)^n/n$. Then $|a_{n+1} - a_n| = (2n+1)/(n^2+n)$, so $\sum_{n=1}^\infty |a_{n+1} - a_n| = \infty$ by the limit comparison test (compare with $\sum_{n=1}^\infty \frac{1}{n}$).

2A. Prove or disprove the statement: Every function $f \colon \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ for all x and y is continuous.

Solution: The statement is false. Let π be an irrational number. Then 1 and π are linearly independent over $\mathbb{Q}$, so we may extend the set $\{1, \pi\}$ to a basis B of $\mathbb{R}$ as a $\mathbb{Q}$-vector space. There exists a $\mathbb{Q}$-linear function $f \colon \mathbb{R} \to \mathbb{Q}$ taking arbitrarily prescribed values on the basis B; choose f such that $f(1) = 1$, $f(\pi) = 0$. The first condition implies $f(x) = x$ for all $x \in \mathbb{Q}$. If f were continuous it would follow that $f(x) = x$ for all $x \in \mathbb{R}$, contradicting $f(\pi) = 0$.

3A. Prove that there is no holomorphic bijection from the punctured disk $0 < |z| < 1$ in $\mathbb{C}$ onto the annulus $r < |z| < R$, where $0 < r < R < \infty$.

Solution: Suppose the analytic function f maps $D \setminus \{0\} = \{z : 0 < |z| < 1\}$ *onto* the annulus A. Then f is bounded in a neighborhood of 0, and therefore f has a removable singularity at 0, so f extends to an analytic function on the open disk D. By the open mapping theorem, $f(0) = p \in A$. Also there is some $z_0 \in D \setminus \{0\}$ with $f(z_0) = p$. Then there are small disjoint neighborhoods U, V of 0 and z_0 respectively, such that $f(U)$ and $f(V)$ are neighborhoods of p.

Hence $f(U \setminus \{0\})$ and $f(V)$ are open sets in A which are not disjoint.

This shows that f is *not* $1-1$ on $D \setminus \{0\}$.

4A. Suppose A and B are commuting $n \times n$ matrices over $\mathbb{R}$. Suppose A and B are each diagonalizable over $\mathbb{R}$. Show that AB is diagonalizable over $\mathbb{R}$.

Solution: Let $V_1, \ldots, V_r$ be the eigenspaces in K^n corresponding to the distinct eigenvalues of A in K. Because A is diagonalizable,

$$K^n = \bigoplus_i V_i.$$

Because A and B commute, $BV_i \subseteq V_i$. Because B is diagonalizable over $\mathbb{R}$, its minimal polynomial is a product of linear factors over $\mathbb{R}$, and the minimal polynomial of $B|_{V_i}$ divides this, so $B|_{V_i}$ is diagonalizable as well. Thus

$$V_i = \bigoplus_j W_{i\,j},$$

where the $W_{i\,j}$ are the eigenspaces of B in V_i corresponding to distinct eigenvalues. Since $W_{i\,j}$ is an eigenspace for AB and

$$\bigoplus_{i\,j} W_{i\,j} = K^n,$$

AB must be diagonalizable.

5A. Let I be an open interval and let $f\colon I \to \mathbb{R}$ have continuous k-th derivatives everywhere on I for all $k \leq n-1$. Let $a \in I$ be such that $f^{(k)}(a) = 0$ for $1 \leq k \leq n-1$, and assume that $f^{(n)}(a)$ is defined and $f^{(n)}(a) > 0$. Prove that if n is even, then f has a local minimum at a, and if n is odd, then f has no local extremum at a.

Solution: By the definition of derivative and the assumption that $f^{(n-1)}(a) = 0$,

$$\lim_{x \to a} \frac{f^{(n-1)}(x)}{x-a} = f^{(n)}(a) > 0.$$

Hence there exists ϵ such that $f^{(n-1)}(x)/(x-a) > 0$ for all $x \in (a-\epsilon, a+\epsilon) - \{a\}$. By Taylor's theorem with remainder, we have

$$f(x) = f(a) + f^{(n-1)}(c)(x-a)^{n-1}/(n-1)!$$

for some $c \in [a, x]$ if $x \geq a$, or $c \in [x, a]$ if $x \leq a$. For $x \in (a-\epsilon, a)$ we have $f^{(n-1)}(c) \leq 0$, so $f(x) \geq f(a)$ if n is even, $f(x) \leq f(a)$ if n is odd. For $x \in (a, a+\epsilon)$, we have $f^{(n-1)}(c) \geq 0$, so $f(x) \geq f(a)$ for all n. This implies that f has a local minimum at a if n is even. If n is odd, it implies that either f has no local extremum, or f is constant on $(a-\epsilon, a+\epsilon)$. But the latter possibility contradicts the assumption that $f^{(n)}(a) > 0$.

6A. For every positive integer n, define $[n]_q = q^{n-1} + q^{n-2} + \cdots + q + 1$. Prove that $[1]_q[2]_q \cdots [r]_q$ divides $[k+1]_q[k+2]_q \cdots [k+r]_q$ in the polynomial ring $\mathbb{Z}[q]$, for all positive integers k and r.

Solution: Both polynomials are monic, so we need only show that every complex root ω of $[1]_q[2]_q \cdots [r]_q$ is also a root of $[1]_q[2]_q \cdots [r]_q$, with equal or greater multiplicity.

The roots of $[n]_q = (q^n - 1)/(q-1)$ are the n-th roots of unity, excluding 1, and they are distinct. In particular, every root ω of $[1]_q[2]_q \cdots [r]_q$ is a root of unity. Let d be the order of ω in the multiplicative group $\mathbb{C}^*$, that is, ω is a primitive d-th root of unity. Then ω is a root of $[n]_q$ if and only if $d \mid n$. It follows that ω has multiplicity $\lfloor r/d \rfloor$ as a root of $[1]_q[2]_q \cdots [r]_q$, and multiplicity $\lfloor (k+r)/d \rfloor - \lfloor k/d \rfloor$ as a root of $[k+1]_q[k+2]_q \cdots [k+r]_q$. To complete the proof, we need the following inequality.

Lemma. $\lfloor (k+r)/d \rfloor \geq \lfloor k/d \rfloor + \lfloor r/d \rfloor$ for all k, r, d.

Proof. Set $a = \lfloor k/d \rfloor$, $b = \lfloor r/d \rfloor$. Then $k \geq ad$, $r \geq bd$, hence $k + r \geq (a+b)d$ and $\lfloor (k+r)/d \rfloor \geq \lfloor (a+b)d/d \rfloor = a + b$, since the floor function is monotone.

(An alternative proof is to show by induction that the Gauss binomial coefficient

$$\begin{bmatrix} k+r \\ r \end{bmatrix}_q := \frac{[k+1]_q[k+2]_q \cdots [k+r]_q}{[1]_q[2]_q \cdots [r]_q}$$

is a polynomial, by using a q-analog of the Pascal's triangle recurrence.)

7A. Let U be a connected open subset of $\mathbb{C}$, and let $f(z)$ be a meromorphic function on U having at least one pole. For each $c \in U$ that is not a pole of f, let $R(c)$ be the radius of convergence of the Taylor series of f centered at c. Prove that $R(c)$ extends to a continuous function defined on all of U.

Solution: Let P be the set of poles of f in U. Each pole is isolated, so P is closed in U, and $U - P$ is open (both in U and in $\mathbb{C}$). For $c \in \mathbb{C}$ and $r > 0$, let $D(c, r) := \{z \in \mathbb{C} : |z - c| < r\}$.

Fix $c \in U - P$. Choose $\epsilon > 0$ such that $D(c, \epsilon) \subseteq U - P$. Then $\epsilon \leq R(c)$, and the function g_c on $D(c, R(c))$ defined by the Taylor series at c agrees with f on $D(c, \epsilon)$. Note that $R(c) < \infty$, since otherwise by connectedness f would equal the restriction to U of an entire function g_c, contradicting the fact that f has a pole. If $c' \in D(c, \epsilon/2)$, then $\{c, c'\} \subseteq D(c', \epsilon/2) \subseteq D(c, \epsilon)$, so the restrictions of g_c and the analogous function $g_{c'}$ to $D(c', \epsilon/2)$ each agree with the restriction of f. Thus $g_c, g_{c'}, f$ have the same Taylor series centered at c, and they have the same Taylor series centered at c'. The restriction of g_c to $D(c', R(c) - |c - c'|) \subseteq D(c, R(c))$ is holomorphic, so $R(c') \geq R(c) - |c - c'|$. Similarly $R(c) \geq R(c') - |c - c'|$, so $|R(c) - R(c')| \leq |c - c'|$. Thus R is continuous at c.

Suppose $p \in P$. We may choose $\epsilon > 0$ such that $D(p, \epsilon) - \{p\} \subseteq U - P$. For $c \in D(p, \epsilon/2) - \{p\}$, the restriction of g_c to $D(c, |c - p|)$ agrees with f, and $g_c(z) \to \infty$ as $z \to p$ within $D(c, |c - p|)$, so $R(c) = |c - p|$. Thus defining $R(p) = 0$ at each $p \in P$ gives an extension of R to a continuous function on U.

Remark: It is not true that $R(c)$ equals the distance from c to the complement of $U - \{\text{poles of } f\}$ in $\mathbb{C}$, even if f does not extend to a larger open subset of $\mathbb{C}$. For example, if f is the standard branch of $\frac{\log z}{z - 100}$ on $\mathbb{C} - \mathbb{R}_{\leq 0}$, then $R(-1 + i) = \sqrt{2}$, not 1.

The correct statement is that if $c \in U$ is not a pole, then $R(c)$ equals the radius of the largest open disk on which there is some holomorphic function that agrees with f on some open neighborhood of c.

8A. Let C and D be two $n \times n$ positive definite Hermitian matrices over $\mathbb{C}$ and let $A = CD$. Prove that all eigenvalues of A are positive real numbers.

Solution: Let $Ax = \lambda x$, $x \neq 0$. Then $CDx = \lambda x$. Since D is positive definite, it is invertible, so $Dx \neq 0$. Let B^H denote the conjugate transpose of a matrix (or column vector) B. Since C is positive definite,

$$0 < \langle Dx, C(Dx)\rangle = (Dx)^H C(Dx) = (Dx)^H \lambda x = \lambda x^H D^H x = \lambda x^H Dx,$$

since D is Hermitian. But $x^H Dx > 0$ since D is positive definite. Dividing, we get $\lambda > 0$.

9A. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be an infinitely differentiable function that is zero outside some bounded subset of $\mathbb{R}^2$. Prove that

$$\lim_{\epsilon\to 0} \iint_{x^2+y^2\geq \epsilon^2} \frac{f(x,y)}{(x+iy)^3}\, dx\, dy$$

exists.

Solution: The answer is positive. We must prove that

$$\lim_{\delta,\epsilon\to 0} \iint_{\delta^2<x^2+y^2<\epsilon^2} \frac{f(x,y)}{(x+iy)^3}\, dx\, dy = 0$$

We write

$$f(x,y) = a + bx + cy + O(x^2+y^2)$$

and consider each of the three terms. For the last one we note that

$$\frac{x^2+y^2}{|x+iy|^3} = (x^2+y^2)^{-\frac{1}{2}}$$

which is integrable at zero. For the constant we compute

$$\iint_{\delta^2<x^2+y^2<\epsilon^2} \frac{1}{(x+iy)^3}\, dx\, dy = \int_0^{2\pi}\int_{\delta<r<\epsilon} r^{-2} e^{-3i\theta}\, dr\, d\theta = 0$$

For $f(x,y) = y$ we have

$$\begin{aligned}\iint_{\delta^2<x^2+y^2<\epsilon^2} \frac{y}{(x+iy)^3}\, dx\, dy &= \int_0^{2\pi}\int_{\delta<r<\epsilon} r^{-1}\cos\theta\; e^{-3i\theta}\, dr\, d\theta \\ &= \frac{1}{2}(\ln\epsilon - \ln\delta)\int_0^{2\pi} e^{-2i\theta} + e^{-4i\theta}\, d\theta \\ &= 0\end{aligned}$$

The case $f(x,y) = x$ is similar by symmetry. This concludes the proof.

1B. Let G be a finite group. Suppose $ab = ba$ holds whenever $a, b \in G$ have prime power order. Prove that G is abelian.

Solution: Let $x, y \in G$. By the Chinese Remainder Theorem, the finite cyclic group generated by x is a product of cyclic groups of prime power order, so we can write $x = x_1 x_2 \cdots x_m$ where each x_i has prime power order. Write $y = y_1 y_2 \cdots y_n$ similarly. By assumption x_1 commutes with each y_j, so x_1 commutes with their product y. Similarly x_i commutes with y for each i, so their product x commutes with y.

2B. Prove that, for any $\varepsilon > 0$, the function $f(z) = \sin z + \frac{1}{z+i}$ has infinitely many zeros in the strip $|\operatorname{Im} z| < \varepsilon$.

Solution: We use Rouché's theorem. Without loss of generality, assume $\varepsilon < \pi$. Let δ be the minimum value of $|\sin z|$ on the compact set $|z| = \varepsilon$. Since the zeros of $\sin z$ in $\mathbb{C}$ are the integer multiples of π, we have $\delta > 0$. By periodicity, we have $|\sin z| \geq \delta$ also on the circle C_n defined by $|z - 2\pi n| = \varepsilon$ for any $n \in \mathbb{Z}$. On the other hand, if n is sufficiently large, then $|\frac{1}{z+i}| < \delta$ on C_n. For such n, Rouché's theorem implies that $f(z)$ has the same number of zeros as $\sin z$ inside C_n, namely 1. Letting n vary, we find infinitely many zeros of $f(z)$ inside the strip.

3B. Let $M_n(F)$ be the ring of $n \times n$ matrices over a field F. Prove that for every $A \in M_n(F)$ there exists $X \in M_n(F)$ such that $AXA = A$.

Solution: Let $\phi\colon F^n \to F^n$ be the linear transformation defined by A. Let $W = \ker \phi$, and let $V \subseteq F^n$ be a complementary subspace, such that $F^n = W \oplus V$. Let $U = \operatorname{im} \phi$. Note that $\dim U = \dim V = \operatorname{rank} A$. The restriction $\overline{\phi}$ of ϕ to V is injective, hence $\overline{\phi}\colon V \to U$ is an isomorphism. Let $\overline{\psi}\colon U \to V$ be its inverse, and let ψ be any extension of $\overline{\psi}$ from U to all of F^n. Take X to be the matrix of ψ. Then for every vector v in the column space U of A, we have $AXv = \phi\psi v = v$, which implies $AXA = A$.

4B. Let D be a subset of $\mathbb{R}$, and let $f\colon D \to \mathbb{R}$ be a function. The graph of f is the subset

$$G := \{(x, y) : x \in D,\ y = f(x)\}$$

of $\mathbb{R}^2$. Prove that if G is compact, then f is continuous.

Solution: It suffices to prove that $f^{-1}(C)$ is closed in D for every closed subset C of $\mathbb{R}$. Let π_1, π_2 be the coordinate projections $\mathbb{R}^2 \to \mathbb{R}$. Then $\pi_2^{-1}(C)$ is closed in $\mathbb{R}^2$. Thus $\pi_2^{-1}(C) \cap G$ is closed in G and hence compact. Now $f^{-1}(C) = \pi_1(\pi_2^{-1}(C) \cap G)$ is the continuous image of a compact set, so it is compact. Thus $f^{-1}(C)$ is closed in $\mathbb{R}$, hence closed in D.

5B. Let $\mathbb{Q}(x)$ be the field of rational functions in one variable over $\mathbb{Q}$. Let $i\colon \mathbb{Q}(x) \to \mathbb{Q}(x)$ be the unique field automorphism such that $i(x) = x^{-1}$. Prove that the fixed subfield $\{r \in \mathbb{Q}(x) : i(r) = r\}$ is equal to $\mathbb{Q}(x + x^{-1})$.

Solution: Let F denote the fixed subfield, and set $y = x + x^{-1}$. Obviously $\mathbb{Q}(y) \subseteq F \neq \mathbb{Q}(x)$. The equation $x^2 - yx + 1 = 0$ shows that $\mathbb{Q}(x)$ is an algebraic extension of $\mathbb{Q}(y)$, and $[\mathbb{Q}(x) : \mathbb{Q}(y)] = 2$. Since the intermediate field F is not equal to $\mathbb{Q}(x)$, we must have $F = \mathbb{Q}(y)$.

6B. Evaluate the integral $\displaystyle\int_0^\infty \frac{x \sin x}{x^2 + a^2}\, dx$, where $a > 0$.

Solution: Let I be the desired integral. Then

$$\begin{aligned} I &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx \\ &= \frac{1}{2} \lim_{R \to \infty} \int_{-R}^{R} \frac{x \sin x}{x^2 + a^2} dx \\ &= \frac{1}{2} \lim_{R \to \infty} \operatorname{Im} \int_{-R}^{R} \frac{x e^{ix}}{x^2 + a^2} dx. \end{aligned}$$

Integrate $\frac{ze^{iz}}{z^2+a^2}$ counterclockwise around the curve $-R \le x \le R$, $z = Re^{i\theta}$, $0 \le \theta \le \pi$, where $R > a$.

The residue of the integrand at $z = ia$ is

$$\frac{iae^{-a}}{2ia} = \frac{e^{-a}}{2}.$$

Moreover, by "Jordan's lemma", the integral over the semicircular part of the curve tends to 0 as $R \to \infty$.

Therefore $I = \frac{1}{2} \cdot \mathrm{Im}(2\pi i e^{-a}/2) = \frac{\pi e^{-a}}{2}$.

7B. Let $\mathbb{F}_p$ be the field of p elements. Let $\mathrm{SL}_2(\mathbb{F}_p)$ be the group of 2×2 matrices over $\mathbb{F}_p$ of determinant 1. Let G be a normal subgroup of $\mathrm{SL}_2(\mathbb{F}_p)$. Suppose G contains a non-identity element γ that fixes a nonzero vector v. Show that any $\gamma' \in \mathrm{SL}_2(\mathbb{F}_p)$ that fixes a nonzero vector v' belongs to G.

Solution: For each vector u, let S_u be the set of elements of $\mathrm{SL}_2(\mathbb{F}_p)$ that fix u. First, we can complete $\{v\}$ to a basis $\{v, w\}$. With respect to this basis, the matrix of γ is upper-triangular and hence is

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

for some $x \neq 0$. Then S_v (where now $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$) consists of the powers $\gamma^k = \begin{pmatrix} 1 & kx \\ 0 & 1 \end{pmatrix}$, so $S_v \subseteq G$.

Now suppose $v' := \begin{pmatrix} a \\ c \end{pmatrix} \in \mathbb{F}_p^2 \setminus 0$. Then we can find $b, d \in \mathbb{F}_p$ so that $ad - bc = 1$ (take $d = 0$ and $c = -1/b$ or $c = 0$ and $d = 1/a$). Thus

$$\rho := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_p)$$

satisfies $\rho v = v'$. Now $S_{v'} = \rho S_v \rho^{-1} \subseteq \rho G \rho^{-1} = G$, which is what we needed to show.

Remark: Many students confused $\mathrm{SL}_2(\mathbb{F}_p)$-conjugacy with similarity, which is $\mathrm{GL}_2(\mathbb{F}_p)$-conjugacy.

8B. Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{R}$. Suppose that $f(0) = 0$, and that $|f'(x)| \le |f(x)|$ for all $x \in \mathbb{R}$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

Solution: Let us show that $f(x) = 0$ for all $x \in [0, 1]$. Let $a = \max\{|f(x)| : x \in [0, 1]\}$. We have to show that $a = 0$. Suppose on the contrary $a > 0$. Let $E = \{x \in [0, 1] : |f(x)| = a\}$. Then E is closed and $\alpha = \inf E \in E$, i.e., $|f(\alpha)| = a > 0 \Rightarrow \alpha > 0$ since $f(0) = 0$. Thus $0 < \alpha \le 1$ and $|f(c)| < a(*)$ for all $0 \le c < \alpha$. We have $a = |f(\alpha)| = |f(\alpha) - f(0)| = |f'(c)| \cdot \alpha$ (for some $c \in (0, \alpha)$ by the Mean Value Theorem) $\le |f(c)| \cdot \alpha \le |f(c)|$ (since $0 < \alpha \le 1$). This contradicts $(*)$. Thus $f \equiv 0$ on $[0, 1]$. In particular, $f(1) = 0$ and we can use the same argument to show $f \equiv 0$ on $[1, 2]$ and on every $[n, n+1]$, $n = \pm 1, \pm 2, \ldots$.

Alternative solution: Suppose $f(x)$ is not identically zero. Replacing $f(x)$ by $\pm f(\pm x)$ we may assume that there exists $b > 0$ such that $f(b) > 0$. Let $a = \sup\{x \in [0, b] : f(x) = 0\}$.

Thus f is positive on (a, b). So $f' \leq f$ on (a, b). Thus the derivative of $g = e^{-x} f$ is ≤ 0 on (a, b). This contradicts $g(a) = 0 < g(b)$.

9B. (a) Prove that if $n > 0$ is even, there does not exist $f(x) \in \mathbb{R}[x]$ such that $f(x)^2 - x$ is divisible by $x^n - 1$.

(b) For odd $n > 0$, find the number of $f(x) \in \mathbb{R}[x]$ of degree $< n$ such that $f(x)^2 - x$ is divisible by $x^n - 1$.

Solution: (a) If $f(x)^2 - x$ is divisible by $x^n - 1$, it is divisible by the factor $x + 1$, so $f(-1)^2 - (-1) = 0$. This is impossible since $f(-1) \in \mathbb{R}$.

(b) Equivalently, we must count the square roots of the image of x in $\mathbb{R}[x]/(x^n - 1)$. If n is odd, only one zero of $x^n - 1$ is real, so $x^n - 1 = (x - 1) \prod_{j=1}^{(n-1)/2} f_j(x)$ where $f_j(x) \in \mathbb{R}[x]$ is irreducible of degree 2. Moreover, the factors are distinct, since $x^n - 1$ shares no zeros with its derivative nx^{n-1}. By the Chinese Remainder Theorem,

$$\mathbb{R}[x]/(x^n - 1) \simeq \frac{\mathbb{R}[x]}{(x-1)} \times \prod_{j=1}^{(n-1)/2} \frac{\mathbb{R}[x]}{(f_j(x))} \simeq \mathbb{R} \times \prod_{j=1}^{(n-1)/2} \mathbb{C}.$$

To choose a square root of the image of x is equivalent to choosing a square of the image of x in each factor. The image of x in the factor $\mathbb{R}$ is 1, and the image in each factor $\mathbb{C}$ is nonzero (since x has an inverse in $\mathbb{R}[x]/(x^n - 1)$, namely x^{n-1}), so there are 2 choices of square root in each of the $(n+1)/2$ factors. Thus the answer is $2^{(n+1)/2}$.

Alternative solution to (b): By Lagrange interpolation, a polynomial $f(x) \in \mathbb{C}[x]$ of degree $< n$ is uniquely specified by its values at the n-th roots of unity. Such a specification gives a polynomial with real coefficients if and only if the prescribed values at complex conjugate roots of unity are complex conjugates. Now $f(x)^2 - x$ is divisible by $x^n - 1$ if and only if $f(w)$ is a square root of w for each n-th root of unity w. We can construct such f by prescribing $f(1) = \pm 1$ and $f(w)$ for each n-th root of unity in the upper half plane, but then we must choose $f(\overline{w}) = \overline{f(w)}$. There are $(n - 1)/2$ n-th roots of unity in the upper half plane, so we have $1 + (n - 1)/2 = (n + 1)/2$ sign choices. Thus there are $2^{(n+1)/2}$ possibilities for f.