Spring 2004 Prelim Solutions

1A. Consider a sequence of functions $f_n\colon [a,b] \to \mathbb{R}$ with the property that for each $x \in [a,b]$ there is an open interval I_x containing x such that $(f_n)_{n\geq 1}$ converges uniformly in $I_x \cap [a,b]$. Show that $(f_n)_{n\geq 1}$ converges uniformly in $[a,b]$.

Solution: For each $x \in [a,b]$, the sequence (f_n) converges uniformly on I_x, and in particular converges pointwise at x. Let $f\colon [a,b] \to \mathbb{R}$ be the pointwise limit of (f_n). The compact set $[a,b]$ is covered by the collection of open intervals I_x, so there is a finite subcovering, say $[a,b] \subset \bigcup_{k=1}^m I_{x_k}$. Given $\epsilon > 0$, there exists N_k such that for $n \geq N_k$, the difference $|f_n - f|$ is bounded by ϵ on I_{x_k}. Let $N := \max(N_1, \ldots, N_m)$. Then for $n \geq N$, the difference $|f_n - f|$ is bounded by ϵ on all of $[a,b]$. Hence by definition, (f_n) converges to f uniformly.

2A. Find a countable abelian group whose endomorphism ring has the same cardinality as the set of real numbers. Justify your answer.

Solution: Let G be a vector space of dimension $\aleph_0$ over $\mathbb{F}_2$. Then G is countable, since it is a countable union of finite subspaces. Let $v_1, v_2, \ldots$ be a basis. For each $S \subseteq \{1,2,3,\ldots\}$, there is an endomorphism of G mapping each v_i to v_i or 0 according to whether $i \in S$. Different subsets S give different endomorphisms, so $\#\operatorname{End} G \geq 2^{\aleph_0}$. On the other hand,

$$\#\operatorname{End} G \leq (\#G)^{\#G} = \aleph_0^{\aleph_0} \leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0\aleph_0} = 2^{\aleph_0}.$$

Thus $\#\operatorname{End} G = 2^{\aleph_0} = \#\mathbb{R}$.

3A. Let $a_1, \ldots, a_n, b_1, \ldots, b_m$ be distinct complex numbers, let $r_1, \ldots, r_n$ be nonnegative integers, and let $c_1, \ldots, c_m$ be complex numbers. Prove that if $m \leq r_1 + \cdots + r_n + 1$, then there exists a rational function $F(z) \in \mathbb{C}(z)$ satisfying all of the following:

1. $F(z)$ is holomorphic at ∞ and everywhere in $\mathbb{C}$ except possibly at $a_1, \ldots, a_n$.
2. $\operatorname{ord}_{z=a_i} F(z) \geq -r_i$
3. $F(b_j) = c_j$ for $j = 1, \ldots, m$.

Solution: Write $F(z) = G(z)/\prod_{i=1}^n (z-a_i)^{r_i}$, where $G(z) \in \mathbb{C}(z)$ is to be determined. The condition that F be holomorphic on $\mathbb{C}$ except for poles of order at most r_i at a_i corresponds to the condition that $G(z)$ be holomorphic on $\mathbb{C}$, hence a polynomial. The condition that $F(z)$ be holomorphic at ∞ corresponds to the condition $\deg G \leq r_1 + \cdots + r_n$. The m conditions $F(b_j) = c_j$ correspond to conditions $G(b_j) = c_j'$ where $c_j' = c_j \prod_{i=1}^n (b_j - a_i)^{r_i}$. These m conditions can be satisfied by a polynomial of degree $m-1$ (which is $\leq r_1 + \cdots + r_n$), by the Lagrange interpolation formula. Alternatively,

$$\begin{aligned} \{ \text{ polynomials of degree} \leq m-1 \} &\to \mathbb{C}^m \\ G(z) &\mapsto (G(b_1), \ldots, G(b_m)) \end{aligned}$$

is a linear map between $\mathbb{C}$-vector spaces of the same finite dimension, and is injective (since a nonzero polynomial of degree $\leq m-1$ has at most $m-1$ zeros), so it is also surjective.

4A. For which positive integers n is it true that every invertible 2×2 matrix A with real entries can be expressed as the n-th power of another 2×2 matrix with real entries?

Solution: The answer is the odd positive integers. If n is even, then $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ cannot be the n-th power of another 2×2 matrix with real entries, because its determinant is not an n-th power of a real number.

Now assume n is odd. Thus every real number is an n-th power of a real number. The question of whether A is an n-th power is not affected by conjugation. Thus if A has distinct real eigenvalues, then without loss of generality we may assume that A is diagonal, in which we take the n-th roots of the diagonal entries to find another diagonal matrix B with $B^n = A$.

If A has equal real eigenvalues, then by conjugation, we may assume

$$A = \lambda \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$$

where $\lambda \in \mathbb{R}^*$ and $c \in \mathbb{R}$. Then $A = B^n$ where

$$B = \lambda^{1/n} \begin{pmatrix} 1 & c/n \\ 0 & 1 \end{pmatrix}.$$

Finally if the eigenvalues of A are not real, then the minimal polynomial of A is a quadratic polynomial $f(x)$ with no real roots, so the $\mathbb{R}$-subalgebra $\mathbb{R}[A]$ of $M_2(\mathbb{R})$ generated by A is isomorphic to $\mathbb{R}[x]/(f(x)) \simeq \mathbb{C}$. Since every element of $\mathbb{C}$ has an n-th root, the matrix A has an n-th root in $\mathbb{R}[A]$.

5A. Suppose $f\colon \mathbb{R} \to \mathbb{C}$ satisfies $f'(t) + 2itf(t) = e^{2it}$ and $f(0) = 0$. Compute

$$\lim_{t \to +\infty} e^{it^2}(f(t) - f(-t)).$$

You may assume $\int_0^\infty e^{-t^2}\,dt = \sqrt{\pi}/2$.

Solution: Multiply the ODE by the integrating factor e^{it^2}, and integrate to get

$$e^{it^2} f(t) = \int_0^t e^{ix^2+2ix}\,dx$$

(The hypothesis $f(0) = 0$ implies that there is no constant of integration.) Substituting $-t$ for t and subtracting, we get

$$\begin{aligned} e^{it^2}(f(t) - f(-t)) &= \int_{-t}^t e^{ix^2+2ix}\,dx \\ &= e^{-i} \int_{-t}^t e^{i(x+1)^2}\,dx \\ &= e^{-i} \int_{-t+1}^{t+1} e^{iz^2}\,dz. \end{aligned}$$

Since e^{iz^2} is an even function, the limit as $t \to +\infty$ equals $2e^{-i}I$, where $I := \lim_{R \to +\infty} \int_0^R e^{iz^2}\,dz$ (assuming for now that the latter limit exists). Apply Cauchy's Theorem to the triangular contour from 0 to R to $R + Ri$ and back to 0. The vertical part contributes

$$\int_R^{R+Ri} e^{iz^2}\,dz = \int_0^R e^{i(R+ti)^2}\,i\,dt,$$

whose absolute value is bounded by

$$\begin{aligned}\int_0^R |e^{i(R+ti)^2}|\,dt &= \int_0^R e^{-2Rt}\,dt \\ &= \frac{1}{2R}\int_0^{2R^2} e^{-u}\,du,\end{aligned}$$

which goes to 0 as $R \to \infty$. Thus

$$\begin{aligned}I &= \lim_{R\to\infty}\int_0^{R+Ri} e^{iz^2}\,dz && \text{(if the limit exists)} \\ &= \lim_{R\to\infty}\int_0^{R} e^{i(e^{i\pi/4}t)^2}e^{i\pi/4}\,dt && \text{(if the limit exists)} \\ &= e^{i\pi/4}\lim_{R\to\infty}\int_0^{R} e^{-t^2}\,dt && \text{(if the limit exists)} \\ &= e^{i\pi/4}\frac{\sqrt{\pi}}{2}.\end{aligned}$$

Thus we now know that all the limits exist, and the answer is $2e^{-i}I = e^{-i+i\pi/4}\sqrt{\pi}$.

6A. For which pairs of integers (a,b) is the quotient ring $\mathbb{Z}[x]/(x^2+ax+b)$ isomorphic (as a ring) to the direct product of rings $\mathbb{Z}\times\mathbb{Z}$?

Solution: Let $A = \mathbb{Z}[x]/(x^2+ax+b)$ and $B = \mathbb{Z}\times\mathbb{Z}$. If x^2+ax+b is irreducible in the UFD $\mathbb{Z}[x]$, then (x^2+ax+b) is a prime ideal, so A is a domain. But B is not a domain. Thus we may assume $x^2+ax+b = (x-c)(x-d)$ for some $c, d \in \mathbb{Z}$.

Suppose p is a prime integer dividing $c-d$. Then $A \simeq B$ implies $A/pA \simeq B/pB$; that is, $\mathbb{F}_p[x]/(x-\bar{c})^2 \simeq \mathbb{F}_p \times \mathbb{F}_p$, where $\bar{c} = \bar{d}$ is the image of c in $\mathbb{F}_p$. The ring on the left has a nonzero element with square 0, namely $x - \bar{c}$, whereas the right hand side has no such element. This contradiction shows that $c-d$ is divisible by no primes, so $c - d = \pm 1$.

Conversely, if $c-d = \pm 1$, then the sum of the ideals $(x-c)$ and $(x-d)$ in $\mathbb{Z}[x]$ is the unit ideal, and their product equals their intersection (since they are generated by non-associate irreducible elements), so the Chinese Remainder Theorem gives

$$\frac{\mathbb{Z}[x]}{((x-c)(x-d))} \simeq \frac{\mathbb{Z}[x]}{(x-c)} \times \frac{\mathbb{Z}[x]}{(x-d)}.$$

Each factor on the right is isomorphic to $\mathbb{Z}$, because each polynomial in $\mathbb{Z}[x]$ is uniquely expressible as $q(x)(x-c)+r$ with $q(x) \in \mathbb{Z}[x]$ and $r \in \mathbb{Z}$. Thus $c - d = \pm 1$ implies $A \simeq B$.

In other words, the answer is the set of (a,b) such that x^2+ax+b has the form $(x-n)(x-(n+1))$; that is,

$$\{\, (-(2n+1), n(n+1)) : n \in \mathbb{Z} \,\}.$$

7A. Evaluate $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$.

Solution: For $R > 1$, let γ_1 be the straight line path from $1/R$ to R, let γ_2 be the straight line path from R to $R+Ri$, let γ_3 be the straight line path from $R+Ri$ to $-R+Ri$, let γ_4 be the straight line path from $-R+Ri$ to $-R$, let γ_5 be the straight line path from $-R$ to

$-1/R$, and let γ_6 be the upper semicircle from $-1/R$ to $1/R$ given by the parameterization $\gamma_6(t) = e^{it}$ for t running from π to 0. Let γ be the closed loop formed by concatenating these six paths. Cauchy's Theorem implies that $\int_\gamma \frac{e^{iz}}{z}\, dz = 0$.

We have

$$\left|\int_{\gamma_2} \frac{e^{iz}}{z}\, dz\right| \le \int_0^R \frac{e^{-t}}{R}\, dt = \frac{1-e^{-R}}{R} \to 0$$

as $R \to \infty$. Similarly $\int_{\gamma_4} \frac{e^{iz}}{z}\, dz \to 0$, and

$$\left|\int_{\gamma_3} \frac{e^{iz}}{z}\, dz\right| \le \int_{-R}^R \frac{e^{-R}}{R}\, dt = 2e^{-R} \to 0.$$

On the other hand, $e^{iz}z$ differs from $1/z$ by a holomorphic function, and γ_6 is shrinking to a point, so

$$\begin{aligned}\lim_{R\to\infty} \int_{\gamma_6} \frac{e^{iz}}{z}\, dz &= \lim_{R\to\infty} \int_{\gamma_6} \frac{1}{z}\, dz \\ &= \lim_{R\to\infty} \int_\pi^0 \frac{1}{(1/R)e^{it}}(1/R)ie^{it}\, dt \\ &= -\pi i.\end{aligned}$$

Thus

$$\int_{\gamma_1} \frac{e^{iz}}{z}\, dz + \int_{\gamma_5} \frac{e^{iz}}{z}\, dz \to \pi i$$

as $R \to \infty$. Taking imaginary parts and using the fact that $(\sin z)/z$ is an even function, we find that

$$2\int_{1/R}^R \frac{\sin z}{z}\, dz \to \pi$$

as $R \to \infty$. Since $(\sin z)/z$ is holomorphic, it does not hurt to replace the lower limit $1/R$ by 0, so $\int_0^\infty \frac{\sin x}{x}\, dx = \pi/2$.

8A. Let V and W be finite-dimensional vector spaces over a field k. Let $f\colon V^n \to W$ be a function such that

(a) For each fixed $i \in \{1, \ldots, n\}$ and fixed $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n \in V$, the map

$$\begin{aligned} V &\to W \\ x &\mapsto f(v_1, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_n)\end{aligned}$$

is a k-linear transformation; and

(b) $f(v_1, \ldots, v_n) = 0$ whenever $v_i = v_{i+1}$ for some $i \in \{1, \ldots, n-1\}$.

Prove that either $\dim V \ge n$ or f is identically zero.

Solution: Fix i, and $v_1, \dots, v_{i-1}, v_{i+2}, \dots, v_n \in V$, and define $g(x, y) = f(v_1, \dots, v_{i-1}, x, y, v_{i+2}, \dots, v_n)$. Then

$$\begin{aligned} 0 &= g(x+y, x+y) \\ &= g(x+y, x) + g(x+y, y) \\ &= g(x, x) + g(y, x) + g(x, y) + g(y, y) \\ &= g(y, x) + g(x, y) \end{aligned}$$

so interchanging adjacent arguments changes the sign of the value of f.

Suppose $v_1, \dots, v_n \in V$ are such that $v_i = v_j$ for some $i < j$. Then we can interchange arguments repeatedly to move v_j to the $i+1$ position, possibly changing the sign of the value of $f(v_1, \dots, v_n)$ as we go along. Since at the end the result is zero, we must have had $f(v_1, \dots, v_n) = 0$ originally. Thus $f(v_1, \dots, v_n) = 0$ whenever $v_i = v_j$ for some $i \neq j$.

We now solve the problem. If the conclusion fails, we have $\dim V < n$ and there exist $v_1, \dots, v_n \in V$ with $f(v_1, \dots, v_n) \neq 0$. Since $\dim V < n$, the vectors $v_1, \dots, v_n$ must be linearly dependent. Thus for some i, we can write $v_i = \sum_{j \neq i} c_j v_j$ for some constants $c_j \in k$ for $j \neq i$. By linearity of f in the i-th argument,

$$\begin{aligned} f(v_1, \dots, v_n) &= \sum_{j \neq i} c_j f(v_1, \dots, v_{i-1}, v_j, v_{i+1}, \dots, v_n) \\ &= \sum_{j \neq i} c_j \cdot 0 \end{aligned}$$

by the previous paragraph, since in each term some v_j appears twice as an argument. Thus $f(v_1, \dots, v_n) = 0$, a contradiction.

9A. Let $f \colon \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable function, and let L be a nonnegative real number. Prove that the following are equivalent:

(i) For every $x, y \in \mathbb{R}^n$,

$$(f(x) - f(y)).(x - y) \leq L|x - y|^2$$

(ii) For every $x, v \in \mathbb{R}^n$,

$$Df(x)v.v \leq L|v|^2,$$

where $Df(x)$ is the derivative of f at x, and . denotes the standard inner product of vectors in $\mathbb{R}^n$.

Solution:

(i) $\implies$ (ii): Let $x = y + tv$. Then (i) says

$$t(f(y + tv) - f(y)).v \leq Lt^2|v|^2.$$

Divide by t^2 and take the limit as $t \to 0$ to deduce $Df(y)v.v \leq L|v|^2$.

(ii) $\implies$ (i): Let $\phi(t) = f(y + t(x - y))$ for $t \in \mathbb{R}$. Then

$$\begin{aligned} f(x) - f(y) &= \phi(1) - \phi(0) \\ &= \int_0^1 \phi'(t)\,dt \\ &= \int_0^1 Df(y + t(x-y))(x-y)\,dt \qquad \text{(by the Chain Rule).} \end{aligned}$$

so

$$\begin{aligned} (f(x) - f(y)).(x-y) &= \int_0^1 Df(y + t(x-y))(x-y).(x-y)\,dt \\ &\le \int_0^1 L|x-y|^2\,dt \qquad \text{(by (ii))} \\ &= L|x-y|^2. \end{aligned}$$

1B. Let F be a field (of arbitrary characteristic). Suppose g is a nonnegative integer, and polynomials $a(x), b(x) \in F[x]$ satisfy $\deg a(x) \le g$ and $\deg b(x) = 2g + 1$. Prove that the polynomial $y^2 + a(x)y + b(x)$ is irreducible over $F(x)$.

Solution: If instead it factors in $F(x)[y]$ into polynomials of y-degree ≥ 1, then by Gauss's Lemma, it factors in $F[x][y] = F[x, y]$ into polynomials of y-degree ≥ 1. Thus we would have

$$y^2 + a(x)y + b(x) = (y + p(x))(y + q(x))$$

for some $p(x), q(x) \in F[x]$. Since $p(x)q(x) = b(x)$ has odd degree, $p(x)$ and $q(x)$ have distinct degrees, so

$$\deg(p(x) + q(x)) = \max(\deg p(x), \deg q(x)) \ge (\deg p(x) + \deg q(x))/2 = (2g+1)/2 > g.$$

This contradictions $\deg a(x) = g$.

2B. Find the maximum possible value of $|f'(1)|$ given that f is holomorphic on an open neighborhood of $\{z \in \mathbb{C} : |z| \le 2\}$ and satisfies $|f(z)| \le 1$ when $|z| = 2$.

Solution: We will use a fractional linear transformation to change the problem to one where the derivative is evaluated at the center of a disk.

The function $z \mapsto \frac{2}{z}\left(\frac{z-1}{\bar{z}-1}\right)$ on $|z| = 2$ has absolute value 1, and it extends to a fractional linear transformation $g(z) = 2\left(\frac{z-1}{4-z}\right)$ Since it also maps $z = 1$ to the interior of the unit disk, it must map the region $|z| \le 2$ bijectively onto the unit disk. We calculate $|g'(1)| = 2/3$.

Now, for any other f mapping the circle $|z| = 2$ into $|z| \le 1$, the composition $h := f \circ g^{-1}$ is holomorphic on a neighborhood of $|z| \le 1$, and maps $|z| = 1$ into $|z| \le 1$. Taking absolute values in

$$h'(0) = \frac{1}{2\pi i} \int_{|z|=1} \frac{h(z)}{z^2}\,dz$$

gives $|h'(0)| \le 1$. Since $g^{-1}(0) = 1$, the Chain Rule gives $h'(0) = f'(1)g'(1)^{-1}$. Thus $|f'(1)| = |h'(0)||g'(1)| \le |g'(1)| = 2/3$. Thus $2/3$ is the maximum possible value of $|f'(1)|$.

3B. Let A be a $d \times d$ matrix with complex entries. Assume that every eigenvalue of A has absolute value 1. Prove that there exists a constant $c \in \mathbb{R}$ independent of n such that

$$\|A^n x\| \leq c n^{d-1} \|x\|$$

for all $n \geq 1$ and $x \in \mathbb{C}^d$. Here $\|x\| := (|x_1|^2 + \cdots + |x_d|^2)^{1/2}$ for all $(x_1, \ldots, x_d) \in \mathbb{C}^d$.

Solution: We may use $|x|_\infty := \max\{|x_1|, \ldots, |x_n|\}$ instead of $\|x\|$, since different norms on a finite-dimensional vector space are bounded by positive constants times each other. Then it suffices to show that the entries of A^n are $O(n^{d-1})$ as $n \to \infty$. This property is unchanged if we conjugate all the A^n by a fixed invertible matrix. Thus we may assume that A is in Jordan canonical form. Thus $A = D + N$ where D is diagonal, N is nilpotent, and D and N commute. By the Cayley-Hamilton theorem, $N^d = 0$. Thus the binomial theorem gives

$$A^n = D^n + \binom{n}{1} D^{n-1} N + \binom{n}{2} D^{n-2} N^2 + \cdots + \binom{n}{d-1} D^{n-d+1} N^{d-1}.$$

The diagonal entries of D are the eigenvalues of A, which have absolute value 1, so the entries of D^m are $O(1)$ for any m. The entries of N, N^2, …, N^{d-1} do not depend on n. The binomial coefficients are $O(n^{d-1})$. Thus the entries of A^n are $O(n^{d-1})$, as desired.

4B. Let $a_1, \ldots, a_n$ be positive real numbers. Let Δ be the set of points $\mathbf{x} \in \mathbb{R}^n$ satisfying the conditions

$$\sum_{i=1}^{n} a_i x_i = 1, \quad x_i > 0 \text{ for all } i.$$

Prove that the function $\log(\prod_{i=1}^n x_i)$ has a unique maximum on Δ and find the point where it occurs.

Solution: The given function is continuous and approaches $-\infty$ at every point on the boundary of Δ (since each x_i is bounded above, and at least one of them approaches zero at every point on the boundary). Hence a maximum exists. By Lagrange multipliers, at a maximum we must have $d \log(\prod_{i=1}^n x_i) = \lambda \, d \sum_{i=1}^n a_i x_i$ for some λ, or $\sum_i dx_i/x_i = \lambda \sum_i a_i \, dx_i$. Hence $(x_1, \ldots, x_n) = (1/\lambda)(1/a_1, \ldots, 1/a_n)$. Combining this with the equation $\sum_i a_i x_i = 1$ shows that $\lambda = n$ and $(x_1, \ldots, x_n) = (1/n)(1/a_1, \ldots, 1/a_n)$. This locates the maximum and proves that it is unique.

Alternative solution: The arithmetic-mean–geometric-mean inequality gives

$$\frac{\sum_{i=1}^n a_i x_i}{n} \geq \left(\prod_{i=1}^n (a_i x_i) \right)^{1/n},$$

with equality if and only if $a_1 x_1 = \cdots = a_n x_n$. On Δ, the left hand side is constant, so we get an upper bound on $\prod_{i=1}^n x_i$, attained exactly when $a_1 x_1 = \cdots = a_n x_n$. It follows that there is a unique maximum where $a_i x_i = 1/n$ for all i; that is, $x_i = 1/(n a_i)$ for all i.

5B. Let $n_1, \ldots, n_r$ be integers ≥ 2. Prove that there is a finite group G containing elements $g_1, \ldots, g_r$ such that g_i has exact order n_i for each i, and $g_i g_j \neq g_j g_i$ for $i \neq j$.

Solution: Let $T_1, \ldots, T_r$ be disjoint sets with $\#T_i = n_i - 1$. Let S be the union of the T_i together with one more element x outside all the T_i. Let G be the set of permutations of S.

Choose $g_i \in G$ such that g_i acts as an n_i-cycle on $T_i \cup \{x\}$, and acts as the identity on the complement. Then g_i has order n_i. If $i \neq j$, then $(g_i g_j)(x) = g_i(g_j(x)) \in g_i(T_j) = T_j$, and similarly $(g_j g_i)(x) \in T_i$, so $g_i g_j \neq g_j g_i$.

6B. Let $(u_n(x,y))_{n\geq 1}$ be a sequence of functions that are defined and harmonic for (x,y) in an open neighborhood of the upper half plane $\mathbb{R} \times \mathbb{R}_{\geq 0}$. Suppose that $\frac{\partial u_n}{\partial y}(x,0) = 0$ for all $x \in \mathbb{R}$, and $u_n(x,0)$ converges to 0 as $n \to \infty$ uniformly for $x \in \mathbb{R}$. Must $u_n(x,y) \to 0$ as $n \to \infty$ for every $(x,y) \in \mathbb{R} \times \mathbb{R}_{>0}$?

Solution: No. Let $u_n = \cosh(ny)\cos(nx)/n$. Since u_n is the real part of the holomorphic function $\cos(nz)/n$, it is harmonic on the entire plane. Then $\frac{\partial u_n}{\partial y}(x,0) = -\sinh(0)\cos(nx) = 0$, and $u_n(x,0) = \cos(nx)/n \to 0$ as $n \to \infty$ uniformly for $x \in R$. But $u_n(0,1) = \cosh(n)/n$ does not tend to 0 as $n \to \infty$.

7B. Let A and B be $n \times n$ matrices with complex entries, such that $AB - BA$ is a linear combination of A and B. Prove that there exists a nonzero vector v that is an eigenvector of both A and B.

Solution: Let $AB - BA = C = \alpha A + \beta B$. If $\alpha = \beta = 0$, then A and B commute. By a theorem of linear algebra, commuting complex matrices have a common eigenvector. Otherwise, assume without loss of generality that $\beta \neq 0$. Then B is a linear combination of A and C, so it suffices to prove that A and C have a common eigenvector. Note that $AC - CA = \beta C$. Since A has finitely many eigenvalues, it must have one, call it λ, such that $\lambda + \beta$ is not an eigenvalue of A. Let v be a nonzero vector with $Av = \lambda v$. Then $ACv = CAv + \beta Cv = (\lambda + \beta)Cv$, so $Cv = 0$. Hence v is a common eigenvector of A and C.

8B. For each real number x, compute

$$\lim_{n\to\infty} n\left(\left(1+\frac{x}{n}\right)^n - e^x\right).$$

Solution: We have

$$\begin{aligned} n\left(\left(1+\frac{x}{n}\right)^n - e^x\right) &= n\left(e^{n\log(1+x/n)} - e^x\right) \\ &= ne^x\left(e^{n\log(1+x/n)-x} - 1\right). \end{aligned}$$

Taylor's Theorem with Remainder gives

$$\log\left(1+\frac{x}{n}\right) = \frac{x}{n} - \frac{1}{2}\left(\frac{x^2}{n^2}\right) + O\left(\frac{1}{n^3}\right)$$

where the constant in the big-O depends on x, but not on n. Substituting, we get

$$ne^x\left(e^{-\frac{x^2}{2n}+O\left(\frac{1}{n^2}\right)} - 1\right).$$

Since $e^y = 1 + y + O(y^2)$ as $y \to 0$, this becomes

$$ne^x\left(-\frac{x^2}{2n} + O\left(\frac{1}{n^2}\right)\right) = -\frac{1}{2}x^2e^x + O\left(\frac{1}{n}\right),$$

so the limit is $-\frac{1}{2}x^2e^x$.

9B. Let S_4 be the group of permutations of $\{1, 2, 3, 4\}$. Determine the order of the automorphism group $\mathrm{Aut}(S_4)$. Justify your answer.

Solution: The center of S_4 is trivial, so S_4 acts faithfully on itself by inner automorphisms. We will then have $|\mathrm{Aut}(S_4)| = |S_4| = 24$, if we can show that every automorphism of S_4 is inner.

Let $\sigma \in \mathrm{Aut}(S_4)$. The group S_4 has exactly four subgroups H_1, H_2, H_3, H_4 of order 3, where H_i contains the identity and the two 3-cycles that fix i. The automorphism σ must permute these subgroups. Since inner automorphisms permute them arbitrarily, we can assume after multiplying σ by an inner automorphism that σ fixes each H_i. The set of transpositions is characterized as the unique conjugacy class consisting of 6 elements of order 2, so σ stabilizes it. Among the transpositions, each one $\tau = (i\ j)$ is characterized by the property that τ and H_k generate S_4 if and only if $k \in \{i, j\}$. Therefore σ fixes every transposition. Since the transpositions generate S_4, σ must be the identity.