Department of Mathematics, University of California, Berkeley

YOUR 1 OR 2 DIGIT EXAM NUMBER ____

GRADUATE PRELIMINARY EXAMINATION, Part A — Fall Semester 2021

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1. Please write your 1- or 2-digit exam number on this cover sheet and on **all** problem sheets (even problems that you do not wish to be graded).
2. Indicate below which six problems you wish to have graded. **Cross out** solutions you may have begun for the problems that you have not selected.
3. Extra sheets should be stapled to the appropriate problem at the upper right corner. Do not put work for problem p on either side of the page for problem q if $p \neq q$.
4. No notes, books, calculators or electronic devices may be used during the exam.

PROBLEM SELECTION

Part A: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

GRADE COMPUTATION (for use by grader—do not write below)

1A. ______	1B. ______	Calculus
2A. ______	2B. ______	Real analysis
3A. ______	3B. ______	Real analysis
4A. ______	4B. ______	Complex analysis
5A. ______	5B. ______	Complex analysis
6A. ______	6B. ______	Linear algebra
7A. ______	7B. ______	Linear algebra
8A. ______	8B. ______	Abstract algebra
9A. ______	9B. ______	Abstract algebra

Part A Subtotal: ______ Part B Subtotal: ______ Grand Total: ______

YOUR EXAM NUMBER ____

Please cross out this problem if you do not wish it graded

Problem 1A. *Score:*

(a) Find a particular solution y_1 of

$$y' = y^2 - ty + 1$$

(b) Find the general solution. (The solution may be expressed using a definite integral.)

Solution:

YOUR EXAM NUMBER ____
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Problem 2A. *Score:*

Let

$$g(x) = \sin^2(\pi x) + \sin^2(\pi x)\cos^2(\pi x) + \sin^2(\pi x)\cos^4(\pi x) + \cdots \sin^2(\pi x)\cos^{2k}(\pi x) + \cdots$$

Evaluate the limit

$$G(x) = \lim_{n\to\infty} g(n!x).$$

Is G Riemann integrable?

Solution:

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Problem 3A. *Score:*

Let f be a differentiable function from $\mathbb{R}$ to $\mathbb{R}$. Suppose $f(0) = 0$ and $f'(t) > f(t)$ for $t \geq 0$. Show that $f(t) > 0$ for all $t > 0$.

Solution:

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Problem 4A. *Score:*

Let $P(z)$ be a monic complex polynomial with zeroes z_1 through z_n.

(a) Show that if all the zeros z_k have non-negative real part, then all the zeros of the derivative have non-negative real part.

(b) Let D be the convex hull of z_1 through z_n. Show that all the zeroes of the derivative $P'(z)$ lie in D.

Solution:

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Problem 5A. *Score:*

Let $u\colon \mathbb{R}^2 \to \mathbb{R}$ be the function

$$u(x, y) = 2x + 2y - x^3 + 3xy^2 \ .$$

(a) Show that u is harmonic.

(b) Use the Cauchy–Riemann equations to find $v\colon \mathbb{R}^2 \to \mathbb{R}$ such that the function $f\colon \mathbb{C} \to \mathbb{C}$ defined by

$$f(x + iy) = u(x, y) + iv(x, y)$$

is analytic.

Solution:

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Problem 6A. *Score:*

Solve the initial value problem

$$y' = Ay = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} y, \qquad y(0) = y_0 = [1, 0, 0]^T.$$

Solution:

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Problem 7A. *Score:*

Let the $n \times n$ complex matrix A have entries

$$A_{pq} = \exp(-2it_p t_q)$$

where t_p is real for $1 \leq p \leq n$ and $i = \sqrt{-1}$. Suppose A has the singular value decomposition

$$A = U\Sigma V^*$$

i.e. U and V are unitary and Σ is diagonal and nonnegative. Find a singular value decomposition of the matrix B with entries

$$B_{pq} = \exp(i(t_p - t_q)^2).$$

Solution:

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Problem 8A. *Score:*

Let G be a group of order 120, and let H be a subgroup of order 24. Assume that there is at least one left coset of H (other than H itself) which is equal to some right coset of H. Prove that H is a normal subgroup of G.

Solution:

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Problem 9A. *Score:*

Let p be an odd prime number and let $\mathbb{F}_p$ denote the field $\mathbb{Z}/p\mathbb{Z}$ with p elements.

(a). How many elements of $\mathbb{F}_p$ have square roots in $\mathbb{F}_p$?

(b). How many have cube roots in $\mathbb{F}_p$?

Solution:

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GRADUATE PRELIMINARY EXAMINATION, Part B — Fall Semester 2021

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PROBLEM SELECTION

Part B: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

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Problem 1B. *Score:*

Find the unique continuous extension g of $f(x) = x^x$ (defined for positive reals x) to the half-open interval $\mathbb{R}^+ = [0, \infty)$ and determine all its fixed points there. Describe the limit x_∞ of the sequence $x_{n+1} = g(x_n)$ as a function of $x_0 \in \mathbb{R}^+$.

Solution:

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Problem 2B. *Score:*

Let $p(t) = t^n - p_1 t^{n-1} - p_2 t^{n-2} - \cdots - p_n$ where all the coefficients $p_j > 0$. Show that p has exactly one positive zero.

Solution:

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Problem 3B. *Score:*

Suppose all the roots x_j of the degree-n polynomial

$$p(x) = x^n - a_1 x^{n-1} + a_2 x^{n-2} - \cdots$$

are real. Show that the largest root has absolute value at most

$$\sqrt{a_1^2 - 2a_2}.$$

and find all polynomials where equality holds.

Solution:

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Problem 4B. *Score:*

Let f be a meromorphic function on $\mathbb{C}$ which is analytic in a neighborhood of 0. Let its Taylor series at 0 be

$$\sum_{n=0}^{\infty} a_n z^n$$

with $a_n \geq 0$ for all n. Suppose that f has a pole at z_0 for z_0 of absolute value $r > 0$, but that f has no pole at any z with z of absolute value $< r$. Prove that there is a pole at $z = r$.

Solution:

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Problem 5B. *Score:*

Show that there is a complex analytic function defined on the set $U = \{z \in \mathbb{C} : |z| > 4\}$ whose derivative is

$$\frac{z}{(z-1)(z-2)(z-3)} \, .$$

Is there a complex analytic function on U whose derivative is

$$\frac{z^2}{(z-1)(z-2)(z-3)} \, ?$$

Solution:

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Problem 6B. *Score:*

Let A be a nonsingular real $n \times n$ matrix. Show that there is an orthogonal matrix Q and an upper triangular matrix R with positive diagonal entries $r_{ii} > 0$ such that $A = QR$.

Solution:

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Problem 7B. *Score:*

Suppose that A and B are positive definite matrices. Show that if $A \geq B$ (meaning $A - B$ is positive semi-definite) then $A^{-1} \leq B^{-1}$.

(Hint: first do the case of diagonal matrices.)

Solution:

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Problem 8B. *Score:*

Let $G_1, G_2, \ldots$ be an infinite sequence of groups, with $G_1 \leq G_2 \leq \ldots$ (here "$\leq$" denotes subgroup). Let

$$G = \bigcup_i G_i \; .$$

Show, carefully and in detail, that there is a group structure on G such that $G_i \leq G$ for all i.

Solution:

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Problem 9B. *Score:*

Let

$$\frac{1}{1-z-z^2} = \sum_{n=0}^{\infty} f_n z^n$$

be a Taylor series expansion convergent near $z = 0$.

(a) Find a and b such that $f_{n+1} = af_n + bf_{n-1}$ for $n \geq 1$.

(b) Show that $\gcd(f_{n+1}, f_n) = 1$.

Solution: