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Problem 1A.

Evaluate the infinite product

\[ \prod_{n=3}^{\infty} \frac{n^2 - 4}{n^2 - 1} \]

**Solution:** The product of the first \( k \) terms is \( \frac{1!}{2!} \cdot \frac{k!(k+4)!}{(k+1)!(k+3)!} \), which tends to \( 1/4 \) as \( k \) tends to infinity.
Let $f: [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function, and let $g: [0,1] \to \mathbb{R}$ be the function

$$g(x) = \min_{0 \leq y \leq 1} f(x,y).$$

Show in detail that $g$ is continuous on $(0,1)$. (It is also continuous at the endpoints, but don’t worry about them.)

**Solution:** Since $[0,1] \times [0,1]$ is compact, $f$ is uniformly continuous. Let $\epsilon > 0$ be given. Then there is a $\delta > 0$ such that

$$|f(x_0, y_0) - f(x_1, y_1)| < \epsilon \quad \text{whenever } |(x_0, y_0) - (x_1, y_1)| < \delta.$$

We claim that

$$|g(x_0) - g(x_1)| < \epsilon \quad \text{whenever } |x_0 - x_1| < \delta.$$

Indeed, suppose $|x_0 - x_1| < \delta$. Pick $y_0$ such that $g(x_0) = f(x_0, y_0)$. Then

$$g(x_1) \leq f(x_1, y_0) < f(x_0, y_0) + \epsilon = g(x_0) + \epsilon,$$

and therefore $g(x_1) - g(x_0) < \epsilon$. A similar argument shows that $g(x_0) - g(x_1) < \epsilon$, so $|g(x_0) - g(x_1)| < \epsilon$, and we are done.

(This shows that $g$ is in fact uniformly continuous on $(0,1)$.)
Problem 3A.

Prove Taylor’s theorem with the remainder on the form of Peano: If a real-valued function on the number line has a well-defined $n$th derivative at $x = 0$, then the error of approximating the function near $x = 0$ by its degree-$n$ Taylor polynomial is $o(x^n)$. (A function $f$ is $o(x^n)$ at a point if $f/x^n$ tends to 0 at this point.) [Do not assume continuity or even existence of the $n$th derivative in any neighborhood of $x = 0$.]

Solution:

Let $f$ be the function, and $g(x) := f(x) - (f(0) + f'(0)x + f''(0)x^2/2! + \cdots + f^{(n)}(0)x^n/n!)$ . To prove that $g(x) = o(x^n)$, compute the limit of $g(x)/x^n$ using L’Hopital’s Rule. Namely, since $g(0) = g'(0) = \cdots = g^{(n-1)}(0) = 0$, the limit of the ratio as $x \to 0$ coincides with the limit of $f^{(n-1)}(0)/n!$ (the expression obtained by differentiating $n - 1$ times the numerator and denominator of $g(x)/x^n$), provided that the latter limit exists. By definition of the derivative, the limit of the left fraction equals $f^{(n)}(0)/n!$ which cancels with the right fraction. By L’Hopital’s Rule, the limit of $g(x)/x^n$ therefore exists and is equal to 0.
Suppose that $f$ is a complex polynomial all of whose roots have real part at most 0. Show that if $r > 0$ then $|f(r)| \geq |f(-r)|$. Give an example to show that can be false if the condition that $f$ is a polynomial is replaced by the condition that $f$ is entire.

**Solution:**

If $f = a(z - b)$ is linear the result is true because $|f(r)|$ is $|a|$ times the distance of $r$ from $b$, and $b$ is closer to $-r$ than to $r$ as the real part of $b$ is at most 0.

The result follows for any polynomial by writing it as a product of linear polynomials.

The function $e^{-z}$ is entire and has no zeros but does not satisfy the condition $|f(r)| \geq |f(-r)|$.
Problem 5A. 

Evaluate the contour integral

\[ \lim_{R \to +\infty} \int_{c-iR}^{c+iR} \frac{1}{z} \, dz \]

where \( c \) is a nonzero real number. (Warning: the answer depends on \( c \).)

Solution:

The indefinite integral of \( 1/z \) is a suitable choice of a branch of \( \log(z) \), so is given by \( \lim_{R \to +\infty} \log(c+iR) - \log(c-iR) \). If \( c \) is positive this is \( \pi i \), while if \( c \) is negative it is \( -\pi i \).
Problem 6A.

A plane passing through the origin in $\mathbb{R}^3$ intersects the ellipsoid $x^2/4 + y^2/9 + z^2/16 = 1$ by an ellipse. Determine how many such sections are circles and find their radiiues.

**Solution:** By the Cauchy interlacing theorem, the semi-axes $a \geq b > 0$ of the ellipse satisfy $2 \leq b \leq 3 \leq a \leq 4$. Namely, all points of the ellipse $x^2/4 + y^2/9 = 1$ in the plane $z = 0$ are at most distance 3 away from the origin (with the equality held only for the points on the $y$-axis), and any other plane passing through the origin intersects the plane $z = 0$ in a line, and thus contains such points. This shows that $b \leq 3$ (with the equality achieved only when the plane contains the $y$-axis). Similarly, $a \geq 3$ follows by intersecting with the plane $x = 0$. Thus, if the section is a circle ($a = b$), then the radius is 3, and the plane must contain the $y$-axis. Since a plane containing the $y$-axis is symmetric (together with the ellipsoid) about the plane $y = 0$, the corresponding ellipse has the $y$-axis as one of its principal ones, and so the other one lies in the plane $y = 0$. Obviously the ellipse $x^2/4 + z^2/16 = 1$ contains two pairs of centrally symmetric points at the distance 3 from the origin, and therefore the ellipsoid has 2 circular sections.
Problem 7A.  
Suppose that the square complex matrix $A$ is similar to $A^n$ for some integer $n > 1$. Prove all eigenvalues of $A$ are either zero or roots of unity.

**Solution:** If $a$ is an eigenvalue then $a^n$ is an eigenvalue of $A^n$ and therefore of $A$ because $A$ and $A^n$ are similar. Similarly $a, a^n, a^{n^2}, \ldots$ are all eigenvalues, so two of these must be equal as the number of eigenvalues is finite. So $a$ is a root of $x^{n^i} = x^{n^j}$ for some distinct integers $i, j$, whose only roots are 0 and roots of unity.
Problem 8A.

Let \( \alpha : G \rightarrow G_1 \) and \( \beta : G \rightarrow G_2 \) be group homomorphisms.

(a). Show that if \( \ker \alpha \subseteq \ker \beta \) and \( \alpha \) is surjective (onto) then there is a well-defined group homomorphism \( \phi : G_1 \rightarrow G_2 \) such that \( \beta = \phi \circ \alpha \).

(b). Show that if \( \ker \alpha \not\subseteq \ker \beta \) then there is no such homomorphism \( \phi \).

Solution: (a). For each \( y \in G_1 \) there is an \( x \in G \) such that \( \alpha(x) = y \). We then define \( \phi(y) = \beta(x) \). This is well defined because if \( \alpha(x') = \alpha(x) = y \), then \( \alpha(x'x^{-1}) = \alpha(x)\alpha(x')^{-1} = yy^{-1} = e \), so \( x'x^{-1} \in \ker \alpha \subseteq \ker \beta \); hence \( \beta(x'x^{-1}) = e \), so \( \beta(x')\beta(x)^{-1} = e \), and therefore \( \beta(x') = \beta(x) \).

It is a homomorphism because if \( \alpha(x_1) = y_1 \) and \( \alpha(x_2) = y_2 \), then \( \alpha(x_1x_2) = y_1y_2 \), and thus

\[
\phi(y_1y_2) = \beta(x_1x_2) = \beta(x_1)\beta(x_2) = \phi(y_1)\phi(y_2)
\]

for all \( y_1, y_2 \in G_1 \). Finally, \( \phi \) satisfies \( \beta = \phi \circ \alpha \) by construction.

(b). We prove the contrapositive.

Assume that \( \phi \) exists. Let \( x \in \ker \alpha \). Then \( \beta(x) = \phi(\alpha(x)) = \phi(e) = e \), so \( x \in \ker \beta \). Thus \( \ker \alpha \subseteq \ker \beta \), and we are done.
Recall that $S_5$ and $A_5$ are the symmetric group and alternating group on 5 letters, respectively.

Prove or give a counterexample: For every $\sigma \in A_5$ there is a $\tau \in S_5$ such that $\tau^2 = \sigma$.

**Solution:**
Any element $\sigma$ of odd order $2n + 1$ in a group is the square of $\tau = \sigma^{-n}$, so we can assume $\sigma$ has even order. The only elements $\sigma \in A_5$ of even order are of the forms

$$\sigma = (ab)(cd)$$

, which is the square of

$$\tau = (acb'd)$$
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Let \( f: (a, b] \rightarrow \mathbb{R} \) be a function. Assume that \( f \) is strictly increasing on \((a, b)\). (This means that \( f(x_1) < f(x_2) \) for all \( x_1 < x_2 \) in \((a, b)\).) Assume also that \( f \) is continuous from the left at \( b \). Then show that \( f \) is strictly increasing on \((a, b)\).

**Solution:** It remains only to show that \( f(x_1) < f(b) \) for all \( x_1 < b \) in \((a, b)\).

Let \( x_1 \) be as above, and pick \( x' \in (x_1, b) \). Since \( f(x) \geq f(x') \) for all \( x \in (x', b) \), we have \( \lim_{x\to b^-} f(x) \geq f(x') \), and therefore

\[
f(x_1) < f(x') \leq \lim_{x\to b^-} f(x) = f(b),
\]

as was to be shown.
Problem 2B.

By definition, the unit cube in the space $C[0,1]$ of continuous real-valued functions on the closed interval $[0,1]$ consists of those functions whose norm $\|f\| := \max_{0 \leq t \leq 1} |f(t)|$ doesn’t exceed 1. Find a linear map $\mathbb{R}^3 \to C[0,1]$ such that the inverse image of the unit cube is the unit ball in $\mathbb{R}^3$.

Solution:

Consider a map $\mathbb{R}^3 \to C[0,1]$, $(a, b, c) \mapsto af + bg + ch$, where $f, g, h$ are three continuous functions on $[0,1]$. These functions define a continuous parametric curve in $\mathbb{R}^3 : x = f(t), y = g(t), z = h(t)$. Our goal is to find a parametric curve, such that it lies in the “layer” $-1 \leq ax + by + cz \leq 1$ exactly when $a^2 + b^2 + c^2 \leq 1$. For this, it is sufficient that the convex hull of the parametric curve is exactly the unit ball $x^2 + y^2 + z^2 \leq 1$. To construct such a curve, take a Peano map, i.e. a continuous mapping of the interval $[0,1]$ surjectively onto the unit square $[0,1] \times [0,1]$, and compose it with any continuous map wrapping the square (e.g. cylindrically) around the unit sphere $x^2 + y^2 + z^2 = 1$. 
Show that the recursive sequence \( x_{n+1} = x_n/2 + 1/x_n \) with the initial value \( x_0 = 1.5 \) converges to \( \sqrt{2} \), and that \( x_{10} \) has at least 1000 correct decimal digits.

**Solution:** The fixed point of \( x/2 + 1/x \) is \( x = \sqrt{2} \) indeed. Moreover, due to the inequality between the arithmetic and geometric means, \( (x + 2/x)/2 \geq \sqrt{x \cdot 2/x} = \sqrt{2} \), i.e. the fixed point is also a critical point of the function. Therefore, if \( x_n = \sqrt{2} + \epsilon_n \) where the (necessarily positive) error \( \epsilon_n < 10^{-k} \), the next error \( \epsilon_{n+1} \) will be of the order \( \epsilon_n^2 < 10^{-2k} \). More precisely, from the geometric series expansion:

\[
\frac{1}{\sqrt{2} + \epsilon_n} = \frac{1}{\sqrt{2}} - \frac{\epsilon_n}{2} + \frac{\epsilon_n^2}{2(\sqrt{2} + \epsilon_n)},
\]

and therefore

\[
\epsilon_{n+1} = \frac{\sqrt{2} + \epsilon_n}{2} + \frac{1}{\sqrt{2} + \epsilon_n} - \sqrt{2} = \frac{\epsilon_n^2}{2(\sqrt{2} + \epsilon_n)}.
\]

Since \( \epsilon_0 < 1.5 - 1.41 < 1/10 \), it follows that \( \epsilon_{10} < 1/10^{2^{10}} = 1/10^{1024} \).
Problem 4B.

Show that there is a function holomorphic on the open unit disc that is a bijection from the open unit disc to the vertical strip $0 < \Re z < 1$, where $\Re z$ is the real part of $z$. You may not use the Riemann mapping theorem.

**Solution:** We give such a function as a composition of several functions as follows.

1. Shift the unit disc by 1 so that the boundary passes through 0.
2. Apply $1/z$ so that the image is a half plane.
3. Apply a linear transformation to make the half plane the half plane with positive real part.
4. Apply the log function to get the strip with imaginary part between $\pm \pi i/2$.
5. Apply a linear transformation to make this the strip with real part between 0 and 1.
Find the radius of convergence of the Taylor series of $1/(e^x + e^{-x})$ at the point $x = 1$.

**Solution:** The radius of convergence is the distance from 1 to the nearest singularity. The singularities are at the points where $e^{2x} = -1$, so $2x = \pm i\pi, \pm 3i\pi, \ldots$. The nearest singularities to 1 are therefore $\pm i\pi/2$ so the radius of convergence is $\sqrt{1 + \pi^2/4}$. 
Problem 6B.  

Prove that if two real square matrices are similar by conjugation by a complex matrix, then they are similar by conjugation by a real matrix.

**Solution:** This follows from the real version of the Jordan canonical form theorem, but can be also derived directly. Namely, suppose $B = CAC^{-1}$ where $C = D + iE$ is an invertible complex matrix and $A, B, D, E$ are real. Then $CA = BC$ and therefore $DA = BD$ and $EA = BE$. Even if neither $D$ nor $E$ is invertible, the polynomial $\det(D + tE)$ cannot be identically zero (since $\det(D + iE) \neq 0$), and so it should be non-zero for some real values of $t$. For such values, $B = (D + tE)A(D + tE)^{-1}$. 
Problem 7B.  

Given an example of two square complex matrices that have the same minimal polynomial and the same characteristic polynomial but are not similar.

Solution:
Take two 4 by 4 matrices in Jordan normal form with all eigenvalues 0, such that the first has Jordan blocks of size 2, 2 and the second has Jordan blocks of size 2, 1, 1.
Let $n$ be a positive integer, and $p$ any prime. Prove that over a finite field $F$ of $p^{\phi(n)}$ elements the polynomial $x^n - 1$ factors into linear factors. (Here $\phi$ is Euler’s totient function.)

**Solution:**

Let $n = p^r m$ where $m$ is coprime to $p$. Over $F$, we have $x^n - 1 = (x^m - 1)^{p^r}$ (since $(\frac{p}{k}) \equiv 0 \mod p$ for $k = 1, \ldots, p - 1$). Since the multiplicative group of a finite field is cyclic, to show that $x^m - 1$ has $m$ distinct roots in $F$, it suffices to check that the order $|F^\times| = p^{\phi(n)} - 1$ is divisible by $m$. But since $p$ and $m$ are coprime, we have $\phi(n) = \phi(p^r)\phi(m)$, and $p^{\phi(n)} \equiv (p^{\phi(m)})^{\phi(p^r)} \equiv 1 \mod m$ by Euler’s theorem.
Problem 9B.  

Give an example of a commutative ring which has an infinite descending chain of distinct prime ideals \( I_1 \supset \cdots \supset I_n \supset \cdots \) (Recall that an ideal \( I \) of a commutative ring \( R \) is called prime if \( R/I \) is an integral domain.)

Solution:

In the ring \( R \) of (say, complex coefficient) polynomials \( \mathbb{C}[x_1, x_2, \ldots, x_n, \ldots] \) in infinitely many variables, take \( I_n \) to be the ideal generated by all \( x_i \) with \( i > n \). The quotient \( R/I_n \) is isomorphic to the polynomial ring \( \mathbb{C}[x_1, \ldots, x_n] \) which has no zero divisors, implying that the ideal \( I_n \) is prime.