1. Answer six of the nine problems each day. You will get no extra credit for attempting more than 6 problems.

2. The exam lasts 3 hours each day. There is an extra half hour to give time to download it and to submit your solutions to gradescope.

3. Do not answer more than one question on any given piece of paper, as this will confuse the examiners.

4. The easiest way to submit your answers is by taking pictures of them with a phone and uploading them to gradescope.

5. The exam is open book: you may use notes or books or calculators or the internet, but should not consult anyone else.

6. In case of questions or unexpected problems during the prelim send email to the chair of the prelim committee. If a correction or announcement is needed during the exam it will be sent as an email to the dummy address you use on gradescope for the prelim, so please keep an eye on this during the prelim.
Problem 1A.  

Find the volume of the solid given by $x^2 + z^2 \leq 1$, $y^2 + z^2 \leq 1$. (Hint: $\int_{-1}^{1} (something) dz$.)

Solution:
Let \( \cdots \subset X_2 \subset X_1 \) be a nested sequence of closed nonempty connected subsets of a compact metric space \( X \). Prove that \( \bigcap_{i=1}^{\infty} X_i \) is nonempty and connected.

Solution:
Problem 3A.

Show that the series

$$\sum_{n=1}^{\infty} \sin \frac{x}{n^2}$$

converges uniformly on any bounded interval in $\mathbb{R}$.

Solution:
If $f$ is an analytic function from the unit disk into itself with $f(0) = 0$, prove that $|f'(0)| \leq 1$.

Solution:
Problem 5A.

Use residues to compute

\[ \int_0^\infty \frac{dx}{x^4 + 1}. \]

Solution:
Let $A$ be an $n$ by $n$ real matrix such that all entries not on the diagonal are positive, and the sum of the entries in each row is negative. Show that the determinant of $A$ is nonzero.

Solution:
Suppose $L$ is a linear operator acting on a nontrivial vector space $V$ over a field $K$. Suppose $P(x) \in K[x]$ is not identically zero and $P(L) = 0$. Show every eigenvalue of $L$ is a root of $P$. Show that if $P$ factors completely over $K$ then some roots of $P$ are eigenvalues of $L$.

Solution:
Find an irreducible polynomial over the integers with $2\cos(2\pi/7)$ as a root, and use this to show that it is not contained in any extension of the rational numbers of degree a power of 2.

Solution:
Problem 9A. 

For $G$ a finite group, $H$ a proper subgroup, show that $G \neq \bigcup \{gHg^{-1} ; g \in G\}$. 

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Problem 1B.

For which pairs of real numbers \((a, b)\) does the series \(\sum_{n=3}^{\infty} n^a (\log n)^b\) converge?

Solution:
Suppose that $X$ is a compact metric space. If $Y$ is another metric space (possibly noncompact), let $p : X \times Y \to Y$ be the map $p(x, y) = y$. Show that if $Z$ is a closed subset of $X \times Y$ then $p(Z)$ is closed in $Y$.

Solution:
Problem 3B.

Prove the existence of the limit

\[
\lim_{n \to \infty} \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n.
\]

Solution:
Problem 4B.

If $0 < r < 1$, find

$$\sum_{k=0}^{\infty} r^k \cos(k\theta).$$

Your final answer should not involve any complex numbers.

Solution:
Problem 5B.

If $a$ and $b$ are points in the open unit disk of the complex plane, show that there is a holomorphic map from the open unit disc onto itself with holomorphic inverse that takes $a$ to $b$.

Solution:
Problem 6B. 

For each of the following 4 statements, give either a counterexample or a reason why it is true.

(a) For every real matrix $A$ there is a real matrix $B$ with $B^{-1}AB$ diagonal.
(b) For every symmetric real matrix $A$ there is a real matrix $B$ with $B^{-1}AB$ diagonal.
(c) For every complex matrix $A$ there is a complex matrix $B$ with $B^{-1}AB$ diagonal.
(d) For every symmetric complex matrix $A$ there is a complex matrix $B$ with $B^{-1}AB$ diagonal.

Solution:
Problem 7B.

Find the eigenvalues of the $n \times n$ matrix with entries $a_{ij}$, where $a_{ij}$ is 1 if $i = j + 1$, $-1$ if $i = j - 1$, and 0 otherwise.

Solution:
Does there exists a homomorphism of commutative rings with unit from $\mathbb{Z}[x]/(x^2 + 3)$ to $\mathbb{Z}[x]/(x^2 - x + 1)$? Either exhibit such a homomorphism, or prove that none exists.

Solution:
Prove that the polynomial $x^4 + x + 2021$ is irreducible over $\mathbb{Q}$.

Solution: