Department of Mathematics, University of California, Berkeley

YOUR 1 OR 2 DIGIT EXAM NUMBER ____

GRADUATE PRELIMINARY EXAMINATION, Part A — Fall Semester 2021

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1. Please write your 1- or 2-digit exam number on this cover sheet and on **all** problem sheets (even problems that you do not wish to be graded).
2. Indicate below which six problems you wish to have graded. **Cross out** solutions you may have begun for the problems that you have not selected.
3. Extra sheets should be stapled to the appropriate problem at the upper right corner. Do not put work for problem p on either side of the page for problem q if $p \neq q$.
4. No notes, books, calculators or electronic devices may be used during the exam.

PROBLEM SELECTION

Part A: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

GRADE COMPUTATION (for use by grader—do not write below)

1A. ______	1B. ______	Calculus
2A. ______	2B. ______	Real analysis
3A. ______	3B. ______	Real analysis
4A. ______	4B. ______	Complex analysis
5A. ______	5B. ______	Complex analysis
6A. ______	6B. ______	Linear algebra
7A. ______	7B. ______	Linear algebra
8A. ______	8B. ______	Abstract algebra
9A. ______	9B. ______	Abstract algebra

Part A Subtotal: ______ Part B Subtotal: ______ Grand Total: ______

YOUR EXAM NUMBER ____
Please cross out this problem if you do not wish it graded

Problem 1A. *Score:*

(a) Find a particular solution y_1 of

$$y' = y^2 - ty + 1$$

(b) Find the general solution. (The solution may be expressed using a definite integral.)

Solution:
(a) By inspection, $y_1(t) = t$ works.
(b) The general solution is $y(t) = y_1(t) + u(t)$ where

$$u' = u^2 + tu$$

so that

$$u(t) = \frac{e^{t^2/2}}{C - \int_0^t e^{s^2/2} ds}$$

where C is an arbitrary constant of integration. Hence

$$y(t) = t + \frac{e^{t^2/2}}{C - \int_0^t e^{s^2/2} ds}.$$

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Problem 2A. *Score:*

Let

$$g(x) = \sin^2(\pi x) + \sin^2(\pi x)\cos^2(\pi x) + \sin^2(\pi x)\cos^4(\pi x) + \cdots \sin^2(\pi x)\cos^{2k}(\pi x) + \cdots$$

Evaluate the limit

$$G(x) = \lim_{n\to\infty} g(n!x).$$

Is G Riemann integrable?

Solution: By the geometric series, $g(x)$ is 0 if x is an integer and 1 otherwise. Hence $G(x)$ is 0 if x is rational and 1 otherwise, which is not Riemann integrable. (Lower sum is 0 and upper sum is $b - a$ on any interval $[a, b]$.)

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Problem 3A. *Score:*

Let f be a differentiable function from $\mathbb{R}$ to $\mathbb{R}$. Suppose $f(0) = 0$ and $f'(t) > f(t)$ for $t \geq 0$. Show that $f(t) > 0$ for all $t > 0$.

Solution:

Since $f'(0) = s > f(0) = 0$, the definition of limit ensures that there is some $\delta > 0$ such that

$$f(h) \geq \frac{s}{2}h$$

for $0 \leq h \leq \delta$.

By integration,

$$f(t) \geq \int_0^t f(s)ds$$

for $t \geq 0$. Let T be the set of $t \geq \delta$ such that $f(t) = 0$. Then T is closed and bounded below, so if it is nonempty then there is a smallest element t_0 of T. This would imply

$$0 = f(t_0) \geq \int_0^{t_0} f(s)ds > 0$$

since $f(s) > 0$ for $0 < s < t_0$, so T must be empty.

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Problem 4A. *Score:*

Let $P(z)$ be a monic complex polynomial with zeroes z_1 through z_n.

(a) Show that if all the zeros z_k have non-negative real part, then all the zeros of the derivative have non-negative real part.

(b) Let D be the convex hull of z_1 through z_n. Show that all the zeroes of the derivative $P'(z)$ lie in D.

Solution: Write
$$P(z) = (z - z_1) \cdots (z - z_n)$$
so that
$$P'(z) = \sum_{j=1}^{n} \prod_{i \neq j} (z - z_i) = \left(\prod_{i=1}^{n} (z - z_i) \right) \sum_{j=1}^{n} \frac{1}{z - z_j} = P(z) \sum_{j=1}^{n} \frac{1}{z - z_j}$$
for almost all z. Hence $P'(z) = 0$ only when $P(z) = 0$ or
$$\sum_{j=1}^{n} \frac{z - z_j}{|z - z_j|^2} = 0$$
i.e.
$$z \sum \frac{1}{|z - z_j|^2} = \sum \frac{z_j}{|z - z_j|^2}.$$
Dividing through represents z as a convex combination of the roots z_j.

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Problem 5A. *Score:*

Let $u\colon \mathbb{R}^2 \to \mathbb{R}$ be the function

$$u(x,y) = 2x + 2y - x^3 + 3xy^2 \ .$$

(a) Show that u is harmonic.

(b) Use the Cauchy–Riemann equations to find $v\colon \mathbb{R}^2 \to \mathbb{R}$ such that the function $f\colon \mathbb{C} \to \mathbb{C}$ defined by

$$f(x+iy) = u(x,y) + iv(x,y)$$

is analytic.

Solution: By definition, u is harmonic if $u_{xx} + u_{yy} = 0$. This is true for u because

$$u_{xx} = \frac{\partial}{\partial x}(2 - 3x^2 + 3y^2) = -6x \qquad \text{and} \qquad u_{yy} = \frac{\partial}{\partial y}(2 + 6xy) = 6x \ ,$$

and the sum of these is zero.

To find its harmonic conjugate $v(x,y)$, we need

$$v_y = u_x = 2 - 3x^2 + 3y^2 \qquad \text{and} \qquad v_x = -u_y = -2 - 6xy \ .$$

Integrating gives

$$v(x,y) = 2y - 3x^2y + y^3 + \alpha(x) = -2x - 3x^2y + \beta(y)$$

for some functions $\alpha, \beta\colon \mathbb{R} \to \mathbb{R}$. Taking $\alpha(x) = -2x$ and $\beta(y) = 2y + y^3$ gives

$$v(x,y) = -2x + 2y - 3x^2y + y^3 \ .$$

(Therefore $u + iv = 2z - 2iz - z^3$ when $z = x + iy$.)

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Problem 6A. *Score:*

Solve the initial value problem

$$y' = Ay = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} y, \qquad y(0) = y_0 = [1, 0, 0]^T.$$

Solution:

The matrix is $A = uu^T$ where $u = [123]^T$, so

$$y' = uu^T y$$

gives

$$(u^T y)' = (u^T u)(u^T y)$$

or

$$u^T y = \exp(tu^T u)u^T y_0.$$

Thus

$$y' = \exp(tu^T u)(u^T y_0)u$$

and

$$y(t) = y_0 + \int_0^t \exp(su^T u)ds(u^T y_0)u.$$

which gives

$$y(t) = (e14t + 13, 2e^{14t} - 2, 3e^{14t} - 3)/14$$

Problem 7A. *Score:*

Let the $n \times n$ complex matrix A have entries

$$A_{pq} = \exp(-2it_p t_q)$$

where t_p is real for $1 \le p \le n$ and $i = \sqrt{-1}$. Suppose A has the singular value decomposition

$$A = U \Sigma V^*$$

i.e. U and V are unitary and Σ is diagonal and nonnegative. Find a singular value decomposition of the matrix B with entries

$$B_{pq} = \exp(i(t_p - t_q)^2).$$

Solution: Since

$$B_{pq} = \exp(it_p^2) \exp(-2it_p t_q) \exp(it_q^2),$$

we have

$$B = (CU)\Sigma(DV)^*$$

where

$$C_{pq} = \exp(it_p^2)\delta_{pq}$$

and

$$D_{pq} = \exp(-it_p^2)\delta_{pq}$$

are diagonal unitary matrices. Since unitary matrices form a group, this is a singular value decomposition of B.

Problem 8A. *Score:*

Let G be a group of order 120, and let H be a subgroup of order 24. Assume that there is at least one left coset of H (other than H itself) which is equal to some right coset of H. Prove that H is a normal subgroup of G.

Solution: Recall that the *normalizer* of H in G is the set

$$N_G(H) = \{x \in G : xH = Hx\} .$$

This is a subgroup of G, and it contains H.

Also, by assumption, there is some $x \in G$ such that $x \notin H$ and $xH = Hx$. Therefore $N_G(H) \supsetneqq H$. Let $n = |N_G(H)|$. Then $24 \mid n$, $n > 24$, and $n \mid 120$. Therefore $m = n/24$ is a positive integer, $m \neq 1$, and $m \mid 5$. Since 5 is prime, this implies $m = 5$, so $n = 120$, which implies $N_G(H) = G$, and therefore H is a normal subgroup.

Problem 9A. *Score:*

Let p be an odd prime number and let $\mathbb{F}_p$ denote the field $\mathbb{Z}/p\mathbb{Z}$ with p elements.

(a). How many elements of $\mathbb{F}_p$ have square roots in $\mathbb{F}_p$?

(b). How many have cube roots in $\mathbb{F}_p$?

Solution: (a). Since $\mathbb{F}_p$ is a field, the polynomial $x^2 - 1$ has at most two roots. It also has at least two roots, namely 1 and -1 (with $1 \neq -1$). Therefore the kernel of the group homomorphism $\mathbb{F}_p^* \to \mathbb{F}_p^*$ given by $x \mapsto x^2$ has two elements, so its image has $(p-1)/2$ elements. Since $0 \in \mathbb{F}_p$ also has a square root, this gives $(p+1)/2$ elements of $\mathbb{F}_p$ that have square roots in $\mathbb{F}_p$.

(b). The multiplicative group $\mathbb{F}_p^*$ is cyclic of order $p-1$. Let $g \in \mathbb{F}_p^*$ be an element that generates this group.

Case 1. Assume that $p \equiv 1 \pmod 3$. Since the polynomial $x^3 - 1$ has at most three roots, the kernel of the group homomorphism $\phi\colon \mathbb{F}_p^* \to \mathbb{F}_p^*$ given by $x \mapsto x^3$ has at most three elements. It also has at least three elements, namely g^a for $a = 0, (p-1)/3, 2(p-1)/3$. Therefore the kernel has exactly three elements, so the image of ϕ has $(p-1)/3$ elements. Since 0 also has a cube root $0 \in \mathbb{F}_p$, this gives $1 + (p-1)/3$ elements of $\mathbb{F}_p$ that have cube roots in $\mathbb{F}_p$.

Case 2. Assume that $p \not\equiv 1 \pmod 3$. Then $p-1$ is not a multiple of 3, so by the Chinese Remainder Theorem there is an integer a such that $3 \mid a$ and $a \equiv 1 \pmod{p-1}$. Let $b = a/3 \in \mathbb{Z}$; then $g = g^a = (g^b)^3$ with $g^b \in \mathbb{F}_p$. Therefore all powers of g have cube roots in $\mathbb{F}_p$, so since 0 also has a cube root in $\mathbb{F}_p$, it follows that all elements of $\mathbb{F}_p$ have cube roots in $\mathbb{F}_p$, so the number of such elements is p.

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GRADUATE PRELIMINARY EXAMINATION, Part B Fall Semester 2021

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PROBLEM SELECTION

Part B: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

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Problem 1B. *Score:*

Find the unique continuous extension g of $f(x) = x^x$ (defined for positive reals x) to the half-open interval $\mathbb{R}^+ = [0, \infty)$ and determine all its fixed points there. Describe the limit x_∞ of the sequence $x_{n+1} = g(x_n)$ as a function of $x_0 \in \mathbb{R}^+$.

Solution: Since $x^x = e^{x \ln x} \to 1$ as $x \to 0$, $g(0) = 1$ extends f continuously from $(0, \infty)$ to the closure $\mathbb{R}^+$ and is therefore unique. Clearly $x = 1$ is a fixed point of g. Since $x^x > x^1 = x$ for $x > 1$, there are no fixed points in $(1, \infty)$. Let $h(x) = g(x)/x = x^{x-1} = e^{(x-1)\ln x}$ so $h(0) = \infty$, $h(1) = 1$, and $h'(x) = (\ln x + 1 - 1/x)x^{x-1} < 0$ for $0 < x < 1$. Thus $h(x) > 1$ for $0 < x < 1$ so there are no fixed points of g there. Hence the only fixed point of g on $\mathbb{R}^+$ is $x = 1$.

If $x_n \to x_\infty$ then because g is continuous we must have $x_\infty = g(x_\infty)$. Hence the only possible limits are $x_\infty = 1$ and $x_\infty = \infty$. Since $g'(x) = (1 + \ln x)x^x = 0$ iff $x = 1/e$, while $g'(1) = 1$, the contraction mapping principle does not apply to the limit at 1. However, x_n is a monotone nondecreasing sequence so $x_\infty = 1$ if $0 \le x_0 \le 1$ and $x_\infty = \infty$ if $x_0 > 1$.

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Problem 2B. *Score:*

Let $p(t) = t^n - p_1 t^{n-1} - p_2 t^{n-2} - \cdots - p_n$ where all the coefficients $p_j > 0$. Show that p has exactly one positive zero.

Solution: Clearly true for $n = 1$. For $n > 1$ the polynomial $p'(t)/n = z^{n-1} - p_1((n-1)/n)z^{n-2} - ... - p_{n-1}/n$ satisfies the same hypothesis at degree $n-1$. By induction on n, $p'(t)$ has exactly one positive zero t_1. Since $p'(0) < 0$ and $p'(+\infty) = +\infty$, $p'(t) < 0$ for $t < t_1$ and $p'(t) > 0$ for $t > t_1$. Since $p(0) < 0$, $p(t)$ starts negative at $t = 0$, decreases steadily to a negative value $p(t_1) < 0$ as t increases to t_1, and increases steadily thereafter. Since $p(+\infty) = +\infty$, $p(t)$ has a single zero between $t_1 > 0$ and $+\infty$ by the intermediate value theorem.

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Problem 3B. *Score:*

Suppose all the roots x_j of the degree-n polynomial

$$p(x) = x^n - a_1 x^{n-1} + a_2 x^{n-2} - \cdots$$

are real. Show that the largest root has absolute value at most

$$\sqrt{a_1^2 - 2a_2}.$$

and find all polynomials where equality holds.

Solution: This follows easily from

$$\sum_i x_i^2 = a_1^2 - 2a_2$$

The only polynomials where equality holds have at most one non-zero root so are of the form $x^n - a_1 x^{n-1}$.

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Problem 4B. *Score:*

Let f be a meromorphic function on $\mathbb{C}$ which is analytic in a neighborhood of 0. Let its Taylor series at 0 be

$$\sum_{n=0}^{\infty} a_n z^n$$

with $a_n \geq 0$ for all n. Suppose that f has a pole at z_0 for z_0 of absolute value $r > 0$, but that f has no pole at any z with z of absolute value $< r$. Prove that there is a pole at $z = r$.

Solution: Since f is analytic on $|z| < r$, the radius of convergence of the above series is at least r. In particular, it converges (to a real number) at all points of the interval $[0, r) \subseteq \mathbb{R}$, and to show that it has a pole at $z = r$ it suffices to show that

$$\lim_{x \to r^-} \sum_{n=0}^{\infty} a_n z^n = \infty \ . \qquad (*)$$

Now, by assumption, there is a $w \in \mathbb{C}$ such that f has a pole at w and $|w| = r$. Then $\lim_{z \to w} |f(z)| = \infty$, so for any $M > 0$ there is a point z such that $|z| < r$ and $|f(z)| > M$ (this holds because w is in the boundary of $\{z : |z| < r\}$).

Since $a_n \geq 0$ for all n, we then have

$$M < |f(z)| \leq \sum_{n=0}^{\infty} a_n |z|^n = f(|z|) \ ;$$

since M was arbitrary we conclude that (*) is true and therefore f has a pole at $z = r$.

Problem 5B. *Score:*

Show that there is a complex analytic function defined on the set $U = \{z \in \mathbb{C} : |z| > 4\}$ whose derivative is

$$\frac{z}{(z-1)(z-2)(z-3)} .$$

Is there a complex analytic function on U whose derivative is

$$\frac{z^2}{(z-1)(z-2)(z-3)} ?$$

Solution: For the first part of the question, for all $z_0 \in U$ let C be a path in U from 5 to z_0, and define

$$f(z_0) = \int_C g(z)\,dz , \qquad \text{where} \qquad g(z) = \frac{z}{(z-1)(z-2)(z-3)} .$$

To check that f is well defined, let C_1 and C_2 be two paths in U from 5 to z_0, and let C be the (closed) path obtained by concatenating C_1 and the reverse of C_2. Then C is a closed path; since it is contained in U, it has the same winding number n for each of the poles $z = 1, 2, 3$ of g. Therefore

$$\begin{aligned}
\int_{C_1} g(z)\,dz - \int_{C_2} g(z)\,dz &= \int_C g(z)\,dz \\
&= 2\pi i n \left(\operatorname{Res}_{z=1} g(z) + \operatorname{Res}_{z=2} g(z) + \operatorname{Res}_{z=3} g(z)\right) \\
&= 2\pi i n \left(\frac{1}{(-1)(-2)} + \frac{2}{(1)(-1)} + \frac{3}{(2)(1)}\right) \qquad (1) \\
&= 2\pi i n \left(\frac{1}{2} - 2 + \frac{3}{2}\right) \\
&= 0 .
\end{aligned}$$

Threfore the function f is well defined, and its derivative equals g by the Fundamental Theorem of Calculus.

For the second question, no such function f exists, because if we let

$$g(z) = \frac{z^2}{(z-1)(z-2)(z-3)}$$

and let C be the positively oriented contour $|z| = 5$, then

$$0 = \oint_C f'(z)\,dz = \oint_C g(z)\,dz = 2\pi i\left(\frac{1}{2} + \frac{4}{-1} + \frac{9}{2}\right) \neq 0\,, \tag{2}$$

a contradiction.

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Problem 6B. *Score:*

Let A be a nonsingular real $n \times n$ matrix. Show that there is an orthogonal matrix Q and an upper triangular matrix R with positive diagonal entries $r_{ii} > 0$ such that $A = QR$.

Solution: Since A is nonsingular, $A^T A > 0$ is symmetric positive definite and therefore enjoys a Cholesky factorization $A^T A = R^T R$ where R is the droid we are looking for. Moreover, $Q = AR^{-1}$ satisfies $QQ^T = AR^{-1}R^{-T}A^T = I$ so Q is orthogonal.

(Or use Gram-Schmidt orthogonalization on the columns of A.)

(Or use Householder reflections or Givens rotations.)

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Problem 7B. *Score:*

Suppose that A and B are positive definite matrices. Show that if $A \geq B$ (meaning $A - B$ is positive semi-definite) then $A^{-1} \leq B^{-1}$.

(Hint: first do the case of diagonal matrices.)

Solution: The case of diagonal matrices is trivial.

The matrices can be simulaneously diagonalized, and the result follows immediately.

Alternative solution: By the Fundamental Theorem of Calculus,

$$A^{-1} - B^{-1} = \int_0^1 \frac{d}{dt} (tA + (1-t)B)^{-1} \, dt.$$

Carrying out the differentiation,

$$A^{-1} - B^{-1} = \int_0^1 (tA + (1-t)B)^{-1} (A - B) (tA + (1-t)B)^{-1} \, dt.$$

Taking inner products on both sides,

$$< x, (A^{-1} - B^{-1})x >= - \int_0^1 < x(t), (A - B)x(t) > dt \leq 0,$$

where

$$x(t) = (tA + (1-t)B)^{-1} \, x.$$

Hence $A^{-1} \leq B^{-1}$.

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Problem 8B. *Score:*

Let $G_1, G_2, \ldots$ be an infinite sequence of groups, with $G_1 \le G_2 \le \ldots$ (here "$\le$" denotes subgroup). Let

$$G = \bigcup_i G_i \ .$$

Show, carefully and in detail, that there is a group structure on G such that $G_i \le G$ for all i.

Solution: First define a group operation on G as follows. Let $a, b \in G$. Then there are positive integers i and j such that $a \in G_i$ and $b \in G_j$. Taking n to be the larger of the two, we have $a, b \in G_n$, and define the product of a and b in G to be their product in G_n. This is well defined (i.e., independent of the choice of n) because if $a, b \in G_m$, then without loss of generality we may assume that $n \le m$, and then the product ab in G_n equals the product in G_m because G_n is a subgroup of G_m.

This operation is associative, because given any $a, b, c \in G$, we may choose $n \in \mathbb{Z}_{>0}$ such that $a, b, c \in G_n$; then $a(bc) = (ab)c$ in G because it's true in G_n.

Now let $e \in G$ be the identity element of G_1. Then e is the identity element of G_n for all n, so $ae = ea = a$ for all $a \in G$, because we may choose some n such that $a \in G_n$ and note that $ae = ea = a$ in G_n.

Finally, let $a \in G$, and choose n such that $a \in G_n$. Let b be the inverse of a in G_n. Then $ab = ba = e$ in G because it holds in G_n, so a has an inverse in G.

Thus G is a group in which all G_n are subgroups.

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Problem 9B. *Score:*

Let

$$\frac{1}{1-z-z^2} = \sum_{n=0}^{\infty} f_n z^n$$

be a Taylor series expansion convergent near $z = 0$.

(a) Find a and b such that $f_{n+1} = af_n + bf_{n-1}$ for $n \geq 1$.

(b) Show that $\gcd(f_{n+1}, f_n) = 1$.

Solution: (a) Cross multiply and shift the index of summation to get $f_{n+1} = f_n + f_{n-1}$.

(b)

$$\gcd(f_{n+1}, f_n) = \gcd(f_n + f_{n-1}, f_n) = \gcd(f_n, f_{n-1}) = \gcd(f_1, f_0) = 1$$

since $f_0 = f_1 = 1$.