Department of Mathematics, University of California, Berkeley

ONLINE GRADUATE PRELIMINARY EXAMINATION, Part A Spring Semester 2021

---

1. Answer six of the nine problems each day. You will get no extra credit for attempting more than 6 problems.
2. The exam lasts 3 hours each day. There is an extra half hour to give time to download it and to submit your solutions to gradescope.
3. Do not answer more than one question on any given piece of paper, as this will confuse the examiners.
4. The easiest way to submit your answers is by taking pictures of them with a phone and uploading them to gradescope.
5. The exam is open book: you may use notes or books or calculators or the internet, but should not consult anyone else.
6. In case of questions or unexpected problems during the prelim send email to the chair of the prelim committee If a correction or announcement is needed during the exam it will be sent as an email to the dummy address you use on gradescope for the prelim, so please keep an eye on this during the prelim.

YOUR EXAM NUMBER ____

Please cross out this problem if you do not wish it graded

Problem 1A. *Score:*

Find the volume of the solid given by $x^2 + z^2 \leq 1$, $y^2 + z^2 \leq 1$. (Hint: $\int_{-1}^{1}(something)dz$.)

Solution:

Problem 2A. *Score:*

Let

$$\ldots \subset X_2 \subset X_1$$

be a nested sequence of closed nonempty connected subsets of a compact metric space X. Prove that $\bigcap_{i=1}^{\infty} X_i$ is nonempty and connected.

Solution:

YOUR EXAM NUMBER ____

Please cross out this problem if you do not wish it graded

Problem 3A. *Score:*

Show that the series

$$\sum_{n=1}^{\infty} \sin \frac{x}{n^2}$$

converges uniformly on any bounded interval in $\mathbb{R}$.

Solution:

YOUR EXAM NUMBER ___

Please cross out this problem if you do not wish it graded

Problem 4A. *Score:*

If f is an analytic function from the unit disk into itself with $f(0) = 0$, prove that $|f'(0)| \leq 1$.

Solution:

YOUR EXAM NUMBER ___
Please cross out this problem if you do not wish it graded

Problem 5A. *Score:*

Use residues to compute

$$\int_0^\infty \frac{dx}{x^4+1}.$$

Solution:

YOUR EXAM NUMBER ____
Please cross out this problem if you do not wish it graded

Problem 6A. *Score:*

Let A be an n by n real matrix such that all entries not on the diagonal are positive, and the sum of the entries in each row is negative. Show that the determinant of A is nonzero.

Solution:

YOUR EXAM NUMBER ____

Please cross out this problem if you do not wish it graded

Problem 7A. *Score:*

Suppose L is a linear operator acting on a nontrivial vector space V over a field K. Suppose $P(x) \in K[x]$ is not identically zero and $P(L) = 0$. Show every eigenvalue of L is a root of P. Show that if P factors completely over K then some roots of P are eigenvalues of L.

Solution:

YOUR EXAM NUMBER ___

Please cross out this problem if you do not wish it graded

Problem 8A. *Score:*

Find an irreducible polynomial over the integers with $2\cos(2\pi/7)$ as a root, and use this to show that it is not contained in any extension of the rational numbers of degree a power of 2.

Solution:

Problem 9A. *Score:*

For G a finite group, H a proper subgroup, show that $G \neq \bigcup\{gHg^{-1}; g \in G\}$.

Solution:

Department of Mathematics, University of California, Berkeley

ONLINE GRADUATE PRELIMINARY EXAMINATION, Part B   Spring Semester 2021

---

1. Answer six of the nine problems each day. You will get no extra credit for attempting more than 6 problems.
2. The exam lasts 3 hours each day. There is an extra half hour to give time to download it and to submit your solutions to gradescope.
3. Do not answer more than one question on any given piece of paper, as this will confuse the examiners.
4. The easiest way to submit your answers is by taking pictures of them with a phone and uploading them to gradescope.
5. The exam is open book: you may use notes or books or calculators or the internet, but should not consult anyone else.
6. In case of questions or unexpected problems during the prelim send email to the chair of the prelim committee If a correction or announcement is needed during the exam it will be sent as an email to the dummy address you use on gradescope for the prelim, so please keep an eye on this during the prelim.

YOUR EXAM NUMBER ____
Please cross out this problem if you do not wish it graded

Problem 1B. *Score:*

For which pairs of real numbers (a, b) does the series $\sum_{n=3}^{\infty} n^a (\log n)^b$ converge?

Solution:

YOUR EXAM NUMBER ____
Please cross out this problem if you do not wish it graded

Problem 2B. *Score:*

Suppose that X is a compact metric space. If Y is another metric space (possibly noncompact), let $p : X \times Y \to Y$ be the map $p(x, y) = y$. Show that if Z is a closed subset of $X \times Y$ then $p(Z)$ is closed in Y.

Solution:

YOUR EXAM NUMBER ___

Please cross out this problem if you do not wish it graded

Problem 3B. *Score:*

Prove the existence of the limit

$$\lim_{n\to\infty} \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n.$$

Solution:

YOUR EXAM NUMBER ___
Please cross out this problem if you do not wish it graded

Problem 4B. *Score:*

If $0 < r < 1$, find

$$\sum_{k=0}^{\infty} r^k \cos(k\theta).$$

Your final answer should not involve any complex numbers.

Solution:

YOUR EXAM NUMBER ___

Please cross out this problem if you do not wish it graded

Problem 5B. *Score:*

If a and b are points in the open unit disk of the complex plane, show that there is a holomorphic map from the open unit disc onto itself with holomorphic inverse that takes a to b.

Solution:

YOUR EXAM NUMBER ____

Please cross out this problem if you do not wish it graded

Problem 6B. *Score:*

For each of the following 4 statements, give either a counterexample or a reason why it is true.

(a) For every real matrix A there is a real matrix B with $B^{-1}AB$ diagonal.

(b) For every symmetric real matrix A there is a real matrix B with $B^{-1}AB$ diagonal.

(c) For every complex matrix A there is a complex matrix B with $B^{-1}AB$ diagonal.

(d) For every symmetric complex matrix A there is a complex matrix B with $B^{-1}AB$ diagonal.

Solution:

YOUR EXAM NUMBER ____

Please cross out this problem if you do not wish it graded

Problem 7B. *Score:*

Find the eigenvalues of the $n \times n$ matrix with entries a_{ij}, where a_{ij} is 1 if $i = j + 1$, -1 if $i = j - 1$, and 0 otherwise.

Solution:

YOUR EXAM NUMBER ____
Please cross out this problem if you do not wish it graded

Problem 8B. *Score:*

Does there exists a homomorphism of commutative rings with unit from $\mathbb{Z}[x]/(x^2+3)$ to $\mathbb{Z}[x]/(x^2-x+1)$? Either exhibit such a homomorphism, or prove that none exists.

Solution:

YOUR EXAM NUMBER ___

Please cross out this problem if you do not wish it graded

Problem 9B. *Score:*

Prove that the polynomial $x^4 + x + 2021$ is irreducible over $\mathbb{Q}$.

Solution: