1. Please write your 1- or 2-digit exam number on this cover sheet and on all problem sheets (even problems that you do not wish to be graded).

2. Indicate below which six problems you wish to have graded. Cross out solutions you may have begun for the problems that you have not selected.

3. Extra sheets should be stapled to the appropriate problem at the upper right corner. Do not put work for problem $p$ on either side of the page for problem $q$ if $p \neq q$.

4. No notes, books, calculators or electronic devices may be used during the exam.

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PROBLEM SELECTION

Part A: List the six problems you have chosen:

_____ , _____ , _____ , _____ , _____ , _____

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GRADE COMPUTATION (for use by grader—do not write below)

1A. _____ 1B. _____ Calculus
2A. _____ 2B. _____ Real analysis
3A. _____ 3B. _____ Real analysis
4A. _____ 4B. _____ Complex analysis
5A. _____ 5B. _____ Complex analysis
6A. _____ 6B. _____ Linear algebra
7A. _____ 7B. _____ Linear algebra
8A. _____ 8B. _____ Abstract algebra
9A. _____ 9B. _____ Abstract algebra

Part A Subtotal: _____ Part B Subtotal: _____ Grand Total: _____
Problem 1A.

For which pairs of real numbers $r, s$ does the following double integral converge?

$$\int_{x=0}^{\infty} \int_{y=0}^{\infty} \frac{dx \, dy}{1 + x^r y^s}$$

Solution:

If $r = 0$ the integral obviously diverges. Otherwise changing $x$ to $x/y^{s/r}$ (to simplify the denominator) the integral becomes

$$\int_{x=0}^{\infty} \int_{y=0}^{\infty} \frac{y^{-s/r} \, dx \, dy}{1 + x^r}$$

which diverges for all $r \neq 0, s$ because the integral over $y$ is always infinite.
Problem 2A.

Let \( p(t) = t^n - p_1 t^{n-1} - p_2 t^{n-2} - \cdots - p_n \) where all the coefficients \( p_j > 0 \). Show that \( p \) has exactly one positive zero.

**Solution:**

This follows from the Descartes rules of signs.

It can also be proved as follows. Clearly true for \( n = 1 \). For \( n > 1 \) the polynomial \( p'(t) = z^{-1} - p_1((n-1)/n) z^{-2} - \cdots - p_{n-1}/n \) satisfies the same hypothesis at degree \( n - 1 \). By induction on \( n \), \( p'(t) \) has exactly one positive zero \( t_1 \). Since \( p'(0) < 0 \) and \( p'(+\infty) = +\infty \), \( p'(t) < 0 \) for \( t < t_1 \) and \( p'(t) > 0 \) for \( t > t_1 \). Since \( p(0) < 0 \), \( p(t) \) starts negative at \( t = 0 \), decreases steadily to a negative value \( p(t_1) < 0 \) as \( t \) increases to \( t_1 \), and increases steadily thereafter. Since \( p(+\infty) = +\infty \), \( p(t) \) has a single zero between \( t_1 > 0 \) and \(+\infty\) by the intermediate value theorem.
For each \( n = 1, 2, 3, \ldots \) let \( f_n : \mathbb{R} \to \mathbb{R} \) be a differentiable function such that \( |f'_n(x)| \leq 1 \) for all \( x \in \mathbb{R} \). Assume also that the sequence \( \{f_n\} \) converges pointwise to a function \( g : \mathbb{R} \to \mathbb{R} \).

Prove that \( g \) is continuous.

**Solution:** Fix \( x_0 \in \mathbb{R} \). Let \( \epsilon > 0 \) be given. Then there is an integer \( N > 0 \) such that \( |f_n(x_0) - g(x_0)| < \epsilon \) for all \( n \geq N \).

Let \( \delta = \epsilon \), and suppose that \( |x - x_0| < \delta \). By pointwise continuity, there is an \( n \geq N \) (depending on \( x \)) such that \( |f_n(x) - g(x)| < \epsilon \). By a straightforward application of the Mean Value Theorem, we have \( |f_n(x) - f_n(x_0)| < \epsilon \). Combining these inequalities, we then have

\[
|g(x) - g(x_0)| \leq |f_n(x) - g(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - g(x_0)| < 3\epsilon ,
\]

which implies that \( g \) is continuous at \( x_0 \). Since \( x_0 \) was arbitrary, \( g \) is continuous.
The Bernoulli numbers $B_n$ are defined by $\sum_n B_n z^n/n! = z/(e^z - 1)$. Find the largest integer $N$ such that the sequence $\frac{B_n N^n}{n!}$ is bounded.

**Solution:** The radius of convergence of the power series above is $2\pi$ as this is the distance to the nearest singularities (at $\pm 2\pi i$). So the integer $N$ is the largest integer less than $2\pi$, which is 6.
Is there an entire function (holomorphic on the complex plane) taking the values 
\[1, 1, 1/2, 1/2, 1/3, 1/3, \ldots\]
at the points \(1, 1/2, 1/3, \ldots\) respectively? Give an example or prove that no such function exists.

**Solution:** If \(f\) was such a function then \(f(z) - 2z\) would vanish at \(1/2, 1/4, \ldots\) so would have to be zero everywhere as this sequence of points has a limit point. This contradicts the fact that \(f(1) = 1\), so no such function exists.
Let $A$ be an $n \times n$ complex invertible matrix and define $X_{k+1} = 2X_k - X_kAX_k$ for integer $k \geq 0$. Show that $X_k$ has a limit $X$ as $k \to \infty$ if $\|I - AX_0\| < 1$ and identify the limit (It may be useful to consider $R_k = I - AX_k$.)

**Solution:** The residual $R_k = I - AX_k$ satisfies $R_{k+1} = -R_k^2$, so

$$\|R_{k+1}\| \leq \|R_k\|^2$$

and therefore $R_k$ goes to zero as $k \to \infty$ if $\|R_0\| < 1$. Since multiplication is continuous, the error $E_k = A^{-1}R_k = A^{-1} - X_k \to 0$ as well.
Problem 7A.  

Let $A$ be a nonsingular real $n \times n$ matrix. Show that there is an orthogonal matrix $Q$ and an upper triangular matrix $R$ with positive diagonal entries $r_{ii} > 0$ such that $A = QR$.

Solution: Since $A$ is nonsingular, $A^T A > 0$ is symmetric positive definite and therefore enjoys a Cholesky factorization $A^T A = R^T R$ where $R$ is the upper triangular matrix we are looking for. Moreover, $Q = AR^{-1}$ satisfies $QQ^T = AR^{-1} R^{-T} A^T = I$ so $Q$ is orthogonal.

(Or use Gram-Schmidt orthogonalization on the columns of $A$.)

(Or use Householder reflections or Givens rotations.)
Let $R = k[x]/(x^n)$ be a ring, where $k$ is a field and $n$ is a positive integer. Find all ideals of $R$.

**Solution:** Any ideal $I$ of $R$ corresponds to an ideal $J$ of $k[x]$ containing $(x^n)$. As $k[x]$ is a PID, $J$ is generated by a single element $\alpha \in k[x]$. As $J$ contains $x^n$, we have $\alpha | x^n$. Then we get $\alpha = x^i$ with $i = 0, \ldots, n$. Therefore, a complete set of ideals of $R$ is

$$x^i R, \quad i = 0, \ldots, n.$$
Let $k$ be a field. Denote by $R$ the subring of the polynomial ring $k[x]$ consisting of all $f(x) \in k[x]$ such that $f(0) = f(1)$. Find a finite set of generators for $R$ as an algebra over $k$. (This means that every element of $R$ can be written as a polynomial in these elements with coefficients in $k$.)

**Solution:** For $f \in R$, the polynomial $f(x) - f(0)$ takes value 0 at $x = 0, 1$. Then $f(x) - f(0)$ is divisible by $x(x - 1)$. Thus

$$R = \{ c + (x^2 - x)g(x) : c \in k, g(x) \in k[x] \}.$$  

We claim that $R$ is generated by $x^2 - x$ and $(x^2 - x)x$. It suffices to prove that these two elements generate $(x^2 - x)x^n$ for any $n \geq 0$. This follows from induction and the identity

$$(x^2 - x)x^n - (x^2 - x)x^{n-1} = (x^2 - x) \cdot (x^2 - x)x^{n-2}.$$
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PROBLEM SELECTION

Part B: List the six problems you have chosen:

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Problem 1B.  

Find the unique continuous extension $g$ of $f(x) = x^x$ to the half-open interval $R^+ = [0, \infty)$ and determine all its fixed points there. Describe the limit $x_\infty$ of the sequence $x_{n+1} = g(x_n)$ as a function of $x_0 \in R^+$.

Solution: Since $x^x = e^{x \ln x} \to 1$ as $x \to 0$, $g(0) = 1$ extends $f$ continuously from $(0, \infty)$ to the closure $R^+$ and is therefore unique. Clearly $x = 1$ is a fixed point of $g$. Since $x^x > x^1 = x$ for $x > 1$, there are no fixed points in $(1, \infty)$. Let $h(x) = g(x)/x = x^{x-1} = e^{(x-1) \ln x}$ so $h(0) = \infty$, $h(1) = 1$, and $h'(x) = (\ln x + 1 - 1/x)x^{x-1} < 0$ for $0 < x < 1$. Thus $h(x) > 1$ for $0 < x < 1$ so there are no fixed points of $g$ there. Hence the only fixed point of $g$ on $R^+$ is $x = 1$.

If $x_n \to x_\infty$ then because $g$ is continuous we must have $x_\infty = g(x_\infty)$. Hence the only possible limits are $x_\infty = 1$ and $x_\infty = \infty$. Since $g'(x) = (1 + \ln x)x^x = 0$ if $x = 1/e$, while $g'(1) = 1$, the contraction mapping principle does not apply to the limit at 1. However, $x_n$ is a monotone nondecreasing sequence so $x_\infty = 1$ if $0 \leq x_0 \leq 1$ and $x_\infty = \infty$ if $x_0 > 1$. 

Score:
Problem 2B.  

Let \( J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-int+ix \sin t} dt \) for \( n \in \mathbb{Z} \). Show \( J_n(x+y) = \sum_{m \in \mathbb{Z}} J_{n-m}(x) J_m(y) \).

**Solution:** Since \( J_n(x+y) \) is the \( n \)th complex exponential Fourier coefficient of the smooth function \( f_{x+y}(t) = \exp(i(x+y)\sin t) = f_x(t)f_y(t) \), the absolutely convergent Fourier series

\[
    f_{x+y}(t) = \sum_{n \in \mathbb{Z}} J_n(x+y) \exp(int) = f_x(t)f_y(t) = \sum_{n \in \mathbb{Z}} J_n(x) \exp(int) \sum_{m \in \mathbb{Z}} J_m(y) \exp(imt)
\]

can be rearranged into

\[
    \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} J_n(x) \exp(imt) J_m(y) \exp(i(n-m)t) = \sum_{n \in \mathbb{Z}} \exp(int) \sum_{m \in \mathbb{Z}} J_m(x) J_{n-m}(y).
\]

Since Fourier series coefficients are unique, equating terms gives

\[
    J_n(x+y) = \sum_{m \in \mathbb{Z}} J_{n-m}(x) J_m(y)
\]

upon interchanging \( x \) and \( y \).
Suppose $n \geq 1$, and there are $n$ real numbers $w_j$ and $n$ distinct real numbers $t_j$ such that

$$\int_{-1}^{1} f(t)dt = \sum_{j=1}^{n} w_j f(t_j)$$

for all polynomials $f$ of degree at most $2n-1$. Show that $w_j > 0$ and $|t_j| \leq 1$ for $j = 1, \ldots, n$.

**Solution:** Let $\omega_i(t) = \prod_{j \neq i} (t - t_j)^2$ so that $\omega_i(t_j)$ is positive iff $i = j$ and 0 otherwise. Since $\omega_i$ is a polynomial of degree $2n - 2$, we have by linearity

$$\int_{-1}^{1} \omega_i(t)dt = w_i \omega_i(t_i) > 0$$

since the integrand is non-negative and vanishes on a set of measure zero. Hence $w_i > 0$ for $i = 1, \ldots, n$.

Since $t\omega_i(t)$ is a polynomial of degree $2n - 1$, we have by linearity

$$\int_{-1}^{1} t\omega_i(t)dt = w_i t_i \omega_i(t_i) = t_i \int_{-1}^{1} \omega_i(t)dt.$$

Hence

$$t_i = \frac{\int_{-1}^{1} t\omega_i(t)dt}{\int_{-1}^{1} \omega_i(t)dt}$$

is the mean value of $t$ over $[-1, 1]$ with a non-negative weight function of total mass 1. Since any mean value of a function lies between its minimum and maximum values we have shown $|t_i| \leq 1$ for $i = 1, \ldots, n$. 
Evaluate

\[ \int_{0}^{2\pi} e^{e^{i\theta}} d\theta . \]

**Solution:** We write it as a path integral of a holomorphic function along the path \( \gamma(\theta) = e^{i\theta} \) (which goes around the circle \( |z| = 1 \) once in the positive direction). If \( z = e^{i\theta} \) then \( dz = ie^{i\theta} d\theta \), so \( d\theta = (1/iz)dz \) and we have

\[
\int_{0}^{2\pi} e^{e^{i\theta}} d\theta = \oint_{|z|=1} e^{z} \frac{dz}{iz} = \frac{1}{i} \oint_{|z|=1} \frac{e^{z}}{z} dz = \frac{1}{i} \cdot 2\pi i \text{Res}_{z=0} \frac{e^{z}}{z} = 2\pi e^{0} = 2\pi .
\]
Problem 5B.

Prove the Fundamental Theorem of Algebra: Every nonconstant polynomial with complex coefficients has a complex root.

Solution: Let \( f \) be such a polynomial. We may assume that \( f \) is monic:

\[
f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0.
\]

Assume, by way of contradiction, that \( n > 0 \) and that \( f \in \mathbb{C}[z] \) has no complex roots. Then \( 1/f \) is an entire function.

Pick \( R \geq 1 \) such that \( R \geq 2n|a_i| \) for all \( i = 0, \ldots, n-1 \). Then

\[
\frac{|a_{n-1}z^{n-1} + \cdots + a_0|}{|z|^n} \leq \frac{|a_{n-1}|}{|z|} + \cdots + \frac{|a_0|}{|z|^n} \\
\leq \frac{|a_{n-1}|}{R} + \cdots + \frac{|a_0|}{R^n} \\
\leq \frac{|a_{n-1}| + \cdots + |a_0|}{R} \\
\leq n \cdot \frac{1}{2n} \leq \frac{1}{2}
\]

for all \( z \in \mathbb{C} \) with \( |z| \geq R \). Therefore, for all such \( z \),

\[
|f(z)| \geq |z^n| - \frac{1}{2} |z^n| \geq \frac{R^n}{2}.
\]

It follows that \( |1/f| \) is bounded on the set \( |z| > R \), and it is also bounded on \( |z| \leq R \) because it is a continuous function and \( |z| \leq R \) is compact.

Therefore \( 1/f \) is a bounded entire function, which by Liouville’s Theorem must be constant. This contradicts the assumption that the polynomial is nonconstant.
Problem 6B.

Put \( u = (1, 2, 3) \) and let

\[
A = u^T u = \begin{bmatrix}
1 & 2 & 3 \\
2 & 4 & 6 \\
3 & 6 & 9
\end{bmatrix}
\]

Compute \( A^n \) for all integers \( n \geq 1 \) and find the largest eigenvalue of \( A \).

**Solution:** Since \( A = u^T u \) where \( u = (1, 2, 3) \), we have \( A^n = u^T (uu^T)^{n-1} u = 14^{n-1} A \) for \( n \geq 1 \). Since \( A^2 = 14A \) the largest eigenvalue is 14.
Problem 7B. 

How many 2 dimensional subspaces are there in the $n$-dimensional vector space over a finite field with $q$ elements?

**Solution:** \( \frac{(q^n-1)(q^n-q)}{(q^2-1)(q^2-q)} \). The numerator is the number of ways to choose 2 linearly independent vectors in an $n$-dimensional space, and the denominator is the number in the 2-dimensional subspace.
Let $p$ be a prime number. View $\mathbb{Z}/p\mathbb{Z}$ as a subgroup of $(\mathbb{Z}/p^5\mathbb{Z}) \oplus (\mathbb{Z}/p^5\mathbb{Z})$ by the homomorphism sending 1 to $(p^4, p^4)$. Prove that the quotient of $(\mathbb{Z}/p^5\mathbb{Z}) \oplus (\mathbb{Z}/p^5\mathbb{Z})$ by $\mathbb{Z}/p\mathbb{Z}$ is isomorphic to $(\mathbb{Z}/p^a\mathbb{Z}) \oplus (\mathbb{Z}/p^b\mathbb{Z})$ for some integers $a, b$ with $a \leq b$. Find $a$ and $b$.

**Solution:** Consider the isomorphism

$$(\mathbb{Z}/p^5\mathbb{Z}) \oplus (\mathbb{Z}/p^5\mathbb{Z}) \rightarrow (\mathbb{Z}/p^5\mathbb{Z}) \oplus (\mathbb{Z}/p^5\mathbb{Z}), \quad (x, y) \mapsto (x, y - x).$$

Under this isomorphism, $\mathbb{Z}/p\mathbb{Z}$ becomes a subgroup of $(\mathbb{Z}/p^5\mathbb{Z}) \oplus (\mathbb{Z}/p^5\mathbb{Z})$ by the homomorphism sending 1 to $(p^4, 0)$. Then it is easy to see that the quotient is isomorphic to $(\mathbb{Z}/p^a\mathbb{Z}) \oplus (\mathbb{Z}/p^b\mathbb{Z})$.

Note that the existence of $a, b$ also follows from the structure theorem of finitely generated abelian groups, since the quotient is generated by two elements.
Let \( f : \mathbb{Z}^n \to \mathbb{Z}^n \) be a homomorphism. Prove that \( f \) is an isomorphism if and only if the homomorphism \( f_p : (\mathbb{Z}/p\mathbb{Z})^n \to (\mathbb{Z}/p\mathbb{Z})^n \) induced from \( f \) by reduction modulo \( p \) is an isomorphism for all prime numbers \( p \).

**Solution:** Let \( M \) be the \( n \times n \) matrix with integer coefficients representing \( f \). Denote by \( M_p \) the reduction of \( M \) modulo \( p \), which represents \( f_p \). Then \( f \) is an isomorphism if and only if \( \det(M) = \pm 1 \), which holds if and only if \( \det(M_p) \neq 0 \) for all \( p \).